ASYMPTOTIC VALUES OF MODULUS 1 OF FUNCTIONS IN THE UNIT BALL OF $H^\infty$

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ABSTRACT. The main purpose of this paper is to prove a theorem concerning a necessary and sufficient condition for an inner function to have a limiting value of modulus 1 along an arc inside the unit disc, terminating at a point of the unit circle.

1. Introduction. A function $h(z)$ defined and analytic in the unit disc $U = \{z: |z| < 1\}$ is said to be in the unit ball $B$ of $H^\infty$ if $|h(z)| \leq 1$ for all $z \in U$. A sequence $\{a_n\}$ of complex numbers is called a Blaschke sequence if $\sum_{n=1}^{\infty} (1 - |a_n|)$ converges and $0 < |a_n| < 1$. If $\{a_n\}$ is a Blaschke sequence of nonzero numbers the associated Blaschke product is defined by the formula

$$B(z, \{a_n\}) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - a_n z}.$$  

We assume that the function $h(z)$ in $B$ has no zeros at the origin. It is well known that the zeros $\{a_n\}$ of $h(z)$ form a Blaschke sequence, and $h(z)$ admits the following factorization:

$$h(z) = c B(z, \{a_n\}) \phi(z)$$

where $c$ is a constant of modulus one. The function $\phi(z)$ is defined such that

$$\phi(z) = \exp \left\{ \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\mu(\theta) \right\}$$

where $\mu$ is a positive finite Borel measure defined on the unit circle $\partial U = \{z: |z| = 1\}$.

A function $h_1(z)$ in $B$ is said to be a divisor of the function $h(z)$ in $B$ provided that

$$h_1(z) = c_1 B(z, \{b_n\}) \phi_1(z)$$

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where \( c_1 \) is a constant of modulus one, \( \{b_n\} \) is a subset of the set \( \{a_n\} \) of zeros of \( h(z) \) and

\[
\phi_E(z) = \exp \left\{ -\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_E(\theta) \right\}
\]

where \( \mu_E \) is the restriction of \( \mu \) to a measurable subset \( E \) of \( \partial U \).

Ahern and Clark [1] have proved the following result.

**Theorem A.** A necessary and sufficient condition for each divisor of a function \( h(z) \) in \( B \) to have a radial limit of modulus one at the point 1 is \( \mu(T(0)) = 0 \) and

\[
\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|} < \infty.
\]

It should be noted that Theorem A has been proved by Frostman [5] in the case \( h(z) = B(z, \{a_n\}) \).

In [6] we obtained the following result for the case \( h(z) = B(z, \{a_n\}) \).

**Theorem B.** Let \( r \) denote a function which is decreasing on an interval \( (0, \theta_0) \), let \( \lim_{\theta \to 0} r(\theta) = 1 \), and let \( r \) satisfy a Lipschitz condition

\[
|r(\theta) - r(\theta')| \leq Q|\theta - \theta'|
\]

for some constant \( Q \), let \( \{r_n e^{i\theta_n}\} \) be a Blaschke sequence, and let \( \Gamma = \{r(\theta)e^{i\theta}, \theta \in (0, \theta_0)\} \). Then \( \lim_{z \to 1, z \in T} B(z, \{b_n\}) \) exists, and has modulus one for each subsequence \( \{b_n\} \) of \( \{r_n e^{i\theta_n}\} \) if and only if both of the conditions

\[
\sum_{n=1}^{\infty} \frac{1 - r_n}{|1 - r_n e^{i\theta_n}|} < \infty
\]

and

\[
\lim_{t \to 0+} \sum_{t/2 < \theta_n < 2t} \frac{1 - r_n}{1 - r(t) + |t - \theta_n|} = 0
\]

hold.

The main purpose of this paper is to extend Theorem B to the general inner functions.

2. We first prove the following lemma.

**Lemma 1.** Let \( \mu \) be a positive Borel measure defined on \( \partial U \) and assume that \( \mu(\{0\}) = 0 \), and

\[
\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} < \infty.
\]
Let $\phi$ be the function in $B$ associated with $\mu$. Let $\Gamma$ be a curve as defined in Theorem B. Then if $0 < \varepsilon < \sqrt{2}/4\pi$, there exists a positive number $\delta$ such that

(2.2) \[ |\phi_E(z) - \phi^*(1)| < 52\varepsilon \]

for each $E = (-\pi, \pi) \setminus (t/2, 2t)$, $z = r(t)e^{it}$, $0 < t < \delta/2$, and $\phi^*(1)$ is the radial limit of $\phi$ at $1$.

**Proof.** We first have, for each number $f$,

(2.3) \[ |1 - e^f| = \left| 1 - \sum_{n=0}^{\infty} \frac{f^n}{n!} \right| \leq \sum_{n=1}^{\infty} \frac{|f|^n}{n!} = |f| \sum_{n=0}^{\infty} \frac{|f|^n}{(n+1)!} \leq |f|e^{|f|} . \]

From (2.1) we can find a positive number $\delta_1 < \pi/2$ such that $r(\delta_1) > \frac{\pi}{2}$ and

(2.4) \[ \int_{-\Phi}^{\Phi} d\mu(\theta)/|\theta| < \varepsilon \]

for all $\Phi \in (0, \delta_1)$.

We denote the set $[-\Phi, \Phi]$ by $E_1$. Let $t$ be a positive number less than $\Phi/2$. Then

(2.5) \[ |\phi_E(z) - \phi^*(1)| \leq |\phi_E(z) - \phi_{\partial U \setminus E_1}(z)| + |\phi_{\partial U \setminus E_1}(z) - \phi^*_{\partial U \setminus E_1}(1)| + |\phi^*_{\partial U \setminus E_1}(1) - \phi^*(1)| . \]

Now

(2.6) \[ |\phi^*_{\partial U \setminus E_1}(1) - \phi^*(1)| = \left| \exp \left\{ -\int_{\partial U \setminus E_1} \frac{e^{it} + 1}{e^{it} - 1} \, d\mu(\theta) \right\} - \exp \left\{ -\int_{-\pi}^{\pi} \frac{e^{it} + 1}{e^{it} - 1} \, d\mu(\theta) \right\} \right| = \left| 1 - \exp \left\{ -\int_{E_1} \frac{e^{it} + 1}{e^{it} - 1} \, d\mu(\theta) \right\} \right| . \]

But

(2.7) \[ \int_{E_1} \left| \frac{e^{it} + 1}{e^{it} - 1} \right| \, d\mu(\theta) < 2\pi \int_{E_1} \frac{e^{it} + 1}{e^{it} - 1} \, d\mu(\theta) < 2\pi e < 1. \]

We therefore have

(2.8) \[ |\phi^*_{\partial U \setminus E_1}(1) - \phi^*(1)| < 21\varepsilon . \]

Since the function $\phi_{\partial U \setminus E_1}$ is analytic at $1$, there exists a positive number $\delta_2$ such that

(2.9) \[ |\phi_{\partial U \setminus E_1}(z) - \phi^*_{\partial U \setminus E_1}(1)| < \varepsilon \]

for all $z = r(t)e^{it}$, $0 < t < \delta_2$. 

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\begin{align*}
|\phi_E(z) - \phi_{\partial U \setminus E_1}(z)| &= \left| \exp \left\{ -\int_{E} e^{i\theta} + z \over e^{i\theta} - z \ d\mu(\theta) \right\} - \exp \left\{ -\int_{\partial U \setminus E_1} e^{i\theta} + z \over e^{i\theta} - z \ d\mu(\theta) \right\} \right| \\
&< \left| 1 - \exp \left\{ -\int_{E \cap E_1} e^{i\theta} + z \over e^{i\theta} - z \ d\mu(\theta) \right\} \right|.
\end{align*}

But
\begin{align*}
\int_{E \cap E_1} \left| e^{i\theta} + z \over e^{i\theta} - z \right| d\mu(\theta) &\leq \int_{E \cap E_1} \frac{2}{|e^{i\theta} - z|} d\mu(\theta) \\
&= \int_{E \cap E_1} \frac{2d\mu(\theta)}{((1 - r(t))^2 + 4r(t)\sin^2(\theta - \tau)/2)^{1/2}} \\
&\leq \int_{E \cap E_1} \frac{2d\mu(\theta)}{((1 - r(t))^2 + 2(2/\pi)^2(\theta - \tau)^2/4)^{1/2}} \\
&\leq \int_{E \cap E_1} \frac{2\pi}{\sqrt{2} |\theta - \tau|} d\mu(\theta) \\
&= \int_{-\phi}^{t/2} \frac{2\pi}{\sqrt{2} |\theta - \tau|} d\mu(\theta) + \int_{2t}^{\Phi} \frac{2\pi}{\sqrt{2} |\theta - \tau|} d\mu(\theta) \\
&\leq \int_{-\phi}^{t/2} \frac{2\pi}{\sqrt{2} |\theta|} d\mu(\theta) + \int_{2t}^{\Phi} \frac{4\pi}{\sqrt{2} |\theta|} d\mu(\theta) \\
&\leq \frac{4\pi}{\sqrt{2}} \int_{E_1} \frac{d\mu(\theta)}{|\theta|} \leq \frac{4\pi}{\sqrt{2}} \epsilon < 1
\end{align*}

for $z = r(t)e^{it}$, $\pi/2 > t > 0$, $r(t) > 1/2$.

Again applying (2.3) we have
\begin{equation}
|\phi_E(z) - \phi_{\partial U \setminus E_1}(z)| < 2^{-1/2} 4\pi \epsilon e^{4\pi \epsilon/2^{1/2}} < 30\epsilon.
\end{equation}

Combining (2.6), (2.7), and (2.8) and taking $\delta = \min(\delta_1, \delta_2)$, we have
\begin{equation}
|\phi_E(z) - \phi^*(1)| < 52\epsilon.
\end{equation}

This completes the proof of Lemma 1.

The following theorem is an extension of Theorem B.

**Theorem 1.** Let $h$ be a function in $B$ with factorization (1.2), let $a_n = r_ne^{i\theta}$, $n = 1, 2, \ldots$, and let $\mu(\{0\}) = 0$. Let $\Gamma$ be a curve as defined in Theorem B. Then, for the limit
\begin{equation}
\lim_{z \to \Gamma; z \in \Gamma} h_1(z)
\end{equation}
to exist and have modulus 1 for each divisor \( h_1 \) of \( h \), it is necessary and sufficient that both of the conditions

\[
\sum_{n=1}^{\infty} \frac{1 - r_n}{|1 - r_ne^{i\theta n}|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|} < \infty
\]

and

\[
\lim_{t \to 0^+} \left( \sum_{t/2 < \theta < 2t} \frac{1 - r_n}{1 - r(t) + |t - \theta_n|} + \int_{t/2}^{2t} \frac{d\mu(\theta)}{1 - r(t) + |t - \theta|} \right) = 0
\]

hold.

**Proof.** Theorem B is the special case of Theorem 1 where \( h(z) = B(z, \{a_n\}) \).

We now prove Theorem 1 for the case that \( h(z) = \phi(z) \).

Let \( z = r(t)e^{it}, t > 0, r(t) > \frac{1}{2} \). Since \( |(e^{i\theta} + z)/(e^{i\theta} - z)| \leq c/(1 - r(t) + |t - \theta|) \) for some positive constant \( c \), we have

\[
\int_{-\pi}^{\pi} \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| d\mu(\theta) \leq \int_{-\pi}^{\pi} \frac{c}{1 - r(t) + |t - \theta|} d\mu(\theta)
\]

\[
\leq c \int_{\pi}^{2\pi} \frac{d\mu(\theta)}{|t - \theta|} + c \int_{-\pi}^{t/2} \frac{d\mu(\theta)}{|t + \theta|} + c \int_{t/2}^{2\pi} \frac{d\mu(\theta)}{|1 - r(t) + |t - \theta||}
\]

\[
\leq 2c \int_{\pi}^{2\pi} \frac{d\mu(\theta)}{|\theta|} + c \int_{-\pi}^{t/2} \frac{d\mu(\theta)}{|\theta|} + c \int_{t/2}^{2\pi} \frac{d\mu(\theta)}{1 - r(t) + |t - \theta|}
\]

\[
\leq 3c \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} + c \int_{t/2}^{2\pi} \frac{d\mu(\theta)}{1 - r(t) + |t - \theta|}
\]

Now (2.10) and (2.11) imply the uniform convergence [8] of the integral \( \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} \) when \( z = r(t)e^{it} \) for all \( t \in (0, C_1) \) for some \( C_1 > 0 \). Therefore the limit

\[
\lim_{z \to 1; \zeta \in \Gamma} \frac{\phi(z)}{r(t)e^{it}}
\]

exists and has modulus one. The same arguments are also valid for the divisors of \( \phi(z) \).

To see the converse, we assume that each divisor of \( \phi(z) \) has a limit of modulus one as \( z \to 1 \), along the curve \( \Gamma \). A theorem of Lindelöf [4, p. 10] implies that each divisor of \( \phi(z) \) has a radial limit of modulus one at 1. By Theorem A, we have \( \mu(\{0\}) = 0 \) and

\[
\int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|\theta|} < \infty.
\]
(2.14) \[ \lim_{r \to 0+} \int_{r/2}^{2r} \frac{d\mu(\theta)}{1 - r(t) + |t - \theta|} = 0. \]

Suppose, on the contrary,

(2.15) \[ \lim_{r \to 0+} \int_{r/2}^{2r} \frac{d\mu(\theta)}{1 - r(t) + |t - \theta|} > 0 \]

then

(2.16) \[ \lim_{r \to 0+} \int_{r/2}^{2r} \frac{1 - r(t) + |t - \theta|}{(1 - r(t) + |t - \theta|)^2} d\mu(\theta) > 0. \]

Hence one of the following two cases must arise

(2.17) (I) \[ \lim_{r \to 0+} \int_{r/2}^{2r} \frac{1 - r(t)}{(1 - r(t) + |t - \theta|)^2} d\mu(\theta) > 0, \]

(2.18) (II) \[ \lim_{r \to 0+} \int_{r/2}^{2r} \frac{|t - \theta|}{(1 - r(t) + |t - \theta|)^2} d\mu(\theta) > 0. \]

In the case (I) there is a positive number \( p > \infty \) such that

(2.19) \[ \int_{t_n/2}^{2t_n} \frac{1 - r(t_n)}{(1 - r(t_n) + |t_n - \theta|)^2} d\mu(\theta) > p \]

for each \( t_n \in F \), where \( F \) is a set of positive numbers with 0 as an accumulation point. But we have for \( z_n = r(t_n)e^{itn} \)

\[ |\phi(z_n)| = \exp \left\{ -\Re \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right\} \]

\[ = \exp \left\{ \int_{-\pi}^{\pi} \frac{1 - r(t_n)^2}{[1 - r(t_n)]^2 + 4r(t_n)\sin^2(t_n - \theta)/2} d\mu(\theta) \right\} \]

\[ \leq \exp \left\{ \int_{-\pi}^{\pi} \frac{1 - r(t_n)}{[1 - r(t_n)]^2 + (t_n - \theta)^2} d\mu(\theta) \right\} \]

\[ \leq \exp \left\{ \int_{-\pi}^{\pi} \frac{1 - r(t_n)}{(1 - r(t_n) + |t_n - \theta|)^2} d\mu(\theta) \right\} \]

\[ \leq \exp \left\{ \int_{t_n/2}^{2t_n} \frac{1 - r(t_n)}{(1 - r(t_n) + |t_n - \theta|)^2} d\mu(\theta) \right\} \]

\[ \leq \exp(-p) < 1. \]
Hence in the case (I) if the limit (2.12) exists, it cannot have modulus one. Therefore

\[
\lim_{t \to 0^+} \int_{t/2}^{2t} \frac{1 - r(t)}{(1 - r(t) + |t - \theta|)^2} \, d\mu(\theta) = 0. \tag{2.20}
\]

It remains to consider the case (II). In this case we construct a divisor of \( \phi(z) \) which does not have the appropriate limit

\[
\lim_{t \to 0^+} \int_{t/2}^{2t} \frac{|t - \theta|}{(1 - r(t) + |t - \theta|)^2} \, d\mu(\theta) > 0; \tag{2.18}
\]

then either

\[
\lim_{t \to 0^+} \int_{t/2}^{2t} \frac{\theta - t}{(1 - r(t) + \theta - t)^2} \, d\mu(\theta) > 0 \tag{2.21}
\]

or

\[
\lim_{t \to 0^+} \int_{t/2}^{2t} \frac{t - \theta}{(1 - r(t) + t - \theta)^2} \, d\mu(\theta) > 0. \tag{2.22}
\]

We suppose without loss of generality that it is the case that (2.21) holds. Then there exist a positive constant \( q \) and a set \( \{t_n\} \subset (0, \pi) \), \( t_n \to 0 \) as \( n \to \infty \), \( 4t_{n+1} < t_n \) for \( n = 1, 2, \ldots \), such that for some \( \xi_n \in [t_n, 2t_n] \)

\[
q < \int_{t_n}^{t_{n+1}} \frac{\theta - t_n}{(1 - r(t_n) + \theta - t_n)^2} \, d\mu(\theta) < 2q < \frac{\pi}{2}. \]

Let \( F_n = [t_n, \xi_n] \) and let \( F = \bigcup_{n=1}^{\infty} F_n \); condition (2.13) implies that if \( \phi_F(z) \) has a limit as \( z \to 1 \) along \( F \), then this limit is \( \phi_F^*(1) \). We can show that \( \phi_F(z) \) does not have this limit by verifying that the argument of \( \phi_F(z_n) \) (\( z_n = r(t_n)e^{itn} \)) does not have 0 as a limit (modulo \( 2\pi \)) as \( n \to \infty \).

Since

\[
\frac{e^{i\theta} + r(t)e^{it}}{e^{i\theta} - r(t)e^{it}} = \frac{1 - r(t)^2 + i[2r(t)\sin(t - \theta)]}{|e^{i\theta} - r(t)e^{it}|^2},
\]

by our choice of the sequence \( \{t_n\} \) we have

\[
\arg\phi_F(z_n) = \int_{-\pi}^{\pi} \frac{2r(t_n)\sin(\theta - t_n)}{|e^{i\theta} - r(t_n)e^{itn}|^2} \, d\mu_{F_n}(\theta) = \int_{t_n}^{\xi_n} \frac{2r(t_n)\sin(\theta - t_n)}{|e^{i\theta} - r(t_n)e^{itn}|^2} \, d\mu(\theta) = [2 - \delta(n)] \int_{t_n}^{\xi_n} \frac{\sin(\theta - t_n)}{|1 - r(t_n) + i(\theta - t_n)|^2} \, d\mu(\theta).
\]
where $\delta(n) > 0$ and $\lim_{n \to \infty} \delta(n) = 0$. Since $2^{-\gamma} |1 - r(t_n) + (\theta - t_n)| < |1 - r(t_n) + i(\theta - t_n)| < 1 - r(t_n) + (\theta - t_n)$ for $\theta \in F$, we see that

$$q < \arg \phi_{F_n}(z_n) < 2q$$

for sufficiently large $n$. Apply Lemma 1 with $\mu$ replaced by $\mu_F$; we have $\mu_E = \mu_{F \setminus F_n}$. So the lemma implies $\phi_{F \setminus F_n}(z_n) \to \phi_{F}(1)$, and this gives a contradiction to (2.23).

This completes the proof of Theorem 1 for the case $h(z) = \phi(z)$.

Combining the proof for the cases $h(z) = B(z, \{a_n\})$, and $h(z) = \phi(z)$, the theorem follows without difficulty.

3. Upon generalizing theorems of Ahern and Clark [1] and Leung and Linden [6], we obtain the following.

**Theorem 2.** Let $h$ be a function in $B$ with factorization (1.2) and $\gamma \geq 1$. A necessary and sufficient condition for $h$ and all its divisors to have $T_\gamma$ limits of modulus 1 at the point 1 is that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|} < \infty$$

and

$$\lim_{t \to 0^+} \left\{ \sum_{t/2 < |\theta_n| < 2t} \frac{1 - r_n}{t^{\gamma} + |t - |\theta_n||^{k+1}} \right\} = 0$$

hold.

**Theorem 3.** Let $\gamma \geq 1$, and $k$ be a nonnegative integer. Then the $k$th derivatives of the function $h(z)$ in $B$ and all its divisors have $T_\gamma$ limits at 1 if

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^{k+1}} + \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{|1 - e^{i\theta}|^{k+1}} < \infty$$

and

$$\lim_{t \to 0^+} \left\{ \sum_{t/2 < |\theta_n| < 2t} \frac{1 - r_n}{(t^{\gamma} + |t - |\theta_n||)^{k+1}} \right\} = 0$$

hold.

4. The method we used in the previous sections also applies to the class $N$ of holomorphic functions of bounded characteristic. A function $f$ holomorphic in the unit disc is said to be in the class $N$ if and only if

$$\sup_{0 < r < 1 - \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta < \infty.$$

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It is well known that if \( f \in N \), then it admits the factorization that

\[
(4.1) \quad f(z) = cz^m B(z, \{a_n\}) Q(z),
\]

where \( m \) is the order of zeros of \( f(z) \) at the origin, \( B(z, \{a_n\}) \) is the Blaschke product associated with the zeros \( \{a_n\} \) of \( f(z) \) with \( a_n \neq 0 \). Hence \( Q(z) \) is defined by the relation

\[
Q(z) = \exp\left\{ -\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta) \right\}, \quad d\sigma(\theta) = -d\mu(\theta) + \frac{(2\pi)^{-1}}{1 - e^{i\theta}} \log |f^*(e^{i\theta})| d\theta,
\]

where \( f^* \) is the boundary function of \( f \), \( \mu \) is a positive singular Borel measure defined on \( \partial U \), and \( c \) is a constant.

Again we assume without loss of generality that \( m = 0 \).

We obtain the following result.

**Theorem A.** Let \( f \in N \) with factorization (4.1), let \( \gamma \geq 1 \), and let \( k \) be a nonnegative integer. If

\[
(4.2) \quad \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^{k+1}} + \int_{-\pi}^{\pi} \frac{d\sigma(\theta)}{|1 - e^{i\theta}|^{k+1}} < \infty
\]

and

\[
(4.3) \quad \lim_{t\to0^+} \left\{ \sum_{t/2 < |\theta_n| < 2t} \frac{1 - r_n}{(t^\gamma + |t - |\theta_n||)^{k+1}} + \left( \int_{-2t}^{-t/2} + \int_{t/2}^{2t} \right) \frac{d\sigma(\theta)}{(t^\gamma + |t - |\theta||)^{k+1}} \right\} = 0
\]

then the \( T_{\gamma} \) limit of \( f^{(k)} \) exists.

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