ON J-CONVEXITY AND SOME ERGODIC SUPER-PROPERTIES
OF BANACH SPACES

BY

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ABSTRACT. Given two Banach spaces $F$ and $X$, write $F \subset X$ iff for each finite-dimensional subspace $F'$ of $F$ and each number $e > 0$, there is an isomorphism $V$ of $F'$ into $X$ such that $|x| - \|Vx\| < e$ for each $x$ in the unit ball of $F'$. Given a property $P$ of Banach spaces, $X$ is called super-$P$ iff $F \subset X$ implies $F$ is $P$. Ergodicity and stability were defined in our articles On B-convex Banach spaces, Math. Systems Theory 7 (1974), 294–299, and C. R. Acad. Sci. Paris Ser. A 275 (1972), 993, where it is shown that super-ergodicity and super-stability are equivalent to super-reflexivity introduced by R. C. James [Canad. J. Math. 24 (1972), 896–904]. $Q$-ergodicity is defined, and it is proved that super-$Q$-ergodicity is another property equivalent with super-reflexivity. A new proof is given of the theorem that $J$-spaces are reflexive [Schaffer-Sundaresan, Math. Ann. 184 (1970), 163–168]. It is shown that if a Banach space $X$ is $B$-convex, then each bounded sequence in $X$ contains a subsequence $(y_n)$ such that the Cesàro averages of $(-1)^j y_j$ converge to zero.

Given two Banach spaces $F$ and $X$, $F$ is said to be finitely representable in $X$, in symbols $F \subset X$, iff for each finite-dimensional subspace $F'$ of $F$ and each number $e > 0$, there is an isomorphism $V$ of $F'$ into $X$ such that $|x| - \|Vx\| < e$ for each $x$ in the unit ball of $F'$. Given a property $P$ of Banach spaces, we say that $X$ is super-$P$ iff $F \subset X$ implies that $F$ has the property $P$. Super-reflexive spaces were introduced by James [12], [13]; the result announced in [4] but implicit in the earlier paper [3] is that the following super-properties are equivalent: Super-ergodicity, super-reflexivity, super-Banach-Saks, super-stability. Here we define $Q$-ergodicity, a notion in appearance weaker than ergodicity, and prove that super-$Q$-ergodicity is another property equivalent with super-reflexivity. At the same time we give a new proof of James's theorem [10] that $(2, e)$-convex spaces are reflexive, and more generally of the recent results of Schaffer-Sundaresan [19], that $J$-spaces are reflexive. We also show that...
$B$-convex spaces are alternate signs Banach-Saks: Each bounded sequence contains a subsequence $(y_n)$ such that the Cesáro averages of $(-1)^jy_i$ converge to zero.

1. Preliminaries. Let $X$ be an arbitrary Banach space with norm $\|\|$. An isometry (contraction) is a linear map $T: X \to X$ such that $\|Tx\| = \|x\|$ ($\|Tx\| \leq \|x\|$) for each $x \in X$. The Cesáro averages $(1/n)(T^n + \cdots + T^{n-1})$ are denoted by $A_n$, or $A_n(T)$. The following simple result seems new.

**Proposition 1.1.** If $T$ is a contraction on a Banach space $X$ then for each $x \in X$ the limit of $\|A_nx\|$ exists.

**Proof.** Let $x \in X$ and set $\alpha = \liminf \|A_nx\|$. It suffices to show that, for each $\delta > 0$,

$$\limsup \|A_nx\| \leq \alpha + \delta. \tag{1.1}$$

Given a $\delta > 0$, choose a fixed integer $N$ such that $\|A_Nx\| \leq \alpha + \delta$. If $m$ and $n$ are positive integers, $mN \leq n < (m + 1)N$, then as $m \to \infty$

$$\|A_{mN} - A_{n}\| \to 0. \tag{1.2}$$

Therefore it suffices to prove that $\limsup_m \|A_{mN}x\| \leq \alpha + \delta$. $\|T\| \leq 1$ implies $\|T^NA_Nx\| \leq \alpha + \delta$ for each $j$. Hence for each $m$

$$\|A_{mN}(T)x\| = \|A_m(T^N)A_N(T)x\| \leq \alpha + \delta. \tag{1.3}$$

This proves that $\lim\|A_nx\|$ exists. It is easy to see that this limit, considered as a function of $x$, is a seminorm. $\square$

Note that applying the proposition to the space of bounded operators on $X$ one obtains: for each contraction $T$, $\lim\|A_n(T)\|$ exists.

We will now define $Q$-ergodicity. Let $S$ be the space of all sequences $a = (a_i)_{i=1,2,\cdots}$ such that $a_i = 0$ but all but finitely many $i$'s. Assuming $T$ fixed, set, for each $x \in X$ and $a \in S$,

$$Q(x; a; n_1, n_2, \cdots) \tag{1.4}$$

$$= \|a_1A_{n_1}x + a_2A_{n_2}T^{n_1}x + a_3A_{n_3}T^{n_1+n_2}x + \cdots\|,$n = \inf(n_i). \tag{1.5}$$

The $\limsup$ above becomes limit if $a$ is one-dimensional (by Proposition 1.1), or if the norm is “invariant under spreading of the sequence $T^n x$” (see Proposition 2.2 below).

Let $r$ be an integer $\geq 2$ and $\epsilon$ a number, $0 \leq \epsilon \leq 1/r$. The space $X$ is called $(r, \epsilon)$-ergodic iff for each isometry $T$, each $x \in X$, any $r$ elements
\[ a^1, \ldots, a^r \text{ of } S \text{ such that } L(x, a^j) \leq 1, \text{ one has} \]
\[ \min_{1 \leq k \leq r} L(x, a^1 + a^2 + \cdots + a^k - a^{k+1} - a^{k+2} - \cdots - a^r) \leq r(1 - \varepsilon). \]

\( X \) is called \( Q \)-ergodic, or qualitatively ergodic, iff it is ergodic for some \( r \) and \( \varepsilon \).

We recall that \( X \) is called ergodic (for isometries) iff \( \lim A_n x \) exists for each isometry \( T \) and each \( x \in X \). We now will show that if \( X \) is ergodic, then it is \( (r, \varepsilon) \)-ergodic for each \( r \) and \( \varepsilon \). It is known and easy to see that the ergodic theorem for \( T \) implies that, for each \( x \), \( \lim_n A_n T^j x \) exists uniformly in \( j \). (Apply, e.g., the decomposition theorem \[6, p. 662\]; uniform in \( j \) converges to the limit is obvious for a \( T \)-invariant \( x \), and also for an \( x \) of the form \( x = y - Ty \).) Let \( \bar{x} = \lim A_n x \), \( a^j = (a^j_1) \), \( \alpha_j = \sum_i a^j_i \|\bar{x}\| \). If \( T \) is ergodic then the \( j \)th summand in (1.4) converges to \( a_j \bar{x} \), hence (1.6) follows from the inequality
\[ \min_{1 \leq k \leq r} |a_1 + a_2 + \cdots + a_k - a_{k+1} - \cdots - a_r| \leq (r-1) \sup_j |a_j| \leq r(1 - \varepsilon), \]
easy to verify by induction on \( r \).

A Banach space \( X \) is called \( J-(r, \varepsilon) \)-convex, where \( r \geq 2, 0 \leq \varepsilon < 1 \), iff for each \( r \)-tuple \((x_1, \ldots, x_r)\) of elements of the unit ball \( U_X \) of \( X \) one has
\[ \min_{1 \leq k \leq r} \|x_1 + \cdots + x_k - x_{k+1} - \cdots - x_r\| \leq r(1 - \varepsilon). \]

\( X \) is called \( J \)-convex iff it is \( J-(r, \varepsilon) \)-convex for some \( r \) and \( \varepsilon \). It follows from a recent unpublished result of R. C. James [13] that \( J \)-convexity is a properly stronger notion than \( B \)-convexity introduced in [2]; cf. §3 below.

It is easy to see that \( J-(r, \varepsilon) \)-convexity, hence \( J \)-convexity, are super-properties; i.e., if \( X \) enjoys them, so does every space finitely representable in \( X \). It has been proven by Schaffer-Sunderasan [19], and will be again shown below, that \( J \)-convex spaces are reflexive; hence, as already noted in [14], super-reflexive.

Since the ergodic theorem holds for reflexive spaces, it follows that \( J \)-convex spaces are ergodic. It would be perhaps of interest to give a direct proof of this result; here we only point out that \( J-(2, \varepsilon) \)-convexity easily implies the relation:
\[ \lim_{n,p \to \infty} \sup \|A_n(T)x - A_n(T)x\| \leq 2(1 - \varepsilon) \lim \|A_n(T)x\| \]
for each contraction \( T \) on \( X \) and each \( x \in X \): Note that for any fixed positive integers \( i, N, m \) one has the identities
\[ A_{2iN} = A_i(T^N)\left[\frac{1}{2}(A_N + T_0^N A_N)\right], \]
Let \( x \in X; \lim \| A_n x \| = \alpha \) exists by Proposition 1.1. Select a fixed number \( \delta, 0 < \delta < \epsilon \alpha / (2 - \epsilon) \). Choose a fixed \( N \) so large that

\[
(1.12) \quad \| A_{N+k} x \| - \alpha < \delta, \quad k = 0, 1, \cdots .
\]

Since \( \| T \| \leq 1 \), either for some integer \( i \)

\[
(1.13) \quad \| A_N x + T^{iN} A_N x \| \leq 2(1 - \epsilon)(\alpha + \delta),
\]

or for all \( i \)

\[
(1.14) \quad \| A_N x - T^{iN} A_N x \| \leq 2(1 - \epsilon)(\alpha + \delta).
\]

In the first case (1.10) implies \( \| A_{2iN} x \| \leq (1 - \epsilon)(\alpha + \delta) \), which contradicts (1.12). Therefore (1.14) must hold for all \( i \), and (1.11) implies \( \| A_N x - A_{mN} x \| \leq 2(1 - \epsilon)(\alpha + \delta) \) for all \( m \). Since \( \delta \) may be chosen arbitrarily small, (1.2) now implies (1.9).

2. Ergodic super-properties. A Banach space \( X \) with norm \( \| \| \) is given. A bounded sequence \( (x_n) \) in \( X \) is called stable iff there is an element \( \overline{x} \) such that

\[
(2.1) \quad \lim_n \left\| \frac{1}{n} \sum_{i=1}^{n} x_{k_i} - \overline{x} \right\| = 0
\]

uniformly in the set \( K \) of all strictly increasing sequences \( (k_n) \) of natural numbers. Actually, the uniformity is an easy consequence of convergence for all \( (k_n) \in K \). A Banach space is called stable iff every bounded sequence contains a stable subsequence; Banach-Saks iff every bounded sequence contains a subsequence which converges Cesáro. Professor Paul Erdös has recently informed us that he had shown jointly with Professor M. Magidor that every space which is Banach-Saks is also stable, the proof being based on the combinatorial fact that every analytic set is Ramsey [20].

We now return to the setting of our papers [3], [4], in which we have attempted to connect ergodic properties of \( X \) with stability, or the Banach-Saks property. We have at first asked the following question: Does an arbitrary bounded sequence \( (x_n) \) in \( X \) admit a subsequence \( (e_n) \) such that the shift \( T \) on \( (e_n) \) is defined and power-bounded? (By a shift on \( (e_n) \) we understand an operator \( T \) satisfying \( T e_n = e_{n+1} \) for all \( n \), and acting on the space spanned by the \( e_n \)'s.) If the answer to this question had been positive, it would follow at once that the ergodic theorem (power-bounded version) for \( X \) and its subspaces
implies the Banach-Saks property—therefore the answer is negative, since there are reflexive spaces which are not Banach-Saks (Baernstein [1]). This showed the need to change the norm. Denoting the space spanned by \((e_n)\) and a new norm \(\|\cdot\|\) by \(F\), we could obtain [3] that the shift on \((e_n)\) be an isometry, and yet \(\|\cdot\|\) be so close to \(\|\cdot\|\) that the ergodic theorem for \(T\) on \(F\) implies that \((e_n)\) contains a stable subsequence in \(X\), and \(F\) fr \(X\). The implication announced in [4], super-ergodic \(\Rightarrow\) super-stable, follows. We recapitulate the construction of \((e_n)\) and \(F\). \(S\) is the space of all sequences \(a = (a_i)_{i=1,2,...}\) with \(a_i = 0\) for all but finitely many \(i\). We have

**Proposition 2.1 (Proposition 1 of [3]).** Each bounded sequence \((x_n)\) in \(X\) contains a subsequence \((e_n)\) with the following property: For each \(a \in S\) there exists a number \(L(a)\) such that \(\|\sum a_i e_{n_i}\| \to L(a)\) as the sequences \((n_1), (n_2), \cdots\) converge to \(\infty\) so that \(n_1 < n_2 < \cdots\).

Now fix \((x_n)\) and let \((e_n)\) be a subsequence of \((x_n)\) satisfying the conditions of the above proposition. Let \(\varphi(S)\) be the space of linear combinations \(\sum a_i e_i, a \in S\). As shown in [3], we may assume without loss of generality that the \(e_n\)'s are algebraically independent in \(X\), and that \(\|\sum a_i e_i\|\) defined as equal to \(L(a)\) is a norm on \(\varphi(S)\). We denote by \(F\) the completion of \(\varphi(S)\) in this norm. We now show that \(F\) fr \(X\): If \(F'\) is an \(n\)-dimensional subspace of \(F\), \(F'\) is topologically isomorphic to \(l_1^{(n)}\), hence we commit a negligible error assuming that \(F'\) is generated by \(e_1, \cdots, e_m\) for \(m\) large. Let the same vectors in \(X\) generate a subspace \(H\). Set \(S_n = T^n: H \to X\). Then \(\|S_n x\| \to |x|\) on \(H\) implies \(M = \sup_n \|S_n\| < \infty\) (uniform boundedness principle), hence \(\|T^n x\| \to |x|\) uniformly on compacts of \(H\); therefore uniformly on \(U_{F'}\). Indeed, if \(Y = \{y_j\}\) is a finite \(\delta\)-net in a compact \(C \subset H\), then \(\|T^n x\| - |x| \leq \delta + 2\delta M\) on \(C\). To see this, note that if \(\|x - y_j\| \leq \delta\) then

\[
\|T^n x\| - |x| \leq \|T^n x - T^n y_j\| + \|T^n y_j\| - |y_j| \leq \delta M + \delta M.
\]

The relation \(F\) fr \(X\) was already implicitly used in Lemma 6 [3] and in [4]. Parting from \(F\) we now propose to introduce a new norm \(\|\cdot\|\) on \((e_n)\), with properties even more pleasing than \(\|\cdot\|\); the space \(G\) generated by \((e_n)\), \(\|\cdot\|\) will still be finitely representable in \(X\). The main virtue of \(\|\cdot\|\) (not included in isometric character of the shift \(T\)) may be described as invariance under spreading, or (IS) property: The norm of any finite combination of the \(e_n\)'s remains the same when the vectors are shifted, even though their mutual distances (but not positions) may change. This property formally stated in [3, Lemma 1],
is an immediate consequence of Proposition 2.1. The norm \( || \) will inherit from the (IS) property, but will also be equal signs additive, in short of type (ESA): In computing the norm of any finite linear combination of the \( e_i \)'s, consecutive terms of equal sign may be combined. Formally, for any vector \( x = a_1e_1 + \cdots + a_qe_q \), any integers \( k, p \) such that \( 1 \leq k < p \leq q \) and \( a_i > 0 \) for \( k \leq i \leq p \), one has

\[
||x|| = \left| \sum_{i=1}^{k-1} a_i e_i + (a_k + \cdots + a_p)e_k + \sum_{i=p+1}^{q} a_i e_i \right|.
\]

It is easy to see that it suffices to verify (2.2) for all \( k \) and \( p \) such that \( p - k = 1 \).

We now let \( A_n(T) \) act on the \( e_i \)'s spread so that different averages have disjoint support. More precisely, given a fixed \( a = (a_i) \in S \) with \( a_i = 0 \) for \( i > q \), we define

\[
P(n_1, \ldots, n_q; s_1, \ldots, s_q) = a_1A_{n_1}e_{s_1} + \cdots + a_qA_{n_q}e_{s_q},
\]

(2.3) \( s_1 > 0, s_2 \geq s_1 + n_1, \ldots, s_q \geq s_{q-1} + n_{q-1} \).

Invariance of \( || \) under spreading implies that the F-norm of the first expression in (2.3) does not depend upon the choice of the \( s_i \)'s; therefore this norm will be denoted by \( Q(n_1, \ldots, n_q) \), or \( Q(e_1; a; n_1, \ldots, n_q) \).

**Proposition 2.2.** For each \( x = a_1e_1 + \cdots + a_qe_q \) in \( \varphi(S) \), the limit of \( Q(e_1; a; n_1, \ldots, n_q) \) as \( \inf(n_1) \) converges to infinity exists. This limit, denoted \( \|x\| \), is a seminorm on \( \varphi(S) \).

**Proof.** The invariance of \( || \) under spreading implies that for any fixed positive integers \( N_1, \ldots, N_q; m_1, \ldots, m_q \)

\[
Q(m_1N_1, \ldots, m_qN_q) \leq Q(N_1, \ldots, N_q).
\]

(2.4) The particular case of (2.4) where \( m_i = 1 \) for \( i = 2, \ldots, q \) is obtained by taking the Cesàro average of

\[
P(a_1T^{kN_1}A_{n_1}e_{s_1} + a_2A_{n_2}e_{s_2} + \cdots + a_qA_{n_q}e_{s_q})
\]

for \( k = 0, 1, \ldots, m_1 - 1 \), since \( s_2 > s_1 + m_1N_1, s_i \geq s_{i-1} + N_i \) for \( i = 3, \ldots, q \) implies that each term has the norm \( = Q(N_1, \ldots, N_q) \). An obvious induction argument, again using invariance under spreading of \( || \), establishes (2.4). We denote by \( \alpha (\beta) \) the limit inferior (limit superior) of \( Q(n_1, \ldots, n_q) \) as \( n_i \) converge independently to infinity. To prove the proposition, it suffices to show that, for each \( \delta > 0, \beta \leq \alpha + \delta \). Choose \( N_1, \ldots, N_q \) fixed such that \( Q(N_1, \ldots, N_q) \leq \alpha + \delta \); (2.4) implies \( Q(m_1N_1, \ldots, m_qN_q) \leq \alpha + \delta \) for all \( m_i \). A computation anal-
ogous to (1.2) shows that if \( m_i N_i \leq n_i < (m_i + 1) N_i \) for all \( i \), then

\[
\lim_{m_i \to \infty} \sup |Q(m_1 N_1, \ldots, m_q N_q) - Q(n_1, \ldots, n_q)| = 0.
\]

(2.5) $\beta \leq \alpha + \delta$ follows. Finally, it is easy to see that $!!$ is a seminorm on $\varphi(S)$.

**Lemma 2.1.** *The seminorm $!!$ is of type (ESA) on the $e_n$'s.*

**Proof.** We verify (2.2) assuming, as we may, that $p = k + 1$. Since $!!$ is a continuous function of coefficients $a_i$, we further may suppose that $a_k/a_{k+1}$ is a rational number, and write $a_k = \alpha r$, $a_{k+1} = \alpha s$, where $r$, $s$ are positive integers. Then for all integers $m > 0$, $t > 0$, one has

\[
(2.6) \quad a_k A_m e_t + a_{k+1} A_m s e_{m t} = (a_k + a_{k+1}) A_m (r+s) e_t.
\]

The relation (2.5) is now applied, with $m_k = m_{k+1} = m$, $N_k = r$, $N_{k+1} = s$ to compute $!!!$, and with $N_k = r + s$ and $m_k = m$ to compute the right-hand side of (2.2) which is thus established. $\Box$

**Lemma 2.2.** If $!e_1 - e_2! = 0$ then $(e_n)$ admits a subsequence stable in $X$.

**Proof.** $!e_1 - e_2! = 0$ implies that

\[
|A_n e_1 - A_r e_{1+n}| \to 0 \quad \text{and} \quad |A_p e_1 - A_r e_{1+p}| \to 0 \quad \text{as} \quad n, p, r \to \infty.
\]

Choosing $r$ so that $r/(n + p) \to \infty$, we have that $|A_r e_{1+n} - A_r e_{1+p}| \to 0$; therefore by the triangular inequality the sequence $A_n e_1$ is Cauchy in $F$. Proposition 3 [3] is now applicable. $\Box$

Since we wish to prove that the space $X$ is stable, we only need to consider the case when $!e_1 - e_2! > 0$; then $!!$ may be easily seen to be a norm on $\varphi(S)$: If $!\sum_{i=1}^q a_i e_i! = 0$, then $!a_1 e_1 + a_3 e_3 + \cdots + a_{q+1} e_{q+1}! = 0$ and also $!a_1 e_2 + a_3 e_3 + \cdots + a_{q+1} e_{q+1}! = 0$; hence $!a_1 (e_1 - e_2)! = 0$ which implies $a_1 = 0$. Similarly one shows that $a_2 = 0, \ldots, a_q = 0$. Denote by $G$ the completion of $\varphi(S)$ in this norm.

We show that $G$ is finitely representable in $F$, hence in $X$. Let $G'$ be a finite-dimensional subspace of $G$; we may assume that $G'$ is generated by $e_1$, $e_2, \ldots, e_q$. Let $V = V_{n_1, \ldots, n_q}$ map each vector $a_1 e_1 + \cdots + a_q e_q$ onto

\[
a_1 A_{n_1} e_1 + a_2 A_{n_2} T e_1 + \cdots + a_q A_{n_q} T^{n_1 + \cdots + n_q - 1} e_1.
\]

Then for all $x \in U_{G'}$, by Proposition 2.2 $|!x! - |V x||$ is small if $n_1, \ldots, n_q$ are large. $G$ fr $F$ easily follows (see the proof of $F$ fr $X$ above).

**Define a seminorm $M$ on $S$ by**

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(2.7) \[ M(a) = !a_1(e_1 - e_2) + a_2(e_3 - e_4) + a_3(e_5 - e_6) + \cdots !. \]

Remark. The proofs of the following Lemma 2.3 and Proposition 2.3 use only the (IS) property of the norm. Thus they remain valid with \(| |\) replacing \(! !\).

Lemma 2.3. \( M(a) \geq M(b) \) if for each \( i \), \( a_i b_i \geq 0 \) and \(| a_i | \geq | b_i |\). Hence \( M \) is orthogonal, i.e., \( M(a) \geq M(a^+) \), \( M(a) \geq M(a^-) \), where \( a^+ \) is a sequence \((a_i^+)\), \( a^- \) the sequence \((a_i^-)\).

Proof. The invariance under spreading of \(| |\) implies that for each \( j \), each \( n \),

\[ M(a) = y + a_j(e_{2j-1} - e_{2j}) + z = y + a_j(e_{2j} - e_{2j+1}) + z \]

\[ = \cdots = y + a_j(e_{2j-1+n} - e_{2j+n}) + z \]

where

\[ y = \sum_{i=1}^{j-1} a_i(e_{2i-1} - e_{2i}), \quad z = \sum_{i=j+1}^{\infty} a_i(e_{2i+n-1} - e_{2i+n}) \]

(Since \( a \in S \), \( z \) has only finitely many summands.) Summing the \( n + 1 \) expressions inside \(! !\) in (2.8) and dividing by \( n + 1 \), one obtains

\[ M(a) \geq y + z - |a_j|(|e_{2j-1} - e_{2j+n}|)/(n + 1)! . \]

Let \( n \to \infty \); it follows that \( M(a) \geq M(a') \), where \( a'_i = 0, a'_i = a_i \) for \( i \neq j \).

The lemma is proved, because \( M \) is a convex function of coordinates.

Proposition 2.3. If \( G \) does not contain an isomorphic copy of \( c_0 \), then

\[ \lim_n !e_1 - e_2 + e_3 - e_4 + \cdots + e_{2n-1} - e_{2n} ! = \infty . \]

Proof. Set \( u_n = e_{2n-1} - e_{2n} \); let \( G' \) be the subspace of \( G \) generated by the \( u_i \)'s. Write \( M(\Sigma a_i u_i) = M(a) \) for \( a \in S \); extended to \( G' \), \( M \) is a norm coinciding with \(! !\). Let \( | a | = a^+ + a^- \), \( N(a) = M(|a|) \), \( N(y) = N(a) \) if \( y = \Sigma a_i u_i \), \( a \in S \). \( N(a) \leq M(a^+) + M(a^-) \leq 2M(a) \) by Lemma 2.3. Therefore

\[ 1/2 N(a) \leq M(a) \leq N(a) . \]

Extended to \( G' \), \( N \) is a norm equivalent with \( M \). This observation will be useful in \( \S 3 \) below. Now if (2.10) fails, Lemma 2.3 gives a \( \beta \) such that, for all \( n \),

\[ !e_1 - e_2 + e_3 - e_4 + \cdots + e_{2n-1} - e_{2n} ! \leq \beta , \]

and also shows that \( M(a^+) \leq \beta \cdot \sup(|a_i|) \), \( M(a^-) \leq \beta \sup(|a_i|) \), \( M(a) \leq 2\beta \sup(|a_i|) \). Also, \( M(a) \geq |a_i(e_{2i-1} - e_{2i})| = |a_i|(e_1 - e_2) \), so that \( M(a) \geq \)
(sup|a_i|)!e_1 - e_2!. Thus G' is a subspace of G that is isomorphic to c_0.

**Proposition 2.4.** If (2.10) holds, then G is not J-convex.

**Proof.** We show that G is not J-(r, ε)-convex by first giving a detailed and "graphic" proof of the case r = 2, then a brief proof of the general case. Set

\[ v_n = e_1 - e_3 + \cdots + e_{4n-3} - e_{4n-1}, \]
\[ w_n = e_2 - e_4 + \cdots + e_{4n-2} - e_{4n}. \]

We have \(|v_n| = |w_n| = |v_n + w_n|/2\), the last equality by (ESA). To prove that G is not J-(2, ε)-convex, it will suffice to prove that \(|v_n - w_n|/2|v_n|\) converges to 1. This follows from (2.10) because

\[ |v_n - w_n|/2|v_n| = (e_1 - (e_2 + e_3) + (e_4 + e_5) - \cdots
\]
\[ -(e_{4n-2} + e_{4n-1}) + (e_{4n} + e_{4n+1}) - e_{4n+1}! \]
\[ \geq -|e_1| + 2|v_n| - |e_{4n+1}|. \]

We now show, by essentially the same argument, that G is not J-(r, ε)-convex, where r ≥ 2 is arbitrary. Set for \(j = 1, 2, \cdots, r; n = 1, 2, \cdots, \)

\[ u_j = e_{j} - e_{j+r} + e_{j+2r} - \cdots + e_{j+(2n-2)r} - e_{j+(2n-1)r}. \]

Then \(|u_j| = |u_j|!\) for each j. In the expression \(d_n^k = u_n^1 + u_n^2 + \cdots + u_n^k - u_n^{k+1} - \cdots - u_n^r\), the terms are arranged as follows: First write \(S_1 = e_1 + e_2 + \cdots + e_k. \)

Then \((2n - 1)r\) terms grouped so that \(r\) consecutive \(e_i\)’s with \(-\) sign alternate with \(r\) consecutive \(e_i\)’s with \(+\) sign:

\[ S_2 = -(e_{k+1} + e_{k+2} + \cdots + e_{k+r}) + (e_{k+r+1} + \cdots + e_{k+2r}) - \cdots \]
\[ + (e_{k+2(n-2)r+1} + \cdots + e_{k+2(n-1)r}). \]

\(S_3\) is composed of the remaining \(r - k\) terms of \(d_n^k. \) Then \(\lim_n |S_i|/r|v_n^1| = 0\) for \(i = 1, 3; = 1\) for \(i = 2. \) Hence \(\lim_n |d_n^k|/r|v_n^1| = 1\) for each \(k = 1, 2, \cdots, r. \) The proposition is proved.

Now assume that X is J-convex; then so is G and G cannot contain an isomorphic copy of \(c_0\) (cf. [10] or [8], where this is proved for B-convex spaces). Propositions 2.3 and 2.4 and Lemma 2.2 now imply the following theorem:

**Theorem 2.1.** A J-convex Banach space is stable (hence super-stable).

**Theorem 2.2.** A super-Q-ergodic Banach space is super-stable.

**Proof.** If X is super-Q-ergodic then G is Q-ergodic, and the proof of Proposition 2.4 yields a contradiction. Lemma 2.2 now implies that \(\langle x_n \rangle\) has a
subsequence stable in $X$; since $(x_n)$ is arbitrary, it follows that $X$ is stable. Thus super-$Q$-ergodicity implies stability; it implies super-stability because the relation "$fr$" is transitive. □

Since a Banach-Saks space, and a fortiori a stable space, is easily seen to be reflexive (cf. [17]), the argument above provides a new proof that $J$-convex spaces are reflexive. We finally observe that in the course of the proof of Theorem 2.1 we establish the following: Any sequence $(x_n)$ admits a subsequence $(e_n)$ such that the sequence $(e_{2n-1} - e_{2n})$ is an unconditional basis for the IS norm $|||$, finitely representable in $||$. (Because an orthogonal norm is unconditional, and, as observed above, the proofs of Proposition 2.3 and Lemma 2.3 are valid for the norm $||$ as well as $! !$.)

3. Alternate signs Banach-Saks property. A Banach space $X$ is called $(r, \epsilon)$-convex iff for any $r$ elements $x_1, \ldots, x_r$ in $U_X$ there is a sequence of signs $\sigma_1, \ldots, \sigma_r$ such that $(1/r)(\sigma_1 x_1 + \cdots + \sigma_r x_r) \leq 1 - \epsilon$. A Banach space is called $B$-convex iff it is $(r, \epsilon)$-convex for some integer $r$ and some $\epsilon > 0$.

**Theorem 3.1.** Every bounded sequence $(x_n)$ in a $B$-convex Banach space admits a subsequence $(y_n)$ such that

$$\frac{1}{n} \sum_{i=1}^{n} (-1)^{i+1} y_i \rightarrow 0. \quad (3.1)$$

**Proof.** We may assume that $(x_n)$ is not stable, since otherwise $(y_n)$ satisfying (3.1) may be obtained as a union of two stable subsequences of $(x_n)$. Let $F'$ be the subspace of $F$ generated by $u_1 = e_1 - e_2, u_2 = e_3 - e_4, \cdots$. If $X$ is $B$-convex, then so is $F'$ and therefore, as it is easy to see, there exists a sequence of signs $\sigma_n$ such that

$$\lim \inf_n \left| \frac{1}{n} \sum_{i=1}^{n} \sigma_i u_i \right| = 0. \quad (3.2)$$

The proof of (3.2) is only sketched since the argument is known. We may assume $|u_i| \leq 1$ for all $i$. Let $X$ be $(r, \epsilon)$-convex. First choose signs $\sigma_1 = +, \sigma_2, \sigma_3, \cdots$ so that if $y_k = r^{-1} \Sigma_{i=1}^{r} \sigma_i u_i$, then $|y_k| \leq 1 - \epsilon$ for $k = 0, 1, \cdots$. Second choose signs $\sigma_1 = +, \sigma_2, \sigma_3, \cdots$ so that if $z_k = r^{-1} \Sigma_{i=1}^{r} r \sigma_i^2 y_i$ then $|z_k| \leq (1 - \epsilon)^2$ for $k = 0, 1, \cdots$. Next take Cesáro averages of successive $r$-tuples of $\sigma_i^2 z_i$, where $\sigma_i^3$ are appropriate signs, etc. This procedure yields a sequence of signs $\sigma_i$ satisfying (3.2). As already observed, the proofs of Lemma 2.3 and Proposition 2.3, in particular (2.11), use only the (IS) property of the norm, hence remain valid with $||$ replacing $! !$. Therefore (3.2) remains valid when all the $\sigma_i$'s are replaced by the sign $\epsilon$, Proposition 1.1 with $T$ replacing $T$ now implies that
$n^{-1}(u_1 + \cdots + u_n)$ converges to zero in $F$. The proof of Proposition 3 [3] remains valid if $(u_n)$ replaces $(e_n)$; hence the sequence $(u_n)$ contains a subsequence stable in $X$. This proves (3.1). □

Applying the theorem that every analytic (or only Borel) set is Ramsey (cf. the remarks in the beginning of §2), one may strengthen Theorem 3.1 to read:

Every bounded sequence $(x_n)$ in a $B$-convex space contains a subsequence $(z_n)$ such that (3.1) holds for each subsequence $(y_n)$ of $(z_n)$.

The alternate signs Banach-Saks property does not characterize $B$-convex spaces since $c_0$ has it, as has been shown to us by Professor A. Pełczyński.

PROPOSITION 3.1. Let $(x_n)$ be a sequence of vectors in $c_0$, $x_n = (x_n^{(i)})_{i=1}^{\infty}$, with $\|x_n\| = \sup_i |x_n^{(i)}| < 1$ for all $n$. Then for each $\varepsilon > 0$ there exists a subsequence $(y_n)$ of $(x_n)$ such that for all integers $m$

\[(3.3) \quad \| \sum_{j=1}^{m} (-1)^{j+1} y_j \| = \sup_i \left\| \sum_{j=1}^{m} (-1)^{j+1} y_j^{(i)} \right\| < 2 + \varepsilon.\]

Hence (3.1) holds.

PROOF. Choose $\varepsilon > 0$. Since we can pass to subsequences and apply the diagonal procedure, we may and do assume that $\lim_{n \to \infty} x_n^{(i)} = a_i$ exists for each $i$ and also that $|x_n^{(i)} - a_i| < 2^{-n} \varepsilon$ if $|x_n^{(k)}| > 2^{-k} \varepsilon$ for some $k < n$. Then, for a subsequence $(y_n)$,

\[\left\| \sum_{j=1}^{m} (-1)^{j+1} y_j \right\| = \sup_i \left\| \sum_{j=1}^{m} (-1)^{j+1} y_j^{(i)} \right\| < 2 + \varepsilon,
\]

since for each $i$ we can replace by $a_i$ each $x_n^{(i)}$ for which there exists $k < n$ such that $|x_k^{(i)}| > 2^{-k} \varepsilon$, and obtain

\[\left| \sum_{j=1}^{m} (-1)^{j+1} y_j^{(i)} \right| < \varepsilon \left( \sum_{k=1}^{\infty} 2^{-n} \right) + |x_k^{(i)}| + |a_i| < 2 + \varepsilon. \Box\]

Note that reflexive spaces need not be alternate signs Banach-Saks: The example in [1] is not alternate signs Banach-Saks.

REFERENCES


13. ———, A nonreflexive Banach space that is uniformly nonoctahedral (to appear).


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