

## RIGHT ORDERS IN FULL LINEAR RINGS<sup>(1)</sup>

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**ABSTRACT.** In this paper a right order  $R$  in an infinite dimensional full linear ring is characterized as a ring satisfying:

- (1)  $R$  is meet-irreducible (with zero right singular ideal) and contains uniform right ideals;
- (2) the closed right ideals of  $R$  are right annihilator ideals, and each such right ideal is essentially finitely generated;
- (3)  $R$  possesses a reducing pair (i.e. a pair  $(\beta_1, \beta_2)$  of elements for which  $\beta_1 R, \beta_2 R$  and  $\beta_1^r + \beta_2^r$  are large right ideals of  $R$ );
- (4) for each  $a \in R$  with  $a^f = 0$ ,  $aR$  contains a regular element of  $R$ .

A second characterization of  $R$  is also given. This is in terms of the right annihilator ideals of  $R$  which have the same (uniform) dimension as  $R_R$ .

The problem of characterizing right orders in (infinite dimensional) full linear rings was posed by Carl Faith. The Goldie theorems settled the finite dimensional case.

**Introduction.** One way to generalize the Goldie theorems [5] on right orders in simple and semisimple Artinian rings is by studying right orders in rings which have zero Jacobson radical but for which the finiteness requirement is relaxed, for example, regular right self-injective rings. The study of right orders in a full linear ring  $\text{Hom}_D(V, V)$ , where  $V$  is a right vector space over a division ring  $D$ , falls into this category. The problem of characterizing such rings was posed by Faith [3, p. 129, problem 12]. Since a regular right self-injective ring whose socle is large as a right ideal is a direct product of full linear rings, it is clear how one could approach the study of right orders in these rings from the study of right orders in full linear rings.

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In this paper we present two characterizations of right orders in (infinite dimensional) full linear rings, thereby solving Faith's problem. The first characterization depends on the more general concept of a right quasi-order. A subring  $R$  of a ring  $S$  with identity is a *right quasi-order* in  $S$  if right regular elements of  $R$  are right invertible in  $S$  and for each  $x \in S$  there exists a right regular  $c \in R$  such that  $xc \in R$  (so that  $x = bc^{-1}$  where  $b = xc \in R$  and  $c^{-1}$  is any right inverse of  $c$  in  $S$ ). Theorem 5.1 characterizes a right quasi-order  $R$  in an infinite dimensional full linear ring as a ring satisfying:

- (1)  $R$  is meet-irreducible (with zero right singular ideal) and contains uniform right ideals;
- (2) the closed right ideals of  $R$  are right annihilator ideals, and each such right ideal is essentially finitely generated;
- (3)  $R$  possesses a reducing pair of elements.

A right order  $R$  in an infinite dimensional full linear ring is then characterized in Theorem 7.1 as a right quasi-order which satisfies:

- (4) For each  $a \in R$  with  $a^l = 0$ ,  $aR$  contains a regular element of  $R$ .

This is our first characterization. The second characterization is given in Theorem 7.2, and this is in terms of the right annihilator ideals of  $R$  which have the same dimension as  $R_R$ .

The term *reducing pair* in (3) refers to a pair  $(\beta_1, \beta_2)$  of elements of  $R$  for which  $\beta_1 R, \beta_2 R$  and  $\beta_1' + \beta_2'$  are large right ideals of  $R$ . A reducing pair enables one to "reduce" the number of elements required to generate a right ideal which is to be essential in a given right ideal. For our purposes, this notion appears to be the infinite dimensional analogue of primeness. We shall have more to say about this in §4.

A brief outline of the paper is as follows. §1 is devoted to preliminaries. In §2 our interest centres on R. E. Johnson's [9] internal conditions for a ring  $R$  to have a full linear ring  $Q$  as its maximal right quotient ring. Here we also show that  $R$  must be meet-irreducible (that is  $A \cap B = 0$  implies  $A = 0$  or  $B = 0$ , for two-sided ideals  $A$  and  $B$ ) if  $R$  is a right quasi-order in  $Q$ . However, a right quasi-order need not be prime unless  $Q$  is of finite dimension, and a right order need not be prime unless  $Q$  is of countable dimension. In §3 we consider a finiteness condition (A) for rings, viz., each closed right ideal is essentially finitely generated. Cateforis introduced and characterized this condition in [2]. Right quasi-orders satisfy (A). §4 is devoted to reducing pairs. The characterization of right quasi-orders appears in §5. In §6 we briefly relate the condition (A) on a ring  $R$ , having a full linear ring  $Q$  as a right quotient ring, to  $Q$  being a left-flat epimorphic extension of  $R$  in the sense of Findlay [4]. §7 contains the two characterizations of right orders, and the final section, §8, is concerned with when the idealizer of a right ideal of  $Q$  is a right quasi-order in  $Q$ .

1. **Preliminaries.** A ring is assumed to be associative but need not possess an identity. The unqualified word “ideal” refers to a two-sided ideal.

We denote the right singular ideal of a ring  $R$  by  $Z_r(R)$ . A *closed right ideal* of  $R$  is a right ideal  $I$  such that  $I_R$  has no proper essential extensions within  $R_R$ . We denote the set of all closed right ideals of  $R$  by  $L_r(R)$ . If  $Z_r(R) = 0$ , then  $L_r(R)$  forms a lattice. If  $R$  is a subring of a ring  $S$ , then  $S$  is a *right quotient ring* of  $R$  if  $S_R$  is an essential extension of  $R_R$ . This notion is due to R. E. Johnson [7]. The results we need concerning it are well known and can be found in Johnson [9]. Here we simply state the results for the reader’s convenience.

1.1. **PROPOSITION.** *Suppose  $S$  is a right quotient ring of  $R$  and  $Z_r(S) = 0$ . Then  $Z_r(R) = 0$  and  $L_r(S)$  is isomorphic to  $L_r(R)$  via the contraction map  $A \rightarrow A \cap R, A \in L_r(S)$ .*

A ring  $R$  with  $Z_r(R) = 0$  has a unique (to within isomorphism over  $R$ ) maximal right quotient ring (MRQ ring). The MRQ ring of  $R$  is a regular right self-injective ring.

For a subset  $X$  of a ring  $R$ , we denote the right annihilator of  $X$  in  $R$  by  $r(X, R)$ , or simply by  $X^r$  if  $R$  is understood. Similarly,  $l(X, R)$ , or  $X^l$ , denotes the left annihilator of  $X$  in  $R$ .

1.2. **PROPOSITION.** *Suppose  $S$  is a right quotient ring of a ring  $R$  with  $Z_r(R) = 0$ . For any submodules  $A$  and  $B$  of  $S_R$ , with  $A \subseteq B$ , if  $A_R$  is essential in  $B_R$  then  $l(A, S) = l(B, S)$ .*

1.3. **PROPOSITION (UTUMI [19, THEOREM 2.2]).** *For a ring  $R$  with  $Z_r(R) = 0$ , the closed right ideals of  $R$  are right annihilator ideals if and only if the MRQ ring of  $R$  is left intrinsic over  $R$ , that is, nonzero left ideals of the MRQ ring have nonzero intersection with  $R$ . In this case, for right ideals  $I$  and  $J$  of  $R$  with  $I \subseteq J, I^l = J^l$  implies  $I_R$  is essential in  $J_R$ .*

A module  $M_R$  over a ring  $R$  is called *uniform* if it is nonzero and each of its nonzero submodules is essential in  $M_R$ . If  $M_R$  is a module containing uniform submodules, the *uniform dimension* of  $M_R$ , which we denote by  $\dim M_R$ , is defined to be the cardinal number of any maximal family of independent uniform submodules of  $M_R$ . See, for example, Miyashita [12].

By a *(left) full linear ring* we shall mean a ring which is isomorphic to the ring of all linear transformations of some right vector space over a division ring, with transformations written on the *left* of vectors. Equivalently, a (left) full linear ring is a prime right self-injective ring with nonzero socle.

NOTATION. In the sequel, the sole use of the letter  $Q$  will be to denote a (left) full

linear ring. For a right ideal  $I$  of  $Q$ , we abbreviate  $\dim I_Q$  to  $\dim I$ .

We recall some basic properties of  $Q$ .

1.4. PROPOSITION. (i)  $Q$  is a regular ring.

(ii) The closed right ideals of  $Q$  are of the form  $eQ$ , where  $e$  is an idempotent.

(iii) For idempotents  $e$  and  $f$  of  $Q$ ,  $eQ \cong fQ$  (as right  $Q$ -modules) if and only if  $\dim eQ = \dim fQ$ .

(iv) Provided  $Q$  is not a division ring,  $Q$  is generated (as a ring) by its idempotents. If  $\dim Q$  is infinite, then  $Q$  is generated by the idempotents  $e$  for which  $eQ \cong (1 - e)Q$ .

PROOF. (i), (ii) and (iii) follow from the injectivity of  $Q_Q$ . (Note that the uniform submodules of  $Q_Q$  are the minimal right ideals of  $Q$ .) If  $1 < \dim Q < \infty$ , then  $Q$  is generated by its primitive idempotents. See, for example, [15, Lemma 2.2]. If  $\dim Q$  is infinite, then  $Q$  is isomorphic to the ring of all  $2 \times 2$  matrices over itself, and it is a straightforward procedure showing that the ring  $T$  of all  $2 \times 2$  matrices over a ring with identity is generated by the idempotents  $e$  for which  $eT \cong (1 - e)T$ . This establishes (iv).

A *right* (resp. *left*) *regular* element of a ring  $R$  is an element  $c$  for which  $l(c, R) = 0$  (resp.  $r(c, R) = 0$ ). A *regular* element is one which is both right and left regular. Now suppose  $R$  is a subring of a ring  $S$  with identity. We say that  $R$  is a *right quasi-order* in  $S$  (or  $S$  is a *quasi-classical* right quotient ring of  $R$ ) if right regular elements of  $R$  have right inverses in  $S$  and for each  $x \in S$  there exists a right regular  $c \in R$  such that  $xc \in R$  (so that  $x = bc^{-1}$  where  $b = xc \in R$  and  $c^{-1}$  is any right inverse of  $c$  in  $S$ ).  $R$  is a *right order* in  $S$  (or  $S$  is a *classical* right quotient ring of  $R$ ) if regular elements of  $R$  have two-sided inverses in  $S$  and the elements of  $S$  can be expressed in the form  $bc^{-1}$ , where  $b \in R$  and  $c$  is a regular element of  $R$ . Clearly, quasi-classical and classical right quotient rings are right quotient rings in the sense of our earlier definition.

It is well known that a ring  $R$  (which contains regular elements) has a classical right quotient ring if and only if  $R$  satisfies the right Ore condition: for each  $b$ , regular  $c \in R$ , there exist  $b_1$ , regular  $c_1 \in R$  such that  $bc_1 = cb_1$ . It is also true that  $R$  (if it contains right regular elements) has a quasi-classical right quotient ring if and only if the above condition holds with "regular" replaced by "right regular". An easy modification of the proof given in Lambek [11, p. 109] reveals this.

As with a classical right quotient ring, a quasi-classical right quotient ring of  $R$  is unique to within isomorphism over  $R$ . For  $R$  with  $Z_r(R) = 0$ , we have:

1.5. PROPOSITION. Let  $T$  be the MRQ ring of  $R$ . Let

$$S = \{x \in T: xc \in R \text{ for some right regular } c \in R\}.$$

If  $R$  possesses a quasi-classical right quotient ring  $S_1$ , then  $S$  is a subring of  $T$  and  $S_1$  is isomorphic to  $S$  via an isomorphism which extends the identity mapping of  $R$ .

PROOF. Quite straightforward.

In general, the existence of a classical right quotient ring does not imply the existence of a quasi-classical right quotient ring, or vice versa. If  $R$  is a prime right Goldie ring, but not a right Ore domain, then  $R$  possesses a classical right quotient ring but not a quasi-classical right quotient ring unless  $R$  is also left Goldie (see remark after Theorem 2.2). If  $\dim Q$  is infinite and  $e$  is an idempotent of  $Q$  such that  $eQ \cong (1 - e)Q$ , then the ring  $R = eQ + Q(1 - e)$  is a right quasi-order in  $Q$  but  $R$  does not possess a classical right quotient ring (see Theorem 2.8). Also, a ring  $R$  can be its own classical right quotient ring (i.e. regular elements of  $R$  are already units) but yet  $R$  possesses a proper quasi-classical right quotient ring. An example of such a ring is  $R = eQ + Q(1 - e)$  where  $\dim Q$  is infinite and  $e$  is a nonzero idempotent of  $Q$  with  $\dim eQ$  finite. Here,  $R$  is a right quasi-order in  $Q$ .

The following proposition plays a key role in later work. Its proof requires only a slight modification of the proof of [15, Lemma 2.3].

1.6. PROPOSITION. *Let  $S$  be a ring with identity,  $E$  a set of generators for  $S$  (as a ring), and  $R$  a subring of  $S$  containing right invertible elements of  $S$ . Let  $U = \{c \in R : c \text{ right invertible in } S\}$  and  $T = \{x \in S : xc \in R \text{ for some } c \in U\}$ . If  $E \subseteq T$  and  $c^{-1}Ec \subseteq E$  for all  $c \in U$ , then  $S = T$  ( $c^{-1}$  denotes any right inverse of  $c$  in  $S$ ).*

REMARK. The particular applications we have in mind occur when  $S = Q$  and  $E$  is the set of all idempotents of  $Q$  or, if  $\dim Q$  is infinite, the idempotents  $e$  for which  $eQ \cong (1 - e)Q$  (see 1.4(iv)). However, it is conceivable that one may wish to take for  $E$  the set of all nilpotent elements of  $Q$  (if  $\dim Q \neq 1$ ) or the set of all units of  $Q$  (if  $\dim Q < \infty$ ).

2. Irreducible rings. Our approach to characterizing right quasi-orders in full linear rings stems from the following observation: a ring  $R$  is a right quasi-order in a full linear ring if and only if  $R$  has  $Z_r(R) = 0$  and the MRQ ring of  $R$  is a full linear ring and a quasi-classical right quotient ring of  $R$ . Thus the problem of characterizing such rings falls into two parts:

- (1) Characterize rings  $R$  which have  $Z_r(R) = 0$  and whose MRQ ring is a full linear ring.
- (2) Given a ring  $R$  which has a full linear ring  $Q$  as a right quotient ring, find necessary and sufficient conditions for  $R$  to be a right quasi-order in  $Q$ .

Part (2) is essentially the subject of §§3 and 4. As regards (1), a solution is given in Johnson [9] (see Hutchinson [6] for an alternative solution). Johnson in [8] calls a ring  $R$  (right) *irreducible* if  $Z_r(R) = 0$  and for each nonzero ideal  $A$  of  $R$ ,  $A \cap A^l = 0$  implies  $A^l = 0$ . This is equivalent to saying that  $Z_r(R) = 0$  and the MRQ ring of  $R$  is a prime ring.

**2.1. THEOREM (JOHNSON).** *Let  $R$  be a ring with  $Z_r(R) = 0$ . Then the MRQ ring of  $R$  is a full linear ring if and only if  $R$  is irreducible and contains uniform right ideals.*

**PROOF.** For the "if" part, see Johnson [9, Theorem 3.1]. Now suppose the MRQ ring of  $R$ ,  $S$  say, is a full linear ring. By 1.1,  $R$  contains minimal closed right ideals and hence uniform right ideals. To show  $R$  is irreducible, it suffices to show that  $A^l \cap A = 0$  implies  $A^l = 0$  for any nonzero two-sided ideal  $A$  of  $R$ ,  $A$  closed as a right ideal. Now  $A^l \cap A = 0$  implies  $A^l$  is a unique complement of  $A$  in  $L_r(R)$ . By 1.1,  $A = eS \cap R$  for some idempotent  $e$  of  $S$ , and  $eS$  has a unique complement in  $L_r(S)$ . Since  $S$  is a prime ring, this implies  $e = 1$ . Thus  $A^l = 0$ .

It is shown in O'Meara [14, Lemma 2.3] that the requirement that regular elements of a right order  $R$  in  $Q$  be units of  $Q$  forces  $Q$  to be left intrinsic over  $R$  if  $\dim Q$  is infinite. A similar proof establishes the following (for  $\dim Q$  now arbitrary).

**2.2. THEOREM.** *Suppose  $R$  is a subring of  $Q$  such that for each  $x \in Q$  there exists  $c \in R$  with  $c$  right invertible in  $Q$  and  $xc \in R$ . Then right regular elements of  $R$  are right invertible in  $Q$  if and only if  $Q$  is left intrinsic over  $R$ , equivalently, the closed right ideals of  $R$  are right annihilator ideals.*

**REMARK.** From the above theorem and O'Meara [15, Theorem 3.3] it follows that the right quasi-orders in a finite dimensional full linear ring  $Q$ ,  $Q$  not a division ring, are precisely the right orders which are also left orders. However, as our terminology would suggest, if  $\dim Q$  is infinite then by [14, Lemma 2.3] any right order in  $Q$  is a right quasi-order, but the converse is not true.

Let us call a ring  $R$  *meet-irreducible* (m-irreducible) if  $Z_r(R) = 0$ , and, for any ideals  $A$  and  $B$  of  $R$ ,  $A \cap B = 0$  implies  $A = 0$  or  $B = 0$ . Clearly, m-irreducible implies irreducible. (The converse is false.) A right quasi-order  $R$  in  $Q$  is m-irreducible. This is partly a consequence of our requirement that right regular elements of  $R$  be right invertible in  $Q$ . However, for a right order in  $Q$ , the corresponding requirement on regular elements need not enter into the proof. It is therefore desirable that we give separate proofs to indicate this.

**2.3. LEMMA.** *Let  $R$  be an irreducible ring and let  $T$  be the MRQ ring of  $R$ . If  $R$  is not m-irreducible, then there exist nonzero orthogonal idempotents  $e, f$ ,*

and  $g$  of  $T$  such that  $1 = e + f + g$  and  $R \subseteq eTe + fTf + Tg$ .

PROOF. Suppose  $R$  is not  $m$ -irreducible. Then  $A \cap B = 0$  for some nonzero (two-sided) ideals  $A$  and  $B$  of  $R$ , and we can suppose that  $A$  and  $B$  are closed as right ideals of  $R$ . By 1.1, there exist orthogonal idempotents  $e$  and  $f$  of  $T$  such that  $A = eT \cap R$  and  $B = fT \cap R$ . Now, by 1.2,  $(1 - e)ReA = 0$  implies  $(1 - e)Re = 0$ . Similarly,  $(1 - f)Tf = 0$ . Let  $g = 1 - e - f$ . Then  $R \subseteq eTe + fTf + Tg$ . Clearly  $e \neq 0, f \neq 0$ . If  $g = 0$ , then  $T$  would be a right quotient ring of  $eTe + fTf$ , which is impossible because  $T$  is prime and  $eTe + fTf$  is not irreducible. Thus  $g \neq 0$ . This establishes the lemma.

2.4. LEMMA. *If  $R$  is an irreducible ring and the closed right ideals of  $R$  are right annihilator ideals, then  $R$  is  $m$ -irreducible.*

PROOF. Suppose  $R$  is not  $m$ -irreducible. Let  $T$  be the MRQ ring of  $R$ . Then, by 2.3, there exist nonzero orthogonal idempotents  $e, f$ , and  $g$  of  $T$  such that  $1 = e + f + g$  and  $R \subseteq eTe + fTf + Tg$ . Let  $S = eTe + fTf + Tg$ . Since  $R$  is irreducible,  $T$  is a prime ring and we can choose  $b \in eTf, b \neq 0$ . Since  $T$  is a regular ring,  $bT = e_1T$  for some idempotent  $e_1 \in eTe$ . Let  $h = e_1 + b$ . Then  $Th \cap S = 0$ . For let  $a \in Th \cap S$ . Then  $a = ah \in S \Rightarrow eahf = 0 \Rightarrow eab = 0 \Rightarrow eae_1 = 0$  (since  $e_1T = bT$ )  $\Rightarrow ea = eah = eae_1 + eab = 0 \Rightarrow ae = eae = 0$ . Thus  $ae = 0$ . Hence  $a = ah = aeh = 0$ , i.e.  $a = 0$ . Thus  $Th \cap S = 0$ . But, by 1.3,  $T$  is left intrinsic over  $R$  and hence over  $S$ . This contradiction shows that  $R$  is  $m$ -irreducible.

2.5. THEOREM. *A right quasi-order in  $Q$  is an  $m$ -irreducible ring.*

PROOF. This follows immediately from 2.1, 2.2 and 2.4.

2.6. THEOREM. *If  $R$  is a right order in  $Q$ , then  $R$  is  $m$ -irreducible.*

PROOF. Suppose  $R$  is not  $m$ -irreducible. By 2.1,  $R$  is irreducible and hence, by 2.3, we have  $R \subseteq eQe + fQf + Qg$  where  $e, f$  and  $g$  are nonzero orthogonal idempotents of  $Q$  with  $e + f + g = 1$ . Without loss of generality, we can suppose  $\dim eQ \leq \dim fQ$ . By 1.4(iii),  $eQ$  can be mapped isomorphically into  $fQ$  and hence there exists  $a \in fQe$  such that  $a^r \cap eQ = 0$ . Since  $R$  is a right order in  $Q$ , so is  $S = eQe + fQf + Qg$ . Hence there exists  $c \in S$  such that  $c$  is a unit of  $Q$  and  $ac \in S$ . Now  $ac \in S \Rightarrow face = 0 \Rightarrow aece = 0 \Rightarrow ece = 0$  (since  $a^r \cap eQ = 0$ ). But  $c \in S \Rightarrow ece = ce \Rightarrow ce = 0$  which contradicts  $c$  being a unit of  $Q$ . Thus  $R$  is  $m$ -irreducible.

A prime ring with zero right singular ideal is clearly  $m$ -irreducible, and the converse holds if and only if the ring contains no nonzero nilpotent ideals (Johnson [8, Theorem 2.1]). The following theorem is proved in O'Meara [14, Corollary 3.3].

2.7. THEOREM. *All right orders in  $Q$  are prime rings if and only if  $\dim Q$  is finite or countably infinite.*

If  $\dim Q$  is finite, then a right quasi-order in  $Q$  (being, in particular, a right order in  $Q$ ) is a prime ring. However, as our next result shows, if  $\dim Q$  is infinite there always exist nonprime right quasi-orders in  $Q$ .

2.8. THEOREM. *Let  $e$  and  $f$  be nonzero orthogonal idempotents of  $Q$ . Let  $R = eQ + Qf$ . Then:*

(1)  *$R$  is a right order in  $Q$  if and only if  $\dim eQ > \dim(1 - e)Q$  and  $\dim fQ = \dim(1 - e)Q \geq \aleph_0$ .*

(2)  *$R$  is a right quasi-order in  $Q$  if and only if  $\dim fQ = \dim(1 - e)Q \geq \aleph_0$ .*

PROOF. (1) See O'Meara [14, Theorem 2.1].

(2) Assume that  $\dim fQ = \dim(1 - e)Q \geq \aleph_0$ . If  $\dim eQ > \dim fQ$ , then, by (1),  $R$  is a right order in  $Q$ , in particular,  $R$  is a right quasi-order in  $Q$ . So we can assume that  $\dim eQ \leq \dim fQ$ . Then, as  $\dim Q$  is infinite, we have  $\dim Q = \dim fQ$ . Hence, by 1.4(iii),  $fQ \cong Q$  (as right  $Q$ -modules) and so there exists  $c \in Qf$  such that  $cQ = Q$ , i.e.  $c$  is right invertible in  $Q$ . Clearly,  $xc \in Qf \subseteq R$  for all  $x \in Q$ . Since  $Q$  is left intrinsic over  $R$ , for  $a \in R$  with  $l(a, R) = 0$  we have  $l(a, Q) = 0$  and so  $a$  is right invertible in  $Q$ . This establishes the "if" part of (2). The proof of the "only if" part follows from the same argument as in the corresponding proof of (1), as the reader can readily verify. This completes the proof of 2.8.

3. Finiteness conditions. If  $\dim Q$  is finite, right orders in  $Q$  are distinguishable from the other subrings of  $Q$  over which  $Q$  is a right quotient ring, by their possession of a property, primeness, which is not a finiteness property. However, if  $\dim Q$  is infinite and  $Q$  is a right quotient ring of  $R$ , then it seems unlikely that there is a reasonable nonfiniteness condition which, if satisfied by  $R$ , will ensure that  $R$  is a right order in  $Q$  (after all, there are regular rings with identity which have  $Q$  as a proper right quotient ring). The relation " $Q$  is a right quotient ring of  $R$ " is rather weak in this case. Specifically, the map which sends a right ideal  $B$  of  $Q$  to  $B \cap R$  need not be one-to-one, and the map which sends a right ideal  $I$  of  $R$  to  $IQ$  need not preserve intersections. We point out now that requiring both of these properties for a ring  $R$  with identity is equivalent, in the language of Findlay [4], to requiring that  $Q$  be the left-flat epimorphic hull of  $R$ .

In this section we consider a finiteness condition (A) of Cateforis [2] for rings. For  $R$  as above, (A) is equivalent to the map  $B \rightarrow B \cap R$  being one-to-one. Following Cateforis [1], we say that a module  $M_R$  is *essentially finitely generated* if there exist  $x_1, \dots, x_n \in M$  such that  $x_1R + \dots + x_nR$  is an essential

submodule of  $M_R$ . We say that  $M_R$  is *essentially principally generated* if there exists  $x \in M$  such that  $xR$  is an essential submodule of  $M_R$ .

DEFINITION. A ring  $R$  is said to satisfy condition (A) (resp.  $(A_1)$ ) if  $Z_r(R) = 0$  and each closed right ideal of  $R$  is essentially finitely generated (resp. essentially principally generated) as a right  $R$ -module.

3.1. PROPOSITION (CATEFORIS). *Let  $R$  have  $Z_r(R) = 0$  and let  $T$  be the MRQ ring of  $R$ . Then  $R$  satisfies (A) if and only if  $(B \cap R)T = B$  for all right ideals  $B$  of  $T$ , equivalently, the map  $B \rightarrow B \cap R$  is one-to-one.*

PROOF. For the “only if” part, see Cateforis [2, Lemma 2.4]. For the “if” part, let  $K$  be a closed right ideal of  $R$ . Then  $K = eT \cap R$  for some  $e \in T$ , and so  $KT = eT$  implies  $eT = x_1T + \dots + x_nT$  for some  $x_1, \dots, x_n \in K$ . It follows that  $x_1R + \dots + x_nR$  is essential in  $K_R$ . Thus  $R$  satisfies (A).

3.2. PROPOSITION. *Suppose  $R$  has  $Z_r(R) = 0$  and closed right ideals of  $R$  are right annihilator ideals. Then the following are equivalent:*

- (i)  $R$  satisfies  $(A_1)$ .
- (ii) The closed right ideals of  $R$  take the form  $a^r, a \in R$ .
- (iii) The left annihilator ideals of  $R$  take the form  $a^l, a \in R$ .

PROOF. (i) implies (ii). Let  $K$  be a closed right ideal of  $R$ . Then there exists  $a \in K$  such that  $aR$  is essential in  $K_R$ . By 1.2,  $a^l = K^l$ . Hence  $K = K^{lr} = a^{lr}$ . This gives (ii).

(ii) implies (iii). Let  $J$  be a left annihilator ideal of  $R$ . By (ii),  $J^r = a^{lr}$  for some  $a \in R$ . Hence  $J = J^{rl} = a^{lr} = a^l$ . This gives (iii).

(iii) implies (i). Let  $K$  be a closed right ideal of  $R$ . Then  $K = K^{lr}$ . Now (iii) implies there exists  $a \in R$  such that  $K^l = a^l = (aR)^l$ . Since  $a \in a^{lr} = K^{lr}$ , we have  $a \in K$ . By 1.3,  $(aR)^l = K^l$  implies  $aR$  is essential in  $K_R$ . This gives (i) and completes the proof of the proposition.

REMARK. In terms of the MRQ ring  $T$  of a ring  $R$  with  $Z_r(R) = 0$ ,  $R$  satisfies  $(A_1)$  if and only if for each  $x \in T$  there exists  $a \in R$  such that  $aT = xT$ .

3.3. PROPOSITION. *If  $R$  is a right quasi-order in  $Q$ , then  $R$  satisfies  $(A_1)$ , and hence (ii) and (iii) of 3.2.*

PROOF. Let  $K$  be a closed right ideal of  $R$ . Then  $K = eQ \cap R$  for some idempotent  $e$  of  $Q$  (1.1 and 1.4(ii)). Choose a right regular  $c \in R$  such that  $ec \in R$ . Let  $a = ec$ . Then  $a \in K$ , and  $aQ = eQ$  implies  $aR$  is essential in  $K_R$ . Thus  $R$  satisfies  $(A_1)$  and hence, by 2.2,  $R$  satisfies (ii) and (iii) of 3.2.

REMARKS. (1) Suppose  $Q$  is a right quotient ring of  $R$ . If  $\dim Q$  is finite,  $R$  satisfies (A) because every right ideal of  $R$  is essentially finitely generated. However,  $R$  need not satisfy  $(A_1)$ . In fact, it is not difficult to show that

$R$  satisfies  $(A_1)$  if and only if  $Q = \{ac^{-1} : a \in R, \text{ unit } c \in Q\}$ . If  $\dim Q$  is infinite, then  $R$  need not satisfy  $(A)$ . For example, take  $R = \text{socle } Q$ .

(2) A prime regular ring which satisfies  $(A)$  is necessarily right self-injective. This is not true for semiprime regular rings.

(3) Johnson in [10] studied rings which have finite right dimension and satisfy  $(A_1)$  (he calls these  $I_r$ -rings).

4. Reducing pairs. Let us suppose for the moment that  $\dim Q$  is finite, say  $\dim Q = n$ , and that  $Q$  is a right quotient ring of a prime ring  $R$ . What does primeness of  $R$  enable us to do insofar as showing  $R$  is a right order in  $Q$ ? It is this. Let  $I$  be any right ideal of  $R$ . Then  $I$  can be essentially generated by a finite number of elements. Primeness of  $R$  enables us to reduce the number of generators to one. Let us briefly recall how this can be done. We first find non-zero elements  $\beta_1, \dots, \beta_n \in R$  such that each  $Q\beta_i$  is a minimal left ideal of  $Q$  and  $Q = Q\beta_1 \dot{+} \dots \dot{+} Q\beta_n$ , where  $\dot{+}$  indicates a direct sum (see, for example, O'Meara [15, Lemma 2.4]). Let  $a_1, \dots, a_k \in I$  be such that each  $a_i R$  is a uniform right ideal and  $a_1 R \dot{+} \dots \dot{+} a_k R$  is essential in  $I_R$  (note  $k \leq n$ ). By primeness of  $R$  we can choose  $y_i \in R$ , for  $i = 1, \dots, k$ , such that  $a_i y_i \beta_i \neq 0$ . Let  $a = a_1 y_1 \beta_1 + \dots + a_k y_k \beta_k$ . Then  $a \in I$ , and  $aQ = a_1 Q + \dots + a_k Q$  implies  $aR$  is essential in  $a_1 R + \dots + a_k R$  and hence in  $I_R$ . The point to notice is that  $\beta_1, \dots, \beta_n$ , once chosen, work for each right ideal  $I$  of  $R$ .

Now suppose  $\dim Q$  is infinite. We make the following observation: if  $\beta_1, \beta_2 \in Q$  are such that  $\beta_1 Q = \beta_2 Q = Q$  and  $Q\beta_1 \cap Q\beta_2 = 0$ , then for any  $x_1, x_2 \in Q$  we have  $x_1 Q + x_2 Q = (x_1 \beta_1 + x_2 \beta_2)Q$ . For want of a better name, we shall refer to  $\beta_1$  and  $\beta_2$  as a (right) reducing pair for  $Q$ . However, because our main concern is with rings  $R$  which have  $Q$  as a right quotient ring, we prefer the following definition which is easily seen to be equivalent to the one above for  $Q$ .

DEFINITION. Let  $R$  be a ring. A pair  $(\beta_1, \beta_2)$  of elements of  $R$  is called a (right) reducing pair for  $R$  if  $\beta_1 R, \beta_2 R$  and  $\beta_1' + \beta_2'$  are large right ideals of  $R$ .

4.1. PROPOSITION. Suppose a ring  $R$  has  $Z_r(R) = 0$  and contains a reducing pair  $(\beta_1, \beta_2)$  of elements. Then, for any  $a_1, a_2 \in R$ ,  $(a_1 \beta_1 + a_2 \beta_2)R$  is essential in  $a_1 R + a_2 R$ . Consequently, any right ideal of  $R$  which is essentially finitely generated is essentially principally generated. In particular, conditions  $(A)$  and  $(A_1)$  are equivalent for  $R$ .

PROOF. Let  $T$  be the MRQ ring of  $R$  (or any regular right quotient ring of  $R$ ). Since  $(\beta_1, \beta_2)$  is a reducing pair for  $R$ , we have  $\beta_1 T = \beta_2 T = T$  and  $r(\beta_1, T) + r(\beta_2, T) = T$ , the latter implying  $T\beta_1 \cap T\beta_2 = 0$ . Let  $e_1$  and  $e_2$  be orthogonal idempotents of  $T$  such that  $T\beta_1 = Te_1$  and  $T\beta_2 = Te_2$ . Then there

exist  $\alpha_1 \in e_1T$  and  $\alpha_2 \in e_2T$  such that

$$\alpha_1\beta_1 = e_1, \quad \beta_1\alpha_1 = 1, \quad \alpha_2\beta_2 = e_2 \quad \text{and} \quad \beta_2\alpha_2 = 1.$$

Thus for any  $t_1, t_2 \in T$  we have  $a_1t_1 + a_2t_2 = (a_1\beta_1 + a_2\beta_2)(\alpha_1t_1 + \alpha_2t_2)$ , which implies  $a_1T + a_2T = (a_1\beta_1 + a_2\beta_2)T$ . It follows that  $(a_1\beta_1 + a_2\beta_2)R$  is essential in  $a_1R + a_2R$ . The remaining statements of the proposition are obvious.

REMARKS. (1) If  $\dim Q$  is finite, then clearly  $Q$  cannot possess a reducing pair of elements. However, if  $\dim Q$  is infinite there is an abundance of reducing pairs. For if  $e_1$  and  $e_2$  are any orthogonal idempotents of  $Q$  with  $e_1Q \cong Q$  and  $e_2Q \cong Q$ , then there exist  $\beta_1 \in Qe_1, \beta_2 \in Qe_2$  such that  $\beta_1Q = \beta_2Q = Q$ , and thus  $(\beta_1, \beta_2)$  is a reducing pair for  $Q$ .

(2) If  $Q$  is a right quotient ring of  $R$ , then, for  $\beta_1, \beta_2 \in R, (\beta_1, \beta_2)$  is a reducing pair for  $R$  if and only if  $(\beta_1, \beta_2)$  is a reducing pair for  $Q$ .

(3) For a general ring  $R$  with  $Z_r(R) = 0$ , the image of  $R$  under the homomorphism  $r \rightarrow (\beta_1r, \beta_2r)$  from  $R_R$  into  $(R \oplus R)_R$ , induced by a pair  $(\beta_1, \beta_2) \in R \oplus R$ , is a large submodule of  $(R \oplus R)_R$  if and only if  $(\beta_1, \beta_2)$  is a reducing pair for  $R$ . Thus if  $R$  has an identity,  $R$  possesses a reducing pair if and only if  $R$  can be homomorphically embedded as a large submodule of  $(R \oplus R)_R$ .

4.2. PROPOSITION. *A right quasi-order  $R$  in  $Q$  possesses a reducing pair of elements if  $\dim Q$  is infinite.*

PROOF. If  $\dim Q$  is infinite,  $Q$  possesses a reducing pair  $(\beta_1, \beta_2)$ . Choose right regular  $c \in R$  such that  $\beta_1c, \beta_2c \in R$ . Then, as a quick check reveals,  $(\beta_1c, \beta_2c)$  is a reducing pair for  $R$ .

If  $\dim Q$  is finite and  $Q$  is a right quotient ring of a ring  $R$  which satisfies  $(A_1)$ , then  $Q = \{ac^{-1} : a \in R, \text{ unit } c \in Q\}$ . In this case,  $R$  is a right order in  $Q$  if and only if  $R$  is prime. The following proposition would therefore suggest that, for our purposes, the notion of a reducing pair is the infinite dimensional analogue of primeness.

4.3. PROPOSITION. *Suppose  $\dim Q$  is infinite and  $Q$  is a right quotient ring of  $R$ . Suppose also that  $Q = \{ac^{-1} : a \in R, \text{ unit } c \in Q\}$  and that regular elements of  $R$  are units of  $Q$ . Then  $R$  is a right order in  $Q$  if and only if  $R$  possesses a reducing pair of elements.*

PROOF. Assume that  $R$  possesses a reducing pair. Let  $e$  be any idempotent of  $Q$ . Let  $I = (1 - e)Q \cap R + eQ \cap R$ . Choose units  $c, d$  of  $Q$  such that  $(1 - e)c \in R$  and  $ed \in R$ . Then  $(1 - e)cR + edR$  is essential in  $I_R$ . By 4.1, there exists  $a \in I$  such that  $aR$  is essential in  $I_R$ . Hence  $aQ =$

$Q$ . Let  $y \in Q$  be such that  $ay = 1$ . Choose a unit  $u \in Q$  such that  $yu \in R$ . Then  $u = a(yu) \in R$  and  $eu \in R$ . Thus  $R$  is a right order in  $Q$  by [15, Lemma 2.3] since  $Q$  is generated by its idempotents (1.4). Conversely, if  $R$  is a right order in  $Q$  then  $R$  contains a reducing pair by 4.2.

EXAMPLE. Suppose  $\dim Q$  is infinite and  $e$  is an idempotent of  $Q$ ,  $e \neq 0, 1$ . Let  $R = eQ + Q(1 - e)$ . By an argument similar to that employed in O'Meara [14, Theorem 2.1], it can be shown that  $Q = \{ac^{-1} : a \in R, \text{ unit } c \in Q\}$  if (and only if)  $\dim eQ > \dim(1 - e)Q$ . Thus if  $\dim(1 - e)Q$  is finite,  $R$  satisfies the hypotheses of 4.3 but  $R$  does not possess a reducing pair because  $R$  is not a right order in  $Q$  (2.8(1)).

## 5. Characterization of right quasi-orders.

5.1. THEOREM. *A ring  $R$  is a right quasi-order in an infinite dimensional full linear ring if and only if  $R$  satisfies each of the following conditions.*

- (1)  *$R$  is an  $m$ -irreducible ring containing uniform right ideals.*
- (2) *The closed right ideals of  $R$  are right annihilator ideals, and each such right ideal is essentially finitely generated.*
- (3)  *$R$  possesses a reducing pair of elements.*

PROOF. Suppose  $R$  is a right quasi-order in a full linear ring  $Q$  with  $\dim Q$  infinite. Then (1) holds by 2.1 and 2.5. By 2.2, the closed right ideals of  $R$  are right annihilator ideals. The second part of (2), which is condition (A) of §3, was established in 3.3. Finally, (3) is shown in 4.2.

Conversely, let us suppose  $R$  satisfies (1), (2) and (3). Then, by 2.1, (1) implies that the MRQ ring of  $R$  is a full linear ring,  $Q$  say. Since  $R$  possesses a reducing pair, so does  $Q$  and this implies  $\dim Q$  is infinite. By 1.3, the first part of (2) implies that  $Q$  is left intrinsic over  $R$ . Hence right regular elements of  $R$  are right invertible in  $Q$ . To show  $R$  is a right quasi-order in  $Q$ , it suffices, by 1.6, to show that for each idempotent  $e \in Q$  there is a right regular  $c \in R$  such that  $ec \in R$ . Let  $e$  be given and let  $I = (1 - e)Q \cap R + eQ \cap R$ . Since  $I$  is the sum of two closed right ideals of  $R$ , (2) implies that  $I_R$  is essentially finitely generated. Hence, by 4.1, there exists  $c \in I$  such that  $cR$  is essential in  $I_R$ . Since  $I_R$  is essential in  $R_R$ , we have  $cR$  is essential in  $R$  and so  $l(c, R) = 0$  because  $Z_r(R) = 0$ . Hence  $c$  is a right regular element of  $R$ . Furthermore,  $c \in I$  implies  $ec \in R$ . This completes the proof of the theorem.

The proof of the above theorem also reveals the following result.

5.2. PROPOSITION. *Suppose  $Q$  is a right quotient ring of  $R$ ,  $\dim Q$  infinite. Then the following are equivalent:*

- (1)  *$R$  satisfies (A) and contains a reducing pair of elements.*

(2) For each  $x \in Q$ , there exists  $c \in R$  such that  $c$  is right invertible in  $Q$  and  $xc \in R$ .

REMARK. A ring  $R$  which satisfies (1) (and hence (2)) of 5.2 need not be a right quasi-order in  $Q$  because right regular elements of  $R$  may not be right regular in  $Q$ . For example, consider  $R = Qe$  where  $e$  is an idempotent of  $Q$ ,  $e \neq 1$ , with  $\dim eQ = \dim Q$ .

For  $R$  as in 5.2(1), we can obtain much the same sort of information concerning the relative right ideal structures of  $R$  and  $Q$  as when  $R$  is a right order in  $Q$ .

5.3. PROPOSITION. Suppose  $Q$  is a right quotient ring of a ring  $R$ , and suppose  $R$  satisfies (A) and contains a reducing pair. Then the following hold:

- (i) For  $x_1, \dots, x_n \in Q$  there exists  $c \in R$  such that  $c$  is right invertible in  $Q$  and  $x_i c \in R$  for  $i = 1, \dots, n$ .
- (ii) For a right ideal  $I$  of  $R$ ,  $IQ = \{ay : a \in I, y \in Q \text{ with } y^r = 0\}$ .
- (iii) For right ideals  $I$  and  $J$  of  $R$ ,  $(I \cap J)Q = IQ \cap JQ$ .

PROOF. This follows easily from 5.2.

6. Left-flat epimorphic extensions. In this section we briefly indicate the role played by condition (A) on a ring  $R$ , having  $Q$  as a right quotient ring, in determining when  $Q$  is a left-flat epimorphic extension of  $R$  in the sense of Findlay [4], that is, when  ${}_R Q$  is flat and the canonical injection of  $R$  into  $Q$  is an epimorphism in the category of rings. Here, for the first time, we assume  $R$  has an identity (which is necessarily the identity of  $Q$ ).

6.1. PROPOSITION. Suppose  $Q$  is a right quotient ring of  $R$ . The following statements are equivalent:

- (i)  $Q$  is a left-flat epimorphic extension of  $R$ .
- (ii)  $Z(Q \otimes_R Q)_R = 0$  and  ${}_R Q$  is flat.
- (iii) For each  $x \in Q$ ,  $(R : x) = \{r \in R : xr \in R\}$  is an essentially finitely generated right ideal of  $R$ , equivalently,  $(R : x)Q = Q$ .
- (iv)  $R$  satisfies (A) and  ${}_R Q$  is flat.
- (v)  $R$  satisfies (A) and  $(I \cap J)Q = IQ \cap JQ$  for all right ideals  $I$  and  $J$  of  $R$ .
- (vi)  $R$  satisfies (A) and for any essentially finitely generated right ideals  $I$  and  $J$  of  $R$ ,  $I \cap J$  is essentially finitely generated.

PROOF. The canonical injection of  $R$  into  $Q$  is an epimorphism in the category of rings if and only if the map  $\sum_i p_i \otimes q_i \rightarrow \sum_i p_i q_i$  from  $Q \otimes_R Q$  into  $Q_R$  is an isomorphism, and the latter is equivalent to  $Z(Q \otimes_R Q)_R = 0$ . Thus (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is given in

Cateforis [2, Theorem 1.6]. Since  $R$  satisfies (A) if and only if (iii) holds for all idempotents of  $Q$ , it is clear that (iii) implies (iv). Also (iv) implies (v) by, for example, Cateforis [1, Lemma 1.10]. By Cateforis [1, Theorem 2.1, (a)  $\Leftrightarrow$  (d)], (iv) is equivalent to (vi). So to complete the proof it suffices to show that (v) implies (iii). We need a lemma.

**LEMMA (CATEFORIS).** *Suppose  $R$  satisfies (v). Let  $I$  and  $J$  be essentially finitely generated right ideals of  $R$ . Then*

- (1)  $I \cap J$  is essentially finitely generated.
- (2) For  $y \in Q$ , if  $(R: y)$  is essentially finitely generated, then  $\{a \in R: ya \in I\}$  is essentially finitely generated.

**PROOF.** (1) follows easily from  $(I \cap J)Q = IQ \cap JQ$  because a right ideal  $K$  of  $R$  is essentially finitely generated if and only if  $KQ$  is a principal right ideal of  $Q$ . (2) follows by a slight modification of the proof in Cateforis [1, Theorem 2.1, (d)  $\Rightarrow$  (c)] since  $r(y, R)$ , being a closed right ideal of  $R$ , is essentially finitely generated and  $yR \cap I = (yR \cap R) \cap I$  is essentially finitely generated by (1).

**PROOF OF (v) IMPLIES (iii).** Assume (v). Let  $S = \{x \in Q: (R: x)Q = Q\}$ . Let  $x, y \in S$ . Since  $(x + y)[(R: x) \cap (R: y)] \subseteq R$  and  $[(R: x) \cap (R: y)]Q = Q$ , we have  $(R: x + y)Q = Q$ , i.e.  $x + y \in S$ . Let  $K = \{a \in R: ya \in (R: x)\}$ . By (2) of the Lemma,  $KQ = Q$ . Hence, since  $(xy)K \subseteq R$ , we have  $(R: xy)Q = Q$ , i.e.  $xy \in S$ . Thus  $S$  is a subring of  $Q$ .

Now if  $Q$  is a division ring (or for that matter, if  $\dim Q$  is finite) then (iii) certainly holds. Suppose  $Q$  is not a division ring. By 1.4,  $Q$  is generated by its idempotents. Since  $R$  satisfies (A), all idempotents of  $Q$  are in  $S$ . Hence  $S = Q$ . This establishes (iii).

**REMARK.** A simple argument, again using the fact that  $Q$  is generated by its idempotents, shows that condition (A) on  $R$  implies  $Q$  is an epimorphic extension of  $R$  (not necessarily left-flat). The converse is false, as shown by  $R = eQ + \text{socle } Q + \langle 1 \rangle$  for  $e$  an idempotent with  $eQ \cong Q$ ,  $1 - e \notin \text{socle } Q$ .

**6.2. PROPOSITION.** *Suppose  $\dim Q$  is infinite and  $Q$  is a right quotient ring of  $R$ . If  $R$  satisfies (A) and contains a reducing pair of elements, then  $Q$  is a left-flat epimorphic extension of  $R$ .*

**PROOF.** This follows easily from 5.2, or from 5.3(iii) and 6.1(v).

**REMARKS.** (1) The converse of 6.2 is false in general. For example, let  $e$  be an idempotent of  $Q$  such that  $e \neq 1$  and  $\dim(1 - e)Q$  is finite, and let  $R = eQ + Q(1 - e)$ . Then  $Q$  is a left-flat epimorphic extension of  $R$  but  $R$  does not contain reducing pairs.

(2) Statement (iii) of 6.1 is easily seen to be equivalent to  $R$  being a

“right quasi-order” in  $Q$  in the sense of Popescu and Spulber [17].

(3) For the general theory of flat epimorphic extensions see, for example, Silver [18], Popescu and Spiricu [16], Findlay [4], and Morita [13].

Cateforis [2, Theorem 1.6] has shown that for a general ring  $R$  with  $Z_r(R) = 0$  the MRQ ring  $T$  of  $R$  is a left-flat epimorphic extension of  $R$  if and only if

(A') every closed submodule of a free module of finite rank is essentially finitely generated.

Condition (A) is very much weaker than (A') for a general  $R$ , and even in the case  $T = Q$  it is weaker (so that in 6.1, (A) is not equivalent to (i)). For let  $Q$  be an infinite dimensional full linear ring whose centre  $C$  is not a prime field, and let  $K$  be any maximal right ideal of  $Q$  containing socle  $Q$ . Let  $R = K + \langle 1 \rangle$ . For any  $x \in C$ ,  $x$  not in the prime subfield of  $C$ ,  $(R : x) = K$  and therefore  $(R : x)Q \neq Q$ . Thus  $R$  fails (A') (see 6.1(iii)). On the other hand  $R$  satisfies (A) because each closed right ideal of  $R$  has the form  $eR$  or  $(1 - e)R$  for some idempotent  $e \in K$ .

7. Characterizations of a right order. Our first characterization of a right order  $R$  in an infinite dimensional full linear ring is a fairly immediate consequence of the characterization of a right quasi-order given in Theorem 5.1.

7.1. THEOREM. A ring  $R$  is a right order in a full linear ring of dimension  $\aleph \geq \aleph_0$  if and only if  $R$  satisfies each of the following conditions.

- (1)  $R$  is an  $m$ -irreducible ring containing uniform right ideals and  $\dim R_R = \aleph$ .
- (2) The closed right ideals of  $R$  are right annihilator ideals, and each such right ideal is essentially finitely generated.
- (3)  $R$  possesses a reducing pair of elements.
- (4) For each  $a \in R$  with  $a^l = 0$ ,  $aR$  contains a regular element of  $R$ .

PROOF. Only if. Suppose  $R$  is a right order in  $Q$  with  $\dim Q = \aleph \geq \aleph_0$ . Then, as  $\dim Q$  is infinite,  $R$  is a right quasi-order in  $Q$  (see Remark after Theorem 2.2). Thus  $R$  satisfies (1), (2) and (3) from Theorem 5.1. ( $\dim R_R = \dim Q = \aleph$  because  $Q$  is a right quotient ring of  $R$ .) Let  $a \in R$  be such that  $l(a, R) = 0$ . Then  $l(a, Q) = 0$  because  $Q$  is left intrinsic over  $R$  [14, Lemma 2.3]. Hence  $ab = 1$  for some  $b \in Q$ . Choose a regular element  $c \in R$  such that  $bc \in R$ . Then  $c = a(bc) \in aR$ , which establishes (4).

If. Suppose  $R$  satisfies (1) to (4). By Theorem 5.1, (1), (2) and (3) imply that  $R$  is a right quasi-order in a full linear ring  $Q$  with  $\dim Q = \dim R_R = \aleph$ . Also, regular elements of  $R$  are units in  $Q$  because the first part of (2) implies  $Q$  is left (as well as right) intrinsic over  $R$ . Let  $x \in Q$  be arbitrary. Since  $R$  is a

right quasi-order in  $Q$ , there exists  $a \in R$  such that  $a^l = 0$  and  $xa \in R$ . By (4),  $aR$  contains a regular element of  $R$ ,  $c$  say. Then  $xc \in R$ . This shows that  $R$  is a right order in  $Q$ .

REMARKS. (i) If  $\aleph = \aleph_0$ , then in (1) "m-irreducible ring" can be replaced by "prime ring with zero right singular ideal". See [14, Corollary 3.3].

(ii) Simple examples show that each of the conditions (1) to (4) is not, in general, implied by the other three.

Our second characterization of a right order  $R$  in  $Q$ ,  $\dim Q$  infinite, is in terms of the right annihilator ideals of  $R$  which have the same dimension as  $R_R$ . In the case of  $Q$  itself, these are the right ideals which are isomorphic to  $Q_Q$ . For a right order  $R$ , they are the closed right ideals which are an essential extension of a right ideal isomorphic to  $R_R$  (and thus are of the form  $(bR)^{lr}$ ,  $b \in R$  with  $b^r = 0$ ).

7.2. THEOREM. *A ring  $R$  is a right order in a full linear ring of dimension  $\aleph \geq \aleph_0$  if and only if each of the following conditions is satisfied.*

(1)  *$R$  is an m-irreducible ring containing uniform right ideals and  $\dim R_R = \aleph$ .*

(2) *The closed right ideals of  $R$  are right annihilator ideals, and, if  $B$  is a right annihilator ideal with  $\dim B_R = \dim R_R$ , then  $B$  has the form  $(bR)^{lr}$  for some  $b \in R$  with  $b^r = 0$ .*

(3)  *$R$  possesses a reducing pair of elements.*

PROOF. *Only if.* Assume  $R$  is a right order in  $Q$ , where  $\dim Q = \aleph \geq \aleph_0$ . Then (1), (3) and the first part of (2) hold by 7.1. Now suppose  $B$  is a right annihilator ideal of  $R$  with  $\dim B_R = \dim R_R$ . Then  $B = eQ \cap R$  for some  $e = e^2 \in Q$ , and  $\dim eQ = \dim B_R = \dim R_R = \dim Q$ . Hence  $Q_Q \cong eQ$  and so  $eQ = xQ$  for some  $x \in Q$  with  $x^r = 0$ . Choose a regular  $c \in R$  such that  $xc \in R$ . Let  $b = xc$ . Then  $b^r = 0$ . Also,  $bQ = eQ$  implies  $bR$  is essential in  $B_R$  and so  $(bR)^l = B^l$  (annihilators taken in  $R$ ). Thus  $B = B^{lr} = (bR)^{lr}$ , which establishes the second part of (2).

*If.* The proof here proceeds along much the same lines as the proof of Theorem 5.1, and so an outline is sufficient. The MRQ ring of  $R$  is a full linear ring, call it  $Q$ . By Proposition 1.4,  $Q$  is generated by the idempotents  $e$  for which  $eQ \cong (1 - e)Q$ , and therefore  $R$  is a right order in  $Q$  if for each such  $e$  there exists a regular  $c \in R$  such that  $ec \in R$  [15, Lemma 2.3]. Let  $e$  be given. Let  $I = (1 - e)Q \cap R + eQ \cap R$ . Proceeding exactly as in Theorem 5.1, we can find  $y \in I$  such that  $yQ = Q$  (we do not need the full force of (2) at this point, only the fact that  $(1 - e)Q \cap R$  and  $eQ \cap R$ , being closed right ideals of  $R$  of dimension  $\dim R_R$ , are essentially finitely generated right ideals of  $R$ ). Now  $Qy = Qf$  for some  $f = f^2 \in Q$ , and since  $fQ \cong Q$  we have  $\dim fQ = \dim Q$ . Let  $B = fQ \cap R$ .

Then  $\dim B_R = \dim fQ = \dim Q = \dim R_R$ . By (2), there exists  $b \in R$  such that  $b^r = 0$  and  $bQ = fQ$ . Let  $c = yb \in R$ . Then  $c$  is a unit of  $Q$  because  $cQ = y(bQ) = y(fQ) = yQ = Q$  and  $Qc = (Qy)b = (Qf)b = Qb = Q$ . In particular,  $c$  is a regular element of  $R$  and, since  $ey \in R$ , we have  $ec = (ey)b \in R$ . The proof is complete.

We close this section with an example to show that, in contrast to the left annihilator ideals of a right order  $R$  in  $Q$ , which we know are of the form  $a^l$ ,  $a \in R$  (see Proposition 3.3), the right annihilator ideals of  $R$  need not have the form  $a^r$ ,  $a \in R$ . Incidentally, this is the case if  $\dim Q$  is finite (Goldie [5, Theorem 3.7]). For a ring  $S$ , we denote the ring of all  $\aleph_0 \times \aleph_0$  (or strictly speaking  $\omega \times \omega$ ) column-finite matrices over  $S$  by  $S_\infty$ .

EXAMPLE. Let  $K$  be a right Ore domain which is not a left Ore domain. Let  $D$  be the right quotient division ring of  $K$ . Let  $T = D_\infty$  and  $A = K_\infty + \text{socle } D_\infty$ . Choose  $d \in D$ ,  $d \neq 0$ , such that  $Kd \cap K = 0$ , and let  $y \in T$  be the scalar matrix corresponding to  $d$ . Then  $Ay \cap A \subseteq \text{socle } T$ . Now let  $Q = T_2$  and  $R = A_2$ . Then  $R$  is a right order in  $Q$ . Let

$$e = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}.$$

Then  $Qe \cap R \subseteq \text{socle } Q$ . Let  $B = (1 - e)Q \cap R$ . Then  $B$  is a right annihilator ideal of  $R$ . Suppose there exists  $a \in R$  such that  $B = r(a, R)$ . Then  $r(a, Q) = (1 - e)Q$  and this implies  $Qa = Qe$ , which contradicts  $Qe \cap R \subseteq \text{socle } Q$ . Hence  $B$  is not of the form  $a^r$ ,  $a \in R$ .

8. **Idealizers.** If  $V_D$  is a vector space over a division ring  $D$  and  $\dim V_D$  is uncountable, then there exist proper right orders in  $Q = \text{Hom}_D(V, V)$ , irrespective of the nature of  $D$  (see 2.8). But what happens if  $V_D$  is of countably infinite dimension? For example, if  $D$  is a finite field, do there exist proper right orders in  $D_\infty$ ? There is one class of subrings of  $Q$  which contains some interesting right quasi-orders in  $Q$  and which may provide candidates for right orders—the idealizers of right ideals of  $Q$ . The idealizer of a right ideal  $K$  of  $Q$  is  $I(K) = \{a \in Q: aK \subseteq K\}$ . In this final section, we take a brief look at when  $I(K)$  is a right quasi-order in  $Q$ , in the case  $K \supseteq \text{socle } Q$ , equivalently,  $I(K)$  is prime. (Notice that now  $I(K) \cong \text{Hom}_Q(K, K)$ .) We shall assume  $\dim Q = \aleph_0$  and that  $K \neq \text{socle } Q$ ,  $K \neq Q$  (i.e.  $I(K) \neq Q$ ). For a subset  $X$  of  $Q$ ,  $\bar{X}$  denotes the canonical image of  $X$  in  $Q/\text{socle } Q$ .

By Theorem 5.1 we know that  $I(K)$  is a right quasi-order in  $Q$  if and only if  $I(K)$  satisfies (A) and possesses a reducing pair.

The case where  $K_Q$  is countably generated as a right ideal (equivalently  $K_Q$  is projective) is easily disposed of: either

(i)  $K = \sum_1^\infty e_i Q$  where  $\{e_i\}_1^\infty$  is a countably infinite set of orthogonal idempotents with each  $\bar{e}_i \neq \bar{0}$  (this is when  $K_Q$  is free), or

(ii)  $K = eQ + \text{socle } Q$  where  $e$  is an idempotent with  $eQ \cong (1 - e)Q$ .

In (i),  $I(K) = \{a \in Q: \text{ for each } j, e_j a e_j = 0 \text{ for almost all } i\}$ ; in (ii),  $I(K) = eQ + Q(1 - e) + \text{socle } Q$ . In both cases  $I(K)$  is a right quasi-order in  $Q$  but not a right order; in fact classical right quotient rings do not exist. It is interesting to note that, in (i),  $I(K) \cong Q_\infty$ , so that  $Q_\infty$  is a (proper) right quasi-order in a full linear ring!

More generally, one sees that if  $I(\bar{K}, \bar{Q}) \neq \bar{0}$  then  $I(K)$  is a right quasi-order in  $Q$  but not a right order.

If  $I(\bar{K}, \bar{Q}) = \bar{0}$  but  $\bar{K}$  is not large in  $\bar{Q}$  then  $I(K)$  is never a right quasi-order because it fails condition (A) although  $I(K)$  can have reducing pairs. Such  $K$  do exist.

The following result applies to an arbitrary  $K$ . Its proof requires little beyond Theorem 5.1 and the observation that  $I(K)/K$  is canonically isomorphic to  $\text{End}(Q/K)_Q$ .

**8.1. THEOREM.** *Let  $K$  be a right ideal of  $Q$ . Let  $W_Q = Q/K$ . Then  $I(K)$  is a right quasi-order in  $Q$  if and only if*

- (1) *each cyclic submodule of  $W_Q$  is a homomorphic image of  $W_Q$ , and*
- (2)  *$I(K)$  possesses a reducing pair.*

One case remains to be settled—that when  $\bar{K}$  is large in  $\bar{Q}$ . I suspect that  $I(K)$  can be a right quasi-order here, but have yet to construct a suitable  $K$ . In view of the following proposition, if such a construction proves possible it will actually provide a right order in  $D_\infty$  for an arbitrary division ring  $D$ , and thereby destroy all hope of any sort of “Faith-Utumi description” for right orders in  $D_\infty$ .

**8.2. PROPOSITION.** *If  $\bar{K}$  is large in  $\bar{Q}$  and  $I(K)$  is a right quasi-order in  $Q$ , then  $I(K)$  is a right order in  $Q$  (and conversely).*

**PROOF.** Let  $R = I(K)$ . We shall call upon Theorem 7.2 and verify condition (2) of that theorem. This requires us to produce for any given idempotent  $e \in Q, \bar{e} \neq \bar{0}$ , an element  $b \in R$  with  $b^r = 0$  and  $bQ = eQ$ . Since  $\bar{K}$  is large in  $\bar{Q}$ , there exists an idempotent  $f \in K \cap eQ, \bar{f} \neq \bar{0}$ . Write

$$eQ = fQ \dot{+} yQ \quad (\text{direct sum}).$$

Since  $R$  is a right quasi-order in  $Q$  we can suppose  $y \in R$ , and that  $y^r \not\subseteq \text{socle } Q$  since, if necessary, we can replace  $y$  by  $y\beta$  for some  $\beta \in R$  with  $\beta Q = Q$  and  $\beta^r \not\subseteq \text{socle } Q$ . Let  $g \in Q$  be an idempotent such that  $Qg = Qy$ . Then  $\overline{1 - g} \neq \bar{0}$  and so, by 1.4(iii),  $(1 - g)Q \cong fQ$ . Hence there exists  $x \in$

$fQ(1 - g)$  with  $xQ = fQ$  and  $Qx = Q(1 - g)$ . Now let  $b = x + y \in R$ . Then  $b^r = 0$ , and  $bQ = b(1 - g)Q + bgQ = fQ + yQ = eQ$ , i.e.  $bQ = eQ$ .

REMARK.  $I(K)$  is not automatically a right order in  $Q$  if  $\bar{K}$  is large in  $\bar{Q}$ , e.g. when  $K$  is a finite intersection of maximal right ideals. Here  $I(K)$  lacks reducing pairs, although it satisfies (A). In fact condition (1) of 8.1 is clearly satisfied because  $(Q/K)_Q$  is completely reducible.

(Added May 1974). Since the appearance of K. R. Goodearl's paper *Prime ideals in regular self-injective rings* (Canad. J. Math. 25 (1973), 829–839), it has become apparent that the methods employed here to characterize right orders in infinite dimensional full linear rings can be applied to characterize right orders in prime regular right self-injective rings  $T$  which satisfy

(\*)  $T$  possesses reducing pairs, equivalently,  $T_T \cong (T \oplus T)_T$ .

In fact the proofs go over almost verbatim if we adopt Goodearl's technique of handling isomorphisms between principal right ideals of  $T$ . In the case  $T = Q$ , we have used the uniform dimension function for this. In Goodearl's language (\*) means

$H(\mathfrak{N}_0) \neq T$  where  $H(\mathfrak{N}_0)$  is the ideal  $\{0\} \cup \{x \in T: xT \cong E[\mathfrak{N}_0(xT)]\}$ .

Now for  $f, g \in T$  with  $f, g \notin H(\mathfrak{N}_0)$ ,  $fT \cong gT$  if and only if  $TfT = TgT$ , that is,  $f$  and  $g$  belong to precisely the same (two-sided) ideals of  $T$ . This observation together with Goodearl's delightful result that the ideals of  $T$  are well-ordered enable us to handle isomorphisms in much the same way as when  $T = Q$ , and we can then interpret Theorem 7.1 as actually showing:

"A ring  $R$  is a right order in some  $T$  with (\*) if and only if  $R$  satisfies (1) to (4), except that in (1) the existence of uniform right ideals is not required ( $T$  will be a full linear ring exactly when  $R$  does contain uniform right ideals)".

Of course a similar statement applies to right quasi-orders in  $T$ .

In this more general set-up, the case " $\dim Q = \mathfrak{N}_0$ " is to be replaced by " $T/H(\mathfrak{N}_0)$  is simple" ( $H(\mathfrak{N}_0)$  now takes over the role of  $\text{socle } Q$ ). Thus, for example, Theorem 2.7 becomes

"All right orders in  $T$  are prime  $\Leftrightarrow T/H(\mathfrak{N}_0)$  is simple".

The interpretation of the other results is left to the reader. (Note that the case  $T = Q$  occurs precisely when  $\text{socle } T \neq 0$ .)

8.1 and 8.2 are also meaningful when  $T/H(\mathfrak{N}_0)$  is simple, in particular if  $T$  is simple (with  $H(\mathfrak{N}_0) = 0$ ). In this connection it is perhaps interesting to note that if  $T$  is the MRQ ring of  $Q/\text{socle } Q$  ( $\dim Q = \mathfrak{N}_0$ ), then it can be shown that there do exist (proper) right ideals of  $T$  whose idealizers are right orders in  $T$ .

The prime regular right self-injective rings  $T$  which fail (\*) are the

"finite" ones (for  $x, y \in T, xy = 1 \Rightarrow yx = 1$ ). Such rings are necessarily simple. All two-sided injective prime regular rings are finite, and non-Artinian examples of these have been constructed by Goodearl in *Simple self-injective rings need not be Artinian* (to appear in Communications in Algebra). Goodearl has also shown (private communication) that the finite  $T$  are those for which  $\dim_T T \leq \aleph_0$  (i.e. every family of independent nonzero left ideals of  $T$  is countable).

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