CONTINUOUS IN WHICH ALL CONNECTED SUBSETS ARE ARCWISE CONNECTED

BY

E. D. TIMCHATYN(*)

ABSTRACT. Let $X$ be a metric continuum such that every connected subset of $X$ is arcwise connected. Some facts concerning the distribution of local cutpoints of $X$ are obtained. These results are used to prove that $X$ is a regular curve.

1. Introduction. Several attempts have been made to characterize the spaces in which all connected subsets are arcwise connected, e.g. Kuratowski and Knaster [1], Whyburn [3], [5], [6] and Tymchatyn [2]. In [10] Mohler conjectured that if $X$ is a metric continuum such that each connected subset of $X$ is arcwise connected, then $X$ is a regular curve. The main purpose of this paper is to resolve this conjecture in the affirmative.

The reader may consult Whyburn [8, V. 2] for a survey of the properties of hereditarily locally connected continua and [8, III. 9] for a treatment of local cutpoints. Our notation follows Whyburn [8]. We collect here some basic definitions for the convenience of the reader. A continuum is a nondegenerate, compact, connected metric space. A continuum is said to be hereditarily locally connected if each of its subcontinua is locally connected. A continuum is said to be regular if it has a basis of open sets with finite boundaries. An arc is a homeomorph of the closed unit interval $[0, 1]$. If $A$ is an arc and $c, d \in A$ then $[c, d]$ denotes the arc in $A$ with endpoints $c$ and $d$. A subset $A$ of a space $X$ is said to be arcwise connected if every pair of points of $A$ can be joined by an arc in $A$. A point $p$ in a continuum $X$ is said to be a local cutpoint or local separating point of $X$ if there is a connected open set $U$ in $X$ such that $U - \{p\}$ is not connected. The term neighbourhood will always mean open neighbourhood. We denote the closure of a set $A$ by $\text{Cl}(A)$.

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2. In this section, we study several conditions that are satisfied by continua whose connected subsets are arcwise connected. The following theorem has played a central role in the study of these continua.

**Theorem 1** (Tymchatyn [2]). Let $X$ be a continuum such that $X = A_0 \cup A_1 \cup \cdots$ where

(i) $A_i$ is not empty and contains no local cutpoints of $X$,

(ii) for each $i = 1, 2, \cdots, A_i$ is a closed set,

(iii) for each $i \neq j$, $A_i \cap A_j$ is void.

Then $X$ contains a connected set that is not arcwise connected.

**Lemma 2.** If $X$ is a locally connected continuum such that $L$, the set of local cutpoints of $X$, is totally disconnected then $L$ can be written as the union of a countable family of pairwise disjoint closed sets.

**Proof.** It is well known (see [8, p. 63]) that $L$ is an $F_\sigma$. Since $L$ is totally disconnected it is easy to see that $L$ can be written as the union of a countable family of pairwise disjoint closed sets.

By a null sequence of sets is meant a sequence of sets whose diameters converge to zero.

**Theorem 3.** Let $X$ be a continuum such that every sequence of disjoint subcontinua of $X$ is a null sequence. Then the following are equivalent:

(a) If $A_1, A_2, \cdots$ is any sequence of pairwise disjoint closed subsets of $X$ and $x$ and $y$ are points that are separated by $X - (A_1 \cup A_2 \cup \cdots)$ then some countable subset of $X - (A_1 \cup A_2 \cup \cdots)$ separates $x$ and $y$ in $X$.

(b) If $A$ is any subcontinuum of $X$ and $x, y \in A$ then there is an arc in $A$ which contains $x$ and $y$ and which contains at most countably many points that are not local cutpoints of $A$.

(c) If $A$ is any subcontinuum of $X$ then the set of local cutpoints of $A$ is not contained in the union of countably many pairwise disjoint closed proper subsets of $A$.

**Proof.** (a) $\Rightarrow$ (c). Suppose (c) fails. Then there is a subcontinuum $A$ of $X$ such that the set of local cutpoints of $A$ is contained in the union of a countable family $A_1, A_2, \cdots$ of closed, proper, pairwise disjoint subsets of $A$. We may suppose without loss of generality that $A_1$ and $A_2$ are nonempty sets. Let $x \in A_1$ and let $y \in A_2$. Since every sequence of pairwise disjoint subcontinua of $X$ is a null sequence we may suppose that the $A_i$ form a null sequence. It follows that the decomposition space $Y$ obtained from $A$ by identifying each of the sets $A_i$ to a point is a compact metric space. The image of the set $A_1 \cup A_2 \cup \cdots$
in $Y$ under the natural projection $\pi$ is a countable set. Hence $A - (A_1 \cup A_2 \cup \cdots)$ separates $x$ and $y$ in $A$ since $Y - \pi(A_1 \cup A_2 \cup \cdots)$ separates $\pi(x)$ from $\pi(y)$ in $Y$. By [8, III. 9.41] every set in $A - (A_1 \cup A_2 \cup \cdots)$ which separates $x$ and $y$ is uncountable. In particular every set in $X$ which separates $x$ and $y$ is uncountable. Thus, (a) also fails.

(b) $\Rightarrow$ (c). Suppose (c) fails. Then there is a subcontinuum $A$ of $X$ such that the set of local cutpoints of $A$ is contained in the union of a countable family $A_1, A_2, \cdots$ of closed proper subsets of $A$. By Sierpinski's theorem there exist $x, y \in A - (A_1 \cup A_2 \cup \cdots)$ and each arc in $A$ which contains $x$ and $y$ contains uncountably many points of $A - (A_1 \cup A_2 \cup \cdots)$. Hence, every arc in $A$ which contains $x$ and $y$ contains uncountably many points that are not local cutpoints of $A$. Thus, (b) also fails.

(c) $\Rightarrow$ (b). Suppose (b) fails. Let $L$ denote the set of local cutpoints of $X$. We may suppose without loss of generality that there exist $x, y \in X$ such that each arc in $X$ which contains both $x$ and $y$ contains uncountably many points of $X - L$. For each $z \in X$ let $A(z) = \bigcup\{A \subset X|A$ is a continuum, $z \in A$ and $A - L$ is countable}. By [7, Theorem 34] each $A(z)$ is closed. Clearly, $A(z)$ is also connected. Define an equivalence relation $\sim$ on $X$ by setting $x \sim y$ if and only if $x \in A(y)$. Since the set of nondegenerate equivalence classes of $\sim$ is a null sequence, it follows that $\sim$ is a closed relation.

Let $\pi$ be the natural projection of $X$ onto the quotient space $X/\sim$. $X$ is hereditarily locally connected since it can contain no continuum of convergence. Hence, $X/\sim$ is also a Peano continuum. A point $z \in X/\sim$ is a local cutpoint of $X/\sim$ only if $\pi^{-1}(z)$ contains a local cutpoint of $X$. Also, if $p$ is a local cutpoint of $X$ and $A(p) = \{p\}$ then $\pi(p)$ is a local cutpoint of $X/\sim$. We shall prove that the set of local cutpoints of $X/\sim$ is totally disconnected. By Lemma 2 it will follow that the set of local cutpoints of $X/\sim$ is the union of a countable family of pairwise disjoint closed sets and hence the set of local cutpoints of $X$ is contained in the union of a countable family of pairwise disjoint closed sets. Thus, it will have proved that (c) also fails.

Just suppose that the set of local cutpoints of $X/\sim$ is not totally disconnected. Since $X/\sim$ is a Peano continuum the set of local cutpoints of $X/\sim$ is an $F_\sigma$. It follows by the Sum Theorem for dimension zero that the set of local cutpoints of $X/\sim$ contains a continuum $A$. Then $\pi^{-1}(A)$ is a continuum in $X$ which is the union of uncountably many equivalence classes of $\sim$. Let $c, d \in \pi^{-1}(A)$ such that $c \neq d$. Let $C$ be an arc in $\pi^{-1}(A)$ with endpoints $c$ and $d$. For convenience, we identify $C$ with the closed unit interval $[0, 1]$ with its usual order and its usual metric. To prove the theorem, it will suffice to prove that there is
an arc $D \subset \pi^{-1}(A)$ such that $c, d \in D$ and $D - L$ is at most countable.

Let $A_1, A_2, \ldots$ denote the nondegenerate equivalence classes of $\sim$. If $x \in C - L$ then $x \in A_1 \cup A_2 \cup \cdots$. Let $m_1$ be an integer such that $\text{diameter } (A_{m_1} \cap C) \geq \text{diameter } (A_i \cap C)$ for each $i = 1, 2, \cdots$. Let $c_i = \min (A_{m_1} \cap C)$ and let $d_i = \max (A_{m_1} \cap C)$. Let $B_i$ be an arc in $A_{m_1}$ with endpoints $c_i$ and $d_i$ such that $B_i - L$ is at most countable. Notice that $A_{m_1} \cap C \subset [c_1, d_1]$ and for each $i = 1, 2, \cdots$ there do not exist $a, b \in A_i \cap C$ such that $a < c_1 < d_1 < b$.

Suppose $A_1, A_2, \ldots, A_{m_n}$ have been selected and for each $j = 1, \ldots, n$, $B_j$ is an arc in $A_{m_j}$ with endpoints $c_j$ and $d_j$ such that $B_j - L$ is at most countable and $(A_{m_j} \cap C) \subset [c_1, d_1] \cup \cdots \cup [c_j, d_j]$. Suppose also that there do not exist $k \in \{1, \ldots, n\}$ and an integer $i \in \{1, 2, \cdots\}$ with $a, b \in (C \cap A_i) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])$ such that $a < c_k < d_k < b$. Suppose the intervals $[c_j, d_j]$ are disjoint.

If, for each $j$, $(A_j \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])$ contains at most one point, then it is easy to check that $D = (C - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])) \cup B_1 \cup \cdots \cup B_n$ is an arc in $\pi^{-1}(A)$ which contains $c$ and $d$ and $D - L$ is at most countable.

Let us suppose, therefore, that there exists an integer $m_{n+1}$ such that

$$0 < \text{diameter } (A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])$$

for each $j = 1, 2, \cdots$.

Let

$$c_{n+1} = \min ((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n]))$$

and let

$$d_{n+1} = \max ((A_{m_{n+1}} \cap C) - ([c_1, d_1] \cup \cdots \cup [c_n, d_n])).$$

Let $B_{n+1}$ be an arc in $A_{m_{n+1}}$ with endpoints $c_{n+1}$ and $d_{n+1}$ such that $B - L$ is at most countable.

Let $D = (C - ([c_1, d_1] \cup [c_2, d_2] \cup \cdots)) \cup B_1 \cup B_2 \cup \cdots$. For each positive integer $i$, $A_i \cap (D - ([c_1, d_1] \cup [c_2, d_2] \cup \cdots))$ contains at most one point. Thus, $D - L$ is at most countable.

For each positive integer $j$ let $h_j: [c_j, d_j] \rightarrow B_j$ be a homeomorphism such that $h_j(c_j) = c_j$ and $h_j(d_j) = d_j$. Define $h : C \rightarrow D$ by

$$h(x) = \begin{cases} x & \text{if } x \notin [c_j, d_j] \text{ for any } j, \\ h_j(x) & \text{if } x \in [c_j, d_j]. \end{cases}$$
Then $h$ is easily seen to be a homeomorphism of the arc $C$ onto $D$. Thus $D$ is an arc in $\pi^{-1}(A)$ which contains $c$ and $d$ and $D - L$ is at most countable. Thus, $\pi(c) = \pi(d)$ which is a contradiction.

(c) $\Rightarrow$ (a). Suppose (a) fails. Then there is a sequence $A_1, A_2, \cdots$ of pairwise disjoint proper closed subsets of $X$ and points $x$ and $y$ that are separated by $X - (A_1 \cup A_2 \cup \cdots)$ such that no countable subset of $X - (A_1 \cup A_2 \cup \cdots)$ separates $x$ and $y$ in $X$. Let $A = A_1 \cup A_2 \cup \cdots$. We shall prove that $X$ contains a subcontinuum $B$ such that $x, y \in B$, no countable subset of $B - A$ separates $x$ and $y$ in $B$, and $B - A$ contains only countably many local cutpoints of $B$. Clearly the set of local cutpoints of $B$ is contained in the union of countably many disjoint proper closed sets so (c) fails.

We define by transfinite induction a nest of subcontinua $B_\alpha$ of $X$ as follows: Let $B_0 = X$. Let $\alpha$ be a countable ordinal number. Suppose that for each $n < \alpha$, $B_n$ has been defined to be a subcontinuum of $X$ such that $x, y \in B$ and no countable subset of $B_n - A$ separates $x$ and $y$ in $B_n$. If, for some $n < \alpha$, $B_n - A$ contains only countably many local cutpoints of $B_n$, we are done. Suppose, therefore, that for each $n < \alpha$, $B_n - A$ has uncountably many local cutpoints of $B_n$.

Case 1. $\alpha$ is the successor of an ordinal number $m$. By assumption $B_m - A$ contains uncountably many local cutpoints of $B_m$. It follows from [8, III. 9.21] that every uncountable set of local cutpoints of a continuum contains a pair of points that separate the continuum. Thus, there exist $a_\alpha, b_\alpha \in B_m - (A \cup \{x, y\})$ such that $B_m - \{a_\alpha, b_\alpha\}$ is not connected. Let $B_\alpha$ be the closure of the component of $B_m - \{a_\alpha, b_\alpha\}$ that contains $x$ and $y$. Then, no countable set of $B_\alpha - A$ separates $x$ and $y$ in $B_\alpha$.

Case 2. $\alpha$ is a limit ordinal. Let $B_\alpha = \bigcap_{n<\alpha} B_n$. We shall show that no countable subset of $B_\alpha - (A \cup \{x, y\})$ separates $x$ and $y$ in $B_\alpha$. Let $C$ be a countable subset of $B_\alpha$.

Let $C' = C \cup \bigcup_{n<\alpha} \{a_{n+1}, b_{n+1}\}$. Since $C'$ is countable $x$ and $y$ lie in the same component $E$ of $X - C'$. There is an arc $D$ in the locally connected, topologically complete, metric space $E$ with endpoints $x$ and $y$. By induction it is easy to see that $D \subset B_n$ for each $n < \alpha$. Hence $D \subset B_\alpha - C$ which implies that $x$ and $y$ lie in the same component of $B_\alpha - C$.

Since $X$ does not contain uncountably many pairwise disjoint nondegenerate subcontinua it follows that for some countable ordinal $\alpha$, $B_\alpha$ has at most countably many local cutpoints of $B_\alpha$ in $B_\alpha - A$. This completes the proof of Theorem 3.

Corollary III. 9.21 in Whyburn [8] asserts that if $X$ is a metric continuum and if $G$ is an uncountable set of local cutpoints of $X$ then there is a countable subset $G_0$ of $G$ such that every point of $G - G_0$ is of order 2 relative to $G - G_0$. In particular, there exist $a, b \in G - G_0$ such that $X - \{a, b\}$ is not connected.
It is easy to see that \( X - \{a, b\} \) has either 2 or 3 components.

If \( X \) is as above and \( C \) is a connected subset of \( X \) then \( \text{Cl}(C) - C \) contains at most countably many local cutpoints of \( \text{Cl}(C) \). For let \( G \) be an uncountable set of local cutpoints of \( \text{Cl}(C) \). Let \( a, b \in G \) such that \( \text{Cl}(C) - \{a, b\} \) is not connected. Since \( C \) is connected and dense in \( \text{Cl}(C) \) it follows that \( C - \{a, b\} \) is not connected. Thus, at least one of \( a \) and \( b \) is in \( C \) and \( G \subset \text{Cl}(C) - C \).

In order that all of the connected subsets of a regular continuum should be arcwise connected it is necessary that the continuum satisfy conditions (a)–(b) of Theorem 3. If \( X \) is a regular continuum that satisfies condition (a) of Theorem 3, let \( Y \) be a connected set in \( X \). Let \( D \) be a countable dense set in \( Y \) and let \( a \in D \). By Theorem 3 for each \( d \in D \) there is an arc \( A_d \) in \( \text{Cl}(Y) \) such that \( a, d \in A_d \) and \( A_d \) contains at most countably many points that are not local cutpoints of \( \text{Cl}(Y) \). By the last paragraph \( A_d - Y \) is at most countable. By [7, Theorem 34] \( Y \cup \bigcup\{A_d | d \in A\} \) is arcwise connected. Thus, there is a countable set \( C = \bigcup\{A_d - Y | d \in A\} \) such that \( Y \cup C \) is arcwise connected.

**Question 1.** If \( X \) is a regular continuum that satisfies condition (a) of Theorem 3, is every connected subset \( Y \) of \( X \) arcwise connected?

**Question 2.** If \( X \) is as in Question 1, is every connected \( F_a \) in \( X \) arcwise connected? In particular is the set of local cutpoints of \( X \) arcwise connected if it is connected?

**Question 3.** Let \( X \) be a regular continuum such that every pair of separated sets in \( X \) can be separated by a countable set. Is every connected subset of \( X \) arcwise connected?

**Theorem 4.** Let \( X \) be a regular continuum. If \( C \) is a connected subset of \( X \) then \( C \) cannot be decomposed into countably infinitely many pairwise disjoint sets that are closed in \( C \).

**Proof.** Let \( C \) be a connected subset of \( X \). Suppose \( C = \bigcup_{i=1}^{\infty} A_i \) where the \( A_i \) are pairwise disjoint proper subsets of \( C \) which are closed in \( C \). Since every sequence of pairwise disjoint connected sets in a regular space is a null sequence we may suppose the sequence \( A_i \) is a null sequence.

Define an equivalence relation \( \sim \) on \( C \) by letting \( x \sim y \) if and only if there is a natural number \( i \) such that \( x, y \in A_i \). Let \( \pi \) be the natural projection of \( C \) onto \( C/\sim \). Now \( C/\sim \) with the quotient topology is a connected countable space. We shall obtain a contradiction by proving that \( C/\sim \) cannot be connected because it is a countable, \( T_1 \), normal space. It is clear that \( C/\sim \) is a \( T_1 \) space since \( \pi^{-1}(p) \) is closed in \( C \) for each \( p \in C/\sim \). It remains to prove only that \( C/\sim \) is normal.

Let \( M \) and \( N \) be disjoint closed sets in \( C/\sim \). Then \( \pi^{-1}(M) \) and \( \pi^{-1}(N) \)
are disjoint closed sets in \( C \). Since \( C \) is normal there exist disjoint open sets \( U \) and \( V \) in \( C \) such that \( \pi^{-1}(M) \subset U \) and \( \pi^{-1}(N) \subset V \). Since the sequence \( A_i \) is a null sequence of closed sets in \( X \) it follows that \( \pi^{-1}(\pi(C - U)) \) is a closed set in \( C \) which misses \( \pi^{-1}(M) \). Hence, \( U - \pi^{-1}(\pi(C - U)) \) and \( V - \pi^{-1}\pi(C - V) \) are neighbourhoods of \( \pi^{-1}(M) \) and \( \pi^{-1}(N) \) respectively whose images under \( \pi \) are disjoint neighbourhoods of \( M \) and \( N \) respectively in \( C/\sim \).

**Corollary 5.** A connected subset of a regular continuum has either one or uncountably many arc components.

**Proof.** The arc components of a connected subset of a regular continuum are closed (see [7, p. 334]); hence Theorem 4 applies.

3. Continua that are not regular. For the remainder of this section let \( X \) be a fixed continuum that is not regular but which contains no nonnull sequence of pairwise disjoint subcontinua.

**Definition.** Let \( M \) be a subcontinuum of a continuum \( X \) and let \( x, y \in M \). We say \( M \) satisfies property \( P(x, y) \) if no finite set separates \( x \) and \( y \) in any neighbourhood of \( M \).

**Theorem 6.** If \( X \) is a continuum that is not regular then \( X \) contains a connected subset that is not arcwise connected.

**Proof.** We may suppose as in [2] that \( X \) is a hereditarily locally connected continuum and that every sequence of pairwise disjoint subcontinua of \( X \) is a null sequence. By Lemma 2 we may suppose that for each subcontinuum \( P \) of \( X \) the set of local cutpoints of \( P \) is not totally disconnected.

Let \( a \in X \) such that \( X \) is not regular at \( a \). Let \( M \) denote the set of points of \( X \) which cannot be separated from \( a \) by a finite set in \( X \). By Whyburn [8, V. 4.4, 4.5] \( M \) is a nondegenerate continuum. By Whyburn [8, III. 9.2] at most a countable number of points of \( M \) are local cutpoints of \( X \).

The first three claims can be proved by contradiction. The proofs are straightforward and are omitted.

**Claim 1.** \( M \) satisfies \( P(a, b) \) for each \( b \in M - \{a\} \).

Let \( b \in M - \{a\} \). Let \( M_{\lambda} \in \Lambda \) be a maximal nest of subcontinua of \( M \) each of which satisfies property \( P(a, b) \) and let \( P_{ab} = \bigcap_{\lambda \in \Lambda} M_\lambda \). Then \( P_{ab} \) is a continuum which is irreducible with respect to satisfying \( P(a, b) \), for if \( U \) is a neighbourhood of \( P_{ab} \) then \( M_\lambda \subset U \) for some \( \lambda \in \Lambda \) and hence no finite set separates \( a \) and \( b \) in \( U \). We shall describe in great detail the structure of the continuum \( P_{ab} \).
In [2, Theorem 6] the author considered the case where \( P_{ab} \) is an arc. To see that \( P_{ab} \) may be quite complicated let \( Y \) be the continuum constructed in the proof of Theorem 6 in [2]. Then \( Y = A_0 \cup A_1 \cup A_2 \cup \cdots \) where \( A_1, A, \cdots \) are pairwise disjoint arcs and \( A_0 \) is also an arc. Let \( Z \) be the decomposition space obtained from \( Y \) by contracting \( A_i \) to a point for each \( i = 1, 2, \cdots \). Then \( Z = P_{ab} \) where \( a \) and \( b \) are the endpoints of the arc \( A_0 \).

**Claim 2.** If \( c, d \in P_{ab} \) then \( P_{ab} \) satisfies \( P(c,d) \).

**Claim 3.** If \( U \) is a neighbourhood of \( x \in P_{ab} \) such that \( b \notin U \) then there is a point \( c \) in the boundary of \( U \) and a continuum \( P \subset P_{ab} - U \) such that \( P \) satisfies \( P(b,c) \).

If \( x \in P_{ab} \) is not a local cutpoint of \( X \) and \( Q \) and \( P \) are continua in \( P_{ab} \) which satisfy \( P(a,x) \) and \( P(x,b) \) respectively then \( P \cup Q \) is a continuum satisfying \( P(a,b) \) and so \( P \cup Q = P_{ab} \) by the irreducibility of \( P_{ab} \).

**Claim 4.** If \( x \notin \{b\} \) then there exists a proper subcontinuum of \( P_{ab} \) which satisfies \( P(a,x) \).

**Proof.** Suppose the claim is false. Let \( x \in P_{ab} - \{b\} \) such that no proper subcontinuum of \( P_{ab} \) satisfies \( P(a,x) \).

If \( U \) is any neighbourhood of \( b \) in \( P_{ab} \) then \( \text{Cl}(U) - U \) contains at most countably many local cutpoints of \( \text{Cl}(U) \) by the argument following Theorem 3. If \( U \) is connected, then by Lemma 2 there exists an arc \( I \) of local cutpoints of \( \text{Cl}(U) \). Now \( I \) intersects the boundary of \( U \) in a set that is compact and at most countable so \( U \) contains an arc of local cutpoints of \( U \) and hence of \( P_{ab} \).

By the last paragraph, there exists a sequence \( I_i \) of pairwise disjoint arcs in \( P_{ab} - \{b\} \) such that \( \lim \sup I_i = \{b\} \) and for each \( i = 1, 2, \cdots \) each point of \( I_i \) is a local cutpoint of \( P_{ab} \).

Let \( c_i \) and \( d_i \) be the endpoints of \( I_i \). Since \( P_{ab} \) contains only countably many local cutpoints of \( X \) we may suppose \( c_i \) and \( d_i \) are not local cutpoints of \( X \). By the argument following Theorem 3 we may suppose that \( c_i \) and \( d_i \) are points of order 2 in \( P_{ab} \), that \( c_i \) and \( d_i \) separate \( P_{ab} \) into either 2 or 3 components and that the component \( K_i \) of \( P_{ab} - \{c_i, d_i\} \) which meets \( I_i \) contains neither \( a \) nor \( b \). By Whyburn [3, §4] we may suppose that every point of \( I_i - \{c_i, d_i\} \) disconnects \( K_i \). Let (by [8, III. 9.21]) \( z_i \in I_i - \{c_i, d_i\} \) be a point of order 2 in \( P_{ab} \) such that \( z_i \) is not a local cutpoint of \( X \). Then \( z_i \) separates \( K_i \) into exactly two components. Finally, we may assume that the sets \( K_i \) are pairwise disjoint. For each \( i \) let \( U_i \) be a neighbourhood of \( K_i \) such that \( \text{Cl}(U_i) \cap \text{Cl}(U_j) \) is empty for \( i \neq j \). Notice that \( \text{Cl}(K_i) = K_i \cup \{c_i, d_i\} \).
By Claim 3 there is a component $P$ of $P_{ab} - K_i$ such that $P$ satisfies either $P(a, c_i)$ or $P(a, d_i)$. We may suppose without loss of generality that $P$ satisfies $P(a, c_i)$. Since $c_i$ is not a local cutpoint of $X$ it follows that $P$ does not satisfy $P(c_i, b)$ for otherwise $P$ would be a proper subcontinuum of $P_{ab}$ which satisfies $P(a, b)$. By Claim 3 it follows that there is a component $Q$ of $P_{ab} - K_i$ such that $Q$ satisfies $P(d_i, b)$. Let $P_{ac_i} \subset P$ and $P_{d_ib} \subset Q$ be continua which are irreducible with respect to satisfying $P(a, c_i)$ and $P(d_i, b)$ respectively. If $P_{ac_i} \cap P_{d_ib}$ is infinite let $U$ be any neighbourhood of $P_{ac_i} \cup P_{ad_i}$. Let $A$ be a finite set in $U$ and let $x \in (P_{ac_i} \cap P_{d_ib}) - A$. By Claim 2, $P_{ac_i}$ satisfies $P(a, x)$ and $P_{ad_i}$ satisfies $P(x, b)$. Since $U$ is a neighbourhood of both $P_{ac_i}$ and $P_{ad_i}$, $A$ does not separate $a$ from $x$ or $x$ from $b$ in $U$. Thus $A$ does not separate $a$ from $b$ in $U$ and so $P_{ac_i} \cup P_{d_ib}$ is a proper subcontinuum of $P_{ab}$ which satisfies $P(a, b)$. With this contradiction we conclude that $P_{ac_i} \cap P_{d_ib}$ is finite. A similar argument can be used to show that every point of $P_{ac_i} \cap P_{d_ib}$ is a local cutpoint of $X$.

By using the facts that $P_{ab} - K_i$ does not satisfy $P(a, b)$ while $P_{ab}$ satisfies $P(a, b)$, that $\text{Cl}(K_i) - K_i = \{c_i, d_i\}$ and that $c_i$ and $d_i$ are not local cutpoints of $X$ one can easily prove by contradiction that $\text{Cl}(K_i)$ satisfies $P(c_i, d_i)$. Since $c_i$ and $d_i$ are not local cutpoints of $X$ it follows from the remark following Claim 3 that $P_{ab} = P_{ac_i} \cup K_i \cup P_{d_ib}$.

Since $z_i$ separates $c_i$ and $d_i$ in $\text{Cl}(K_i)$ it follows from Claim 3 that $\text{Cl}(K_i)$ satisfies $P(c_i, z_i)$. Since $c_i$ is not a local cutpoint of $X$, $P_{ac_i} \cup \text{Cl}(K_i)$ satisfies $P(a, z_i)$. Let $P_{az_i} \subset P_{ac_i} \cup \text{Cl}(K_i)$ be a continuum which is irreducible with respect to satisfying $P(a, z_i)$.

Since $b \notin P_{ac_i} \cup \text{Cl}(K_i)$ we may suppose that, for each $i$, $z_{i+1} \in P_{d_ib}$. Hence, $\text{Cl}(K_{i+1}) \subset P_{d_ib}$. By Claim 2 we may suppose that $P_{d_{i+1}b} \subset P_{d_ib} - K_{i+1}$. It now follows that $P_{az_i} \subset P_{ac_i} \cup \text{Cl}(K_i) \subset P_{ac_{i+1}} \subset P_{az_{i+1}}$.

Since $P_{ab}$ is irreducible with respect to satisfying $P(a, x)$ it follows that $x \notin P_{az_i}$ for each $i$ but $x \in \limsup P_{az_i} = P_{ab}$. We may suppose that for each $i$ there is $x_i \in P_{az_i} - P_{az_{i-1}}$ such that $\lim x_i = x$. Since $P_{ab}$ contains at most countably many local cutpoints of $X$ we may suppose each $x_i$ is not a local cutpoint of $X$.

For each $i = 2, 3, \cdots$ there is a continuum $P_{d_{i-1}c_{i+1}} \subset P_{d_{i-1}b} \cap P_{ac_{i+1}}$ such that $P_{d_{i-1}c_{i+1}}$ is irreducible with respect to satisfying $P(d_{i-1}, c_{i+1})$. Notice that $x_i, z_i \in P_{d_{i-1}c_{i+1}}$.

For each $i$ let $W_i$ be a neighbourhood of $P_{ab} - K_i$ and $A_i$ a finite set such that $A_i$ separates $a$ and $b$ in $W_i$ and $W_{i+j} \subset W_i \cup U_i$ for all $i, j = 1, 2, \cdots$. Since $x_i$ and $z_i$ are not local cutpoints of $X$ we may suppose that $x_i, z_i \notin A_j$ for all $i$ and $j$. Note that for $j = 1, 2, \cdots, z_{i+j}$ and $x_{i+j}$ lie in the component of $W_i - A_i$.
which contains $b$ and, for $j = 1, \cdots, n - 1$, $z_{i-j}$ and $x_{i-j}$ lie in the component of $W_i - A_i$ which contains $a$.

For each $i = 2, 4, 6, \cdots$, $P_{i-1} \c_{i+1} \subset W_i \cap \cdots \cap W_i \cap W_{i+1}$ so there exists an arc $C_i$ joining $x_i$ to $z_i$ in $(W_i \cap \cdots \cap W_{i-1} \cap W_{i+1}) - (A_i \cup \cdots \cup A_{i-1} \cup A_{i+1})$. The arcs $C_i$ are pairwise disjoint by construction. This contradicts our assumption that every sequence of disjoint subcontinua of $X$ is a null sequence. The claim is proved.

**Claim 5.** If $x \in P_{ab}$ then there is a unique continuum $P_{ax}$ (resp. $P_{xb}$) in $P_{ab}$ which is irreducible with respect to satisfying $P(a, x)$ (resp. $P(x, b)$).

**Proof.** Just suppose $A$ and $B$ are two subsets of $P_{ab}$ which are irreducible with respect to satisfying $P(a, x)$. Let $y \in A - B$ and let $z \in B - A$ such that $y$ and $z$ are not local cutpoints of $X$. Let $P_{yb} \subset P_{ab}$ be a continuum which is irreducible with respect to satisfying $P(y, b)$. By the remark following Claim 3 $P_{ab} = A \cup P_{yb}$ thus $z \in P_{yb}$. By Claim 4 there exists a continuum $P_{zb} \subset P_{yb} - \{y\}$ such that $P_{zb}$ satisfies $P(z, b)$. Thus, $B \cup P_{zb}$ is a proper subcontinuum of $P_{ab}$ which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

**Claim 6.** If $x, y \in P_{ab}$ then either $P_{ax} \subset P_{ay}$ or $P_{ay} \subset P_{ax}$.

**Proof.** Suppose $P_{ax} \not\subset P_{ay}$ and $P_{ay} \not\subset P_{ax}$. By Claims 2 and 5, $x \not\in P_{ay}$ and $y \not\in P_{ax}$. Since $P_{ab}$ contains at most countably many local cutpoints of $X$, we may suppose $x$ and $y$ are not local cutpoints of $X$.

Since $P_{ax} \cup P_{xb} = P_{ab}, y \in P_{xb}$. By Claim 4, $P_{yb} \not\subset P_{xb}$. Thus, $x \not\in P_{yb}$ and $P_{ay} \cup P_{yb}$ is a proper subcontinuum of $P_{ab}$ which satisfies $P(a, b)$. This is a contradiction. The claim is proved.

Let $2^{P_{ab}}$ denote the space of closed subsets of $P_{ab}$ with the Hausdorff metric topology. Let $C$ denote the closure in $2^{P_{ab}}$ of $\{P_{ax} | x \in P_{ab}\}$.

**Claim 7.** $C$ is homeomorphic to the closed unit interval $[0, 1]$.

**Proof.** $2^{P_{ab}}$ is a compact metric space that is partially ordered by inclusion. This partial order is a closed relation on $2^{P_{ab}}$. Since $C$ is compact, we need only prove that $C$ is connected and totally ordered under inclusion (Ward [9]).

Let $C \subset C$. We start by showing that if $x \in C$ then $P_{ax} \subset C$. For let $x_i$ be a sequence in $P_{ab}$ such that $P_{ax_i}$ converges to $C$ in $C$. If, for arbitrarily large $i$, $x \in P_{ax_i}$ then $P_{ax} \subset P_{ax_i}$ for all such $i$ by Claims 2 and 5 and hence $P_{ax} \subset \limsup P_{ax_i} = C$. If, on the other hand, $P_{ax_i} \subset P_{ax}$ for arbitrarily large $i$ and if $U$ is a neighbourhood of $C$ such that $a$ and $x$ can be separated by a finite
set in \( U \), then for some sufficiently large \( i \) \( P_{ax_i} \) can be separated in \( U \) by that same finite set. This contradicts Claim 2. By Claim 6, these are the only two cases we need consider. Thus, \( x \in C \) implies \( P_{ax} \subset C \).

Let \( C, D \in \mathcal{C} \) such that \( C \nsubseteq D \). Let \( x \in C - D \) such that \( x \) is not a local cutpoint of \( X \) and let \( y \in D \). Then \( x \notin P_{ay} \) since \( P_{ay} \subset D \). By Claim 6 \( y \in P_{ax} \subset D \). Thus \( D \subset C \) and \( C \) is totally ordered by inclusion.

If \( C \) is not connected then there exist \( C, D \in \mathcal{C} \) with \( C \nsubseteq D \) such that for each \( E \in \mathcal{C} \) either \( E \subset C \) or \( D \subset E \). Now, \( D - C \) contains at least two points \( x \) and \( y \) such that \( x \) and \( y \) are not local cutpoints of \( X \). Then \( P_{ax} = P_{ay} = D \). By Claim 4 \( x = y \). With this contradiction we conclude that \( C \) is connected. The claim is proved.

**Claim 8.** If \( x_i \) is a sequence in \( P_{ab} \) such that the sequence \( P_{ax_i} \) converges in \( C \) then \( x_i \) converges in \( P_{ab} \).

**Proof.** Just suppose that \( x_i \) and \( y_i \) are two sequences in \( P_{ab} \) which converge to \( x \) and \( y \) respectively such that the sequences \( P_{ax_i} \) and \( P_{ay_i} \) both converge to \( C \in \mathcal{C} \). We may suppose that, for each \( i \), \( x_i \) and \( y_i \) are not local cutpoints of \( X \) and neither sequence is constant.

By Claim 6 we need to consider only three cases.

**Case 1.** For each \( i \), \( P_{ax_i} \subset P_{ax_{i+1}} \) and \( P_{ay_i} \subset P_{ay_{i+1}} \). If, for each \( i \) and \( j \), \( P_{ax_i} \subset P_{ay_j} \) then \( C = P_{ay_1} \) and the sequence \( y_i \) would have to be constant. We may suppose, therefore, that for each \( i \) \( P_{ax_i} \subset P_{ay_i} \subset P_{ax_{i+1}} \). As in the proof of Claim 4 if \( y \in P_{ax_i} \cap P_{xb} \) and \( y \) is not a local cutpoint of \( X \) then \( y = x_i \). Since \( P_{ab} \) contains at most countably many local cutpoints of \( X \), \( P_{ax_i} \cap P_{xb} \) is at most a countable set. Since \( P_{ay_{i+1}} \nsubseteq P_{ax_i} \) it follows that \( P_{ab} - (P_{xb} \cup P_{ay_{i+1}}) \) is a nonempty open subset of \( P_{ab} \) which is contained in \( P_{ax_i} \). As in Claim 4 there exists an arc \( I_i \) of local cutpoints of \( P_{ab} \) such that \( I_i \subset P_{ab} - (P_{xb} \cup P_{ay_{i+1}}) \subset P_{ax_i} - P_{ay_{i+1}} \). Now construct as in Claim 4 a sequence \( A_i \) of pairwise disjoint arcs such that for each \( i \) \( x_i, y_i \in A_i \). Since every sequence of disjoint continua in \( X \) is null, \( x = y \).

**Case 2.** For each \( i \) \( P_{ax_i} \supset P_{ax_{i+1}} \) and \( P_{ay_i} \supset P_{ay_{i+1}} \).

Argue as in Case 1.

**Case 3.** For each \( i \) \( P_{ax_i} \subset P_{ax_{i+1}} \) and \( P_{ay_i} \supset P_{ay_{i+1}} \). Then \( P_{ax_i} \subset P_{ay_j} \) for each \( i \) and \( j \). Let \( U \) be a neighbourhood of \( x \) in \( P_{ab} \) such that the boundary of \( U \) contains no local cutpoints of \( X \) and \( y \notin U \). For each \( i \) let \( z_i \in P_{xi} \cap (\text{boundary of } U) \). Then, \( P_{ax_i} \subset P_{az_i} \subset P_{ay_i} \) so that \( \lim P_{az_i} = C \).

If for some subsequence \( z_{ij} \) \( P_{az_{ij}} \subset P_{az_{ij+1}} \) for each \( j \), then we are in Case 1 with the sequences \( x_i \) and \( z_{ij} \). If for some subsequence \( z_{ij} \) \( P_{az_{ij}} \supset P_{az_{ij+1}} \)
for each \( j \) then we are in Case 2 with the sequences \( z_{i_j} \) and \( y_{i_j} \). If the sequence \( z_{i_j} \) contains a constant subsequence we may suppose \( z_i = z \) for each \( i \). Let \( w_{i_j} \) be a sequence in \( P_{az} \) which converges to \( z \). Then \( \lim P_{aw_{i_j}} = P_{az} = C \). We may now apply Case 1 to the sequences \( w_{i_j} \) and \( x_{i_j} \).

The claim is now proved.

We are now in a position to adapt the proof of Theorem 6 in [2] to the continuum \( P_{ab} \).

We shall attach to \( P_{ab} \) an infinite sequence of pairwise disjoint closed sets \( A_i \) such that no pair of points of \( P_{ab} \) can be separated by a finite set in \( P_{ab} \cup (\bigcup A_i) \).

Let \( h \) be a homeomorphism of the closed unit interval \([0, 1]\) onto \( C \) such that \( h(0) = \{a\} \) and \( h(1) = P_{ab} \). Define \( f: [0, 1] \to P_{ab} \) by letting

\[
\begin{cases}
  x, & \text{if } h(r) = P_{ax}, \\
  \lim x_i, & \text{if } h(r) = \lim P_{ax_i}.
\end{cases}
\]

By Claim 8 \( f \) is a continuous function. Notice that if \( x \in P_{ab} \) is not a local cutpoint of \( X \) then \( f^{-1}(x) \) is a singleton.

If \( A \) is a set in \( X \) and \( \varepsilon > 0 \) we let \( S(A, \varepsilon) \) denote the \( \varepsilon \)-neighbourhood of \( A \) in \( X \).

By the proof of Claim 4 if \( r \in [0, 1] \), then \( h(r) = \{f(r)\} \cup (\bigcup \{h(s) | s < r\}) \).

Let \( 0 < r < 1 \). For \( \varepsilon > 0 \) there exists, by the proof of Claim 4, \( \delta > 0 \) such that \( r < s < \delta + r \) implies \( h(s) \subset h(r) \cup S(f(r), \varepsilon) \).

Let \( Y_r = \bigcup \{P_{xb} | x \in P_{ab} \setminus h(r)\} \). Let \( r_i \) be a sequence in \([0, 1]\) which is strictly decreasing to \( r \) such that, for each \( i \), \( f(r_i) \) is not a local cutpoint of \( X \). It can be shown as in the proof of Claim 4 that \( Y_r = \bigcup P_{f(r_i)b} \) and \( Y_r \cup \{f(r)\} \) is compact.

If \( 0 < s < 1 \) and \( U \) is any neighbourhood of \( h(s) \) then no pair of points of \( h(s) \) can be separated in \( U \) by a finite set. We may suppose therefore that either

\( 1) \) \(
\text{for each } i \text{ there exist } c_i, d_i \in [0, 1] \text{ with } c_i < r < d_i \text{ such that } f(c_i) = f(d_i) \in (h(r) \cap Y_r \cap S(f(r), 1/\delta)) \setminus \{f(r)\}, \text{ or}
\)

\( \star \) \(\)

\( 2) \) \(\text{for each } i \text{ there exists an arc } C_i \subset S(f(r), 1/\delta) \text{ such that } C_i \cap h(1) \text{ consists precisely of the two endpoints of } C_i, \text{ one endpoint of } C_i \text{ is in } h(r) \setminus \{f(r)\} \text{ and the other is in } h(1) \setminus h(r). \text{ The arcs } C_i \text{ may be taken to be pairwise disjoint.} \)

Suppose \( 1) \) holds. Let \( E_i = \{f(c_i)\} \). We wish to show that \( \lim c_i = r \).

Just suppose that for each \( i \) \( c_i \leq s < r \). Then \( h(s) \cup Y_r \) is a continuum in \( P_{ab} \).
which satisfies $P(a, b)$. If $s < t < r$ and $f(t)$ is not a local cutpoint of $X$ then $f(t) \notin h(s) \cup Y_r$. This contradicts the assumption that $P_{ab}$ is an irreducible continuum with respect to satisfying $P(a, b)$. Thus, $\lim c_i = r$. Similarly, $\lim d_i = r$.

Now suppose (2) holds. Since every sequence of disjoint subcontinua of $X$ is a null sequence, there exists a sequence $\varepsilon_i$ of positive numbers converging to zero such that if $D_i$ is the component of $\{S(h(1), \varepsilon_i) \cup C_i\} - h(1)$ which meets $C_i$ then $D_i \cap D_j = \emptyset$ for $i \neq j$ and the diameters of the $D_i$ converge to 0.

Let $M_i = \text{Cl}(D_i) \cap (h(r) - \{f(r)\})$ and $N_i = \text{Cl}(D_i) \cap (Y_r - \{f(r)\})$. If $M_i$ (resp. $N_i$) has an isolated point let $c_i$ (resp. $d_i$) be an isolated point of $M_i$ (resp. $N_i$). If $M_i$ (resp. $N_i$) has no isolated points let $c_i = \inf \{s \in [0, 1] | \text{Cl}(D_i) \cap h(s) \text{ is uncountable}\}$ (resp. $d_i = \sup \{s \in [0, 1] | \text{Cl}(D_i) \cap P_{f(s)b} \text{ is uncountable}\}$). Then $h(c_i) \cap \text{Cl}(D_i)$ is at most countable.

As in (1) $\lim c_i = \lim d_i = r$. Let $E_i$ be an arc in $D_i \cup \{c_i, d_i\}$ with endpoints $c_i$ and $d_i$.

We wish to prove that for each $\varepsilon > 0$ and each $s \in [0, 1]$ either

\[(***) \quad (h(s) \cap Y_s \cap S(f(s), \varepsilon)) - \{f(s)\} \neq \emptyset\]

or there exists an arc in $S(f(s), \varepsilon) - E_i$ which joins $h(s)$ to $h(1) - h(s)$. We need only consider $s \in [0, 1]$ such that $f(s) = f(c_i)$ or $f(s) = f(d_i)$.

Clearly, (*** ) is satisfied if both $M_i$ and $N_i$ have an isolated point. Suppose, therefore, that $M_i$ does not have an isolated point. Let $s \in [0, 1]$ such that $f(s) = f(c_i)$ and suppose $\varepsilon > 0$ is given such that

\[(h(s) \cap Y_s \cap S(f(s), \varepsilon)) - \{f(s)\} = \emptyset.\]

By (**) there is a sequence of arcs $F_j \subset S(f(s), \varepsilon) - \{f(s)\}$ which join $h(s)$ to $h(1) - h(s)$ such that $\lim F_j = \{f(s)\}$. Let $e_j$ and $f_j$ be the endpoints of $F_j$. We may suppose $F_j \cap h(1) = \{e_j, f_j\}$ where $e_j \in h(s)$ and $f_j \in h(1) - h(s)$.

Just suppose that for each $j$ $E_i \cap F_j \neq \emptyset$. Then each $e_j \in \text{Cl}(D_i)$. By the choice of $c_i$ and by the assumption that $M_i$ is a perfect set each neighbourhood of $e_j$ contains uncountably many points of $Y_{c_i}$. Since $Y_{c_i} \cup \{f(c_i)\}$ is compact $e_j \in Y_{c_i}$. If $s < c_i$ then

\[e_j \in (h(s) \cap Y_s \cap S(f(s), \varepsilon)) - \{f(s)\}\]

which is a contradiction. If $c_i < s$ then for each $j$ let $e'_j \in [0, 1]$ such that $f(e'_j) = e_j$. We get as in (1) that $\lim e'_j = s$ and hence eventually $e_j \in Y_s - h(c_i)$ which is again a contradiction. We conclude that for all sufficiently large $j$

$E_i \cap F_j = \emptyset$. Thus, (*** ) is satisfied.

If there exist $c < c_i < d$ such that $f(c) = f(d) \in S(f(\varepsilon), 1) - \{f(\varepsilon)\}$ let $C(\varepsilon, 1) = \{f(c)\}$. Otherwise, let $C(\varepsilon, 1)$ be an arc in $S(h(1), 1)$ with endpoints
Let $A_1 = C(\frac{1}{2}, 1)$. Suppose $A_1, \ldots, A_{n-1}$ have been constructed to be pairwise disjoint closed sets such that for each $i = 1, \ldots, n - 1$

(i) $A_i$ is the union of a finite number of arcs and points each of which was obtained as $E_i$ above,

(ii) for each $a \in [1/2^i, 1 - 1/2^i]$ there is a component $C$ of $A_i$ such that $C$ meets both $f([1/2^i, a]) - \{f(a)\}$ and $f([a, 1 - 1/2^i]) - \{f(a)\}$,

(iii) $P_{ab} \cap A_i \supset P_{ab} - f([0, 1/2^{i+1}] \cup [1 - 1/2^{i+2}, 1])$.

For each $x \in [1/2^n, 1 - 1/2^n]$ let $C(x, n) \subset S(h(1), 1/2^{n-1}) - (A_1 \cup \cdots \cup A_{n-1})$ be an (possibly degenerate) arc chosen as was $E_i$ with endpoints $f(c(x, n))$ and $f(d(x, n))$ where $1/2^{n+1} < c(x, n) < x < d(x, n) < 1 - 1/2^{n+1}$, $C(x, n)$ minus its endpoints lies in $X - h(1)$ and $C(x, n)$ satisfies (**). The set of open intervals $[c(x, n), d(x, n)]$ such that $x \in [1/2^n, 1 - 1/2^n]$ is an open cover for the compact set $[1/2^n, 1 - 1/2^n]$ hence there exists a minimal finite set $\{x_1, \cdots, x_k\} \subset [1/2^n, 1 - 1/2^n]$ such that

$$[c(x_1, n), d(x_1, n)] \cup \cdots \cup [c(x_k, n), d(x_k, n)]$$

covers $[1/2^n, 1 - 1/2^n]$. It is clear that conditions (i)—(iii) are satisfied for $n$.

Let $A_n = C(x_1, n) \cup \cdots \cup C(x_k, n)$.

Let $Y = P_{ab} \cup A_1 \cup A_2 \cup \cdots$. Since every sequence of disjoint subcontinua of $X$ is a null sequence $Y$ is a continuum.

To prove that $Y$ contains a connected set that is not arcwise connected it suffices by Theorem 1 and the fact that $P_{ab}$ contains only countably many local cutpoints of $X$ to prove that if $z$ is a local cutpoint of $Y$ then either $z \in A_i$ for some $i$ or $z$ is a local cutpoint of $X$. Let $z \in P_{ab}$ such that $z$ is not a local cutpoint of $X$ and $z \notin A_i$ for any $i$. It follows from the fact that $f^{-1}(z)$ is a singleton that if $U$ is any connected neighbourhood of $z$ in $P_{ab}$ such that $U - \{z\}$ has more than one component then $U - \{z\}$ has exactly two components one of which is contained in $\{f(y) \mid y < f^{-1}(z)\}$ and the other is in $\{f(y) \mid f^{-1}(z) < y\}$. It is now easy to see that $z$ is not a local cutpoint of $Y$ and the theorem is proved.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATCHEWAN, SASKATOON, SASKATCHEWAN S7N 0W0 CANADA