S-OPERATIONS IN REPRESENTATION THEORY(1)

BY

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ABSTRACT. For $G$ a group and $A^G$ the category of $G$-objects in a category $A$, a collection of functors, called "$S$-operations," is introduced under mild restrictions on $A$. With certain assumptions on $A$ and with $G$ the symmetric group $S_k$, one obtains a unigeneration theorem for the Grothendieck ring formed from the isomorphism classes of objects in $A^{S_k}$. For $A = \text{finite-dimensional vector spaces over } \mathbb{C}$, the result says that the representation ring $R(S_k)$ is generated, as a $\lambda$-ring, by the canonical $k$-dimensional permutation representation. When $A = \text{finite sets}$, the $S$-operations are called "$\beta$-operations," and the result says that the Burnside ring $B(S_k)$ is generated by the canonical $S_k$-set if $\beta$-operations are allowed along with addition and multiplication.

I

A. Introduction. In the theory of linear representations of a finite group $G$, representations can be added, multiplied, and formed into a ring $R(G)$, the representation ring of $G$. In addition, $n$th symmetric power operations can be applied to any representation, and these operations can be extended to all elements of $R(G)$. Knutson [5] gives a detailed account of these operations in $R(G)$; Atiyah [1] discusses similar operations in the setting of vector bundles.

This paper attempts to generalize these notions. For any group $G$, a collection of operations on the category $A^G$ is defined under mild restrictions on $A$. In the case of linear representations of a finite group, these operations are combinations of symmetric powers, but, in general, they include other operations as well. Letting $G = S_k$ and with certain assumptions on $A^{S_k}$, one obtains the main result:

**Corollary I.22.** $\langle X_k \rangle = K_0(A^{S_k})$.

Here, $K_0(A^{S_k})$ is the Grothendieck ring formed from the isomorphism classes of objects in $A^{S_k}$, $X_k$ is a particular object in $A^{S_k}$, and $\langle X_k \rangle$ is the
subring of $K_0(A^S_k)$ obtained by applying the operations to $X_k$ and taking sums and products of the results. A principal application of this corollary is that $R(S_k)$ is generated by the canonical permutation representation $X_k$ if symmetric powers are included along with addition and multiplication. For the reader familiar with $\lambda$-rings, this statement says that $R(S_k)$ is generated by one element as a $\lambda$-ring [2], [5].

§§I.B and I.C present some background on the two principal examples, the Burnside and representation rings of a finite group. Chapter II introduces the $S$-operations and explores their behavior; the main theorem and its Corollary II.22 are proved in §C. In Chapter III, Corollary II.22 is used to prove that $R(S_k) = \langle X_k \rangle$ and $B(S_k) = \langle X_k \rangle$ for all $k \geq 1$. It is also shown that, in general, neither $B(S_k)$ nor $R(S_k)$ is unigenerated as a ring. Moreover, although $B(S_k) = \langle X_k \rangle$, if one allows only symmetric power operations rather than all $S$-operations, one does not necessarily obtain all of $B(S_k)$.

B. The Burnside ring, $B(G)$. Let $G$ be a finite group. A $G$-set is a finite set $T$ together with a mapping $G \times T \to T$ such that $(g_1g_2)t = g_1(g_2t)$, $1t = t$, for all $g_1, g_2 \in G$, $t \in T$. A morphism of $G$-sets, or $G$-map, is a set map $f: T \to T'$, with $T$ and $T'$ $G$-sets, such that $f(gt) = gf(t)$ for all $g \in G$, $t \in T$. Two $G$-sets are said to be isomorphic if there is a $G$-map between them which is a set isomorphism. $G$-sets and $G$-maps clearly form a category.

Examples 1.1. (i) Let $G$ be any finite group, $T$ any finite set. Then $T$ can be given the trivial action $gt = t$ for all $g \in G$, $t \in T$.

(i') In example (i), if $T$ has only one element, $T$ is denoted by $1_G$.

(Of course, all one-element $G$-sets are isomorphic.)

(ii) Let $H$ be a subgroup of a finite group $G$. Then $G/H$, the set of left cosets of $H$ in $G$, is a $G$-set by the action $g(xH) = (gx)H$.

(iii) Let $S_n$ be the symmetric group on the symbols $1, 2, \ldots, n$. Let $X_n$ be the set $\{x_1, x_2, \ldots, x_n\}$, and let $S_n$ act on $X_n$ by $\alpha x_i = x_{\sigma(i)}$. $X_n$ will be called the canonical $S_n$-set.

(iv) Let $G$ be any finite group. The empty set $\emptyset$ is clearly a $G$-set.

If $T_1$ and $T_2$ are $G$-sets, then the disjoint union $T_1 \sqcup T_2$ is a $G$-set, under the obvious action. On the other hand, every $G$-set can be decomposed into its $G$-orbits:

**Proposition 1.2.** Every $G$-set $T \neq \emptyset$ is of the form $\bigsqcup_{i=1}^n T_i$, where $T_i$ is a transitive $G$-set. The $T_i$'s are unique up to order. (A $G$-set $T$ is transitive if $T \neq \emptyset$ and if given $t_1, t_2 \in T$ there is a $g \in G$ such that $gt_1 = t_2$.)

**Proposition 1.3.** If $H$ is a subgroup of $G$, then $G/H$ is a transitive
G-set. Conversely, every transitive G-set is of the form G/H for some subgroup H of G.

Proof. Given \( g_1H, g_2H \in G/H \), \( (g_1g_2^{-1})g_2H = g_1H \). Hence G/H is a transitive G-set.

Suppose \( T \) is a transitive G-set. Let \( t \in T \). Then \( T = Gt \). Let \( G_t \) be the isotropy group of \( t \), i.e., \( G_t = \{ g \in G | gt = t \} \). Then the map \( T \rightarrow G/G_t \) defined by \( gt \mapsto gG_t \) is a G-isomorphism.

Proposition 1.4. \( G/H \cong G/K \) as G-sets if and only if H and K are conjugate subgroups of G.

Proof. Suppose H and K are conjugate, i.e., \( K = g_1^{-1}Hg_1 \) for some \( g_1 \in G \). Then the maps

\[ \phi: G/H \rightarrow G/K, \quad \psi: G/K \rightarrow G/H, \]

\[ gH \mapsto (g_1)K, \quad gK \mapsto (g_1^{-1})H, \]

are G-maps, and \( \phi \circ \psi = 1_{G/K}, \ \psi \circ \phi = 1_{G/H} \). So \( G/H \cong G/K \).

Conversely, assume \( G/H \cong G/K \). Then there exist G-maps \( \phi: G/H \rightarrow G/K, \ \psi: G/K \rightarrow G/H \) such that \( \phi \circ \psi = 1_{G/K}, \ \psi \circ \phi = 1_{G/H} \). If \( \phi(1_H) = g_1K \), then \( g_1K = hg_1K \) for all \( h \in H \), so \( g_1^{-1}Hg_1 \subseteq K \). Similarly \( \phi(1_K) = g_2H \) gives \( g_2^{-1}Kg_2 \subseteq H \). Thus \( g_2^{-1}g_1^{-1}Hg_1g_2 \subseteq g_2^{-1}Kg_2 \subseteq H \). Since \( g_2^{-1}g_1^{-1}Hg_1g_2 \) has the same number of elements as \( H \), \( g_2^{-1}g_1^{-1}Hg_1g_2 = H \). □

If \( T_1 \) and \( T_2 \) are G-sets, then the cartesian product \( T_1 \times T_2 \) is a G-set under the obvious action. The Burnside ring of G, \( B(G) \), consists of all finite formal sums, \( \Sigma t_n [T_n] \) (\( n \in \mathbb{Z} \)), of G-sets \( T_n \), modulo the relations

(i) \( [T_1] = [T_2] \) if \( T_1 \cong T_2 \) as G-sets,

(ii) \( [T_1 \sqcup T_2] = [T_1] + [T_2] \).

\( B(G) \) is clearly an abelian group; the cartesian product, together with \( 1_G \), gives \( B(G) \) the structure of a commutative ring with identity, i.e., \( [T_1][T_2] = [T_1 \times T_2] \). Whenever no confusion could arise, the brackets will be omitted.

Propositions 1.2, 1.3, and 1.4 imply

Proposition 1.5. Let \( \{ H_\alpha \} \) be a set of representatives of the conjugacy classes of subgroups of G. Then \( B(G) \) is a free \( \mathbb{Z} \)-module with basis \( \{ [G/H_\alpha] \} \).

The rest of this section is devoted to defining a set map \( h_n: B(G) \rightarrow B(G) \) for each integer \( n \geq 0 \). For any G-set \( T \), the set \( T_G \) is defined to be the collection of elements of \( T \) with the identification \( t_1 \sim t_2 \) iff \( Gt_1 = Gt_2 \).

Let \( T \) be a G-set. Then \( T^n = T \times T \times \cdots \times T \) (n times) is a G-set and also an \( S_n \)-set via \( \sigma(t_1, \cdots, t_n) = (t_{\sigma^{-1}(1)}, \cdots, t_{\sigma^{-1}(n)}) \) for \( \sigma \in S_n \).

For each integer \( n \geq 1 \), let \( h_n(T) \) denote \( (T^n)_{S_n} \); \( h_n(T) \) is thus the nth
symmetric power of $T$. Since the $G$- and $S_n$- actions on $T^n$ commute, $h_n(T) = (T^n)_{S_n}$ is actually a $G$-set. Clearly $h_n$ sends isomorphic $G$-sets to isomorphic $G$-sets. Finally, define $h_0(T)$ to be $1_G$ for all $G$-sets $T$.

If $T_1$, $T_2$ are $G$-sets, then

$$h_n(T_1 \sqcup T_2) = \prod_{i=0}^{n} (h_i(T_1) \times h_{n-i}(T_2)).$$

$h_n$ can now be defined on any element of $B(G)$ by the following construction:

Define

$$H_n: (G\text{-sets}) \times (G\text{-sets}) \to B(G)$$

inductively by

$$H_0(T_1, T_2) = 1_G,$$

$$H_n(T_1, T_2) = h_n(T_1) - \sum_{i=1}^{n-1} H_i(T_1, T_2)h_{n-i}(T_2) \quad \text{for } n > 0.$$ 

Clearly,

$$T_1 \cong U_1, \ T_2 \cong U_2 \Rightarrow H_n(T_1, T_2) = H_n(U_1, U_2) \quad \text{for all } n \geq 0.$$ 

In addition, an induction argument and the "addition formula" above give

$$H_n(T_1 \sqcup T, \ T_2 \sqcup T) = H_n(T_1, T_2), \quad H_n(T, \emptyset) = h_n(T)$$

for all $n \geq 0$ and $G$-sets $T_1, T_2, T$.

An arbitrary element of $B(G)$ looks like $T_1 - T_2$, where $T_1$ and $T_2$ are $G$-sets. If $T_1 - T_2 = U_1 - U_2$, then $T_1 \sqcup U_2 \cong U_1 \sqcup T_2$, so

$$H_n(T_1, T_2) = H_n(T_1 \sqcup U_2, \ T_2 \sqcup U_2) = H_n(U_1 \sqcup T_2, \ T_2 \sqcup U_2) = H_n(U_1, U_2).$$

Thus $H_n(T_1, T_2)$ depends only on $T_1 - T_2$. Therefore, define $h_n(T_1 - T_2) = H_n(T_1, T_2)$. Then $h_n: B(G) \to B(G)$ is a well-defined set map and coincides with its former definition if $T \in B(G)$ is actually a $G$-set.

C. The representation ring, $R(G)$. Let $G$ be a finite group. A (linear) representation of $G$ (over $C$) is a finite-dimensional vector space $V$ over $C$, together with a group homomorphism $\rho: G \to \text{Aut} V$. $V$ is called a $G$-module, and $\rho$ gives an action of $G$ on $V$. One usually writes

$$V \xrightarrow{\rho} V, \quad v \mapsto g v$$

instead of

$$V \xrightarrow{\rho(g)} V, \quad v \mapsto \rho(g)v.$$
A \( G \)-module map is a linear transformation \( f: V \to V' \), with \( V \) and \( V' \) \( G \)-modules, such that \( f(gv) = gf(v) \) for all \( g \in G, v \in V \). Two \( G \)-modules are said to be isomorphic if there exists a \( G \)-module map between them which is also a vector space isomorphism. \( G \)-modules and \( G \)-module maps clearly form a category.

**Examples I.6.** (i) Let \( G \) be a finite group, \( V \) a finite-dimensional vector space. \( V \) can be given the trivial action \( gv = v \) for all \( g \in G, v \in V \).

(i') A special case of example (i) is \( V = 0 \).

(i'') In example (i), if \( \dim V = 1 \), \( V \) is denoted by \( 1_G \).

(ii) Let \( G = S_n \). Let \( V \) have basis \( \{v_1, \ldots, v_n\} \), and let \( S_n \) act by \( \sigma v_i = v_{\sigma(i)} \) for \( \sigma \in S_n \). This representation \( V \) will be called the canonical \( S_n \)-module, and denoted \( X_n \).

(ii') More generally, suppose \( \rho: G \to S_n \) is a group homomorphism. (\( \rho \) is called a permutation representation.) By composing this homomorphism with the one in example (ii), one obtains a linear representation of \( G, G \xrightarrow{\rho} S_n \to \text{Aut } X_n \). Since a \( G \)-set \( T \) consisting of \( n \) elements is a group homomorphism \( G \to S_n \), the concept of \( G \)-set is the same as the concept of permutation representation of \( G \).

A \( G \)-module \( V \) is reducible if \( V = 0 \) or if there is a subspace \( W \) of \( V \) such that \( GW \subseteq W \), with \( W \neq 0 \) and \( W \neq V \). If \( V \) is not reducible, it is called irreducible.

If \( V_1, V_2 \) are \( G \)-modules, then the vector space coproduct \( V_1 \amalg V_2 \) is a \( G \)-module via the obvious action. A \( G \)-module \( V \) is said to be decomposable if \( V \cong V_1 \amalg V_2 \) as a \( G \)-module, where \( V_i \neq 0 \). Propositions I.7–I.9 can be found in any book on group representation theory (see [5], [8]).

**Proposition 1.7 (Maschke).** If \( V \neq 0 \) is reducible, then \( V \) is decomposable.

**Proposition 1.8.** Every \( G \)-module \( V \neq 0 \) can be expressed as a finite coproduct \( V = \coprod_{i=1}^{n} V_i \), where each \( V_i \) is an irreducible \( G \)-module. The \( V_i \)'s are unique (up to order).

**Proposition 1.9.** The number of irreducible representations of \( G \) is equal to the number of conjugacy classes of \( G \).

For \( G \)-modules \( V_1, V_2, V_1 \otimes V_2 \) is a \( G \)-module via \( g(v_1 \otimes v_2) = gv_1 \otimes gv_2 \). The representation ring of \( G, R(G) \), consists of all finite formal sums \( \Sigma n_i[V_i] \) (\( n_i \in \mathbb{Z} \)), of \( G \)-modules \( V_i \), modulo the relations

(i) \( [V_1] = [V_2] \) if \( V_1 \cong V_2 \) as \( G \)-modules,

(ii) \( [V_1 \amalg V_2] = [V_1] + [V_2] \).

\( R(G) \) is clearly an abelian group, the tensor product, together with \( 1_G \), gives
$R(G)$ the structure of a commutative ring with identity, that is $[V_1][V_2] = [V_1 \otimes V_2]$. The brackets will usually be omitted.

Propositions 1.8 and 1.9 imply

**Proposition 1.10.** Let $\text{Irrep } G = \text{the set of isomorphism classes of irreducible } G\text{-modules. Then } R(G) \text{ is a free } \mathbb{Z}\text{-module with basis } \{[V] \mid V \in \text{Irrep } G\}. \text{ The rank of } R(G) = \text{the number of conjugacy classes of } G.$

As in the case of $B(G)$, symmetric power operations $h_n: R(G) \to R(G)$ can be introduced. For any $G$-module $V$, define the vector space $V_G$ to be $V/W$, where $W$ is the subspace of $V$ generated by $\{v - gv \mid v \in V, g \in G\}$. The vector spaces $V_G$ and $V^G$, where $V^G$ is the subspace of $V$ fixed by $G$, are seen to be isomorphic by the fact that the linear transformation $Y: V \to V$ defined by

$$Y(v) = \frac{1}{|G|} \sum_{g \in G} gV$$

has image $V^G$ and kernel $W$. In the case of sets, however, the corresponding objects $T_G$ and $T^G$ are not generally isomorphic.

For any $G$-module $V$, $V^\otimes n = V \otimes \cdots \otimes V$ ($n$ times) is a $G$-module and also an $S_n$-module via $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in S_n$. For each positive integer $n$, let $h_n(V)$ denote $(V^\otimes n)_{S_n}$; $h_n(V)$ is thus the $n$th symmetric power of $V$. Since the $G$- and $S_n$- actions on $V^\otimes n$ commute, $h_n(V) = (V^\otimes n)_{S_n}$ is a $G$-module. Define $h_0(V)$ to be $1_G$ for all $G$-modules $V$. Clearly $h_n$ sends isomorphic $G$-modules to isomorphic $G$-modules.

For $G$-modules $V_1$ and $V_2$,

$$h_n(V_1 \boxplus V_2) = \prod_{i=0}^n (h_i(V_1) \otimes h_{n-i}(V_2)).$$

As in the $G$-set case, $h_n$ can be defined on any element $V_1 - V_2$ of $R(G)$ by defining $H_n: (G\text{-modules}) \times (G\text{-modules}) \to R(G)$ inductively by

$$H_0(V_1, V_2) = 1_G,$$

$$H_n(V_1, V_2) = h_n(V_1) - \sum_{i=0}^{n-1} H_i(V_1, V_2)h_{n-i}(V_2) \text{ for } n > 0,$$

and then using the "addition formula" above to show that $H_n(V_1, V_2)$ depends only on $V_1 - V_2$.

$G$-sets and $G$-modules are examples of the category discussed in Chapter II. There, a family of functors, called $S$-operations, is defined. In the case of
$G$-modules, these $S$-operations turn out to be sums and products of symmetric powers $h_n$. In fact, by applying these operations to the canonical $S_k$-module $X_k$, one can obtain every element in $R(S_k)$ (see III, §A).

In the case of $G$-sets, however, the $S$-operations include more than symmetric powers. In III, §B, one sees that applying sums and products of symmetric power operations $h_n$ to the canonical $S_k$-set $X_k$ does not always give all of $B(S_k)$, whereas applying all the $S$-operations to $X_k$ does.

II. $S$-Operations

A. The category $A^G$ and functors $\phi_{w_n}: A^G \to A^G$. Let $G$ be a group, and $A$ a category. A $G$-object in $A$ is an object $A$ in $A$, together with morphisms $A \xrightarrow{\rho_g} A$ for all $g \in G$, satisfying $\rho_{gh} = \rho_g \circ \rho_h$, $\rho_1 = 1_A$. $A \xrightarrow{\rho_g} A$ is usually written $A \xrightarrow{g} A$.

A $G$-map, or $G$-morphism, is a morphism $f: A \to B$ in $A$, with $A$ and $B$ $G$-objects, such that $f_g = gf$ for all $g \in G$. The category of $G$-objects and $G$-maps in $A$ is denoted $A^G$.

The aim of this section is to define a collection of functors from $A^G$ to $A^G$, under certain assumptions on $A$. The reader is referred to [6] for a reference on category theory.

Recall that given two morphisms $\alpha, \beta: A \to B$, $\mu: B \to K$ is a coequalizer for $\alpha$ and $\beta$ if $\mu \alpha = \mu \beta$, and if whenever $\mu': B \to K'$ satisfies $\mu' \alpha = \mu' \beta$, then there is a unique morphism $\gamma: K \to K'$ such that $\gamma \mu = \gamma \mu'$. Given two morphisms $f_1: A \to B_1$, $f_2: A \to B_2$, a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f_2} & B_2 \\
\downarrow{f_1} & & \downarrow{\mu_2} \\
B_1 & \xrightarrow{\mu_1} & P
\end{array}
$$

is called a pushout for $f_1$ and $f_2$ if for every commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f_2} & B_2 \\
\downarrow{f_1} & & \downarrow{\mu_2} \\
B_1 & \xrightarrow{\mu_1'} & P'
\end{array}
$$

there is a unique morphism $\gamma: P \to P'$ such that $\mu_1' = \gamma \mu_1$ and $\mu_2' = \gamma \mu_2$.

Lemma II.1. Let $A$ be a category with coequalizers and finite coproducts. Then $A$ has pushouts.
PROOF. Consider

\[
\begin{array}{c}
A \\
\downarrow \mu \circ i_1 \\
B_1
\end{array}
\begin{array}{c}
\xrightarrow{f_2} \\
\downarrow f_1 \\
B_2
\end{array}
\]

The coproduct \( B_1 \amalg B_2 \), together with the canonical morphisms \( i_j: B_j \to B_1 \amalg B_2 \), \( j = 1, 2 \), gives morphisms \( i_j \circ f_j: A \to B_1 \amalg B_2 \), \( j = 1, 2 \). Let \( \mu: B_1 \amalg B_2 \to K \) be the coequalizer for \( i_1 \circ f_1 \) and \( i_2 \circ f_2 \). Then \( \mu \circ (i_1 \circ f_1) = \mu \circ (i_2 \circ f_2) \) gives a commutative diagram

\[
\begin{array}{c}
A \\
\downarrow f_1 \\
B_1
\end{array}
\begin{array}{c}
\xrightarrow{f_2} \\
\downarrow \mu \circ i_2 \\
K
\end{array}
\]

The fact that this diagram is actually a pushout follows from the definitions of coproduct and coequalizer. \( \square \)

Given a family \( \{\mu_i: A \to A_i\}_{i \in I} \) of epimorphisms, \( \mu: A \to A' \) is the cointersection of the family if for each \( i \in I \) there exist morphisms \( \nu_i: A_i \to A' \) such that \( \mu = \nu_i \mu_i \), and if every morphism \( A \to B \) which factors through each \( \mu_i \) factors uniquely through \( \mu \).

**Lemma II.2.** If \( A \) has pushouts, then \( A \) has finite cointersections.

**Proof.** It suffices to show existence for a family of two epimorphisms \( \mu_1: A \to A_1 \), \( \mu_2: A \to A_2 \). Let

\[
\begin{array}{c}
A \\
\downarrow \mu_1 \\
A_1
\end{array}
\begin{array}{c}
\xrightarrow{\mu_2} \\
\downarrow \nu_1 \\
P
\end{array}
\begin{array}{c}
\xrightarrow{A_2} \\
\downarrow \nu_2 \\
P
\end{array}
\]

be the pushout for \( \mu_1 \) and \( \mu_2 \). Then \( \nu_1 \mu_1 = \nu_2 \mu_2: A \to P \) is the cointersection of \( \mu_1 \) and \( \mu_2 \) by the definition of pushout. \( \square \)

Let \( F_G: A \to A^G \) be the functor which sends \( A \in A \) to \( A \in A^G \) by letting \( A \xrightarrow{g} A \) be \( A \xrightarrow{1_A} A \) for all \( g \in G \). Let \( V \in A^G \). A \( G \)-orbit space of \( V \) is a pair \((O, \pi)\), where \( O \in A \) and \( \pi \in \text{Mor}_A(V, F_G(O)) \), such that whenever \( X \in A \) and \( f \in \text{Mor}_A(V, F_G(X)) \) there is a unique \( \phi \in \text{Mor}_A(O, X) \) such that \( F_G(\phi) \circ \pi = f \). When such an \( O \) exists, it is of course unique up to natural isomorphism and is denoted \( V_G \).
PROPOSITION II.3. Let $G$ be a finite group and let $A$ have coequalizers and finite coproducts. Then $(V_G, \pi)$ exists for all $V \in A^G$.

PROOF. For each pair of distinct elements $g_i, g_j \in G$, let $\mu_{g_i, g_j} : V \rightarrow K_{g_i, g_j}$ be a coequalizer for the morphisms $g_i, g_j : V \rightarrow V$. Each $\mu_{g_i, g_j}$ is an epimorphism since every coequalizer is. Let $\pi : V \rightarrow O$ be the coequalizer (exists by Lemmas II.1, II.2) of the finite family \{ $\mu_{g_i, g_j}$ $g_i, g_j \in G$ \} in $A$. This construction gives $(V_G, \pi)$:

For each $g \in G$, $\mu_{g, 1} \cdot V = \mu_{g, 1}$, so \( \pi g = \pi \) for all $g \in G$. Hence $\pi \in \text{Mor}_{AG}(V, F_G(O))$. If $f \in \text{Mor}_{AG}(V, F_G(X))$, then $f g_i = f g_j$ for all $g_i, g_j \in G$, so $f$ factors through each $\mu_{g_i, g_j}$. Thus there is a unique $\phi \in \text{Mor}_{A}(O, X)$ such that $\phi \circ \pi = f$.

REMARK II.4. Proposition II.3 says there is a functor $(\ )^G : A^G \rightarrow A$ left adjoint to $F_G : A \rightarrow A^G$, i.e., $\text{Mor}_{AG}(V, F_G(X)) \cong \text{Mor}_{A}(V_G, X)$, natural in arguments $V$ and $X$.

For each integer $n \geq 1$ and each $W_n \in A^{S_n}$ ($S_n$ is the symmetric group), a functor $\phi_{W_n} : A^G \rightarrow A^G$ will be defined. To do so, assume that $A$ has not only coequalizers and finite coproducts but also a "tensor product" $\otimes$, that is, a functor $\otimes : A \times A \rightarrow A$ which is coherently associative and commutative (see [7, Chapter I]), and which distributes with the coproduct. This insures natural isomorphisms

\[(A_1 \otimes A_2) \otimes A_3 \approx A_1 \otimes (A_2 \otimes A_3),\]
\[A_1 \otimes A_2 \approx A_2 \otimes A_1,\]
\[A_1 \otimes (A_2 \otimes A_3) \approx (A_1 \otimes A_2) \otimes (A_1 \otimes A_3),\]

such that isomorphisms between products of several factors, obtained by successively applying the above, are the same.

Fix $W_n \in A^{S_n}$. Let $T \in A$ and let $T^{\otimes n} = T \otimes T \otimes \cdots \otimes T$ ($n$ times). $T^{\otimes n} \in A^{S_n}$ via the natural isomorphisms which permute its factors. Since $\otimes$ is a functor $A \times A \rightarrow A$ it induces a functor $\otimes : A^G \times A^G \rightarrow A^G$ for any group $G$; that is if $A \xrightarrow{g} A, B \xrightarrow{g} B$, then $A \otimes B \xrightarrow{g_1 g} A \otimes B$ gives $A \otimes B$ a well-defined $G$-action by the functoriality of $\otimes$. Hence $W_n \otimes T^{\otimes n} \in A^{S_n}$. Defining $\phi_{W_n}(T)$ to be $(W_n \otimes T^{\otimes n})_{S_n}$, one obtains a functor $\phi_{W_n} : A \rightarrow A$. (For $f : T \rightarrow T', \phi_{W_n}(f) : (W_n \otimes T^{\otimes n})_{S_n} \rightarrow (W_n \otimes T'^{\otimes n})_{S_n}$ is the obvious map.)

If $T \in A^G$, $T$ comes with morphisms $T \xrightarrow{g} T$ for all $g \in G$, which induce morphisms

\[\phi_{W_n}(T) \xrightarrow{\phi_{W_n}(g)} \phi_{W_n}(T)\]
for all $g \in G$. Since $\phi_{W_n}$ is a functor, the maps $\phi_{W_n}(g)$ define a $G$-action on $\phi_{W_n}(T)$. Thus one has a functor $\phi_{W_n} : A^G \rightarrow A^G$ for any group $G$.

In conclusion, then, if $G$ is any group and if $A$ has coequalizers, finite coproducts, and a “tensor product” $\perp$, then for each positive integer and each $W_n \in A^{S_n}$, one has a functor $\phi_{W_n} : A^G \rightarrow A^G$ defined by $\phi_{W_n}(T) = (W_n \perp T^{1n})_{S_n}$.

B. The behavior of the functors $\phi_{W_n}$. The purpose of this section is to investigate the behavior of the functors $\phi_{W_n}$. To do so, one first introduces induced objects.

Suppose $H \subset G$ are groups. $A \in A^G$ may be viewed as an $H$-object via the inclusion $H \hookrightarrow G$, giving rise to a functor $Res^G_H : A^G \rightarrow A^H$. Let $W \in A^H$. An induced object of $W$ is a pair $(V, \psi)$, where $V \in A^G$ and $\psi \in Mor_{A^H}(W, Res^G_H V)$ such that whenever $X \in A^G$ and $f \in Mor_{A^G}(W, Res^G_H X)$ there is a unique $\phi \in Mor_{A^G}(V, X)$ satisfying $(Res^G_H \phi) \circ \psi = f$. When such a $V$ exists, it is unique up to natural isomorphism and is denoted $\text{Ind}_H^G W$.

**Proposition II.5.** Let $H \subset G$ be finite groups and let $A$ have coequalizers and finite coproducts. Then $(\text{Ind}_H^G W, \psi)$ exists for all $W \in A^H$.

**Proof.** Let $W \in A^H$. Form the coproduct of $W$ with itself $|G|$ times to obtain the object $\prod_{x \in G} W_x$ in $A$, which is in $A^G$ via the maps $\prod W_x \xrightarrow{\ast g} \prod W_x$, for all $g \in G$, which permute the factors; more precisely, $\ast g$ is induced by the maps $\ast g_x : W_x \rightarrow W_{gx} \subset \prod_{x \in G} W_x$, where the first morphism is $1_{W_x}$ and the second is the canonical map associated with the coproduct. In the future, $W_x \xrightarrow{1_{W_x}} W_y$ will be denoted $1_y^x$. For $h \in H$, let $\prod W \xrightarrow{h^*} \prod W_x$ be the map induced from maps

$$h^* : W_x \xrightarrow{1^x_h} W_{xh} \hookrightarrow \prod W_x,$$

and let $\prod W \xrightarrow{h} \prod W_x$ be induced from the maps $h_x : W_x \xrightarrow{h} W_x \hookrightarrow \prod W_x$, where the first map is just the action of $H$ on $W$.

Observe that $h^{**g} = \ast gh^*$ and $h(\ast g) = \ast gh$ for all $g \in G$, $h \in H$. For $h \in H$, let $\mu_h : \prod W_x \rightarrow K_h$ be the coequalizer of $h^*$ and $h$. Since $\theta(\mu_{hg}^*)h^* = \mu_h h^{**g} = (\mu_h h)^* g = (\mu_{hg}) h$ for $g \in G$, there is a unique map $K_h \xrightarrow{\theta_g} K_h$ such that the triangle

$$\prod W_x \xrightarrow{\mu_{hg}^*} K_h$$

commutes. One thus obtains a map $\theta_g$ for each $g \in G$. The uniqueness of each
\( \theta_g \) implies \( \theta_1 = 1_{K_h} \) and \( \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2} \). Hence \( K_h \in A^G \) and \( \mu_h \) is a G-map.

Let \( \mu \colon \coprod W_x \to K \) be the cointersection (exists by Lemmas II.1, II.2) of the finite family of epimorphisms \( \{\mu_h\}_{h \in H} \) in \( A \). Since \( \mu \) factors through each \( \mu_h, \mu^*g \) does also; hence for each \( g \in G \), there is a unique map \( K \xrightarrow{\rho_g} K \) such that the triangle

\[
\begin{array}{ccc}
\coprod W_x & \xrightarrow{\mu^*g} & K \\
\downarrow & & \searrow \\
K & \xrightarrow{\rho_g} & K
\end{array}
\]

commutes. The uniqueness of each \( \rho_g \) makes \( K \) a G-object and therefore \( \mu \) a G-map.

Let \( \psi \colon W \to \text{Res}^G_H K \) be the map \( \mu_1 \colon W_1 \to K \) which comes from \( \mu \colon \coprod x \in G W_x \to K \). (Here and elsewhere, \( W \) is identified with \( W_1 \).) \( K = \text{Ind}^G_H W \) by the following argument:

To show that \( \psi \) is an \( H \)-map, one must show that \( \rho_h \psi = \psi \mu_h \). The last commutative triangle and the definition of \( \mu \) give \( \rho_h \psi = \rho_h \mu_1 = \mu^*h_1 \mu_1 = \mu_1 h = \psi h \).

Suppose \( f : W \to \text{Res}^G_H X \) is an \( H \)-map. One can show that there is a unique G-map \( \tau : \coprod W_x \to X \) such that the triangle

\[
\begin{array}{ccc}
W_1 & \xrightarrow{f} & X \\
\downarrow & & \searrow \tau \\
\coprod W_x & \xrightarrow{\tau} & X
\end{array}
\]

commutes, as follows:

If such a \( \tau \) exists, then for each \( x \in G \), the square

\[
\begin{array}{ccc}
W_1 & \xrightarrow{f} & X \\
\downarrow & & \downarrow x \tau \\
\coprod W_x & \xrightarrow{\tau} & X
\end{array}
\]

commutes. Hence \( \tau_x = xf1^1_x \) for all \( x \in G \). Thus \( \tau \) is unique. For existence, define \( \tau \) by \( \tau_x = xf1^1_x \) for all \( x \in G \).

Moreover, \( \tau h^* = \tau h \) for all \( h \in H \) since \( (\tau h^*)_x = \tau_{xh}1^x_{xh} = xhf1^1_{xh}1_{xh} = xhf1^1_x \), \( (\tau h)_x = \tau_x h = xfh1^1_x = xfh1^1 \), and \( f \) is an \( H \)-map. Hence \( \tau \) factors through \( \mu_h \) for all \( h \in H \). It follows that there is a unique \( \phi : K \to X \) satisfying \( \tau = \phi \mu \). By the fact that \( \mu \) and \( \tau \) are G-maps and by the uniqueness of \( \phi, \) \( \phi \) is itself a G-map: \( (g\phi \varphi g^{-1})_{h^*} = g\phi \varphi g^{-1} = g\varphi g^{-1} \tau = \tau = g\varphi \rho g^{-1} = \rho \phi \rho g^{-1} \).

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The commutativity of the two small triangles in the figure

implies the commutativity of the large triangle. Hence $\phi$ is a $G$-map satisfying $(\text{Res}_H^G \phi) \circ \psi = f$. In addition, if $\phi_1$ makes the diagram

commute, then $\phi_1 \mu = \tau$ by uniqueness of $\tau$, and so $\phi_1 = \phi$ by uniqueness of $\phi$.

**Remark II.6.** Proposition II.5 says there is a functor $\text{Ind}_H^G : \text{AH} \to \text{AG}$ left adjoint to $\text{Res}_H^G : \text{AG} \to \text{AH}$, i.e., $\text{Mor}_{\text{AH}}(W, \text{Res}_H^G V) \approx \text{Mor}_{\text{AG}}(\text{Ind}_H^G W, V)$, natural in arguments $W$ and $V$.

**Proposition II.7.** Let $G$ be a group and $A$ a category with finite co-products. Suppose $V \in \text{AG}$ and $V = \coprod_{i=1}^n W_i$ as an object in $A$. Assume $G$ permutes the $W_i$'s transitively, that is, each $g_i : W_i \to \coprod_{i=1}^n W_i$ looks like $W_i \to W_j \subset \coprod_{i=1}^n W_i$ for some $j$ and some morphism $W_i \to W_j$, and given any $i, j$, there is a $g \in G$ such that $g_i : W_i \to W_j \subset \coprod_{i=1}^n W_i$.

Let $W_{i_0}$ be one of the $W_i$'s and let $H$ be its isotropy group, i.e., $H = \{ g \in G | g_{i_0} : W_{i_0} \to W_{i_0} \subset \coprod W_i \}$. Then as an object in $\text{AG}$, $V = \text{Ind}_H^G W_{i_0}$.

**Proof.** If $g_i : W_i \to W_j \subset \coprod W_i$, denote the map $W_i \to W_j$ by $g_i$. Since $gg^{-1}g = 1_w$, $g_i : W_i \to W_j \subset \coprod W_i$ implies $g \cdot W_i \to W_i \subset \coprod W_i$, and $g_i(g^{-1})_j = 1_{W_j}$. Let $\psi : W_{i_0} \to \text{Res}_H^G V$ be the canonical map $W_{i_0} \subset \coprod_{i=1}^n W_i$. $\psi$ is clearly an $H$-map. To show $V = \text{Ind}_H^G W_{i_0}$, one need only show that $V$ satisfies the appropriate universal property.

Suppose $f \in \text{Mor}_{\text{AH}}(W_{i_0}, \text{Res}_H^G X)$. If there is a $G$-map $\phi : V \to X$ such that $(\text{Res}_H^G \phi) \circ \psi = f$, then for each $g \in G$ there is a commutative diagram
Given \( i \), let \( g \in G \) be such that \( g_{i_0} : W_{i_0} \rightarrow W_i \hookrightarrow \amalg W_i \). Then \( \phi_i = \phi_{g_{i_0}}(g^{-1})_{i_0} = gf(g^{-1})_{i_0} \). Thus such a \( \phi \) is unique.

To show existence, define \( \phi \) by \( \phi_i = gf(g^{-1})_{i_0} \), where \( g \in G \) such that \( g_{i_0} : W_{i_0} \rightarrow W_i \hookrightarrow \amalg W_i \). \( \phi \) is well defined: If \( \tilde{g} , g : W_{i_0} \rightarrow W_i \hookrightarrow \amalg W_i \), then \( \tilde{g}^{-1}g \in H \), so that \( \tilde{g}^{-1}gf = f(\tilde{g}^{-1})_{i_0} = f(g^{-1})_{i_0}g_i \); hence \( gf(g^{-1})_{i_0} = \tilde{g}f(\tilde{g}^{-1})_{i_0} \).

**Lemma II.8.** Let \( A \) have coequalizers and finite coproducts, and let \( G \) be a finite group. Then

\[
(A \amalg B)_G \cong A_G \amalg B_G,
\]

natural in arguments \( A \) and \( B \).

**Proof.** There is a natural isomorphism

\[
\text{Mor}_A((A \amalg B)_G, X) \cong \text{Mor}_{A G}(A \amalg B, F_G(X))
\]

(Remark II.4). Since adjoints are unique, one need only show

\[
\text{Mor}_A(A_G \amalg B_G, X) \cong \text{Mor}_{A G}(A \amalg B, F_G(X)).
\]

But

\[
\text{Mor}_A(A_G \amalg B_G, X) \cong \text{Mor}_{A G}(A_G, X) \times \text{Mor}_{A G}(B_G, X)
\]

\[
\cong \text{Mor}_{A G}(A, F_G(X)) \times \text{Mor}_{A G}(B, F_G(X)) \cong \text{Mor}_{A G}(A \amalg B, F_G(X)).
\]

**Theorem II.9.** Let \( G \) be a group, and let \( A \) have finite coproducts, coequalizers, and a "tensor product" \( \otimes \). Then if \( W_n, W'_n \in A^{S_n} \),

\[
\phi_{W_n \amalg W'_n}(T) = \phi_{W_n}(T) \amalg \phi_{W'_n}(T)
\]

for all \( T \in A^G \).

**Proof.**

\[
\phi_{W_n \amalg W'_n}(T) = (W_n \amalg W'_n) \downarrow T^{\amalg n})_{S_n}
\]

\[
\cong (W_n \downarrow T^{\amalg n}) \amalg (W'_n \downarrow T^{\amalg n}))_{S_n}
\]

\[
\cong (W_n \downarrow T^{\amalg n})_{S_n} \amalg (W'_n \downarrow T^{\amalg n})_{S_n} \quad \text{(by Lemma II.8)}
\]

\[
= \phi_{W_n}(T) \amalg \phi_{W'_n}(T)
\]

**Lemma II.10.** Let \( A \) have finite coproducts and coequalizers, and let
Let \( K \subset H \subset G \) be finite groups. Let \( U \in A^K \) and \( W, W' \in A^H \). Then

\[(i) \text{ Ind}_H^G(W \uparrow W') \approx \text{ Ind}_H^G(W \downarrow \text{ Ind}_H^G(W'), \]
\[(ii) \text{ Ind}_H^G(\text{ Ind}_K^H(U)) \approx \text{ Ind}_H^G(U), \]
\[(iii) (\text{ Ind}_H^G(W))_G \approx W_H. \]

All the above isomorphisms are natural.

Proof. (i) The result follows from the adjointness of \( \text{ Res}_K^G \) and \( \text{ Ind}_H^G \) (Remark II.6) and an argument analogous to the one for Lemma II.8.

(ii) This result follows from the uniqueness of adjoints and the obvious fact that \( \text{ Res}_K^G(\text{ Res}_K^G(V)) \approx \text{ Res}_K^G(V). \)

(iii) The proof, which is similar to the preceding one, uses the adjointness of \( \text{ Res}_K^G \) and \( \text{ Ind}_H^G \) and of \( F_G \) and \( (\ )_G \), and the fact that \( \text{ Res}_K^G(F_G(X)) \approx F_H(X). \)

Lemma II.11 (Frobenius Reciprocity). Let \( H \subset G \) be finite groups, and let \( A \) have finite coproducts, coequalizers, and a "tensor product" \( \bot \). Assume there is a functor \( \text{ Hom} : A^\circ \times A \rightarrow A \) such that

\[\text{ Mor}_A(A \bot B, C) \approx \text{ Mor}_A(A, \text{ Hom}(B, C)), \]

natural in \( A, B, C \). Then for \( W \in A^H, V \in A^G, \)

\[(\text{ Ind}_H^G(W) \bot V) \approx \text{ Ind}_H^G(W \downarrow \text{ Res}_K^G(V)). \]

Proof. The functoriality of \( \text{ Hom} : A^\circ \times A \rightarrow A \) induces the functor \( \text{ Hom} : (A^G)^\circ \times A \rightarrow A \), and clearly

\[\text{ Res}_K^G(\text{ Hom}(V, X)) \approx \text{ Hom}(\text{ Res}_K^G(V), \text{ Res}_K^G(X)). \]

The lemma now follows from the standard argument using uniqueness of adjoints.

The following theorem gives a useful simplification for some of the functors \( \phi_{W_n} \) in the special case of the existence of an object \( 1 \) in \( A \) such that \( A \bot 1 \approx A \), natural and coherent in the sense of II, §A. The object \( F_G(1) \in A^G \) will be denoted \( 1_G \), or simply \( 1 \).

Theorem II.12. Let \( G \) be a group, let \( A \) have an object \( 1 \) and be as in Lemma II.11, and let \( \text{ Hom} \) exist. Let \( H \subset S_n, \ T \in A^G \), and \( W_n = \text{ Ind}_H^S(1) \). Then \( \phi_{W_n}(T) = (\text{ Res}_H^S(T^\bot n))_H \).

Proof.

\[\phi_{W_n}(T) = ((\text{ Ind}_H^S(1) \bot T^\bot n))_S \]

\[\approx (\text{ Ind}_H^S(1 \bot \text{ Res}_H^S(T^\bot n)))_S \] (by Lemma II.11)

\[\approx (\text{ Ind}_H^S(\text{ Res}_H^S(T^\bot n)))_S \approx (\text{ Res}_H^S(T^\bot n))_H \] (by Lemma II.10).
Examples 11.13. (i) If $W_n = \text{Ind}_1^{S_n} 1$, then $\phi_{W_n}(T) = (\text{Res}_1^{S_n}(T^{1_n}))_1 = T^{1_n}$.

(ii) If $W_n = \text{Ind}_1^{S_n} 1$, then $\phi_{W_n}(T) = (\text{Res}_1^{S_n}(T^{1_n}))_{S_n} = (T^{1_n})_{S_n}$ is the $n$th symmetric power of $T$.

If $G$ and $H$ are groups, $A \in A^G, B \in A^H$, then the morphisms $A \xrightarrow{g} A, B \xrightarrow{h} B$ induce the morphism $A \perp B \xrightarrow{g \perp h} A \perp B$, thereby making $A \perp B \in A^G \times H$ ($G \times H$ is the direct product of $G$ and $H$). In this setting, one has the following lemma:

Lemma 11.14. Let $G$ and $H$ be finite groups, and let $A$ have finite coproducts, coequalizers, and a "tensor product" $\perp$. Assume there is a functor $\text{Hom}$, as in Lemma 11.11. If $A \in A^G, B \in A^H$, then $(A \perp B)_G \times H \approx A_G \perp B_H$.

Proof. Because of the uniqueness of adjoints, one need only show $\text{Mor}_{A^G \times H}(A \perp B, F_G \times H(X)) \approx \text{Mor}_A(A_G \perp B_H, X)$.

This follows from the following chain of natural isomorphisms, each easily verifiable:

$\text{Mor}_{A^G \times H}(A \perp B, F_G \times H(X))$

$\approx \text{Mor}_{(A^H)_G}(F_H(A) \perp F_G(B), F_G(F_H(X)))$

$\approx \text{Mor}_{(A^H)_G}(F_H(A), \text{Hom}(F_G(B), F_G(F_H(X))))$

$\approx \text{Mor}_{(A^H)_G}(F_H(A), F_G(\text{Hom}(B, F_H(X))))$

$\approx \text{Mor}_{A^H}((F_H(A))_G, \text{Hom}(B, F_H(X)))$

$\approx \text{Mor}_{A^H}(F_H(A_G), \text{Hom}(B, F_H(X)))$

$\approx \text{Mor}_{A^H}(F_H(A_G) \perp B, F_H(X))$

$\approx \text{Mor}_{A^H}((B \perp F_H(A_G)), F_H(X))$

$\approx \text{Mor}_{A^H}((B \perp F_H(A_G), F_H(X))$

$\approx \text{Mor}_{A^H}(B \perp F_H(A_G), F_H(X))$

$\approx \text{Mor}_{A^H}(B \perp F_H(A_G), F_H(X))$

$\approx \text{Mor}_{A^H}(B, F_H(\text{Hom}(A_G, X))) \approx \text{Mor}_A(B_H, \text{Hom}(A_G, X))$

$\approx \text{Mor}_A(B_H \perp A_G, X) \approx \text{Mor}_A(A_G \perp B_H, X)$

In the next theorem, $S_n \times S_m$ is viewed as a subgroup of $S_{n+m}$ by viewing $S_n$ as permuting the symbols $1, 2, \ldots, n$, $S_m$ the symbols $n + 1, n + 2, \ldots, n + m$, and $S_{n+m}$ the symbols $1, 2, \ldots, n + m$.

Theorem 11.15. Let $G$ be a group, let $A$ have finite coproducts, coequal-
izers, a “tensor product” ⊥, and an object 1. Assume there is a functor \( \text{Hom} \) as in Lemma II.11. Let \( W_n \in A^{S_n}, W_m \in A^{S_m}, \) and \( T \in A^G \). If \( W_{n+m} = \text{Ind}^{S_{n+m}}_{S_n \times S_m} W_n \perp W_m \), then \( \phi_{W_{n+m}}(T) = \phi_{W_n}(T) \perp \phi_{W_m}(T) \).

**Proof.**

\[
\phi_{W_{n+m}}(T) = (\text{Ind}^{S_{n+m}}_{S_n \times S_m} W_n \perp W_m \perp T^{\perp_{n+m}})_{S_{n+m}}
\]

\[
\approx (\text{Ind}^{S_{n+m}}_{S_n \times S_m} (W_n \perp W_m \perp \text{Res}^{S_{n+m}}_{S_n \times S_m} T^{\perp_{n+m}}))_{S_{n+m}} \quad \text{(by Lemma II.11)}
\]

\[
\approx (W_n \perp W_m \perp \text{Res}^{S_{n+m}}_{S_n \times S_m} T^{\perp_{n+m}})_{S_n \times S_m} \quad \text{(by Lemma II.10)}
\]

\[
\approx ((W_n \perp T^{\perp_n}) \perp (W_m \perp T^{\perp_m}))_{S_n \times S_m}
\]

\[
\approx (W_n \perp T^{\perp_n})_{S_n} \perp (W_m \perp T^{\perp_m})_{S_m} \quad \text{(by Lemma II.14)}
\]

\[
= \phi_{W_n}(T) \perp \phi_{W_m}(T). \quad \square
\]

C. The main theorem and corollary. Let \( G \) be a group and \( A \) have finite coproducts, a “tensor product” \( \perp \), and an object 1. Define the Grothendieck ring \( K_0(A^G) \) to consist of all finite formal sums \( \sum n_i[T_i] \) \((n_i \in \mathbb{Z})\) of \( G \)-objects \( T_i \) in \( A \), modulo the relations

(i) \( [T_1] = [T_2] \) if \( T_1 \cong T_2 \) as \( G \)-objects,

(ii) \( [T_1] \perp [T_2] = [T_1] + [T_2] \).

Clearly, \( K_0(A^G) \) is an abelian group; the “tensor product” \( \perp \), together with the object 1 in \( A^G \), gives \( K_0(A^G) \) the structure of a commutative ring with identity, i.e. \( [T_1][T_2] = [T_1 \perp T_2] \). When the meaning is clear, brackets will be omitted, e.g., \( [T_1] - [T_2] \) will appear as \( T_1 - T_2 \).

**Examples** II.16. (i) Let \( G \) be a finite group and \( A \) the category of finite sets. Then \( A^G = G\text{-sets} \). Let \( \perp \) be the cartesian product, and 1 be any one-element \( G \)-set. Then \( K_0(A^G) \) is the Burnside ring of \( G \), \( B(G) \). (See I, §B.)

(ii) Let \( G \) be a finite group and \( A \) the category of finite-dimensional vector spaces over \( C \). Then \( A^G = G\text{-modules} \). Let \( \perp \) be the tensor product \( \otimes \), and 1 be the one-dimensional \( G \)-module with trivial \( G \)-action. Then \( K_0(A^G) \) is the representation ring of \( G \), \( R(G) \) (see I, §C).

**Remark** II.17. In the above examples, \( [T_1] = [T_2] \) implies \( T_1 \cong T_2 \) as \( G \)-objects (see I, §B, §C). This is not the case in general; in particular, if \( A \) is the category of vector bundles over a space \( X \), then \( [E] = [F] \) implies only that \( E \oplus n \cong F \oplus n \), where \( n \) is the trivial bundle of dimension \( n \) [3, Appendix].
Let $H \subset G$ be finite groups. Let $A$ have finite coproducts, coequalizers, a “tensor product” $\bot$, and an object $1$. $P(A^G)$ is defined to be the subring of $K_0(A^G)$ generated by $\{\text{Ind}_H^G 1|H \text{ a subgroup of } G\}$.

**Proposition II.18.** Let $H \subset G$ be finite groups. If $W = 1_H$, then $\text{Ind}_H^G W = \prod_{x \in G/H} 1_x$ with $G$-action given by

$$g : 1_x \xrightarrow{1^g} 1_{g \cdot x} \xrightarrow{\prod} \prod 1_x.$$

**Proof.** Let $\psi : 1_\Gamma \hookrightarrow \prod 1_x$. One need only show that $(\prod 1_x, \psi)$ satisfies the appropriate universal property. Clearly, $\psi : 1_\Gamma \rightarrow \text{Res}_G^H(\prod 1_x)$ is an $H$-map. If $f \in \text{Mor}_{A^H}(1_H, \text{Res}_G^H X)$, there is a unique $f \in \text{Mor}_{A^G}(\prod 1_x, X)$ such that $(\text{Res}_G^H f) \circ \psi = f$, namely $f_{x \cdot \overline{g}}$ for all $x \in G/H$.

**Proposition II.19.** Every element in $P(A^G)$ is of the form $\sum n_i \text{Ind}_H^{G_i} 1$, where $n_i \in \mathbb{Z}$ and $H_i$ is a subgroup of $G$.

**Proof.**

$$\text{(Ind}_H^{G_1}) \bot (\text{Ind}_K^{G_2}) \approx \left( \prod_{x \in G/H} 1_x \right) \bot \left( \prod_{y \in G/K} 1_y \right) \quad \text{(by Proposition II.18)}$$

$$\approx \prod_{x, y} (1_x \bot 1_y)$$

$$\approx \prod_{x, y} 1_{(x, y)} \approx \prod_{G\text{-orbits } \alpha} \left( \prod_{(x, y) \in \alpha} 1_{(x, y)} \right)$$

$$\approx \prod_{G\text{-orbits } \alpha} \text{Ind}_H^G 1 \quad \text{(by Proposition II.7).} \square$$

The canonical $S_k$-object in $A$, denoted $X_k$, is defined to be $\text{Ind}_{S_1 \times S_{k-1}} 1$.

**Remark II.20.** From Proposition II.18, it follows that

$$X_k = \prod_{\overline{\sigma} \in S_k/(S_1 \times S_{k-1})} 1_{\overline{\sigma}}.$$

Since $\overline{\sigma} = \overline{\tau}$ in $S_k/(S_1 \times S_{k-1})$ iff $\tau(1) = \sigma(1)$, each $S_1 \times S_{k-1}$-orbit of $S_k$ consists of precisely those $\sigma \in S_k$ which send 1 to the same symbol $j$.

Hence $X_k = \prod_{j=1}^k 1_j$, where $\sigma \in S_k$ acts by

$$1_j \xrightarrow{1^\sigma} 1_{\sigma(j)} \xrightarrow{\prod} \prod 1_j.$$

For examples, see I.1(iii) and I.6(ii).

Let $G$ be a group, and $A$ have finite coproducts, coequalizers, a “tensor product” $\bot$, and object 1. Let $\phi : A^G \rightarrow A^G$ be the functor sending $A \in$
$A^G$ to $I_G$. $\phi_0$ and the functors $\phi_{W_n}$ arising from all positive integers $n$ and all $W_n \in A^{S_n}$ (see II, §A) will be called $S$-operations. If $A = G$-modules (see I, §C), the $S$-operations generate what are known as $\lambda$-operations. If $A = G$-sets (see I, §B), the $S$-operations will be referred to as $\beta$-operations.

For $T \in A^G$, let $\langle T \rangle$ denote the subring of $K_0(A^G)$ generated by $\{[\phi(T)] | \phi$ an $S$-operation\}. If there is a functor $\text{Hom}$ as in Lemma II.11, then Theorem II.15 says that every element in $\langle T \rangle$ is a finite sum $\sum q_{\alpha} [\phi_{\alpha}(T)]$, where $q_{\alpha} \in \mathbb{Z}$, $\phi_{\alpha}$ an $S$-operation.

Summarizing, the ring $K_0(A^G)$ and subrings $P(A^G)$ and $\langle T \rangle$ have been constructed. The main theorem and its immediate corollary apply when $G = S_k$ and $T = X_k$:

**Main Theorem II.21.** Let $A$ have finite coproducts, coequalizers, a "tensor product" $\perp$, and an object $1$. Assume there is a functor $\text{Hom}$ as in Lemma II.11. Then for each positive integer $k$, $P(A^{S_k}) \subset \langle X_k \rangle$.

**Corollary II.22.** Same hypothesis as above. Suppose $P(A^{S_k}) = K_0(A^{S_k})$. Then $\langle X_k \rangle = K_0(A^{S_k})$.

**Lemma II.23.** Same hypothesis as above. Let $H \subset S_n$ and $W_n = \text{Ind}_{H}^{S_n} 1$. Then

$$\phi_{W_n}(X_k) = \bigcup_{\gamma} \text{Ind}_{H}^{S_k} 1,$$

for some collection of subgroups $H_\gamma$ of $S_k$. Here, $\gamma_1 \neq \gamma_2$ need not imply $H_{\gamma_1} \neq H_{\gamma_2}$.

This lemma does not imply $\langle X_k \rangle \subset P(A^{S_k})$, since $\langle X_k \rangle$ is obtained from all $S$-operations $\phi_{W_n}$, and if $P(A^{S_n}) \neq K_0(A^{S_n})$, $W_n$ need not be a linear combination of objects $\text{Ind}_{H}^{S_n} 1$.

**Proof of Lemma.** By Theorem II.12, $\phi_{W_n}(X_k) = (\text{Res}_{H}^{S_n}(X_k^{1^n}))_H$. Since

$$X_k^{1^n} = \left( \bigotimes_{j=1}^{k} 1_j \right)^{1^n} \approx \bigotimes_{1 \leq i \leq k} (1_{j_1} \perp \cdots \perp 1_{j_n})_{1 \leq j_i \leq k},$$

we have

$$\phi_{W_n}(X_k) = \left( \text{Res}_{H}^{S_n} \left( \bigotimes_{1 \leq j_i \leq k} 1_{j_1, \ldots, j_n} \right) \right)_H.$$
σ ∈ S_n acts on \( \Pi 1_{j_1, \ldots, j_n} \) by

\[
\sigma: 1_{\{j_1, \ldots, j_n\}} \xrightarrow{1_1} 1_{\sigma^{-1}(\{j_1, \ldots, j_n\})} \to \Pi 1_{\{j_1, \ldots, j_n\}}
\]

and \( g ∈ S_k \) by

\[
g: 1_{\{j_1, \ldots, j_n\}} \xrightarrow{1_1} 1_{\{g(j_1), \ldots, g(j_n)\}} \to \Pi 1_{\{j_1, \ldots, j_n\}}
\]

moreover, \( \sigma g = g \sigma: \Pi 1_{\{j_1, \ldots, j_n\}} \to \Pi 1_{\{j_1, \ldots, j_n\}} \).

Let \( J = \{(j_1, \ldots, j_n) | 1 ≤ j_i ≤ k\} \). By the above, \( H ∈ S_n \) acts on the set \( J \). For \( j ∈ J \), let \( \bar{j} \) denote the orbit \( H_j \). Using equation (1), it is not hard to show that \( \phi_{w^n}(x_k) = \Pi h\text{-orbits}_j 1_f \), where \( g ∈ S_k \) acts by \( g: 1_f \xrightarrow{1_1} 1_{gj} \to \Pi 1_f \). Let

\[
π: \Pi 1_f \to \Pi 1_f
\]

be the map induced from \( 1_f \xrightarrow{1_1} 1_f \to \Pi 1_f \). It is straightforward to show that \( (\Pi 1_f, π) \) satisfies the universal property defining \( (\Pi_{j∈J} 1_f)_H \). Hence as an object in \( A \), \( \phi_{w^n}(x_k) = \Pi 1_f \). Since, for \( g ∈ S_k \), \( (ng)h = nhg = ng: \Pi 1_f \to \Pi 1_f \) for all \( h ∈ H \), there is a unique map \( \Pi 1_f \xrightarrow{g} \Pi 1_f \) such that the diagram

\[
\begin{array}{ccc}
\Pi 1_f & \xrightarrow{g} & \Pi 1_f \\
\downarrow & & \downarrow \\
\Pi 1_f & \xrightarrow{g} & \Pi 1_f
\end{array}
\]

commutes, and hence is determined by the commutative diagram:

\[
\begin{array}{ccc}
1_f & \xrightarrow{1_g} & 1_{fg} \\
| & & | \\
1_f & \xrightarrow{1_g} & 1_{fg}
\end{array}
\]

Thus \( \phi_{w^n}(x_k) = \Pi h\text{-orbits}_j 1_f \), and \( S_k \) permutes the \( 1_f \)'s by permuting the \( H \)-orbits \( \bar{f} \). Therefore,

\[
\phi_{w^n}(x_k) \approx \Pi_{j∈J} 1_f \approx \bigcup_{S_k\text{-orbits}_γ} \left( \bigcup_{f∈γ} 1_f \right) \approx \text{Ind}_{H,γ}^{S_k} 1_f_0
\]

(by Proposition II.7), where \( 1_f_0 \) is one of the \( 1_f \)'s and \( H_γ \) is its isotropy group. □

Let \( H ⊂ S_k \) and \( m \) be a nonnegative integer. \( H \) is said to be divisible.
by $S_m$ if $H$ is conjugate to $M \times S_m$ (as subgroups of $S_k$) for some subgroup $M$ of $S_{k-m}$. Here, "$H$ conjugate to $M \times S_0$" means $H$ is conjugate to some subgroup $M$ of $S_k$; "$H$ conjugate to $M \times S_k$" means $H$ is conjugate to $S_k$.

Clearly $H$ is always divisible by $S_0$.

**Proof of main theorem.** It is enough to show that if $H \subseteq S_k$ is divisible by $S_m$ for some $m$, $0 \leq m \leq k$, then $\text{Ind}_{H}^{S_k} 1 \in \langle X_k \rangle$. The proof is by induction (backwards) on $m$:

(i) If $m = k$, then $H = S_k$ and $\text{Ind}_{H}^{S_k} 1 = 1_{S_k} = \phi_0(X_k) \in \langle X_k \rangle$.

(ii) Suppose $m < k$ and assume that if $m < m' < k$, then $H$ divisible by $S_{m'}$, $\Rightarrow \text{Ind}_{H}^{S_k} 1 \in \langle X_k \rangle$. Let $H$ be divisible by $S_m$. Then $\text{Ind}_{H}^{S_k} 1 = \text{Ind}_{M \times S_m}^{S_k} 1$ for some $M \subseteq S_{k-m}$. Let $W_{k-m} = \text{Ind}_{M \times S_m}^{S_k} 1$. Lemma II.23 gives

$$\phi_{W_{k-m}}(X_k) = \prod_{S_k\text{-orbits } \gamma} \left( \prod_{j \in \gamma} 1_j \right) = \prod_{\gamma} \text{Ind}_{H_{\gamma}}^{S_k} 1.$$ 

Recall that $J = \{(i_1, \cdots, i_{k-m}) | 1 \leq i_j \leq k\}$ is an $S_k$-set and an $M$-set, that $S_k$ permutes the $M$-orbits $j$ of $J$, and that $\gamma$ runs through the $S_k$-orbits of the set of $M$-orbits of $J$.

Direct computation shows that the $M$-orbit $(1, \cdots, k-m)$, which is in some $S_k$-orbit $\gamma_0$, has isotropy group $H_{\gamma_0} = M \times S_m$. Moreover, $(i_1, \cdots, i_{k-m}) \in \gamma_0$ whenever all the $j_i$'s are distinct, since $S_k$ is $(k-m)$-fold transitive. Thus if $(i_1, \cdots, i_{k-m}) \in \gamma \neq \gamma_0$, $i_i = i_t$ for some $i \neq t$; hence its isotropy group $H_{\gamma}$ is of the form $K \times S_{m'}$, for some $m' > m$ and $K \subseteq S_{k-m}$. Since $H_{\gamma}$ is divisible by $S_{m'}$, for some $m' > m$ if $\gamma \neq \gamma_0$, $\text{Ind}_{H_{\gamma}}^{S_k} 1 \in \langle X_k \rangle$ for all $\gamma \neq \gamma_0$ by induction hypothesis. Thus

$$\text{Ind}_{H}^{S_k} 1 = \text{Ind}_{M \times S_m}^{S_k} 1 = \phi_{W_{k-m}}(X_k) - \sum_{\gamma \neq \gamma_0} \text{Ind}_{H_{\gamma}}^{S_k} 1 \in \langle X_k \rangle.$$ 

The proof is completed by induction.\(\square\)

### III. Applications and Open Questions

**A. Unigeneration of the $\lambda$-ring $R(S_k)$.** It is well known that $R(S_k)$ is a free $\mathbb{Z}$-module with basis $\{\text{Ind}_{S_{k_1} \times S_{k_2} \times \cdots \times S_{k_j}}^{S_k} 1 | k_i \geq 1, \Sigma_i k_i = k\}$ [5, Chapter III]. Therefore, $P(A^{S_k}) = K_0(A^{S_k}) = R(S_k)$, where $A = \text{finite-dimensional vector spaces over } \mathbb{C}$. Corollary II.22 now implies $R(S_k) = \langle X_k \rangle$, and Theorem II.15 gives that every element of $R(S_k)$ is a linear combination of $\{[\phi(X_k)] | \phi$ an $S$-operation$\}$.

Moreover, every $S$-operation is a linear combination of symmetric power operations:

$$W_n \in R(S_n) \Rightarrow [W_n] = [W'_n] - [W''_n],$$

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where
\[ W'_n = \sum \alpha_j \text{Ind}_{S_{n_1} \times \cdots \times S_{n_s}}^{S^n} 1, \quad W''_n = \sum \beta\nu \text{Ind}_{S_{\nu_1} \times \cdots \times S_{\nu_t}}^{S^n} 1, \]

with \( \alpha_j, \beta\nu \) positive integers.

\[ [W'_n] = [W_n] + [W''_n] = [W_n \amalg W'_n] \Rightarrow W'_n \cong W_n \amalg W''_n \quad \text{(See Remark II.17)} \]

\[ \Rightarrow \phi_{W'_n}(T) = \phi_{W_n}(T) \amalg \phi_{W''_n}(T) \quad \text{(by Theorem II.9)} \]

\[ \Rightarrow [\phi_{W'_n}(T)] = [\phi_{W_n}(T)] \amalg [\phi_{W''_n}(T)]. \]

Since
\[ \text{Ind}_{S_{n_1} \times S_{n_2}}^{S^n} 1 = \text{Ind}_{S_{n_1} \times S_{n_2}}^{S^n} (1 \otimes 1), \]

etc., Theorem II.15 implies \( \phi_{W'_n}(T) \) is a linear combination of \( \{h_{n_1}(T) \otimes h_{n_2}(T) \otimes \cdots \otimes h_{n_s}(T)|n_i \geq 0\} \), where \( h_0 = \phi_0 \), and \( h_n = \phi_{W_n} \) for \( n > 0 \) and \( W_n = \text{Ind}_{S^n}^{S^n} 1 \). The \( h_i \)'s are, of course, symmetric power operations (see Example II.13(ii)).

Combining the two paragraphs above, one obtains the result that every element of \( R(S_k) \) is a linear combination of \( \{h_{n_1}(X_k) \otimes \cdots \otimes h_{n_s}(X_k)|n_i \geq 0\} \). Thus \( R(S_k) \) is generated by the single element \( X_k \) if symmetric powers are included with the standard ring operations. Since \( \lambda \)-operations generate symmetric power operations \([2],[5]\), \( X_k \) generates \( R(S_k) \) as a \( \lambda \)-ring.

Remark III.1. Although \( R(S_k) \) is unigenerated as a \( \lambda \)-ring, it is not unigenerated as a ring, i.e., \( R(S_k) \neq \mathbb{Z}[T] \) for all \( T \in R(S_k) \). The first counter-example is \( R(S_4) \):

If \( R(S_4) = \mathbb{Z}[T] \), then the ring \( \mathbb{Z}/2 \otimes_{\mathbb{Z}} R(S_4) \) is unigenerated as a \( \mathbb{Z}/2 \)-module. Since \( R(S_4) \) is a free \( \mathbb{Z} \)-module of rank 5 (see Proposition I.9), \( \mathbb{Z}/2 \otimes_{\mathbb{Z}} R(S_4) \) is a free \( \mathbb{Z}/2 \)-module of rank 5. By writing out its multiplication table \( (\mathbb{Z}/2 \otimes_{\mathbb{Z}} R(S_4) \) has only \( 2^5 \) elements), one can show that no element generates all of \( \mathbb{Z}/2 \otimes_{\mathbb{Z}} R(S_4) \).

B. A unigeneration theorem for \( B(S_k) \). \( B(G) \) is a free \( \mathbb{Z} \)-module with basis \( \{G/H_a\} \), where \( \{H_a\} \) is a set of representatives of the conjugacy classes of subgroups of \( G \) (Proposition I.5). Clearly, if \( A = \text{finite sets} \), then \( \text{Ind}_H^G 1 = \) the \( G \)-set \( G/H \) (see Proposition II.18). Thus \( P(A^G) = K_0(A^G) = B(G) \). Hence Corollary II.22 implies that \( B(S_k) = \langle X_k \rangle \). Thus \( S \)-operations (here called “\( \beta \)-operations”) applied to \( X_k \) generate all of \( B(S_k) \).

Remark III.2. \( B(S_k) \) is not, in general, generated by one element as a ring, since the ring homomorphism \( B(G) \rightarrow R(G) \) defined by \( T \mapsto \text{vector} \).
space with basis \( \{ \nu_t \}_{t \in T} \) (see Examples 1.6(ii), (ii')) is onto if \( G = S_k \) [5, Chapter III], and therefore the ring \( B(S_4) \) is not unigenerated since \( R(S_4) \) is not (see Remark III.1).

**Remark III.3.** Applying sums and products of symmetric power operations \( h_n \) to \( X_k \) does not, in general, give all of \( B(S_k) \). \( B(S_3) \) is a counterexample:

The nonconjugate subgroups of \( S_3 \) are 1, \( S_1 \times S_2 \), \( S_3 \) and \( A_3 \) (= the even permutations in \( S_3 \)). Proposition 1.5 now says that \( B(S_3) \) is a free \( \mathbb{Z} \)-module with basis \( S_3/1, S_3/S_1 \times S_2, S_3/S_3, S_3/A_3 \). Note that \( S_3/S_1 \times S_2 = X_3 \) (see Proposition II.19) and \( S_3/S_3 = 1 \). \( B(S_3) \) is now completely described by the following multiplication table, which is obtained easily by direct calculation using Propositions I.2, I.3, and I.4:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>( S_3/A_3 )</th>
<th>( X_3 )</th>
<th>( S_3/1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( S_3/A_3 )</td>
<td>( X_3 )</td>
<td>( S_3/1 )</td>
</tr>
<tr>
<td>( S_3/A_3 )</td>
<td>( S_3/A_3 )</td>
<td>2( S_3/A_3 )</td>
<td>( S_3/1 )</td>
<td>2( S_3/1 )</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>( X_3 )</td>
<td>( S_3/1 )</td>
<td>( S_3/1 + X_3 )</td>
<td>3( S_3/1 )</td>
</tr>
<tr>
<td>( S_3/1 )</td>
<td>( S_3/1 )</td>
<td>2( S_3/1 )</td>
<td>3( S_3/1 )</td>
<td>6( S_3/1 )</td>
</tr>
</tbody>
</table>

Now suppose that the symmetric power operations \( h_n \) applied to \( X_3 \) give all of \( B(S_3) \). Then, in particular, \( S_3/A_3 \) could be expressed as a finite sum

\[
\sum a_{i_1 \ldots i_s} h_{n_{i_1}}(X_3) h_{n_{i_2}}(X_3) \cdots h_{n_{i_s}}(X_3), \quad \text{where } a_{i_1 \ldots i_s} \in \mathbb{Z}.
\]

From the multiplication table, it is clear that one of the \( h_n(X_3) \)'s above must be of the form \( n_1 1 + n_2 S_3/A_3 + n_3 X_3 + n_4 S_3/1 \), with \( n_2 \neq 0 \). But for all \( n \geq 0 \), \( h_n(X_3) = n_1 1 + n_3 X_3 + n_4 S_3/1 \) for some \( n_i \in \mathbb{Z} \): An element \( (x_1, x_2, \ldots, x_n) \) in an \( S_n \)-orbit of \( X_3^n \) is made up of \( \mu_1 1's, \mu_2 2's, \mu_3 3's \), where \( \mu_1 + \mu_2 + \mu_3 = n \), and the 3-tuple \((\mu_1, \mu_2, \mu_3) \) uniquely determines the \( S_n \)-orbit. If the \( \mu_i \)'s are all different, then the \( S_3 \)-orbit of \( (X_3^n)_{S_n} \) which contains the \( S_n \)-orbit corresponding to \((\mu_1, \mu_2, \mu_3) \) is \( S_3/1 \). If exactly two of the \( \mu_i \)'s are the same, then the \( S_3 \)-orbit is \( S_3/S_1 \times S_2 \). If \( \mu_1 = \mu_2 = \mu_3 \), then the \( S_3 \)-orbit is \( S_3/S_3 \). Therefore, \( S_3/A_3 \) never arises.

Thus \( B(S_k) \) is generated by \( X_k \) if all the \( \beta \)-operations are used, but is, in general, not generated by \( X_k \) if only symmetric power operations are used. Hence \( \beta \)-operations include, but are not the same as, symmetric powers.
C. Some open questions. Since the $S$-operations in the linear representation theory case are generated by symmetric power operations (see III, §A), which are defined on all of $R(G)$ (see I, §C), $S$-operations extend to operations on $R(G)$, thus making $R(G)$ a "$\lambda$-ring". The unigeneration of $R(S_k)$ can be phrased:

There is an onto "$\lambda$-ring homomorphism"

$$\Lambda \rightarrow R(S_k), \quad a_1 \mapsto \lambda_k,$$

where $\Lambda$ is the "free $\lambda$-ring on one generator" $a_1 \in \Lambda$. Hence $R(S_k) \cong \Lambda/I$, for some $\lambda$-ideal $I$. A reasonable description of this $\lambda$-ideal, in particular, a canonical set of generators, is unknown.

In the case of permutation representations, i.e., $G$-sets, the corresponding theory of "$\beta$-ring" which would allow extending the $\beta$-operations to all of $B(G)$ is not known.

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