S-OPERATIONS IN REPRESENTATION THEORY(1)

BY

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ABSTRACT. For $G$ a group and $A^G$ the category of $G$-objects in a category $A$, a collection of functors, called "$S$-operations," is introduced under mild restrictions on $A$. With certain assumptions on $A$ and with $G$ the symmetric group $S_k$, one obtains a unigeneration theorem for the Grothendieck ring formed from the isomorphism classes of objects in $A^{S_k}$. For $A = \text{finite-dimensional vector spaces over } C$, the result says that the representation ring $R(S_k)$ is generated, as a $\lambda$-ring, by the canonical $k$-dimensional permutation representation. When $A = \text{finite sets}$, the $S$-operations are called "$\beta$-operations," and the result says that the Burnside ring $B(S_k)$ is generated by the canonical $S_k$-set if $\beta$-operations are allowed along with addition and multiplication.

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A. Introduction. In the theory of linear representations of a finite group $G$, representations can be added, multiplied, and formed into a ring $R(G)$, the representation ring of $G$. In addition, $n$th symmetric power operations can be applied to any representation, and these operations can be extended to all elements of $R(G)$. Knutson [5] gives a detailed account of these operations in $R(G)$; Atiyah [1] discusses similar operations in the setting of vector bundles.

This paper attempts to generalize these notions. For any group $G$, a collection of operations on the category $A^G$ is defined under mild restrictions on $A$. In the case of linear representations of a finite group, these operations are combinations of symmetric powers, but, in general, they include other operations as well. Letting $G = S_k$ and with certain assumptions on $A^{S_k}$, one obtains the main result:

COROLLARY II.22. $\langle X_k \rangle = K_0(A^{S_k})$.

Here, $K_0(A^{S_k})$ is the Grothendieck ring formed from the isomorphism classes of objects in $A^{S_k}$, $X_k$ is a particular object in $A^{S_k}$, and $\langle X_k \rangle$ is the
subring of \( K_0(\mathbb{A} S^k) \) obtained by applying the operations to \( X_k \) and taking sums and products of the results. A principal application of this corollary is that \( R(S_k) \) is generated by the canonical permutation representation \( X_k \) if symmetric powers are included along with addition and multiplication. For the reader familiar with \( \lambda \)-rings, this statement says that \( R(S_k) \) is generated by one element as a \( \lambda \)-ring [2], [5].

§§I.B and I.C present some background on the two principal examples, the Burnside and representation rings of a finite group. Chapter II introduces the \( S \)-operations and explores their behavior; the main theorem and its Corollary II.22 are proved in §C. In Chapter III, Corollary II.22 is used to prove that \( R(S_k) = \langle X_k \rangle \) and \( B(S_k) = \langle X_k \rangle \) for all \( k \geq 1 \). It is also shown that, in general, neither \( B(S_k) \) nor \( R(S_k) \) is unigenerated as a ring. Moreover, although \( B(S_k) = \langle X_k \rangle \), if one allows only symmetric power operations rather than all \( S \)-operations, one does not necessarily obtain all of \( B(S_k) \).

B. The Burnside ring, \( B(G) \). Let \( G \) be a finite group. A \( G \)-set is a finite set \( T \) together with a mapping \( G \times T \to T \) such that \( (g_1 g_2) t = g_1 (g_2 t) \), \( 1 t = t \), for all \( g_1, g_2 \in G \), \( t \in T \). A morphism of \( G \)-sets, or \( G \)-map, is a set map \( f: T \to T' \), with \( T \) and \( T' \) \( G \)-sets, such that \( f( g t ) = g f( t ) \) for all \( g \in G \), \( t \in T \). Two \( G \)-sets are said to be isomorphic if there is a \( G \)-map between them which is a set isomorphism. \( G \)-sets and \( G \)-maps clearly form a category.

Examples I.1. (i) Let \( G \) be any finite group, \( T \) any finite set. Then \( T \) can be given the trivial action \( g t = t \) for all \( g \in G \), \( t \in T \).

(ii) In example (i), if \( T \) has only one element, \( T \) is denoted by \( 1_G \).

(iii) Let \( H \) be a subgroup of a finite group \( G \). Then \( G/H \), the set of left cosets of \( H \) in \( G \), is a \( G \)-set by the action \( g(xH) = (gx)H \).

(iv) Let \( S_n \) be the symmetric group on the symbols \( 1, 2, \ldots, n \). Let \( X_n \) be the set \( \{ x_1, x_2, \ldots, x_n \} \), and let \( S_n \) act on \( X_n \) by \( \alpha x_i = x_{\sigma(i)} \). \( X_n \) will be called the canonical \( S_n \)-set.

(v) Let \( G \) be any finite group. The empty set \( \emptyset \) is clearly a \( G \)-set.

If \( T_1 \) and \( T_2 \) are \( G \)-sets, then the disjoint union \( T_1 \uplus T_2 \) is a \( G \)-set, under the obvious action. On the other hand, every \( G \)-set can be decomposed into its \( G \)-orbits:

**Proposition I.2.** Every \( G \)-set \( T \neq \emptyset \) is of the form \( \bigsqcup_{i=1}^n T_i \), where \( T_i \) is a transitive \( G \)-set. The \( T_i \)'s are unique up to order. (A \( G \)-set \( T \) is transitive if \( T \neq \emptyset \) and if given \( t_1, t_2 \in T \) there is a \( g \in G \) such that \( g t_1 = t_2 \).

**Proposition I.3.** If \( H \) is a subgroup of \( G \), then \( G/H \) is a transitive \( G \)-set.
G-set. Conversely, every transitive G-set is of the form G/H for some subgroup H of G.

Proof. Given $g_1H, g_2H \in G/H$, $(g_1g_2^{-1})g_2H = g_1H$. Hence G/H is a transitive G-set.

Suppose $T$ is a transitive G-set. Let $t \in T$. Then $T = Gt$. Let $G_t$ be the isotropy group of $t$, i.e., $G_t = \{g \in G | gt = t\}$. Then the map $T \rightarrow G/G_t$ defined by $gt \mapsto gG_t$ is a G-isomorphism.

Proposition 1.4. $G/H \cong G/K$ as G-sets if and only if $H$ and $K$ are conjugate subgroups of $G$.

Proof. Suppose $H$ and $K$ are conjugate, i.e., $K = g_1^{-1}Hg_1$ for some $g_1 \in G$. Then the maps

$$\phi: G/H \rightarrow G/K, \quad \psi: G/K \rightarrow G/H,$$

are G-maps, and $\phi \circ \psi = 1_{G/K}$, $\psi \circ \phi = 1_{G/H}$. So $G/H \cong G/K$.

Conversely, assume $G/H \cong G/K$. Then there exist G-maps $\phi: G/H \rightarrow G/K$, $\psi: G/K \rightarrow G/H$ such that $\phi \circ \psi = 1_{G/K}$, $\psi \circ \phi = 1_{G/H}$. If $\phi(1H) = g_1K$, then $g_1K = h_1g_1K$ for all $h \in H$, so $g_1^{-1}Hg_1 \subset K$. Similarly $\phi(1K) = g_2H$ gives $g_2^{-1}Kg_2 \subset H$. Thus $g_2^{-1}g_1^{-1}Hg_1g_2 \subset g_2^{-1}Kg_2 \subset H$. Since $g_2^{-1}g_1^{-1}Hg_1g_2$ has the same number of elements as $H$, $g_2^{-1}g_1^{-1}g_2 \subset H$.

If $T_1$ and $T_2$ are G-sets, then the cartesian product $T_1 \times T_2$ is a G-set under the obvious action. The Burnside ring of G, $B(G)$, consists of all finite formal sums, $\sum n_i[T_i]$ ($n_i \in \mathbb{Z}$), of G-sets $T_i$, modulo the relations

(i) $[T_1] = [T_2]$ if $T_1 \cong T_2$ as G-sets,
(ii) $[T_1 \sqcup T_2] = [T_1] + [T_2]$.

$B(G)$ is clearly an abelian group; the cartesian product, together with $1_G$, gives $B(G)$ the structure of a commutative ring with identity, i.e., $[T_1][T_2] = [T_1 \times T_2]$. Whenever no confusion could arise, the brackets will be omitted.

Propositions 1.2, 1.3, and 1.4 imply

Proposition 1.5. Let $\{H_\alpha\}$ be a set of representatives of the conjugacy classes of subgroups of G. Then $B(G)$ is a free $\mathbb{Z}$-module with basis $\{[G/H_\alpha]\}$.

The rest of this section is devoted to defining a set map $h_n: B(G) \rightarrow B(G)$ for each integer $n \geq 0$. For any G-set $T$, the set $T^G$ is defined to be the collection of elements of $T$ with the identification $t_1 \sim t_2$ iff $Gt_1 = Gt_2$.

Let $T$ be a G-set. Then $T^n = T \times T \times \cdots \times T$ ($n$ times) is a G-set and also an $S_n$-set via $\sigma(t_1, \cdots, t_n) = (t_\sigma(1), \cdots, t_\sigma(n))$ for $\sigma \in S_n$.

For each integer $n \geq 1$, let $h_n(T)$ denote $(T^n)_{S_n}$. $h_n(T)$ is thus the $n$th
symmetric power of $T$. Since the $G$- and $S_n$- actions on $T^n$ commute, $h_n(T) = (T^n)^{S_n}$ is actually a $G$-set. Clearly $h_n$ sends isomorphic $G$-sets to isomorphic $G$-sets. Finally, define $h_0(T)$ to be $1_G$ for all $G$-sets $T$.

If $T_1, T_2$ are $G$-sets, then

$$h_n(T_1 \sqcup T_2) = \prod_{i=0}^{n} (h_i(T_1) \times h_{n-i}(T_2)).$$

$h_n$ can now be defined on any element of $B(G)$ by the following construction:

Define

$$H_n: (G\text{-sets}) \times (G\text{-sets}) \rightarrow B(G)$$

inductively by

$$H_0(T_1, T_2) = 1_G,$$

$$H_n(T_1, T_2) = h_n(T_1) - \sum_{i=1}^{n-1} H_i(T_1, T_2) h_{n-i}(T_2) \text{ for } n > 0.$$

Clearly,

$$T_1 \cong U_1, \ T_2 \cong U_2 \Rightarrow H_n(T_1, T_2) = H_n(U_1, U_2) \text{ for all } n \geq 0.$$

In addition, an induction argument and the "addition formula" above give

$$H_n(T_1 \sqcup T_2) = H_n(T_1, T_2), \quad H_n(T, \varnothing) = h_n(T)$$

for all $n \geq 0$ and $G$-sets $T_1, T_2, T$.

An arbitrary element of $B(G)$ looks like $T_1 - T_2$, where $T_1$ and $T_2$ are $G$-sets. If $T_1 - T_2 = U_1 - U_2$, then $T_1 \sqcup U_2 \cong U_1 \sqcup T_2$, so

$$H_n(T_1, T_2) = H_n(T_1 \sqcup U_2, T_2 \sqcup U_2).$$

Thus $H_n(T_1, T_2)$ depends only on $T_1 - T_2$. Therefore, define $h_n(T_1 - T_2) = H_n(T_1, T_2)$. Then $h_n: B(G) \rightarrow B(G)$ is a well-defined set map and coincides with its former definition if $T \in B(G)$ is actually a $G$-set.

C. The representation ring, $R(G)$. Let $G$ be a finite group. A (linear) representation of $G$ (over $C$) is a finite-dimensional vector space $V$ over $C$, together with a group homomorphism $\rho: G \rightarrow \text{Aut } V$. $V$ is called a $G$-module, and $\rho$ gives an action of $G$ on $V$. One usually writes

$$V \xrightarrow{\rho} V, \quad v \mapsto gv$$

instead of

$$V \xrightarrow{\rho(g)} V, \quad v \mapsto \rho(g)v.$$
A $G$-module map is a linear transformation $f: V \to V'$, with $V$ and $V'$ $G$-modules, such that $f(gv) = gf(v)$ for all $g \in G$, $v \in V$. Two $G$-modules are said to be isomorphic if there exists a $G$-module map between them which is also a vector space isomorphism. $G$-modules and $G$-module maps clearly form a category.

**Examples I.6.** (i) Let $G$ be a finite group, $V$ a finite-dimensional vector space. $V$ can be given the trivial action $gv = v$ for all $g \in G$, $v \in V$.

(i') A special case of example (i) is $V = 0$.

(ii) Let $G = S_n$. Let $V$ have basis $\{v_1, \ldots, v_n\}$, and let $S_n$ act by $\sigma v_i = v_{\sigma(i)}$ for $\sigma \in S_n$. This representation $V$ will be called the canonical $S_n$-module, and denoted $X_n$.

(ii') More generally, suppose $\rho: G \to S_n$ is a group homomorphism. ($\rho$ is called a permutation representation.) By composing this homomorphism with the one in example (ii), one obtains a linear representation of $G$, $G \to S_n \to \text{Aut} X_n$. Since a $G$-set $T$ consisting of $n$ elements is a group homomorphism $G \to S_n$, the concept of $G$-set is the same as the concept of permutation representation of $G$.

A $G$-module $V$ is reducible if $V = 0$ or if there is a subspace $W$ of $V$ such that $GW \subseteq W$, with $W \neq 0$ and $W \neq V$. If $V$ is not reducible, it is called irreducible.

If $V_1, V_2$ are $G$-modules, then the vector space coproduct $V_1 \amalg V_2$ is a $G$-module via the obvious action. A $G$-module $V$ is said to be decomposable if $V \cong V_1 \amalg V_2$ as a $G$-module, where $V_i \neq 0$. Propositions I.7–I.9 can be found in any book on group representation theory (see [5], [8]).

**Proposition I.7 (Maschke).** If $V \neq 0$ is reducible, then $V$ is decomposable.

**Proposition I.8.** Every $G$-module $V \neq 0$ can be expressed as a finite coproduct $V = \bigsqcup_{i=1}^n V_i$, where each $V_i$ is an irreducible $G$-module. The $V_i$'s are unique (up to order).

**Proposition I.9.** The number of irreducible representations of $G$ is equal to the number of conjugacy classes of $G$.

For $G$-modules $V_1, V_2$, $V_1 \otimes V_2$ is a $G$-module via $g(v_1 \otimes v_2) = gv_1 \otimes gv_2$. The representation ring of $G$, $R(G)$, consists of all finite formal sums $\Sigma_i n_i[V_i]$ ($n_i \in \mathbb{Z}$) of $G$-modules $V_i$, modulo the relations

(i) $[V_1] = [V_2]$ if $V_1 \cong V_2$ as $G$-modules,

(ii) $[V_1 \amalg V_2] = [V_1] + [V_2]$.

$R(G)$ is clearly an abelian group; the tensor product, together with $1_G$, gives
$R(G)$ the structure of a commutative ring with identity, that is $[V_1][V_2] = [V_1 \otimes V_2]$. The brackets will usually be omitted.

Propositions 1.8 and 1.9 imply

**Proposition 1.10.** Let $\text{Irrep } G$ = the set of isomorphism classes of irreducible $G$-modules. Then $R(G)$ is a free $\mathbb{Z}$-module with basis $\{[V] | V \in \text{Irrep } G\}$. The rank of $R(G)$ = the number of conjugacy classes of $G$.

As in the case of $B(G)$, symmetric power operations $h_n: R(G) \rightarrow R(G)$ can be introduced. For any $G$-module $V$, define the vector space $V_G$ to be $V/W$, where $W$ is the subspace of $V$ generated by $\{v - gv | v \in V, g \in G\}$. The vector spaces $V_G$ and $V^G$, where $V^G$ is the subspace of $V$ fixed by $G$, are seen to be isomorphic by the fact that the linear transformation $Y: V \rightarrow V$ defined by

$$Y(v) = \frac{1}{|G|} \sum_{g \in G} gV$$

has image $V^G$ and kernel $W$. In the case of sets, however, the corresponding objects $T_G$ and $T^G$ are not generally isomorphic.

For any $G$-module $V$, $V^\otimes n = V \otimes \cdots \otimes V$ ($n$ times) is a $G$-module and also an $S_n$-module via $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ for $\sigma \in S_n$. For each positive integer $n$, let $h_n(V)$ denote $(V^\otimes n)_{S_n}$; $h_n(V)$ is thus the $n$th symmetric power of $V$. Since the $G$- and $S_n$-actions on $V^\otimes n$ commute, $h_n(V) = (V^\otimes n)_{S_n}$ is a $G$-module. Define $h_0(V)$ to be $1_G$ for all $G$-modules $V$. Clearly $h_n$ sends isomorphic $G$-modules to isomorphic $G$-modules.

For $G$-modules $V_1$ and $V_2$,

$$h_n(V_1 \oplus V_2) = \bigoplus_{i=0}^n (h_i(V_1) \otimes h_{n-i}(V_2)).$$

As in the $G$-set case, $h_n$ can be defined on any element $V_1 - V_2$ of $R(G)$ by defining $H_n: (G\text{-modules}) \times (G\text{-modules}) \rightarrow R(G)$ inductively by

$$H_0(V_1, V_2) = 1_G,$$

$$H_n(V_1, V_2) = h_n(V_1) - \sum_{i=0}^{n-1} H_i(V_1, V_2)h_{n-i}(V_2) \text{ for } n > 0,$$

and then using the "addition formula" above to show that $H_n(V_1, V_2)$ depends only on $V_1 - V_2$.

$G$-sets and $G$-modules are examples of the category discussed in Chapter II. There, a family of functors, called $S$-operations, is defined. In the case of
G-modules, these S-operations turn out to be sums and products of symmetric powers \( h_n \). In fact, by applying these operations to the canonical \( S_k \)-module \( X_k \), one can obtain every element in \( R(S_k) \) (see III, §A).

In the case of \( G \)-sets, however, the S-operations include more than symmetric powers. In III, §B, one sees that applying sums and products of symmetric power operations \( h_n \) to the canonical \( S_k \)-set \( X_k \) does not always give all of \( B(S_k) \), whereas applying all the S-operations to \( X_k \) does.

II. S-Operations

A. The category \( A^G \) and functors \( \phi_{w_n} \): \( A^G \to A^G \). Let \( G \) be a group, and \( A \) a category. A \( G \)-object in \( A \) is an object \( A \) in \( A \), together with morphisms \( A \xrightarrow{\rho_g} A \) for all \( g \in G \), satisfying \( \rho_{gh} = \rho_g \circ \rho_h, \rho_1 = 1_A \). \( A \xrightarrow{\rho_g} A \) is usually written \( A \xrightarrow{g} A \).

A \( G \)-map, or \( G \)-morphism, is a morphism \( f: A \to B \) in \( A \), with \( A \) and \( B \) \( G \)-objects, such that \( f_g = g^f \) for all \( g \in G \). The category of \( G \)-objects and \( G \)-maps in \( A \) is denoted \( A^G \).

The aim of this section is to define a collection of functors from \( A^G \) to \( A^G \), under certain assumptions on \( A \). The reader is referred to [6] for a reference on category theory.

Recall that given two morphisms \( \alpha, \beta: A \to B \), \( \mu: B \to K \) is a coequalizer for \( \alpha \) and \( \beta \) if \( \mu \alpha = \mu \beta \), and if whenever \( \mu': B \to K' \) satisfies \( \mu' \alpha = \mu' \beta \), then there is a unique morphism \( \gamma: K \to K' \) such that \( \gamma \mu = \mu' \). Given two morphisms \( f_1: A \to B_1 \), \( f_2: A \to B_2 \), a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f_2} & B_2 \\
\downarrow{f_1} & & \downarrow{\mu_2} \\
B_1 & \xrightarrow{\mu_1} & P 
\end{array}
\]

is called a pushout for \( f_1 \) and \( f_2 \) if for every commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f_2} & B_2 \\
\downarrow{f_1} & & \downarrow{\mu_2} \\
B_1 & \xrightarrow{\mu_1} & P' 
\end{array}
\]

there is a unique morphism \( \gamma: P \to P' \) such that \( \mu'_1 = \gamma \mu_1 \) and \( \mu'_2 = \gamma \mu_2 \).

**Lemma II.1.** Let \( A \) be a category with coequalizers and finite coproducts. Then \( A \) has pushouts.
Proof. Consider

\[ A \xrightarrow{f_2} B_2 \]
\[ \downarrow f_1 \]
\[ B_1 \]

The coproduct \( B_1 \amalg B_2 \), together with the canonical morphisms \( i_j: B_j \rightarrow B_1 \amalg B_2 \), \( j = 1, 2 \), gives morphisms \( i_j \circ f_j: A \rightarrow B_1 \amalg B_2 \), \( j = 1, 2 \). Let \( \mu: B_1 \amalg B_2 \rightarrow K \) be the coequalizer for \( i_1 \circ f_1 \) and \( i_2 \circ f_2 \). Then \( \mu \circ (i_1 \circ f_1) = \mu \circ (i_2 \circ f_2) \) gives a commutative diagram

\[ A \xrightarrow{f_2} B_2 \]
\[ \downarrow f_1 \]
\[ B_1 \]
\[ \downarrow \mu \circ i_2 \]
\[ \downarrow \mu \circ i_1 \]
\[ K \]

The fact that this diagram is actually a pushout follows from the definitions of coproduct and coequalizer. \( \square \)

Given a family \( \{\mu_i: A \rightarrow A_i\}_{i \in I} \) of epimorphisms, \( \mu: A \rightarrow A' \) is the cointersection of the family if for each \( i \in I \) there exist morphisms \( \nu_i: A \rightarrow A' \) such that \( \mu = \nu_i \mu_i \), and if every morphism \( A \rightarrow B \) which factors through each \( \mu_i \) factors uniquely through \( \mu \).

Lemma II.2. If \( A \) has pushouts, then \( A \) has finite cointersections.

Proof. It suffices to show existence for a family of two epimorphisms \( \mu_1: A \rightarrow A_1 \), \( \mu_2: A \rightarrow A_2 \). Let

\[ A \xrightarrow{\mu_2} A_2 \]
\[ \downarrow \mu_1 \]
\[ A_1 \]
\[ \downarrow \nu_2 \]
\[ \nu_1 \]
\[ P \]

be the pushout for \( \mu_1 \) and \( \mu_2 \). Then \( \nu_1 \mu_1 = \nu_2 \mu_2: A \rightarrow P \) is the cointersection of \( \mu_1 \) and \( \mu_2 \) by the definition of pushout. \( \square \)

Let \( F_G: A \rightarrow A^G \) be the functor which sends \( A \in A \) to \( A \in A^G \) by letting \( A \xrightarrow{g} A \) be \( A^{1_A} \) for all \( g \in G \). Let \( V \in A^G \). A \( G \)-orbit space of \( V \) is a pair \( (O, \pi) \), where \( O \in A \) and \( \pi \in \text{Mor}_{A^G}(V, F_G(O)) \), such that whenever \( X \in A \) and \( f \in \text{Mor}_{A^G}(V, F_G(X)) \) there is a unique \( \phi \in \text{Mor}_A(O, X) \) such that \( F_G(\phi) \circ \pi = f \). When such an \( O \) exists, it is of course unique up to natural isomorphism and is denoted \( V_G \).
PROPOSITION II.3. Let $G$ be a finite group and let $A$ have coequalizers and finite coproducts. Then $(V_G, \pi)$ exists for all $V \in A^G$.

PROOF. For each pair of distinct elements $g_i, g_j \in G$, let $\mu_{g_i, g_j}: V \to K_{g_i, g_j}$ be a coequalizer for the morphisms $g_i, g_j: V \to V$. Each $\mu_{g_i, g_j}$ is an epimorphism since every coequalizer is. Let $\pi: V \to O$ be the co-intersection (exists by Lemmas II.1, II.2) of the finite family $\{\mu_{g_i, g_j}: V \to K_{g_i, g_j}\}$ in $A$. This construction gives $(V_G, \pi)$:

For each $g \in G$, $\mu_{g, 1} = \mu_{g, 1} = 1_V = \mu_{g, 1}$, so $\pi g = \pi$ for all $g \in G$. Hence $\pi \in \text{Mor}_{A^G}(V, F_G(O))$. If $f \in \text{Mor}_{A^G}(V, F_G(X))$, then $f g_i = f g_j$ for all $g_i, g_j \in G$, so $f$ factors through each $\mu_{g_i, g_j}$. Thus there is a unique $\phi \in \text{Mor}_A(O, X)$ such that $\phi \circ \pi = f$.\]

REMARK II.4. Proposition II.3 says there is a functor $(\cdot)_G: A^G \to A$ left adjoint to $F_G: A \to A^G$, i.e., $\text{Mor}_{A^G}(V, F_G(X)) \approx \text{Mor}_A(V_G, X)$, natural in arguments $V$ and $X$.

For each integer $n \geq 1$ and each $W_n \in A^{S_n}$ ($S_n$ is the symmetric group), a functor $\phi_{W_n}: A^G \to A^G$ will be defined. To do so, assume that $A$ has not only coequalizers and finite coproducts but also a "tensor product" $\cdot$, that is, a functor $\cdot: A \times A \to A$ which is coherently associative and commutative (see [7, Chapter I]), and which distributes with the coproduct. This insures natural isomorphisms

$$(A_1 \cdot (A_2 \cdot A_3)) \cdot (A_4 \cdot (A_2 \cdot A_3)) \approx \approx (A_1 \cdot A_2) \cdot (A_1 \cdot A_3),$$

$A_1 \cdot A_2 \approx A_2 \cdot A_1,$

$A_1 \cdot (A_2 \cdot (A_3 \cdot A_4)) \approx (A_1 \cdot A_2) \cdot (A_1 \cdot A_3),$

such that isomorphisms between products of several factors, obtained by successively applying the above, are the same.

Fix $W_n \in A^{S_n}$. Let $T \in A$ and let $T^{\cdot n} = T \cdot T \cdot \cdots \cdot T$ ($n$ times). $T^{\cdot n} \in A^{S_n}$ via the natural isomorphisms which permute its factors. Since $\cdot$ is a functor $A \times A \to A$ it induces a functor $\cdot: A^G \times A^G \to A^G$ for any group $G$; that is if $A \xrightarrow{g} A, B \xrightarrow{g} B$, then $A \cdot B \xrightarrow{g \cdot g} A \cdot B$ gives $A \cdot B$ a well-defined $G$-action by the functoriality of $\cdot$. Hence $W_n \cdot T^{\cdot n} \in A^{S_n}$. Defining $\phi_{W_n}(T)$ to be $(W_n \cdot T^{\cdot n})_{S_n}$, one obtains a functor $\phi_{W_n}: A \to A$. (For $f: T \to T'$, $\phi_{W_n}(f): (W_n \cdot T^{\cdot n})_{S_n} \to (W_n \cdot T'^{\cdot n})_{S_n}$ is the obvious map.)

If $T \in A^G$, $T$ comes with morphisms $T \xrightarrow{g} T$ for all $g \in G$, which induce morphisms

$$\phi_{W_n}(g): \phi_{W_n}(T) \to \phi_{W_n}(T).$$
for all $g \in G$. Since $\phi_{W_n}$ is a functor, the maps $\phi_{W_n}(g)$ define a $G$-action on $\phi_{W_n}(T)$. Thus one has a functor $\phi_{W_n} : A^G \to A^G$ for any group $G$.

In conclusion, then, if $G$ is any group and if $A$ has coequalizers, finite coproducts, and a "tensor product" $\perp$, then for each positive integer and each $W_n \in A^{S_n}$, one has a functor $\phi_{W_n} : A^G \to A^G$ defined by $\phi_{W_n}(T) = (W_n \perp T^{1,n})_{S_n}$.

B. The behavior of the functors $\phi_{W_n}$. The purpose of this section is to investigate the behavior of the functors $\phi_{W_n}$. To do so, one first introduces induced objects.

Suppose $H \subseteq G$ are groups. $A \in A^G$ may be viewed as an $H$-object via the inclusion $H \hookrightarrow G$, giving rise to a functor $Res^G_H : A^G \to A^H$. Let $W \in A^H$. An induced object of $W$ is a pair $(V, \psi)$, where $V \in A^G$ and $\psi \in \text{Mor}_{A^H}(W, \text{Res}^G_H V)$ such that whenever $X \in A^G$ and $f \in \text{Mor}_{A^H}(W, \text{Res}^G_H X)$ there is a unique $\phi \in \text{Mor}_{A^G}(V, X)$ satisfying $(\text{Res}^G_H \phi) \circ \psi = f$. When such a $V$ exists, it is unique up to natural isomorphism and is denoted $\text{Ind}^G_H W$.

**PROPOSITION II.5.** Let $H \subseteq G$ be finite groups and let $A$ have coequalizers and finite coproducts. Then $(\text{Ind}^G_H W, \psi)$ exists for all $W \in A^H$.

**PROOF.** Let $W \in A^H$. Form the coproduct of $W$ with itself $|G|$ times to obtain the object $\coprod_{x \in G} W_x$ in $A$, which is in $A^G$ via the maps $\coprod W_x \overset{*g}{\longrightarrow} \coprod W_x$, for all $g \in G$, which permute the factors; more precisely, $*g$ is induced by the maps $*g_x : W_x \to W_{gx} \hookrightarrow \coprod_{x \in G} W_x$, where the first morphism is $1_W$ and the second is the canonical map associated with the coproduct. In the future, $W_x \to W_y$ will be denoted $1^y_x$. For $h \in H$, let $\coprod W \overset{h}{\longrightarrow} \coprod W$ be the map induced from maps $h^* : W_x \overset{1^x_h}{\longrightarrow} W_{xh} \hookrightarrow \coprod W_x$,

and let $\coprod W \overset{h}{\longrightarrow} \coprod W$ be induced from the maps $h_x : W_x \overset{h}{\longrightarrow} W_{xh} \hookrightarrow \coprod W_x$, where the first map is just the action of $H$ on $W$.

Observe that $h ** g = *gh^*$ and $h(*g) = *gh$ for all $g \in G$, $h \in H$. For $h \in H$, let $\mu_{h^*} : \coprod W \to K_h$ be the coequalizer of $h^*$ and $h$. Since $\theta (\mu_{h^*}) h^* = \mu_{h^*} h = (\mu_h) g = (\mu_{h^*}) h$ for $g \in G$, there is a unique map $K_h \to \coprod W$ such that the triangle

\[
\begin{array}{ccc}
\coprod W & \overset{\mu_{h^*}}{\longrightarrow} & K_h \\
\mu_h & \searrow & \\
& K_h & \\
\end{array}
\]

commutes. One thus obtains a map $\theta_g$ for each $g \in G$. The uniqueness of each
\( \theta_g \) implies \( \theta_1 = 1_K \) and \( \theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2} \). Hence \( K_h \in A^G \) and \( \mu_h \) is a \( G \)-map.

Let \( \mu: \bigoplus W_x \to K \) be the cointersection (exists by Lemmas II.1, II.2) of the finite family of epimorphisms \( \{\mu_h\}_{h \in H} \) in \( A \). Since \( \mu \) factors through each \( \mu_h, \mu^*g \) does also; hence for each \( g \in G \), there is a unique map \( K \xrightarrow{\rho_g} K \) such that the triangle

\[
\begin{array}{c}
\bigoplus W_x \\
\downarrow \mu \\
K \\
\end{array} \xrightarrow{\mu^*g} \begin{array}{c}K \\
\downarrow \rho_g \\
K \\
\end{array}
\]

commutes. The uniqueness of each \( \rho_g \) makes \( K \) a \( G \)-object and therefore \( \mu \) a \( G \)-map.

Let \( \psi: W \to \text{Res}_H^G K \) be the map \( \mu_1: W_1 \to K \) which comes from \( \mu: \bigoplus x \in G W_x \to K \). (Here and elsewhere, \( W \) is identified with \( W_1 \).) \( K = \text{Ind}_H^G W \) by the following argument:

To show that \( \psi \) is an \( H \)-map, one must show that \( \rho_h \psi = \psi h \). The last commutative triangle and the definition of \( \mu \) give \( \rho_h \psi = \rho_h \mu_1 = \mu^* h_1 = \mu h_1 = \mu_1 h = \psi h \).

Suppose \( f: W \to \text{Res}_H^G X \) is an \( H \)-map. One can show that there is a unique \( G \)-map \( \tau: \bigoplus W_x \to X \) such that the triangle

\[
\begin{array}{c}
W_1 \\
\downarrow f \\
\bigoplus W_x \\
\end{array} \xrightarrow{\tau} \begin{array}{c}X \\
\end{array}
\]

commutes, as follows:

If such a \( \tau \) exists, then for each \( x \in G \), the square

\[
\begin{array}{c}
W_1 \\
\downarrow f \\
\bigoplus W_x \\
\downarrow \tau \\
\end{array} \xrightarrow{\tau} \begin{array}{c}X \\
\end{array}
\]

commutes. Hence \( \tau_x = x f 1^1_x \) for all \( x \in G \). Thus \( \tau \) is unique. For existence, define \( \tau \) by \( \tau_x = x f 1^1_x \) for all \( x \in G \).

Moreover, \( \tau h^* = \tau h \) for all \( h \in H \) since \( (\tau h^*)_x = \tau x h 1^1_x = x h f 1^1_x 1^1_x \)

\( = x h f 1^1_x \), \( (\tau h)_x = \tau x h = x f 1^1 x h = x f h 1^1_x \), and \( f \) is an \( H \)-map. Hence \( \tau \) factors through \( \mu_h \) for all \( h \in H \). It follows that there is a unique \( \phi: K \to X \) satisfying \( \tau = \phi \mu \). By the fact that \( \mu \) and \( \tau \) are \( G \)-maps and the uniqueness of \( \phi \), \( \phi \) is itself a \( G \)-map: \( g \phi \rho_{g^{-1}} \mu = g \phi \mu^* g^{-1} = g \tau g^{-1} = g g^{-1} \tau = \tau \Rightarrow g \phi \rho_{g^{-1}} = \phi \Rightarrow g \phi = \phi g \).
The commutativity of the two small triangles in the figure

\[
\begin{aligned}
\begin{array}{ccc}
W_1 & \xrightarrow{f} & X \\
\mu & \Downarrow & \\
\Pi W_x & \xrightarrow{\phi} & K \\
\end{array}
\end{aligned}
\]

implies the commutativity of the large triangle. Hence \( \phi \) is a \( G \)-map satisfying \((\text{Res}_H^G \phi) \circ \psi = f\). In addition, if \( \phi_1 \) makes the diagram

\[
\begin{aligned}
\begin{array}{ccc}
W'_1 & \xrightarrow{f} & V \\
\mu & \Downarrow & \\
\Pi W_x & \xrightarrow{\phi_1} & K \\
\end{array}
\end{aligned}
\]

commute, then \( \phi_1 \mu = \tau \) by uniqueness of \( \tau \), and so \( \phi_1 = \phi \) by uniqueness of \( \phi_0 \).

**Remark II.6.** Proposition II.5 says there is a functor \( \text{Ind}^G_H : \mathcal{A}^H \to \mathcal{A}^G \) left adjoint to \( \text{Res}^G_H : \mathcal{A}^G \to \mathcal{A}^H \), i.e., \( \text{Mor}_{\mathcal{A}^H}(W, \text{Res}^G_H V) \approx \text{Mor}_{\mathcal{A}^G}(\text{Ind}^G_H W, V) \), natural in arguments \( W \) and \( V \).

**Proposition II.7.** Let \( G \) be a group and \( A \) a category with finite coproducts. Suppose \( V \in \mathcal{A}^G \) and \( V = \bigoplus_{i=1}^n W_i \) as an object in \( A \). Assume \( G \) permutes the \( W_i \)'s transitively, that is, each \( g_i : W_i \to \bigoplus_{i=1}^n W_i \) looks like \( W_i \to W_j \subset \bigoplus_{i=1}^n W_i \) for some \( j \) and some morphism \( W_i \to W_j \), and given any \( i, j \), there is a \( g \in G \) such that \( g_i : W_i \to W_j \subset \bigoplus_{i=1}^n W_i \).

Let \( W_{i_0} \) be one of the \( W_i \)'s and let \( H \) be its isotropy group, i.e., \( H = \{g \in G[g_{i_0} : W_{i_0} \to W_{i_0} \subset \bigoplus W_i] \}. \) Then as an object in \( \mathcal{A}^G \), \( V = \text{Ind}^G_H W_{i_0} \).

**Proof.** If \( g_i : W_i \to W_j \subset \bigoplus W_i \), denote the map \( W_i \to W_j \) by \( g_i \). Since \( gg^{-1} = g^{-1}g = 1_W \), \( g_i : W_i \to W_j \subset \bigoplus W_i \) implies \( g_i^{-1} : W_j \to W_i \subset \bigoplus W_i \) and \( g_i(g_i^{-1})_j = 1_{W_j} \), \( (g_i^{-1})_j g_i = 1_{W_j} \).

Let \( \psi : W_{i_0} \to \text{Res}^G_H V \) be the canonical map \( W_{i_0} \subset \bigoplus_{i=1}^n W_i \). \( \psi \) is clearly an \( H \)-map. To show \( V = \text{Ind}^G_H W_{i_0} \), one need only show that \( V \) satisfies the appropriate universal property.

Suppose \( f \in \text{Mor}_{\mathcal{A}^H}(W_{i_0}, \text{Res}^G_H X) \). If there is a \( G \)-map \( \phi : V \to X \) such that \((\text{Res}^G_H \phi) \circ \psi = f\), then for each \( g \in G \) there is a commutative diagram
Given $i$, let $g \in G$ be such that $g_{i_0} : W_{i_0} \rightarrow W_i \hookrightarrow \coprod W_i$. Then $\phi_i = \phi_{g_{i_0}}(g^{-1})_{i_0} = gf(g^{-1})_{i_0}$. Thus such a $\phi$ is unique.

To show existence, define $\phi$ by $\phi_i = gf(g^{-1})_{i_0}$, where $g \in G$ such that $g_{i_0} : W_{i_0} \rightarrow W_i \hookrightarrow \coprod W_i$. $\phi$ is well defined: If $\tilde{g}, g : W_{i_0} \rightarrow W_i \hookrightarrow \coprod W_i$, then $\tilde{g}^{-1}g \in H$, so that $\tilde{g}^{-1}gf = f(\tilde{g}^{-1}g)_{i_0} = f(g^{-1})_{i_0}g_{i_0}$; hence $gf(g^{-1})_{i_0} = \tilde{g}f(\tilde{g}^{-1})_{i_0}$.

**Lemma II.8.** Let $A$ have coequalizers and finite coproducts, and let $G$ be a finite group. Then

$$(A \coprod B)_G \approx A_G \coprod B_G,$$

natural in arguments $A$ and $B$.

**Proof.** There is a natural isomorphism

$$\text{Mor}_A((A \coprod B)_G, X) \approx \text{Mor}_{A_G}(A \coprod B, F_G(X))$$

(Remark II.4). Since adjoints are unique, one need only show

$$\text{Mor}_A(A_G \coprod B_G, X) \approx \text{Mor}_{A_G}(A \coprod B, F_G(X)).$$

But

$$\text{Mor}_A(A_G \coprod B_G, X) \approx \text{Mor}_A(A_G, X) \times \text{Mor}_A(B_G, X) \\ \approx \text{Mor}_{A_G}(A, F_G(X)) \times \text{Mor}_{A_G}(B, F_G(X)) \approx \text{Mor}_{A_G}(A \coprod B, F_G(X)).$$

**Theorem II.9.** Let $G$ be a group, and let $A$ have finite coproducts, coequalizers, and a "tensor product" $\otimes$. Then if $W_n, W'_n \in A^{S_n}$,

$$\phi_{W_n \coprod W'_n}(T) = \phi_{W_n}(T) \otimes \phi_{W'_n}(T)$$

for all $T \in A^G$.

**Proof.**

$$\phi_{W_n \otimes W'_n}(T) = ((W_n \otimes W'_n) \otimes T)_{S_n}$$

$$\approx ((W_n \otimes T) \otimes (W'_n \otimes T))_{S_n}$$

$$\approx (W_n \otimes T)_{S_n} \otimes (W'_n \otimes T)_{S_n} \quad \text{(by Lemma II.8)}$$

$$= \phi_{W_n}(T) \otimes \phi_{W'_n}(T)$$

**Lemma II.10.** Let $A$ have finite coproducts and coequalizers, and let
$K \subset H \subset G$ be finite groups. Let $U \in A^K$ and $W, W' \in A^H$. Then

(i) $\text{Ind}_H^G(W \oplus W') \approx \text{Ind}_G^H(W \oplus \text{Ind}_H^G W')$,

(ii) $\text{Ind}_H^G(\text{Ind}_K^H U) \approx \text{Ind}_K^G U$,

(iii) $(\text{Ind}_H^G W)_G \approx W_H$.

All the above isomorphisms are natural.

**Proof.** (i) The result follows from the adjointness of $\text{Res}_H^G$ and $\text{Ind}_H^G$ (Remark II.6) and an argument analogous to the one for Lemma II.8.

(ii) This result follows from the uniqueness of adjoints and the obvious fact that $\text{Res}_K^H(\text{Res}_G^H V) \approx \text{Res}_K^G V$.

(iii) The proof, which is similar to the preceding one, uses the adjointness of $\text{Res}_H^G$ and $\text{Ind}_H^G$ and of $F_G$ and $(\ )_G$, and the fact that $\text{Res}_H^G(F_G(X)) \approx F_H(X)$. □

**Lemma II.11 (Frobenius Reciprocity).** Let $H \subset G$ be finite groups, and let $A$ have finite coproducts, coequalizers, and a “tensor product” $\perp$. Assume there is a functor $\text{Hom}: A^A \times A \to A$ such that

$\text{Mor}_A(A \perp B, C) \approx \text{Mor}_A(A, \text{Hom}(B, C))$,

natural in $A, B, C$. Then for $W \in A^H, V \in A^G$,

$(\text{Ind}_H^G W) \perp V \approx \text{Ind}_H^G(W \perp \text{Res}_H^G V)$.

**Proof.** The functoriality of $\text{Hom}: A^A \times A \to A$ induces the functor $\text{Hom}: (A^G)^A \times A^G \to A^G$, and clearly

$\text{Res}_H^G \text{Hom}(V, X) \approx \text{Hom}(\text{Res}_H^G V, \text{Res}_H^G X)$.

The lemma now follows from the standard argument using uniqueness of adjoints. □

The following theorem gives a useful simplification for some of the functors $\phi_{W_n}$ in the special case of the existence of an object $1$ in $A$ such that $A \perp 1 \approx A$, natural and coherent in the sense of II, §A. The object $F_G(1) \in A^G$ will be denoted $1_G$, or simply $1$.

**Theorem II.12.** Let $G$ be a group, let $A$ have an object $1$ and be as in Lemma II.11, and let $\text{Hom}$ exist. Let $H \subset S_n, T \in A^G$, and $W_n = \text{Ind}_H^{S_n} 1$. Then $\phi_{W_n}(T) = (\text{Res}_H^{S_n}(T^{\perp n}))_H$.

**Proof.**

$\phi_{W_n}(T) = ((\text{Ind}_H^{S_n} 1) \perp T^{\perp n})_{S_n}$

$\approx (\text{Ind}_H^{S_n}(1 \perp \text{Res}_H^{S_n}(T^{\perp n})))_{S_n}$ (by Lemma II.11)

$\approx (\text{Ind}_H^{S_n}(\text{Res}_H^{S_n}(T^{\perp n})))_{S_n} \approx (\text{Res}_H^{S_n}(T^{\perp n}))_H$ (by Lemma II.10). □
Examples 11.13. (i) If \( W_n = \text{Ind}_{S_n}^G 1 \), then \( \phi_{W_n}(T) = (\text{Res}_{S_n}^G (T^{\downarrow n}))_1 = T^{\downarrow n} \).

(ii) If \( W_n = \text{Ind}_{S_n}^G 1 \), then \( \phi_{W_n}(T) = (\text{Res}_{S_n}^G (T^{\downarrow n}))_{S_n} = (T^{\downarrow n})_{S_n} \) is the \( n \)th symmetric power of \( T \).

If \( G \) and \( H \) are groups, \( A \in A^G, B \in A^H \), then the morphisms \( A \xrightarrow{f} A, B \xrightarrow{h} B \) induce the morphism \( A \downarrow B \xrightarrow{g \times h} A \downarrow B \), thereby making \( A \downarrow B \in A^G \times H \) (\( G \times H \) is the direct product of \( G \) and \( H \)). In this setting, one has the following lemma:

Lemma 11.14. Let \( G \) and \( H \) be finite groups, and let \( A \) have finite coproducts, coequalizers, and a "tensor product" \( \downarrow \). Assume there is a functor \( \text{Hom} \), as in Lemma 11.11. If \( A \in A^G, B \in A^H \), then \( (A \downarrow B)^{G \times H} \approx A_G \downarrow B_H \).

Proof. Because of the uniqueness of adjoints, one need only show

\[ \text{Mor}_{A^G \times H} (A \downarrow B, F_G \times H(X)) \approx \text{Mor}_{A} (A_G \downarrow B_H, X). \]

This follows from the following chain of natural isomorphisms, each easily verifiable:

\[ \text{Mor}_{A^G \times H} (A \downarrow B, F_G \times H(X)) \]
\[ \approx \text{Mor}_{(A_H)^G} (F_H(A) \downarrow F_G(B), F_G(F_H(X))) \]
\[ \approx \text{Mor}_{(A_H)^G} (F_H(A), \text{Hom}(F_G(B), F_G(F_H(X)))) \]
\[ \approx \text{Mor}_{(A_H)^G} (F_H(A), F_G(\text{Hom}(B, F_H(X)))) \]
\[ \approx \text{Mor}_{A_H} (F_H(A), \text{Hom}(B, F_H(X))) \]
\[ \approx \text{Mor}_{A_H} (F_H(A_G), \text{Hom}(B, F_H(X))) \]
\[ \approx \text{Mor}_{A_H} (F_H(A_G) \downarrow B, F_H(X)) \]
\[ \approx \text{Mor}_{A_H} (B \downarrow F_H(A_G), F_H(X)) \]
\[ \approx \text{Mor}_{A_H} (B \downarrow \text{Hom}(A_G, F_H(X))) \]
\[ \approx \text{Mor}_{A_H} (B, F_H(\text{Hom}(A_G, X))) \approx \text{Mor}_{A} (B_H, \text{Hom}(A_G, X)) \]
\[ \approx \text{Mor}_{A} (B_H \downarrow A_G, X) \approx \text{Mor}_{A} (A_G \downarrow B_H, X). \square \]

In the next theorem, \( S_n \times S_m \) is viewed as a subgroup of \( S_{n+m} \) by viewing \( S_n \) as permuting the symbols \( 1, 2, \cdots, n, S_m \) the symbols \( n+1, n+2, \cdots, n+m, \) and \( S_{n+m} \) the symbols \( 1, 2, \cdots, n+m. \)

Theorem 11.15. Let \( G \) be a group, let \( A \) have finite coproducts, coequal-
izers, a "tensor product" \( \perp \), and an object \( 1 \). Assume there is a functor \( \text{Hom} \) as in Lemma II.11. Let \( W_n \in A^S_n, W_m \in A^S_m, \) and \( T \in A^G \). If \( W_{n+m} = \text{Ind}_{S_n \times S_m}^{S_n+m} W_n \perp W_m, \) then \( \phi_{W_{n+m}}(T) = \phi_{W_n}(T) \perp \phi_{W_m}(T). \)

**Proof.**

\[
\phi_{W_{n+m}}(T) = ((\text{Ind}_{S_n \times S_m}^{S_n+m} W_n \perp W_m) \perp T^{\perp n+m})_{S_{n+m}}
\]

\[
\approx (\text{Ind}_{S_n \times S_m}^{S_n+m} (W_n \perp W_m \perp \text{Res}_{S_n \times S_m}^{S_n+m} T^{\perp n+m}))_{S_{n+m}} \quad \text{(by Lemma II.11)}
\]

\[
\approx (W_n \perp W_m \perp \text{Res}_{S_n \times S_m}^{S_n+m} T^{\perp n+m})_{S_n \times S_m} \quad \text{(by Lemma II.10)}
\]

\[
\approx ((W_n \perp T^{\perp n}) \perp (W_m \perp T^{\perp m}))_{S_n \times S_m}
\]

\[
\approx (W_n \perp T^{\perp n})_{S_n} \perp (W_m \perp T^{\perp m})_{S_m} \quad \text{(by Lemma II.14)}
\]

\[
= \phi_{W_n}(T) \perp \phi_{W_m}(T). \Box
\]

C. The main theorem and corollary. Let \( G \) be a group and \( A \) have finite coproducts, a "tensor product" \( \perp \), and an object \( 1 \). Define the Grothendieck ring \( K_0(A^G) \) to consist of all finite formal sums \( \Sigma n_i[T_i] \) \((n_i \in \mathbb{Z})\) of \( G \)-objects \( T_i \) in \( A \), modulo the relations

1. \([T_1] = [T_2]\) if \( T_1 \cong T_2 \) as \( G \)-objects,
2. \([T_1 \perp T_2] = [T_1] + [T_2].\)

Clearly, \( K_0(A^G) \) is an abelian group; the "tensor product" \( \perp \), together with the object \( 1 \in A^G \), gives \( K_0(A^G) \) the structure of a commutative ring with identity, i.e. \([T_1][T_2] = [T_1 \perp T_2]\). When the meaning is clear, brackets will be omitted, e.g., \([T_1] - [T_2]\) will appear as \(T_1 - T_2\).

**Examples II.16.** (i) Let \( G \) be a finite group and \( A \) the category of finite sets. Then \( A^G = G\text{-sets}. \) Let \( \perp \) be the cartesian product, and \( 1 \) be any one-element \( G \)-set. Then \( K_0(A^G) \) is the Burnside ring of \( G, B(G). \) (See I, §B.)

(ii) Let \( G \) be a finite group and \( A \) the category of finite-dimensional vector spaces over \( \mathbb{C}. \) Then \( A^G = G\text{-modules}. \) Let \( \perp \) be the tensor product \( \otimes, \) and \( 1 \) be the one-dimensional \( G \)-module with trivial \( G \)-action. Then \( K_0(A^G) \) is the representation ring of \( G, R(G) \) (see I, §C).

**Remark II.17.** In the above examples, \([T_1] = [T_2]\) implies \( T_1 \cong T_2 \) as \( G \)-objects (see I, §B, §C). This is not the case in general; in particular, if \( A \) is the category of vector bundles over a space \( X, \) then \([E] = [F]\) implies only that \( E \oplus n \cong F \oplus n, \) where \( n \) is the trivial bundle of dimension \( n \) [3, Appendix].
Let $H \subset G$ be finite groups. Let $A$ have finite coproducts, coequalizers, a "tensor product" $\perp$, and an object $1$. $P(A^G)$ is defined to be the subring of $K_0(A^G)$ generated by \{Ind_{H}^{G}\mid H \text{ a subgroup of } G\}.

Proposition II.18. Let $H \subset G$ be finite groups. If $W = 1_{H}$, then $\text{Ind}_{H}^{G}W = \underset{x \in G/H}{\Pi}1_{x}$ with $G$-action given by

$$g \cdot 1_{x} \rightarrow 1_{gx} \rightarrow \Pi 1_{x}.$$

Proof. Let $\psi : 1_{H} \rightarrow \Pi 1_{x}$. One need only show that $(\Pi 1_{x}, \psi)$ satisfies the appropriate universal property. Clearly, $\psi : 1_{H} \rightarrow \text{Res}^{G}_{H}(\Pi 1_{x})$ is an $H$-map. If $f \in \text{Mor}_{A^{H}}(1_{H}, \text{Res}^{G}_{H}X)$, there is a unique $\phi \in \text{Mor}_{A^{G}}(\Pi 1_{x}, X)$ such that $(\text{Res}^{G}_{H}\phi) \circ \psi = f$, namely $\phi_{x}f_{1}^{H}$ for all $x \in G/H$.

Proposition II.19. Every element in $P(A^G)$ is of the form $\sum_{i}n_{i}\text{Ind}_{H_{i}}^{G}1$, where $n_{i} \in \mathbb{Z}$ and $H_{i}$ is a subgroup of $G$.

Proof.

$$\text{Ind}_{H}^{G}1 \perp \text{Ind}_{K}^{G}1 \cong \left(\underset{x \in G/H}{\Pi}1_{x}\right) \perp \left(\underset{y \in G/K}{\Pi}1_{y}\right) \quad \text{(by Proposition II.18)}$$

$$\cong \underset{x, y}{\Pi}(1_{x} \perp 1_{y})$$

$$\cong \underset{x, y}{\Pi}1_{x, y} \cong \underset{\alpha \in G/H \times G/K}{\Pi} \left(\underset{(x, y) \in \alpha}{\Pi}1_{(x, y)}\right)$$

$$\cong \underset{\alpha}{\Pi}\text{Ind}_{H_{\alpha}}^{G}1 \quad \text{(by Proposition II.7)}.$$

The canonical $S_{k}$-object in $A$, denoted $X_{k}$, is defined to be $\text{Ind}_{S_{1} \times S_{k-1}}^{S_{k}}1$.

Remark II.20. From Proposition II.18, it follows that

$$X_{k} = \underset{\overline{\sigma} \in S_{k}/(S_{1} \times S_{k-1})}{\Pi}1_{\overline{\sigma}}.$$

Since $\overline{\sigma} = \overline{\tau}$ in $S_{k}/(S_{1} \times S_{k-1})$ iff $\tau(1) = \sigma(1)$, each $S_{1} \times S_{k-1}$-orbit of $S_{k}$ consists of precisely those $\sigma \in S_{k}$ which send $1$ to the same symbol $j$.

Hence $X_{k} = \Pi_{j=1}^{k}1_{j}$, where $\sigma \in S_{k}$ acts by

$$1_{j} \rightarrow 1_{\sigma(j)} \rightarrow \Pi 1_{j}.$$

For examples, see I.1(iii) and I.6(ii).

Let $G$ be a group, and $A$ have finite coproducts, coequalizers, a "tensor product" $\perp$, and object $1$. Let $\phi_{0} : A^{G} \rightarrow A^{G}$ be the functor sending $A \in$
A^G to 1_G. \( \phi_0 \) and the functors \( \phi_{W_n} \) arising from all positive integers \( n \) and all \( W_n \in A^{S_n} \) (see II, §A) will be called \( S \)-operations. If \( A = G \)-modules (see I, §C), the \( S \)-operations generate what are known as \( \lambda \)-operations. If \( A = G \)-sets (see I, §B), the \( S \)-operations will be referred to as \( \beta \)-operations.

For \( T \in A^G \), let \( \langle T \rangle \) denote the subring of \( K_0(A^G) \) generated by \{\( [\phi(T)] \mid \phi \) an \( S \)-operation\}. If there is a functor \( \text{Hom} \) as in Lemma II.11, then Theorem II.15 says that every element in \( \langle T \rangle \) is a finite sum \( \Sigma \alpha q_\alpha [\phi_\alpha(T)] \), where \( q_\alpha \in \mathbb{Z}, \phi_\alpha \) an \( S \)-operation.

Summarizing, the ring \( K_0(A^G) \) and subrings \( P(A^G) \) and \( \langle T \rangle \) have been constructed. The main theorem and its immediate corollary apply when \( G = S_k \) and \( T = X_k \):

**Main Theorem II.21.** Let \( A \) have finite coproducts, coequalizers, a “tensor product” \( \perp \), and an object \( 1 \). Assume there is a functor \( \text{Hom} \) as in Lemma II.11. Then for each positive integer \( k \), \( P(A^{S_k}) \subset \langle X_k \rangle \).

**Corollary II.22.** Same hypothesis as above. Suppose \( P(A^{S_k}) = K_0(A^{S_k}) \). Then \( \langle X_k \rangle = K_0(A^{S_k}) \).

**Lemma II.23.** Same hypothesis as above. Let \( H \subset S_n \) and \( W_n = \text{Ind}_{H}^{S_n}1 \). Then

\[
\phi_{W_n}(X_k) = \bigcup_{\gamma} \text{Ind}_{H\gamma}^{S_k}1,
\]

for some collection of subgroups \( H_\gamma \) of \( S_k \). Here, \( \gamma_1 \neq \gamma_2 \) need not imply \( H_{\gamma_1} \neq H_{\gamma_2} \).

This lemma does not imply \( \langle X_k \rangle \subset P(A^{S_k}) \), since \( \langle X_k \rangle \) is obtained from all \( S \)-operations \( \phi_{W_n} \), and if \( P(A^{S_n}) \neq K_0(A^{S_n}) \), \( W_n \) need not be a linear combination of objects \( \text{Ind}^{S_n}_{H}1 \).

**Proof of Lemma.** By Theorem II.12, \( \phi_{W_n}(X_k) = (\text{Res}_{H}^{S_n}(X^{\perp l_n}))_H \). Since

\[
X^{\perp l_n} = \left( \prod_{j=1}^{k} I_j \right)^{l_n} \approx \prod_{j \leq n} \prod_{1 \leq j_i < k} (I_{j_1} \cdots 1_{i_{j_{n}}})
\]

we have

\[
(1) \quad \phi_{W_n}(X_k) = \left( \text{Res}_{H}^{S_n} \left( \prod_{1 \leq i \leq k} 1_{j_1 \cdots j_n} \right) \right)_H.
\]
\( \sigma \in S_n \) acts on \( \prod_{j=1}^{m} (\cup_{j=1}^{n}) \) by

\[
\sigma: \prod_{j=1}^{m} (\cup_{j=1}^{n}) \xrightarrow{1_{\sigma^{-1}(1), \ldots, \sigma^{-1}(n)}} \prod_{j=1}^{m} (\cup_{j=1}^{n})
\]

and \( g \in S_k \) by

\[
g: \prod_{j=1}^{m} (\cup_{j=1}^{n}) \xrightarrow{1_{g(1), \ldots, g(n)}} \prod_{j=1}^{m} (\cup_{j=1}^{n})
\]

moreover, \( \sigma g \in S_k \)

\[
(\sigma g) = g \sigma
\]

Let \( J = \{ (j_1, \ldots, j_n) | 1 \leq j_i \leq k \} \). By the above, \( H \subset S_n \) acts on the set \( J \). For \( j \in J \), let \( f_j \) denote the orbit \( H_j \). Using equation (1), it is not hard to show that \( \phi_{\nu_n}(X_k) = \prod_{j \in f_j} H_j \)-orbits \( \prod_{j \in f_j} \), where \( g \in S_k \) acts by \( g: \prod_{j \in f_j} \prod_{j \in f_j} \).

Let \( \pi: \prod_{j \in f_j} \prod_{j \in f_j} \rightarrow \prod_{j \in f_j} \prod_{j \in f_j} \).

be the map induced from \( \prod_{j \in f_j} \prod_{j \in f_j} \rightarrow \prod_{j \in f_j} \prod_{j \in f_j} \). It is straightforward to show that \( (\prod_{j \in f_j}, \pi) \) satisfies the universal property defining \( (\prod_{j \in f_j}, \pi)_H \). Hence as an object in \( A \), \( \phi_{\nu_n}(X_k) = \prod_{j \in f_j} H_j \)-orbits \( \prod_{j \in f_j} \). Since, for \( g \in S_k \), \( (ng)h = ng = ng: \prod_{j \in f_j} \rightarrow \prod_{j \in f_j} \) for all \( h \in H \), there is a unique map \( \prod_{j \in f_j} \rightarrow \prod_{j \in f_j} \) such that the diagram

\[
\begin{array}{ccc}
\prod_{j} & \xrightarrow{g} & \prod_{j} \\
\downarrow & & \downarrow \\
\prod_{j} & \xrightarrow{g} & \prod_{j}
\end{array}
\]

commutes, and hence is determined by the commutative diagram:

\[
\begin{array}{ccc}
1_{f_j} & \rightarrow & 1_{gj} \\
\downarrow & & \downarrow \\
1_{f_j} & \rightarrow & 1_{gj}
\end{array}
\]

Thus \( \phi_{\nu_n}(X_k) = \prod_{j \in f_j} H_j \)-orbits \( \prod_{j \in f_j} \), and \( S_k \) permutes the \( 1_{f_j} \)'s by permuting the \( H \)-orbits \( f_j \). Therefore,

\[
\phi_{\nu_n}(X_k) \approx \prod_{j \in f_j} \approx \bigcup_{S_k\text{-orbits } \gamma} \left( \prod_{j \in \gamma} 1_{f_j} \right) \approx \text{Ind}^{S_k}_{H} 1_{f_0}
\]

(by Proposition II.7), where \( 1_{f_0} \) is one of the \( 1_{f_j} \)'s and \( H_\gamma \) is its isotropy group. \( \square \)

Let \( H \subset S_k \) and \( m \) be a nonnegative integer. \( H \) is said to be divisible.

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by $S_m$ if $H$ is conjugate to $M \times S_m$ (as subgroups of $S_k$) for some subgroup $M$ of $S_{k-m}$. Here, "$H$ conjugate to $M \times S_0$" means $H$ is conjugate to some subgroup $M$ of $S_k$; "$H$ conjugate to $M \times S_k$" means $H$ is conjugate to $S_k$. Clearly $H$ is always divisible by $S_0$.

**Proof of Main Theorem.** It is enough to show that if $H \subseteq S_k$ is divisible by $S_m$ for some $m$, $0 \leq m \leq k$, then $\text{Ind}_{H}^{S_k} 1 \in \langle X_k \rangle$. The proof is by induction (backwards) on $m$:

(i) If $m = k$, then $H = S_k$ and $\text{Ind}_{H}^{S_k} 1 = 1_{S_k} = \phi_0(X_k) \in \langle X_k \rangle$.

(ii) Suppose $m < k$ and assume that if $m < m' < k$, then $H$ divisible by $S_{m'} \Rightarrow \text{Ind}_{H}^{S_k} 1 \in \langle X_k \rangle$. Let $H$ be divisible by $S_m$. Then $\text{Ind}_{H}^{S_k} 1 = \text{Ind}_{M \times S_m}^{S_k} 1$ for some $M \subseteq S_{k-m}$. Let $W_{k-m} = \text{Ind}_{M \times S_m}^{S_k} 1$. Lemma II.23 gives

$$\phi_{W_{k-m}}(X_k) = \prod_{S_k \text{-orbits } \gamma} \left( \prod_{j \in \gamma} 1_{j} \right) = \prod_{\gamma} \text{Ind}_{H_{\gamma}}^{S_k} 1.$$

Recall that $J = \{(i, \cdots, k-m) | 1 \leq i \leq k\}$ is an $S_k$-set and an $M$-set, that $S_k$ permutes the $M$-orbits $j$ of $J$, and that $\gamma$ runs through the $S_k$-orbits of the set of $M$-orbits of $J$.

Direct computation shows that the $M$-orbit $(1, \cdots, k-m)$, which is in some $S_k$-orbit $\gamma_0$, has isotropy group $H_{\gamma_0} = M \times S_m$. Moreover, $(i_1, \cdots, i_{k-m}) \in \gamma_0$ whenever all the $i_j$'s are distinct, since $S_k$ is $(k-m)$-fold transitive. Thus if $(i_1, \cdots, i_{k-m}) \in \gamma \neq \gamma_0$, $j = i_t$ for some $i \neq t$; hence its isotropy group $H_{\gamma}$ is of the form $K \times S_{m'}$, for some $m' > m$ and $K \subseteq S_{k-m}$. Since $H_{\gamma}$ is divisible by $S_{m'}$, for some $m' > m$ if $\gamma \neq \gamma_0$, $\text{Ind}_{H_{\gamma}}^{S_k} 1 \in \langle X_k \rangle$ for all $\gamma \neq \gamma_0$ by induction hypothesis. Thus

$$\text{Ind}_{H}^{S_k} 1 = \text{Ind}_{M \times S_m}^{S_k} 1 = \phi_{W_{k-m}}(X_k) - \sum_{\gamma \neq \gamma_0} \text{Ind}_{H_{\gamma}}^{S_k} 1 \in \langle X_k \rangle.$$

The proof is completed by induction.\(\square\)

**III. Applications and Open Questions**

A. **Unigeneration of the $\lambda$-ring $R(S_k)$.** It is well known that $R(S_k)$ is a free $\mathbb{Z}$-module with basis $\{\text{Ind}_{S_{k_1} \times S_{k_2} \times \cdots}^{S_k} 1 \mid k_i \geq 1, \Sigma i k_i = k\}$ [5, Chapter III]. Therefore, $P(A^{S_k}) = K_0(A^{S_k}) = R(S_k)$, where $A$ is **finite-dimensional vector spaces over $\mathbb{C}$**. Corollary II.22 now implies $R(S_k) = \langle X_k \rangle$, and Theorem II.15 gives that every element of $R(S_k)$ is a linear combination of $\{[\phi(X_k)] \mid \phi$ an $S$-operation$\}$.

Moreover, every $S$-operation is a linear combination of symmetric power operations:

$$W_n \in R(S_n) \Rightarrow [W_n] = [W'_n] - [W''_n],$$
where
\[ W'_n = \sum \alpha_\mu \text{Ind}^{S_{\mu_1} \times \cdots \times S_{\mu_s}}_{S\mu_s} 1, \quad W''_n = \sum \beta_\nu \text{Ind}^{S_{\nu_1} \times \cdots \times S_{\nu_t}}_{S\nu_t} 1, \]
with \( \alpha_\mu, \beta_\nu \) positive integers.

\[ [W'_n'] = [W_n] + [W''_n'] \Rightarrow [W'_n \cup W''_n] \Rightarrow W'_n \cup W''_n \]  (See Remark II.17)

\[ \Rightarrow \phi_{W'_n}(T) = \phi_{W_n}(T) \cup \phi_{W''_n}(T) \] (by Theorem II.9)

\[ \Rightarrow [\phi_{W'_n}(T)] = [\phi_{W_n}(T)] \cup [\phi_{W''_n}(T)]. \]

Since
\[ \text{Ind}^{S_{\nu_1} \times S_{\nu_2}}_{S\nu_1 \times S\nu_2} 1 = \text{Ind}^{S_{\nu_1} \times S_{\nu_2}}_{S\nu_1 \times S\nu_2} (1 \otimes 1), \]

etc., Theorem II.15 implies \( \phi_{W'_n}(T) \) is a linear combination of \( \{ h_{n_1}(T) \otimes h_{n_2}(T) \otimes \cdots \otimes h_{n_t}(T) | n_i \geq 0 \} \), where \( h_0 = \phi_0 \), and \( h_n = \phi_{W'_n} \) for \( n > 0 \) and \( W'_n = \text{Ind}^{S_{\nu_1}}_{S\nu_1} 1 \). The \( h_i \)'s are, of course, symmetric power operations (see Example II.13(ii)).

Combining the two paragraphs above, one obtains the result that every element of \( R(S_k) \) is a linear combination of \( \{ h_{n_1}(X_k) \otimes \cdots \otimes h_{n_s}(X_k) | n_i \geq 0 \} \). Thus \( R(S_k) \) is generated by the single element \( X_k \) if symmetric powers are included with the standard ring operations. Since \( \lambda \)-operations generate symmetric power operations [2], [5], \( X_k \) generates \( R(S_k) \) as a \( \lambda \)-ring.

**Remark III.1.** Although \( R(S_k) \) is unigenerated as a \( \lambda \)-ring, it is not unigenerated as a ring, i.e., \( R(S_k) \neq Z[T] \) for all \( T \in R(S_k) \). The first counter-example is \( R(S_2) \):

If \( R(S_4) = Z[T] \), then the ring \( Z/2 \otimes Z R(S_4) \) is unigenerated as a \( Z/2 \)-module. Since \( R(S_4) \) is a free \( Z \)-module of rank 5 (see Proposition I.9), \( Z/2 \otimes Z R(S_4) \) is a free \( Z/2 \)-module of rank 5. By writing out its multiplication table \( (Z/2 \otimes Z R(S_4) \) has only \( 2^5 \) elements), one can show that no element generates all of \( Z/2 \otimes Z R(S_4) \).

**B. A unigeneration theorem for** \( B(S_k) \). \( B(G) \) is a free \( Z \)-module with basis \( \{ G/H_\alpha \} \), where \( \{ H_\alpha \} \) = a set of representatives of the conjugacy classes of subgroups of \( G \) (Proposition I.5). Clearly, if \( A = \text{finite sets} \), then \( \text{Ind}^G_H 1 = \) the \( G \)-set \( G/H \) (see Proposition II.18). Thus \( P(A^G) = K_0(A^G) = B(G) \). Hence Corollary II.22 implies that \( B(S_k) = \langle X_k \rangle \). Thus \( S \)-operations (here called "\( \beta \)-operations") applied to \( X_k \) generate all of \( B(S_k) \).

**Remark III.2.** \( B(S_k) \) is not, in general, generated by one element as a ring, since the ring homomorphism \( B(G) \rightarrow R(G) \) defined by \( T \mapsto \text{vector} \).
space with basis \( \{v_r\}_{r \in T} \) (see Examples 1.6(ii), (ii')) is onto if \( G = S_k \) [5, Chapter III], and therefore the ring \( B(S_4) \) is not unigenerated since \( R(S_4) \) is not (see Remark III.1).

**Remark III.3.** Applying sums and products of symmetric power operations \( h_n \) to \( X_k \) does not, in general, give all of \( B(S_k) \). \( B(S_3) \) is a counterexample:

The nonconjugate subgroups of \( S_3 \) are 1, \( S_1 \times S_2 \), \( S_3 \) and \( A_3 \) (= the even permutations in \( S_3 \)). Proposition 1.5 now says that \( B(S_3) \) is a free \( \mathbb{Z} \)-module with basis \( S_3/1, S_3/S_1 \times S_2, S_3/S_3, S_3/A_3 \). Note that \( S_3/S_1 \times S_2 = X_3 \) (see Proposition II.19) and \( S_3/S_3 = 1 \). \( B(S_3) \) is now completely described by the following multiplication table, which is obtained easily by direct calculation using Propositions I.2, I.3, and I.4:

\[
\begin{array}{cccc}
1 & S_3/A_3 & X_3 & S_3/1 \\
S_3/A_3 & 1 & S_3/A_3 & X_3 & S_3/1 \\
X_3 & S_3/A_3 & 2S_3/A_3 & S_3/1 & 2S_3/1 \\
S_3/1 & S_3/A_3 & 2S_3/1 & 3S_3/1 & 6S_3/1 \\
\end{array}
\]

Now suppose that the symmetric power operations \( h_n \) applied to \( X_3 \) give all of \( B(S_3) \). Then, in particular, \( S_3/A_3 \) could be expressed as a finite sum

\[
\sum a_{i_1 \ldots i_g} h_{n_{i_1}}(X_3) h_{n_{i_2}}(X_3) \cdots h_{n_{i_g}}(X_3), \quad \text{where} \quad a_{i_1 \ldots i_g} \in \mathbb{Z}.
\]

From the multiplication table, it is clear that one of the \( h_n(X_3) \)'s above must be of the form \( n_1 1 + n_2 S_3/A_3 + n_3 X_3 + n_4 S_3/1 \), with \( n_2 \neq 0 \). But for all \( n \geq 0 \), \( h_n(X_3) = n_1 1 + n_2 X_3 + n_4 S_3/1 \) for some \( n \in \mathbb{Z} \): An element \( (x_1, x_2, \ldots, x_n) \) in an \( S_n \)-orbit of \( X_3^n \) is made up of \( \mu_1 \) 1's, \( \mu_2 \) 2's, \( \mu_3 \) 3's, where \( \mu_1 + \mu_2 + \mu_3 = n \), and the 3-tuple \( (\mu_1, \mu_2, \mu_3) \) uniquely determines the \( S_n \)-orbit. If the \( \mu_i \)'s are all different, then the \( S_3 \)-orbit of \( (X_3^n)_{S_n} \) which contains the \( S_n \)-orbit corresponding to \( (\mu_1, \mu_2, \mu_3) \) is \( S_3/1 \). If exactly two of the \( \mu_i \)'s are the same, then the \( S_3 \)-orbit is \( S_3/S_1 \times S_2 \). If \( \mu_1 = \mu_2 = \mu_3 \), then the \( S_3 \)-orbit is \( S_3/S_3 \). Therefore, \( S_3/A_3 \) never arises.

Thus \( B(S_k) \) is generated by \( X_k \) if all the \( \beta \)-operations are used, but is, in general, not generated by \( X_k \) if only symmetric power operations are used. Hence \( \beta \)-operations include, but are not the same as, symmetric powers.
C. Some open questions. Since the $S$-operations in the linear representation theory case are generated by symmetric power operations (see III, §A), which are defined on all of $R(G)$ (see I, §C), $S$-operations extend to operations on $R(G)$, thus making $R(G)$ a "$\lambda$-ring". The unigeneration of $R(S_k)$ can be phrased:

There is an onto "$\lambda$-ring homomorphism"

$$\Lambda \to R(S_k), \quad a_1 \mapsto X_k,$$

where $\Lambda$ is the "free $\lambda$-ring on one generator" $a_1 \in \Lambda$. Hence $R(S_k) \cong \Lambda/I$, for some $\lambda$-ideal $I$. A reasonable description of this $\lambda$-ideal, in particular, a canonical set of generators, is unknown.

In the case of permutation representations, i.e., $G$-sets, the corresponding theory of "$\beta$-ring" which would allow extending the $\beta$-operations to all of $B(G)$ is not known.

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