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SOME POLYNOMIALS DEFINED BY GENERATING RELATIONS(1)

BY

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ABSTRACT. In an attempt to present a unified treatment of the various polynomial systems introduced from time to time, new generating functions are given for the sets of polynomials $\{S_{n,q}^{(\alpha,\beta)}(\lambda; x)\}$ and $\{T_{n,q}^{(\alpha,\beta)}(\lambda; x)\}$, defined respectively by (6) and (29) below, and for their natural generalizations in several complex variables. This paper also indicates relevant connections of the results derived here with different classes of generating relations which have appeared recently in the literature.

1. Introduction. Recently, H. B. Mittal [6] demonstrated some applications of the differential operator

$$T_k = x(k + xD), \quad D = d/dx, \quad k \text{ a constant}, \tag{1}$$

which evidently has the property that

$$T_k^n \{x^\alpha\} = (\alpha + k)_n x^{\alpha+n}, \tag{2}$$

where n is a positive integer and, in general, $(\alpha)_\lambda = \Gamma(\alpha + \lambda)/\Gamma(\alpha)$. His main result may be stated as

THEOREM 1 (MITTAL [6, p. 81]). *Corresponding to every power series*

$$\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \qquad \gamma_0 \neq 0, \tag{3}$$

one can define a set of polynomials $\{f_{n,p,q}^{(a,m)}(c; x)\}$ *by*

$$f_{n,p,q}^{(a,m)}(c; x) = \sum_{k=0}^{[n/q]} \frac{(-n)_{qk}(a + mn)_{(p-mq)k}}{(a + (m-1)n)_{(p-mq+q)k}} (-1)^{qk} \gamma_k c^k x^{(p-q)k}, \tag{4}$$

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such that

$$(5)\quad \sum_{n=0}^{\infty} \frac{(a+1)_{mn}}{(a+1)_{mn-n}} f_{n,p,q}^{(a+1,m)}(c;x)\frac{x^n t^n}{n!} = \frac{(1+v)^{a+1}}{1-(m-1)v}\psi(ct^q x^p(1+v)^p),$$

where a and c are constants, m, p and q are positive integers, and $v = xt(1+v)^m$.

A closer look at the defining relation (4) would evidently expose a number of superfluous parameters on its right-hand side. For instance, one can replace, without any loss of generality, x by $((-1)^q x/c)^{1/(p-q)}$. Also, in the generating relation (5), t can conveniently be replaced throughout by t/x. Furthermore, it would suggest the following interesting extension of Theorem 1.

THEOREM 2. *Corresponding to the power series* $\psi(u)$ *given by* (3), *let*

$$(6)\qquad S_{n,q}^{(\alpha,\beta)}(\lambda;x) = \sum_{k=0}^{[n/q]} \frac{(-n)_{qk}(1+\alpha+(\beta+1)n)_{\lambda k}}{(1+\alpha+\beta n)_{(\lambda+q)k}}\gamma_k x^k,$$

where α, β *and* λ *are arbitrary constants, real or complex, and* q *is an arbitrary positive integer.*

Then

$$(7)\quad \sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n} S_{n,q}^{(\alpha,\beta)}(\lambda;x)t^n = \frac{(1+w)^{\alpha+1}}{1-\beta w}\psi(x(-w)^q(1+w)^\lambda),$$

where w *is a function of* t *defined by*

$$(8)\qquad w = t(1+w)^{\beta+1},\quad w(0)=0.$$

(We assume throughout that α and β take on such values that equations like (7) make sense.)

Since it is readily seen that

$$(9)\qquad f_{n,p,q}^{(a,m)}(c;x) = S_{n,q}^{(a-1,m-1)}(p-mq;(-1)^q c x^{p-q}),$$

Theorem 1 would follow from Theorem 2 in the special case when, for instance, λ and $\beta \geqslant 0$ take on integral values only.

Our proof of Theorem 2 is direct; it does not make use of the differential operator T_k defined by (1). Indeed, in view of the elementary relationship $(-n)_k = (-1)^k n!/(n-k)!$, $0 \leqslant k \leqslant n$, we observe that

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} S_{n,q}^{(\alpha,\beta)}(\lambda; x) t^n$$

$$(10) \qquad = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} (-1)^{qk} \binom{\alpha + (\beta + 1)n + \lambda k}{n - qk} \gamma_k x^k$$

$$= \sum_{k=0}^{\infty} (-1)^{qk} \gamma_k x^k t^{qk} \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)qk + \lambda k + (\beta + 1)n}{n} t^n.$$

The inner series can be expressed in a closed form by using the following consequence of Lagrange's expansion formula [7, p. 302, Problem 216]:

$$(11) \qquad \sum_{n=0}^{\infty} \binom{\alpha + (\beta + 1)n}{n} t^n = \frac{(1 + w)^{\alpha+1}}{1 - \beta w},$$

where w is given by (8), and thus the generating relation (7) follows at once.

2. **Applications.** At the outset, we remark that in an earlier work [13] Srivastava has shown how Theorem 2, in the special case $\lambda = 0$, can be applied to derive a large number of generating relations for various special functions of interest. As a matter of fact, he observed (cf. [11, p. 593 (16)]) that formula (8.5) (Mittal [6, p. 80]), involving Jacobi polynomials, is contained in his generating relation [11, p. 591 (9)], and hence also in his subsequent result [13, p. 233 (12)]. Note that Mittal's formula [6, p. 79 (7.4)], involving Bessel polynomials, which was given earlier by Calvez and Génin [2, p. 654 (22)], follows also as a limiting case of Srivastava's generating relation [11, p. 594 (21)] evidently contained in [11, p. 591 (9)] and [13, p. 233 (12)]. On the other hand, Mittal's formula [6, p. 82 (9.9)] is a special case of a result of Chaundy [3, p. 62 (25)], which is also contained in the aforementioned generating relations [11, p. 591 (9)] and [13, p. 233 (12)].

In this section we consider applications of Theorem 2 when λ is a positive or negative integer. First of all we observe that, in the special case $\lambda = -1$, if we set $q = \beta = 1$, $\alpha = c - 1$, $\gamma_k = \delta_k/k!$ and replace x, t by $1/x$ and $-xt$, respectively, Theorem 2 would reduce fairly readily to the known generating relation (14) of Rainville [9, p. 296]. Note that this formula of Rainville [9] would follow also from Theorem 2 if we set $\lambda = 0$, $q = 1$, $\beta = -2$, $\alpha = -c$, $\gamma_k = \delta_k/k!$, and replace x, t by $-1/x$ and xt, respectively.

On the other hand, for $\lambda = m - 2$, m being a positive integer, $q = \beta = 1$, $\alpha = a$, Theorem 2 with x replaced by $(-1)^m x$, and γ_k by $\gamma_k/k!$, was given recently by Brown [1, p. 58, Theorem H].

Next we specialize the power series in (3) by means of

$$\psi(u) = {}_rF_s[a_1, \cdots, a_r; b_1, \cdots, b_s; u], \tag{12}$$

giving us

$$\gamma_k = \left\{\prod_{j=1}^{r} (a_j)_k\right\}\left\{k! \prod_{j=1}^{s} (b_j)_k\right\}^{-1}, \qquad k \geqslant 0. \tag{13}$$

The most interesting special cases would seem to occur when $\beta = 1$ and $\lambda = \pm(p-1)$, p being an arbitrary positive integer. We are thus led to the hypergeometric generating functions

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{\alpha+2n}{n} {}_{r+p+q-1}F_{s+p+q-1} \\
&\cdot \left[\begin{matrix} a_1, \cdots, a_r, & \Delta(q;-n), \Delta(p-1; 1+\alpha+2n); \\ b_1, \cdots, b_s, & \Delta(p+q-1; 1+\alpha+n); \end{matrix} \; \frac{q^q(p-1)^{p-1}}{(p+q-1)^{p+q-1}} x\right] t^n \\
&= (1-4t)^{-1/2}\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{\alpha} {}_rF_s\left[\begin{matrix} a_1, \cdots, a_r; \\ b_1, \cdots, b_s; \end{matrix} x(-t)^q\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{p+2q-1}\right];
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{\alpha+2n}{n} {}_{r+q}F_{s+q}\left[\begin{matrix} a_1, \cdots, a_r, \Delta(q;-n); \\ b_1, \cdots, b_s, \Delta(q-p+1; 1+\alpha+n), \Delta(p-1; -\alpha-2n); \end{matrix} \; \frac{q^q x}{(1-p)^{p-1}(q-p+1)^{q-p+1}}\right] t^n \\
&= (1-4t)^{-1/2}\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{\alpha} {}_rF_s\left[\begin{matrix} a_1, \cdots, a_r; \\ b_1, \cdots, b_s; \end{matrix} x(-t)^q\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{2q-p+1}\right],
\end{aligned} \tag{15}$$

if $p \leqslant q+1$, or

$$\begin{aligned}
&\sum_{n=0}^{\infty} \binom{\alpha+2n}{n} {}_{r+p-1}F_{s+p-1}\left[\begin{matrix} a_1, \cdots, a_r, \Delta(q;-n), \Delta(p-q-1; \alpha-n); \\ b_1, \cdots, b_s, \Delta(p-1; -\alpha-2n); \end{matrix} \; \frac{(-q)^q(p-q-1)^{p-q-1}}{(p-1)^{p-1}} x\right] t^n \\
&= (1-4t)^{-1/2}\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{\alpha} {}_rF_s\left[\begin{matrix} a_1, \cdots, a_r; \\ b_1, \cdots, b_s; \end{matrix} x(-t)^q\left(\frac{2}{1+(1-4t)^{1/2}}\right)^{2q-p+1}\right],
\end{aligned} \tag{16}$$

if $p \geqslant q+1$, where, for convenience, $\Delta(m; \lambda)$ is taken to abbreviate the sequence of

m parameters $\lambda/m, (\lambda+1)/m, \cdots, (\lambda+m-1)/m$, $m \geqslant 1$, it being understood that the set $\Delta(0;\lambda)$ is empty.

In the special case $p=q$, if we let $q \longrightarrow 1$, both (14) and (15) would lead to an interesting generating relation for a generalization of the pseudo Laguerre polynomials given earlier by Shively [10, p. 54 (48)]. (See also Rainville [9, p. 298 (5)].) On the other hand, if in our formula (16) we set $p=2q$, $\alpha=a-1$, and replace x, t by $(-1)^q z x^{1-q}$ and xt, respectively, it would provide us with the corrected version of the generating relation (9.11) of Mittal [6, p. 82].

Now we let $\beta=0$ and $\lambda=-p$, where p is a positive integer $\leqslant q$, and from Theorem 2 we obtain

$$\sum_{n=0}^{\infty}\binom{\alpha+n}{n}{}_{r+q}F_{s+q}\left[\begin{matrix} a_1,\cdots,a_r,\Delta(q;-n); \\ b_1,\cdots,b_s,\Delta(q-p;1+\alpha),\Delta(p;-\alpha-n); \end{matrix}\ \frac{q^q x}{(-p)^p(q-p)^{q-p}}\right]t^n$$
$$(17)\qquad = (1-t)^{-\alpha-1}{}_rF_s\left[\begin{matrix} a_1,\cdots,a_r; \\ b_1,\cdots,b_s; \end{matrix}\ \frac{x(-t)^q}{(1-t)^{q-p}}\right].$$

The generating relation (17) is a generalization, for instance, of formulas (25) and (27) of Chaundy [3, p. 62], and of the more recent result (9.8) in [6, p. 82].

For $\beta=-1$ and $\lambda=-p$, p being an arbitrary positive integer, Theorem 2 yields the generating functions

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}{}_{r+q}F_{s+q}\left[\begin{matrix} a_1,\cdots,a_r,\Delta(q;-n); \\ b_1,\cdots,b_s,\Delta(p;-\alpha),\Delta(q-p;1+\alpha-n); \end{matrix}\ \frac{q^q x}{(-p)^p(q-p)^{q-p}}\right]t^n$$
$$(18)\qquad = (1+t)^{\alpha}{}_rF_s\left[\begin{matrix} a_1,\cdots,a_r; \\ b_1,\cdots,b_s; \end{matrix}\ \frac{x(-t)^q}{(1+t)^p}\right],\quad \text{if } p \leqslant q;$$

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}{}_{r+p}F_{s+p}\left[\begin{matrix} a_1,\cdots,a_r,\Delta(p-q;-\alpha+n),\Delta(q;-n); \\ b_1,\cdots,b_s,\Delta(p;-\alpha); \end{matrix}\ \frac{(-q)^q(p-q)^{p-q}}{p^p}x\right]t^n$$
$$(19)\qquad = (1+t)^{\alpha}{}_rF_s\left[\begin{matrix} a_1,\cdots,a_r; \\ b_1,\cdots,b_s; \end{matrix}\ \frac{x(-t)^q}{(1+t)^p}\right],\quad \text{if } p \geqslant q.$$

Formula (18) is substantially the same as the generating function (17) above. On the other hand, formula (19) incorporates, as its special cases, a fairly large number of generating functions for various special functions of interest. As an example, we cite the generating relation (26) of Chaundy [3, p. 62], which was rediscovered, a decade later, by Rainville [8, p. 106 (6)].

Finally, we set $\beta=-2$ and $\lambda=p$, p being a positive integer, and Theorem 2 gives us

$$\sum_{n=0}^{\infty}\binom{\alpha-n}{n}{}_{r+p+q}F_{s+p+q}\left[\begin{matrix}a_1,\cdots,a_r,\Delta(p;1+\alpha-n),\Delta(q;-n);\\ b_1,\cdots,b_s,\Delta(p+q;1+\alpha-2n);\end{matrix}\ \frac{p^pq^qx}{(p+q)^{p+q}}\right]t^n$$

(20)

$$=(1+4t)^{-1/2}\left(\frac{2}{1+(1+4t)^{1/2}}\right)^{-\alpha-1}{}_rF_s\left[\begin{matrix}a_1,\cdots,a_r;\\ b_1,\cdots,b_s;\end{matrix}\ x(-t)^q\left(\frac{2}{1+(1+4t)^{1/2}}\right)^{q-p}\right],$$

which provides a generalization of the generating function (4) of Rainville [8, p. 104]. (See also Brown [1, p. 44, Theorem E].)

Similar specializations will evidently lead us to several other applications of Theorem 2.

In view of (6) it seems natural to consider also the function defined by the series

$${}_r\Psi_s^*[(a_1,A_1),\cdots,(a_r,A_r);(b_1,B_1),\cdots,(b_s,B_s);x]$$

(21)

$$=\sum_{k=0}^{\infty}\left\{\prod_{j=1}^{r}(a_j)_{A_jk}\right\}\left\{k!\prod_{j=1}^{s}(b_j)_{B_jk}\right\}^{-1}x^k,$$

where all of the A_j and B_j are positive, and for convergence,

(22)

$$1+\sum_{j=1}^{s}B_j-\sum_{j=1}^{r}A_j\geqslant 0,$$

the equality holds only for suitably restricted values of $|x|$. This function is closely related to the generalized hypergeometric function of Wright. In the case where all of the A_j and B_j are equal to 1 it reduces to the generalized hypergeometric function ${}_rF_s(a_1,\cdots,a_r;b_1,\cdots,b_s;x)$. If in Theorem 2 we set

(23)

$$\gamma_k=\left\{\prod_{j=1}^{r}(a_j)_{A_jk}\right\}\left\{k!\prod_{j=1}^{s}(b_j)_{B_jk}\right\}^{-1},$$

instead of as in (13), then ${}_r\Psi_s^*$ replaces ${}_rF_s$ in (12) as the generating function and for $\lambda>0$ we then obtain

$$\sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}{}_{r+q+1}\Psi_{s+1}^*[(-n/q,1),\cdots,((-n+q-1)/q,1),(1+\alpha+(\beta+1)n,\lambda),$$

(24)

$$(a_1,A_1),\cdots,(a_r,A_r);(1+\alpha+\beta n,\lambda+q),(b_1,B_1),\cdots,(b_s,B_s);q^qx]\,t^n$$

$$=\frac{(1+w)^{\alpha+1}}{1-\beta w}\,{}_r\Psi_s^*[(a_1,A_1),\cdots,(a_r,A_r);(b_1,B_1),\cdots,(b_s,B_s);x(-w)^q(1+w)^\lambda],$$

where w is given by (8). If $\lambda=0$ then in (24) ${}_{r+q+1}\Psi_{s+1}^*$ is to be replaced by ${}_{r+q}\Psi_{s+1}^*$ and the parameter pair $(1+\alpha+(\beta+1)n,\lambda)$ deleted. This is a generalization of formulas of the form of (14); similar generalizations can be obtained for (15) through (20).

3. A generalization of Theorem 2. Making use of Gould's identity [4, p. 196],

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n} t^n = (1+w)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha-\gamma}{n}\binom{n+\gamma/(\beta+1)}{n}^{-1}\left(\frac{w}{1+w}\right)^n, \tag{25}$$

where α, β and γ are arbitrary complex numbers, and w is given by (8), we can derive the following generalization of Theorem 2.

THEOREM 3. *With the power series $\psi(u)$ defined by* (3), *let*

$$\theta(n, q; \alpha, \beta, \gamma, \lambda; u) = \sum_{k=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)qk}\binom{\alpha-\gamma+\lambda k}{n}\binom{n+qk+\gamma/(\beta+1)}{n}^{-1} \gamma_k u^k, \tag{26}$$

where α, β, γ and λ are arbitrary complex numbers, q is a positive integer, and $n = 0, 1, 2, \cdots$.

Then

$$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n} S_{n,q}^{(\alpha,\beta)}(\lambda; x) t^n = (1+w)^{\alpha}\phi(x(-w)^q(1+w)^{\lambda}, -w/(1+w)), \tag{27}$$

where, for convenience,

$$\phi(u, v) = \sum_{n=0}^{\infty} \theta(n, q; \alpha, \beta, \gamma, \lambda; u) v^n, \tag{28}$$

and, as before, $S_{n,q}^{(\alpha,\beta)}(\lambda; x)$ is given by (6) *and w is given by* (8).

For $\gamma = \alpha$, the generating relation (27) would simplify considerably. On the other hand, its limiting case as $\gamma \longrightarrow \infty$ corresponds formally to our generating function (7). Thus it would seem obvious that, for finite γ, $\gamma \neq \alpha$, Theorem 3 may be looked upon as being independent of Theorem 2.

Yet another interesting special case of the generating relation (27) would occur when we set $\lambda = 0$ and choose γ so that $(\alpha - \gamma)$ is a positive integer. We are thus led to what is essentially the same as Theorem 3 of Zeitlin [15, p. 410].

4. An associated set of polynomials. We define the set of polynomials

$$T_{n,q}^{(\alpha,\beta)}(\lambda; x) = \sum_{k=0}^{[n/q]} \frac{(-n)_{qk}(\alpha+(\beta+1)n)_{\lambda k}}{(1+\alpha+\beta n)_{(\lambda+q)k}} \gamma_k x^k, \tag{29}$$

where, as before, α, β and λ are arbitrary complex numbers, q is a positive integer, and the γ_k are given by (3). For these associated polynomials we give here a class of generating functions of type (27) with $\gamma = \alpha$. Indeed, instead of the identity (25) we make use of the following consequence of the Lagrange expansion formula (cf. [7, p. 301, Problem 212])

$$(1+w)^{\alpha+1} = 1 + (\alpha+1)\sum_{n=1}^{\infty}\binom{\alpha+(\beta+1)n}{n-1}\frac{t^n}{n}, \tag{30}$$

which follows also from (25) when $\gamma = \alpha$ and wherein w is given by equation (8), and we obtain

THEOREM 4. *With the γ_k given by* (3), *let*

$$\zeta(a;u) = \sum_{k=0}^{\infty}\frac{a}{a+k}\gamma_k u^k. \tag{31}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}T_{n,q}^{(\alpha,\beta)}(\lambda;x)t^n \\ = (1+w)^{\alpha}\zeta\left(\frac{\alpha}{\lambda+(\beta+1)q};x(-w)^q(1+w)^{\lambda}\right),\end{aligned} \tag{32}$$

where α, β and λ are arbitrary constants, real or complex, q is a positive integer, and w is a function of t defined by (8).

For $\beta = 1$, the generating relation (32) would reduce to the elegant form

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+2n}\binom{\alpha+2n}{n}T_{n,q}^{(\alpha,1)}(\lambda;x)t^n \\ = \left(\frac{2}{1+(1-4t)^{1/2}}\right)^{\alpha}\zeta\left(\frac{\alpha}{\lambda+2q};x(-t)^q\left[\frac{2}{1+(1-4t)^{1/2}}\right]^{(\lambda+2)q}\right),\end{aligned} \tag{33}$$

where $\zeta(a;u)$ is defined by (31).

A special case of our formula (33) when $q = 1$ and $\lambda = m-2$, m being a positive integer, was given earlier by Brown [1, p. 61, Theorem I].

We remark in passing that even though the generating function in (32) is not contained in Theorem 2 or Theorem 3 of this paper, it is essentially an integrated form of the generating function given by Theorem 2.

5. Polynomials in several variables. In order to give a multidimensional extension of the generating relations (7) and (27), we define a function $F[z_1,\cdots,z_r]$ of several complex variables $z_1,\cdots,z_r$ by the formal series

$$F[z_1, \cdots, z_r] = \sum_{k_1, \cdots, k_r=0}^{\infty} C(k_1, \cdots, k_r) z_1^{k_1} \cdots z_r^{k_r}, \tag{34}$$

where the coefficients $C(k_1, \cdots, k_r)$, $k_j \geqslant 0$, $1 \leqslant j \leqslant r$, are arbitrary constants, real or complex. Corresponding to every multiple series of this type, we introduce a set of polynomials in several variables $z_1, \cdots, z_r$ defined by

$$\begin{aligned} &\Omega_n^{(\alpha,\beta)}[\lambda_1, \cdots, \lambda_r; q_1, \cdots, q_r; z_1, \cdots, z_r] \\ &= \sum_{k_1, \cdots, k_r=0}^{q_1k_1+\cdots+q_rk_r \leqslant n} \frac{(-n)_{q_1k_1+\cdots+q_rk_r}(1+\alpha+(\beta+1)n)_{\lambda_1k_1+\cdots+\lambda_rk_r}}{(1+\alpha+\beta n)_{(\lambda_1+q_1)k_1+\cdots+(\lambda_r+q_r)k_r}} \\ &\qquad \cdot C(k_1, \cdots, k_r) z_1^{k_1} \cdots z_r^{k_r}, \end{aligned} \tag{35}$$

where α, β and $\lambda_1, \cdots, \lambda_r$ are arbitrary complex numbers, and $q_1, \cdots, q_r$ are positive integers.

As a consequence of the identity (25) we have a multidimensional extension of Theorem 3 given by

THEOREM 5. *Corresponding to the multiple series $F[z_1, \cdots, z_r]$ defined by* (34), *let*

$$\begin{aligned} &G(n, q_1, \cdots, q_r; \alpha, \beta, \gamma, \lambda_1, \cdots, \lambda_r; u_1, \cdots, u_r) \\ &= \sum_{k_1, \cdots, k_r=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)(q_1k_1+\cdots+q_rk_r)} \binom{\alpha-\gamma+\lambda_1k_1+\cdots+\lambda_rk_r}{n} \\ &\quad \cdot \binom{n+q_1k_1+\cdots+q_rk_r+\gamma/(\beta+1)}{n}^{-1} C(k_1, \cdots, k_r) u_1^{k_1} \cdots u_r^{k_r}, \end{aligned} \tag{36}$$

where α, β, γ and $\lambda_1, \cdots, \lambda_r$ are arbitrary complex numbers, $q_1, \cdots, q_r$ are arbitrary positive integers, and $n = 0, 1, 2, \cdots$.

Then

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1)n} \binom{\alpha+(\beta+1)n}{n} \Omega_n^{(\alpha,\beta)}[\lambda_1, \cdots, \lambda_r; q_1, \cdots, q_r; z_1, \cdots, z_r] t^n \\ &= (1+w)^{\alpha} H[z_1(-w)^{q_1}(1+w)^{\lambda_1}, \cdots, z_r(-w)^{q_r}(1+w)^{\lambda_r}; -w/(1+w)], \end{aligned} \tag{37}$$

where w is given by (8) *and, for convenience,*

$$H[u_1, \cdots, u_r; v] = \sum_{n=0}^{\infty} G(n, q_1, \cdots, q_r; \alpha, \beta, \gamma, \lambda_1, \cdots, \lambda_r; u_1, \cdots, u_r) v^n. \tag{38}$$

Some special cases of this last generating relation are worthy of note. For instance, if $\lambda_1 = \cdots = \lambda_r = 0$, it would reduce to a known result [14, p. 484 (5)].

On the other hand, its limiting case when $\gamma \longrightarrow \infty$ would correspond formally to the elegant generating relation

$$\sum_{n=0}^{\infty} \binom{\alpha + (\beta+1)n}{n} \Omega_n^{(\alpha,\beta)}[\lambda_1, \cdots, \lambda_r; q_1, \cdots, q_r; z_1, \cdots, z_r] t^n \tag{39}$$
$$= \frac{(1+w)^{\alpha+1}}{1-\beta w} F[z_1(-w)^{q_1}(1+w)^{\lambda_1}, \cdots, z_r(-w)^{q_r}(1+w)^{\lambda_r}],$$

where $F[z_1, \cdots, z_r]$ is defined by (34).

Finally, we state without proof the following multidimensional extension of the generating function (32).

THEOREM 6. *For arbitrary complex coefficients $C(k_1, \cdots, k_r)$, $k_j \geqslant 0$, $1 \leqslant j \leqslant r$, let*

$$\Lambda_n^{(\alpha,\beta)}[\lambda_1, \cdots, \lambda_r; q_1, \cdots, q_r; z_1, \cdots, z_r]$$
$$= \sum_{k_1, \cdots, k_r = 0}^{q_1 k_1 + \cdots + q_r k_r \leqslant n} \frac{(-n)_{q_1 k_1 + \cdots + q_r k_r} (\alpha + (\beta+1)n)_{\lambda_1 k_1 + \cdots + \lambda_r k_r}}{(1+\alpha+\beta n)_{(\lambda_1+q_1)k_1 + \cdots + (\lambda_r + q_r)k_r}} \tag{40}$$
$$\cdot C(k_1, \cdots, k_r) z_1^{k_1} \cdots z_r^{k_r},$$

where α, β and $\lambda_1, \cdots, \lambda_r$ are arbitrary constants, real or complex, and $q_1, \cdots, q_r$ are positive integers.

Then

$$\sum_{n=0}^{\infty} \frac{\alpha}{\alpha + (\beta+1)n} \binom{\alpha + (\beta+1)n}{n} \Lambda_n^{(\alpha,\beta)}[\lambda_1, \cdots, \lambda_r; q_1, \cdots, q_r; z_1, \cdots, z_r] t^n \tag{41}$$
$$= (1+w)^{\alpha} H[\alpha/\{\lambda + (\beta+1)q\}; z_1(-w)^{q_1}(1+w)^{\lambda_1}, \cdots, z_r(-w)^{q_r}(1+w)^{\lambda_r}],$$

where w is given by equation (8) *and, for convenience,*

$$H[a; u_1, \cdots, u_r] = \sum_{k_1, \cdots, k_r = 0}^{\infty} \frac{a}{a + k_1 + \cdots + k_r} C(k_1, \cdots, k_r) u_1^{k_1} \cdots u_r^{k_r}. \tag{42}$$

It may be of interest to remark here that formula (39), which evidently provides a multidimensional extension of the generating relation (7), is essentially equivalent to the special case $\gamma = \alpha$ of its parent formula (37). Note also that by specializing the coefficients $C(k_1, \cdots, k_r)$ by means of equation (10) in the earlier paper [14, p. 485], formulas (37), (39) and (41) can be applied to derive generating relations for polynomial systems associated with the generalized Lauricella functions of several complex variables (cf. [12, p. 454 et seq.]). We omit details which may be left as an exercise for the interested reader.

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