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Φ-LIKE HOLOMORPHIC FUNCTIONS IN C^n AND BANACH SPACES

BY

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ABSTRACT. In a recent paper, L. Brickman introduced the concept of Φ-like holomorphic functions as a complete generalization of starlike and spiral-like functions of a single complex variable. In the present paper, the author extends this work to locally biholomorphic mappings of several complex variables and then to locally biholomorphic mappings defined in an arbitrary Banach space. Complete characterizations of univalency and starlikeness of locally biholomorphic maps in general Banach spaces are obtained.

1. **Introduction.** The object of this paper is to extend to locally biholomorphic mappings of several complex variables the theory of Φ-like holomorphic functions of a single variable as developed by L. Brickman [1]. We also show how this theory may be extended to locally biholomorphic mappings defined in a Banach space X.

Suppose $f(z) = (f_1(z), \cdots, f_n(z))$, $z \in C^n$, is a locally biholomorphic mapping from the unit ball B^n into C^n such that $f(0) = 0$ and $Df(0) = I$. Let Ω be a region in C^n. For such mappings and regions in C^n, we respectively generalize Brickman's one-dimensional definitions of Φ-like holomorphic functions and Φ-like regions (Definitions 1 and 2) in §2. Extending a result of Brickman [1, Lemma 1, p. 556], we develop differential equations in C^n and show that the origin is an asymptotically stable critical point of the equation (Lemma 2).

This result is used in §3 to obtain generalizations in C^n (with Euclidean norm) of Brickman's one-dimensional univalence criteria. In Theorems 1 and 2, we establish that Φ-like functions have Φ-like images and, conversely, univalent normalized holomorphic functions having Φ-like images are Φ-like functions. We show for a locally biholomorphic function f on the unit ball in C^n normalized at the origin, that f being univalent is equivalent to f being Φ-like for some Φ (Corollary 1).

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A necessary and sufficient condition (Theorem 3) for starlikeness of locally biholomorphic maps $f: B^n \longrightarrow C^n$ is proven in §4. We also introduce the definitions of a spiral-like function of several variables and show that a spiral-like function is necessarily univalent.

The remaining sections consist of generalizations to an arbitrary Banach space X of the results contained in the first part of the paper. These generalizations are relatively straightforward when we consider X as a semi-inner product space. (See [6] or [7].) We completely characterize univalence for locally biholomorphic maps on the unit ball in X (Theorem 5) and starlikeness for locally biholomorphic maps from the unit ball of one Banach space into another Banach space (Theorem 7).

I am indebted to Professor John A. Pfaltzgraff for many stimulating discussions on this subject and his patient assistance during the preparation of this paper. The idea of exploiting results from Pfaltzgraff [10] for the generalization of Φ-like holomorphic functions originated with him.

2. **Preliminaries.** Let C^n denote the space of n complex variables $z = (z_1, \cdots, z_n)$, $z_j \in C$ $(j = 1, \cdots, n)$, with the standard orthonormal basis. The Euclidean inner product and norm are denoted by $\langle z, w\rangle = \Sigma_{j=1}^n z_j \bar{w}_j$ and $\|z\| = \langle z, z\rangle^{1/2}$ respectively. The open ball $\{z: \|z - a\| < r\}$ is denoted by $B_r^n(a)$: the unit ball is abbreviated by $B_1^n(0) = B^n$. The closure of any set $E \subset C^n$ is denoted by $\bar{E}$.

The space of continuous linear operators from C^n into C^n, i.e. the $n \times n$ complex matrices $A = (A_{jk})$, is symbolized by $L(C^n)$. The letter I will always represent the identity map on C^n. The standard norm on $L(C^n)$ is used: $\|A\| = \sup\{\|Az\|: z \in \bar{B}^n\}$, $A \in L(C^n)$. The adjoint or conjugate transpose of A is denoted A^*, i.e. $\langle Au, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in C^n$.

The class of holomorphic mappings

$$f(z) = (f_1(z), \cdots, f_n(z)), \qquad z \in \Omega,$$

from a region Ω (contained in C^n) into C^n is denoted by $H(\Omega)$. A function $f \in H(\Omega)$ is said to be *locally biholomorphic* in Ω if the differential

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k}\right), \qquad 1 \leqslant j, k \leqslant n,$$

is nonsingular at each point $z \in \Omega$, or equivalently f has a *local* holomorphic inverse at each point in Ω.

A mapping $v(z) \in H(B^n)$ is called a *Schwarz function* if $\|v(z)\| \leqslant \|z\|$ for all $z \in B^n$, or equivalently (by the Schwarz Lemma) $v(0) = 0$ and $\|v(z)\| \leqslant 1$ for all $z \in B^n$.

We define the following two classes of mappings:

$$N = \{h \in H(B^n): h(0) = 0, \operatorname{Re}\langle h(z), z\rangle > 0, z \in B^n, z \neq 0\},$$

$$M = \{h \in N: Dh(0) = I\}.$$

Clearly, $M \subset N$. The properties of M have been investigated in the paper [10] by Pfaltzgraff and are shown there to be valuable in the study of subordination chains and the Loewner-Pommerenke differential equation for holomorphic mappings from B^n into C^n. The class N is more suitable in extending Brickman's work [1] to several complex variables, and hence results about M from [10] depending upon the normalization of $Dh(0)$ will be modified as needed in this paper.

Note that in the single variable case, the class N consists of functions $h(z) = zp(z)$ $(z \in B^1)$ where $\operatorname{Re} p(z) > 0$ for $z \in B^1$. Recall that functions with positive real parts satisfy the classical inequalities

$$\frac{1-|\lambda|}{1+|\lambda|} \operatorname{Re} p(0) \leqslant \operatorname{Re} p(\lambda) \leqslant \frac{1+|\lambda|}{1-|\lambda|} \operatorname{Re} p(0), \qquad \lambda \in B^1. \tag{1}$$

Corresponding inequalities for N are established in Lemma 1.

LEMMA 1. *Let $h \in N$. Then $A \equiv (Dh(0) + Dh(0)^*)/2 \in L(C^n)$ is a positive definite operator, i.e. $\langle Az, z\rangle > 0$ for $z \in C^n$, $z \neq 0$, and h satisfies the following inequalities*:

$$\|z\|^2 \frac{1-\|z\|}{1+\|z\|} k_h \leqslant \operatorname{Re}\langle h(z), z\rangle \leqslant \|z\|^2 \frac{1+\|z\|}{1-\|z\|} m_h \tag{2}$$

where k_h is the minimum eigenvalue of A and m_h is the maximum eigenvalue of A.

PROOF. Fix $z \in B^n$, $z \neq 0$, and consider the function

$$p(\lambda) = \frac{1}{\|z\|\lambda} \left\langle h\left(\frac{z}{\|z\|}\lambda\right), z\right\rangle, \qquad \lambda \in C.$$

Then $p(\lambda)$ is defined and holomorphic in $|\lambda| < 1$ and

$$\operatorname{Re} p(\lambda) = \frac{1}{|\lambda|^2} \operatorname{Re}\left\langle h\left(\frac{z}{\|z\|}\lambda\right), \frac{z}{\|z\|}\lambda\right\rangle > 0 \qquad (|\lambda| < 1).$$

Furthermore,

$$\begin{aligned} \operatorname{Re} p(0) &= \operatorname{Re}\left(\frac{1}{\|z\|} \lim_{\lambda \to 0} \left\langle \frac{1}{\lambda} h\left(\frac{z}{\|z\|}\lambda\right), z\right\rangle\right) \\ &= \frac{1}{\|z\|^2} \operatorname{Re}\langle Dh(0)z, z\rangle, \end{aligned}$$

$$\operatorname{Re} p(0) = \frac{1}{\|z\|^2} \langle Az, z\rangle > 0, \tag{3}$$

which shows that A is a positive definite operator. By applying (1) to $p(\lambda)$ and taking $\lambda = \|z\|$, we obtain

$$\frac{1-\|z\|}{1+\|z\|}\langle Az, z\rangle \leqslant \operatorname{Re}\langle h(z), z\rangle \leqslant \frac{1+\|z\|}{1-\|z\|}\langle Az, z\rangle, \qquad z \in B^n. \tag{4}$$

A is selfadjoint with numerical range $W(A) = \{\langle Az, z\rangle : \|z\| = 1\}$ bounded below by $k_h = \inf\{r \in W(A)\}$ and above by $m_h = \sup\{r \in W(A)\}$. Finally we have

$$k_h\|z\|^2 \leqslant \langle Az, z\rangle \leqslant m_h\|z\|^2, \qquad z \in C^n,$$

and (2) follows from (4).

In a recent paper [1] Brickman introduced the notion of a complex-valued Φ-like function of a single variable as follows: Let $f \in H(B^1)$ such that $f(0) = 0$, $f'(0) \neq 0$. Let $\Phi \in H(f(B^1))$ such that $\Phi(0) = 0$ and $\operatorname{Re}\Phi'(0) > 0$. Then f is *Φ-like* in B^1 if $\operatorname{Re}(zf'(z)/\Phi(f(z))) > 0$, or equivalently, $\operatorname{Re}(\Phi(f(z))/zf'(z)) > 0$, $z \in B^1$.

If f is Φ-like in B^1, then $\Phi(f(z))/f'(z) \in N$. Conversely, if $f \in H(B^1)$, $f(0) = 0$, $f'(0) \neq 0$, $\Phi \in H(f(B^1))$, and if there exists $h \in N$ such that $\Phi(f(z)) = f'(z)h(z)$, then $\Phi(0) = 0$, $\operatorname{Re}\Phi'(0) > 0$, and f is Φ-like. We shall extend Brickman's work by introducting the notion of Φ-likeness for vector-valued functions in C^n.

DEFINITION 1. Let $f \in H(B^n)$ such that $f(0) = 0$, $Df(0) = I$, and f is locally biholomorphic in B^n. Let Φ be analytic on $f(B^n)$. Then f is *Φ-like* if there exists $h \in N$ such that

$$\Phi(f(z)) = Df(z)h(z), \qquad z \in B^n, \tag{5}$$

or equivalently $(Df(z))^{-1}\Phi(f(z)) \in N$.

Note that since the open mapping theorem fails for holomorphic maps from C^n into C^n, the condition that $Df(z)$ be nonsingular for all $z \in B^n$ in Definition 1 is necessary to insure that $f(B^n)$ is an open set in C^n. The geometric counterpart of Definition 1 is given next; it is the n-dimensional generalization of Brickman's definition of a Φ-like region in C^1.

DEFINITION 2. Let Ω be a region in C^n containing 0 and let $\Phi \in H(\Omega)$ such that $\Phi(0) = 0$ and $(D\Phi(0) + D\Phi(0)^*)$ is a positive definite operator. Then Ω is Φ-like if for any $\alpha \in \Omega$, the initial value problem

$$\frac{dw}{dt} = \begin{pmatrix} dw_1(t)/dt \\ \vdots \\ dw_n(t)/dt \end{pmatrix} = -\Phi(w(t)), \qquad w(0) = \alpha, \tag{6}$$

has a solution defined for all $t \geqslant 0$ such that $w(t) \in \Omega$ for all $t \geqslant 0$ and $w(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

If there exists a solution for (6), it is unique.

As indicated in [1] for the case in C^1, any Φ satisfying the conditions imposed in Definition 1, i.e. $\Phi(0) = 0$ and $\operatorname{Re} \Phi'(0) > 0$, must be of the form $\Phi(w) = wp(w)$ where $\operatorname{Re} p(w) > 0$ for all $w \in B_\epsilon^1(0)$, a sufficiently small ball about the origin. According to a theorem due to Bony [11, Theorem 1, p. 741], the closure of $B_\delta^1(0)$, $\delta < \epsilon$, is a flow-invariant set for the vector field $-\Phi$ in (6); that is every trajectory $w(t)$ which meets $\overline{B_\delta^1(0)}$ at t_0 must remain in $\overline{B_\delta^1(0)}$ for $t > t_0$. Furthermore if these trajectories approach zero as $t \longrightarrow +\infty$, then $B_\epsilon^1(0)$ is Φ-like itself. If $n \geqslant 2$, $\operatorname{Re}\langle h(w), w\rangle > 0$ and $\Phi(w) = h(w)$ for all $w \in B_\epsilon^n(0)$, the same conclusions follow. These considerations motivate us to prove the following vector-valued generalization of Brickman's Lemma 1 in [1].

LEMMA 2. *Let $h(z) \in \mathcal{N}$. Then for each $z \in B^n$, the initial value problem*

$$\partial v/\partial t = -h(v), \qquad v(0) = z, \tag{7}$$

has a unique solution $v(t) = v(z, t)$ defined for all $t > 0$. For fixed t, $v_t(z) = v(z, t)$ is a univalent Schwarz function on B^n whose norm satisfies the following bound

$$\|v(z, t)\| \leqslant \|z\| \exp\left\{-\frac{1 - \|z\|}{1 + \|z\|} k_h t\right\}, \tag{8}$$

where $k_h > 0$.

PROOF. The existence and uniqueness of $v(t)$, the solution of (7), follow readily from the standard successive approximation techniques applied to $v(z, t) = v(z, 0) + \int_0^t h(v(z, \tau))\, d\tau$, and the details are given in Theorem 2.1 of [10]. Theorem 2.1 involves a family of functions $h(z, t) \in \mathcal{M}$, $0 \leqslant t < \infty$. In (7) we have a single $h(z)$ with no explicit t dependence and this merely simplifies the existence and uniqueness proof. Furthermore $v(z, t)$ is a univalent-Schwarz function as in [10, Theorem 2.1]. Discarding the normalization $Dh(0) = I$ merely changes the form of $Dv(z, t)$ at $z = 0$.

It is easy to see that $\|v(z, t)\|(\partial \|v(z, t)\|/\partial t)$ exists a.e. for $t \in [0, \infty)$ and that

$$\|v(z, t)\| \frac{\partial \|v(z, t)\|}{\partial t} = \operatorname{Re}\left\langle \frac{\partial v(z, t)}{\partial t}, v(z, t)\right\rangle, \quad \text{a.e. } t \geqslant 0. \tag{9}$$

For details see [10, Theorem 2.1] or [4, Lemma 1.3, p. 510].

Thus

$$\|v(z, t)\| \frac{\partial \|v(z, t)\|}{\partial t} = -\mathrm{Re}\langle h(v(z, t)), v(z, t)\rangle, \quad \text{a.e. } t \geqslant 0,$$

and from (2) we obtain a.e. $t \geqslant 0$

$$\|v(z, t)\| \frac{\partial \|v(z, t)\|}{\partial t} \leqslant -\|v(z, t)\|^2 \frac{1 - \|v(z, t)\|}{1 + \|v(z, t)\|} k_h$$

for some $k_h > 0$. Therefore

$$\frac{\partial}{\partial t} \log\|v(z, t)\| \leqslant -\frac{1 - \|v(z, t)\|}{1 + \|v(z, t)\|} k_h$$

$$\leqslant -\frac{1 - \|z\|}{1 + \|z\|} k_h, \quad \text{a.e. } t \geqslant 0.$$

Hence

$$\log\left(\frac{\|v(z, t)\|}{\|z\|}\right) = \int_0^t \frac{1}{\|v(z, s)\|} \frac{\partial \|v(z, s)\|}{\partial s} ds \leqslant -\frac{1 - \|z\|}{1 + \|z\|} k_h t$$

and exponentiation yields (8).

3. **Φ-likeness and univalence criteria in C^n.** The next two theorems and one corollary generalize Brickman's results in [1]: Φ-like functions are necessarily univalent in B^n and have Φ-like images (Theorem 1). Moreover every function analytic and univalent in B^n and normalized at the origin is Φ-like for some Φ (Corollary 1). Hence Φ-likeness is equivalent to univalence. In Theorem 2 we have the converse result to Theorem 1, that is, f is a Φ-like function in the sense of Definition 1 if f is analytic, univalent in B^n, $f(0) = 0$, $Df(0) = I$ and $f(B^n)$ is a Φ-like region in the sense of Definition 2.

THEOREM 1. *Let f be Φ-like in B^n. Then f is univalent in B^n and $f(B^n)$ is Φ-like.*

PROOF. By hypothesis there exists $h \in N$ such that

$$\Phi(f(z)) = Df(z)h(z), \qquad z \in B^n. \tag{10}$$

Letting α approach zero in $\Phi(f(\alpha z))/\alpha = Df(\alpha z)h(\alpha z)/\alpha$ and using the normalization of f, we see that $D\Phi(0) = Dh(0)$ and consequently by Lemma 1 that $D\Phi(0) + D\Phi(0)^*$ is a positive definite operator. This is the reason for replacing Brickman's condition $f'(0) \neq 0$ by $Df(0) = I$ in Definition 1. Fix $z \in B^n$ and define $v(t) = v_z(t)$ for all $t \geqslant 0$ where $v_z(t)$ is the unique solution to (7) in Lemma 2. Since $v_z(t) \in B^n$, we can define

$$w(t) = w_z(t) \equiv f(v_z(t)), \qquad t \geqslant 0, \tag{11}$$

and note that

$$w_z(0) = f(z). \tag{12}$$

By Lemma 2, we see that

$$\begin{aligned} dw_z(t)/dt &= Df(v_z(t))\partial v_z(t)/\partial t \\ &= -Df(v_z(t))h(v_z(t)) \\ &= -\Phi(f(v_z(t))), \end{aligned}$$

$$dw_z(t)/dt = -\Phi(w_z(t)). \tag{13}$$

Therefore $w_z(t) = f(v_z(t))$ satisfies the initial value problem of Definition 2 with $\alpha = f(z)$. Clearly $w_z(t) \in f(B^n)$ and in fact $\|v_z(t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$ because of (8). Hence

$$\lim_{t\to+\infty} w_z(t) = \lim_{t\to+\infty} f(v_z(t)) = f(0) = 0.$$

Therefore $f(B^n)$ is Φ-like.

To show f is univalent, we let $a, b \in B^n$ and suppose $f(a) = f(b)$. By (12) this may be rewritten as $w_a(0) = w_b(0)$. Thus $w_a(t)$ and $w_b(t)$ are two solutions of the same initial value problem defined by (12) and (13). Hence $w_a(t) = w_b(t)$ for all $t \geqslant 0$ by the uniqueness of solutions to (12) and (13). Equivalently $f(v_a(t)) = f(v_b(t))$ for all $t \geqslant 0$. Since f is locally biholomorphic in B^n, f has a local inverse at 0, and consequently $v_a(t)$ and $v_b(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ implies there exists M such that $v_a(t) = v_b(t)$ for all $t \geqslant M$. Therefore $v_a(t) = v_b(t)$ for all $t \geqslant 0$ since both are solutions to the initial value problem

$$dv(t)/dt = -h(v(t)), \qquad v(M) = v_a(M) = v_b(M),$$

which has a unique solution for all $t \geqslant 0$. Hence $a = v_a(0) = v_b(0) = b$, and f is univalent.

COROLLARY 1. *Let $f \in H(B^n)$ with $f(0) = 0$ and $Df(0) = I$. Then f is univalent in B^n if and only if f is Φ-like for some Φ.*

PROOF. Theorem 1 establishes that Φ-likeness implies univalence. Conversely suppose f is univalent and select any function $h \in N$. If we define Φ on $f(B^n)$ by

$$\Phi(f(z)) = Df(z)h(z) \tag{14}$$

then Φ is holomorphic on $f(B^n)$ and f is Φ-like.

THEOREM 2. *Let $f \in H(B^n)$ be univalent in B^n with $f(0) = 0$, $Df(0) = I$,*

and $f(B^n)$ a Φ-like region. Then f is Φ-like in B^n (Definition 1).

PROOF. Since $f(B^n)$ is Φ-like, we define $w_z(t)$ (for $z \in B^n$ and $t \geqslant 0$) to be the solution of the initial value problem

$$dw_z(t)/dt = -\Phi(w_z(t)), \qquad w_z(0) = f(z). \tag{15}$$

The univalence of f insures that $v_z(t) \equiv f^{-1}(w_z(t))$ is well defined. Hence

$$Df(v_z(t))\partial v_z(t)/\partial t = dw_z(t)/dt = -\Phi(w_z(t))$$

and setting $t = 0$, we obtain

$$-\Phi(f(z)) = Df(z)\partial v_z(0)/\partial t. \tag{16}$$

We must show that the holomorphic (in B^n) function $h(z) = -\partial v_z(0)/\partial t$ belongs to N. We shall prove that $\operatorname{Re}\langle h(z), z\rangle \geqslant 0$ and then show the inequality is strict for $z \neq 0$.

For $z = 0$, $w_z(t) = 0$ for all $t \geqslant 0$ by the uniqueness of solutions to (15); thus $v_z(t) = 0$ for all $t \geqslant 0$ if $z = 0$. If $z \neq 0$, then $w_z(t) \neq 0$ for all $t \geqslant 0$ which implies $v_z(t) = f^{-1}(w_z(t)) \neq 0$ for all $t \geqslant 0$ since f is univalent. An easy computation using the differentiability of the norm, $\| \ \|$, in $C^n - \{0\}$ yields

$$\|v_z(t)\|\, \partial \|v_z(t)\|/\partial t = \operatorname{Re}\langle \partial v_z(t)/\partial t, v_z(t)\rangle, \qquad z \in B^n,\ t \geqslant 0. \tag{17}$$

(In both the definition of h and in (16) for $t = 0$, we mean the right-hand derivative of $v_z(t)$ and $\|v_z(t)\|$ at $t = 0$ respectively.) We shall prove that $\operatorname{Re}\langle h(z), z\rangle \geqslant 0$ by showing that $\partial \|v_z(0)\|/\partial t \leqslant 0$.

The analyticity of $\Phi(z)$ and $f(z) = w_z(0)$ insures that $w_z(t)$ is analytic in z for each fixed $t \geqslant 0$. Therefore $v_z(t) = f^{-1}(w_z(t))$ is analytic in z for fixed t, and

$$\|v_z(t)\| < 1, \qquad z \in B^n,\ t \geqslant 0. \tag{18}$$

By uniqueness of the solution to (15), we know $w_0(0) = 0$ implies $w_0(t) \equiv 0$ for all $t \geqslant 0$ and consequently

$$v_0(t) = 0 \quad \text{for all } t \geqslant 0. \tag{19}$$

The Schwarz Lemma applied to (18) and (19) shows that $\|v_z(t)\| \leqslant \|z\|$ for all $z \in B^n$, $t \geqslant 0$. Hence

$$\frac{\partial \|v_z(0)\|}{\partial t} = \lim_{t\to 0^+} \frac{\|v_z(t)\| - \|v_z(0)\|}{t}$$

$$= \lim_{t\to 0^+} \frac{\|v_z(t)\| - \|z\|}{t} \leqslant 0$$

and $\operatorname{Re}\langle h(z), z\rangle \geqslant 0$.

As in Theorem 1, the relation $\Phi(f(z)) = Df(z)h(z)$ forces $D\Phi(0) = Dh(0)$, and therefore $(Dh(0) + Dh(0)^*)/2 = (D\Phi(0) + D\Phi(0)^*)/2$ is positive definite. Using a method from Lemma 1, we know that

$$p(\lambda) = \frac{1}{\|z\|\lambda}\left\langle h\left(\frac{z}{\|z\|}\lambda\right), z\right\rangle,$$

for fixed $z \in B^n$, is a complex-valued holomorphic function of λ, $|\lambda| < 1$, such that

$$\operatorname{Re} p(\lambda) = \frac{1}{|\lambda|^2} \operatorname{Re}\left\langle h\left(\frac{z}{\|z\|}\lambda\right), \frac{z}{\|z\|}\lambda\right\rangle \geqslant 0$$

since $\operatorname{Re}\langle h(z), z\rangle \geqslant 0$ for all $z \in B^n$. The operator $A = (Dh(0) + Dh(0)^*)/2$ is positive definite and therefore

$$\operatorname{Re} p(0) = \frac{1}{\|z\|^2} \operatorname{Re}\langle Dh(0)z, z\rangle = \frac{1}{\|z\|^2}\langle Az, z\rangle > 0, \qquad z \neq 0.$$

By the minimum principle for harmonic functions, $\operatorname{Re} p(\lambda) > 0$ for all $|\lambda| < 1$. Taking $\lambda = \|z\|$, we see that $\operatorname{Re}\langle h(z), z\rangle > 0$, $z \neq 0$, and $h(z) \in N$.

REMARK. The last paragraph in the proof of Theorem 2 is necessary to show that $h(z) \in N$; in general it does not follow that $\operatorname{Re}\langle h(z), z\rangle > 0$ if $h(z) \in H(B^n)$ and $\operatorname{Re}\langle h(z), z\rangle \geqslant 0$. Suffridge's example $h(z_1, z_2) = (-z_2, z_1)$ [14, p. 577] is a counterexample.

4. Φ-like, starlike, and spiral-like criteria. A holomorphic mapping $f\colon B^n \longrightarrow C^n$ is *starlike* if f is univalent, $f(0) = 0$, and for all $t \in [0, 1]$, $(1 - t)f(B^n) \subset f(B^n)$; that is, the region $f(B^n)$ is starlike with respect to 0. If $\Phi = I$ (the identity map) in Definition 2, then the solution to (6) is $w(t) = \alpha e^{-t}$. Hence Ω is I-like if and only if Ω is starlike.

THEOREM 3. *f is starlike if and only if $f(0) = 0$, f is locally biholomorphic in B^n and $(Df(z))^{-1}f(z) \in N$.*

PROOF. If $Df(0) = I$, then this is an immediate consequence of Theorems 1 and 2 and our preceding remarks that identify I-like and starlike regions. The results of Theorems 1 and 2 where $Df(0) = I$ depend upon the equation

$$D\Phi(0) = Dh(0) \tag{20}$$

which follows from $\Phi(f(z)) = Df(z)h(z)$. ((10) in Theorem 1 and (16) in Theorem 2.) In Theorem 1 (20) is used to show that $(D\Phi(0) + D\Phi(0)^*)/2$ is positive definite and in Theorem 2 (20) establishes the positive definiteness of $(Dh(0) + Dh(0)^*)/2$.

If the normalization $Df(0) = I$ is dropped, but $\Phi = I$ then (20) is replaced by

$$Df(0) = Df(0)Dh(0) \tag{20'}$$

which implies $Dh(0) = I$. Hence Theorems 1 and 2 are still true in the case where $Df(0) \neq I$ and $\Phi = I$ since $D\Phi(0) = I = Dh(0)$. This completes the proof of Theorem 3.

Theorem 3 was first obtained by Matsuno [8]. Suffridge has obtained similar results characterizing starlike mappings in C^n [13]. In fact, our Theorem 3 above is a special case of [13, Theorem 4, p. 247]. In §5 we shall generalize the methods used in §§2 and 3 to obtain Suffridge's results in [14] concerning starlike maps in Banach spaces.

We note the role of the class M in studying starlike functions. Since $(Df(z))^{-1}f(z) \in N$ if and only if $(Df(z))^{-1}f(z) \in M$, Theorem 3 could be restated using M instead of N.

In Definition 1 for $n = 1$, the choice $\Phi(w) = \lambda w$ with $\operatorname{Re} \lambda > 0$ yields the classical definition,

$$\operatorname{Re}\{e^{i\beta} zf'(z)/f(z)\} > 0, \qquad |z| < 1,$$

of a spiral-like function of type $\beta = -\arg \lambda$. In Definition 2 for $n = 1$, the same choice, $\Phi(w) = \lambda w$, yields $w(t) = \alpha e^{-\lambda t}$ as the solution to (6). Thus Ω in C^1 is Φ-like for some $\Phi(w) = \lambda w$ if and only if Ω contains all spirals $\{\alpha e^{-\lambda t}: t \geqslant 0\}$ where $\alpha \in \Omega$. This motivates the following extension to C^n of the notion of a spiral-like function.

DEFINITION 3. Let f be locally biholomorphic in B^n, $f(0) = 0$ and $Df(0) = I$. Let $\Phi \in L(C^n)$ be a normal operator for which $\operatorname{Re} \lambda > 0$ for each eigenvalue λ of Φ. Then f is *spiral-like* if $(Df(z))^{-1}(\Phi(f(z))) \in N$.

It is clear from Theorem 1 that a spiral-like function f is necessarily univalent. By Theorem 1 and Definition 2, the initial value problem (for a fixed $z \in B^n$)

$$dw/dt = -\Phi(w), \qquad w(0) = f(z), \tag{21}$$

has a solution $w(t)$ such that $w(t) \in f(B^n)$ for all $t \geqslant 0$ and $w(t) \longrightarrow 0$ as $t \longrightarrow +\infty$. Since Φ is normal, there exists a unitary operator $U \in L(C^n)$ that diagonalizes Φ,

$$U\Phi U^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \lambda_n \end{bmatrix},$$

and $\operatorname{Re} \lambda_j > 0$, $1 \leqslant j \leqslant n$, since f is spiral-like. By a change of variables, $v = Uw$, we know that (21) is equivalent to

$$dU^{-1}v/dt = -\Phi U^{-1}v, \qquad v(0) = Uf(z).$$

Applying U to each side of this differential equation, we obtain the initial value problem

$$dv/dt = -U\Phi U^{-1}v, \qquad v(0) = Uf(z), \tag{22}$$

equivalent to (21). The unique solution to (22) is $v(t)$ where $v_j(t) = (Uf)_j(z) \exp\{-\lambda_j t\}$. From these elementary considerations, and Theorems 1 and 2 we have the following theorem.

THEOREM 4. *Let f be locally biholomorphic in B^n, $f(0) = 0$, and $Df(0) = I$. Then f is spiral-like if and only if there exist $\{\lambda_j\colon \operatorname{Re} \lambda_j > 0, j = 1, \cdots, n\}$ and a unitary transformation U such that each of the coordinate projections $(Uf)_j(B^n)$, $j = 1, \cdots, n$, of B^n into C contains all of the spirals*

$$\{(Uf(z))_j \exp\{-\lambda_j t\}\colon t \geqslant 0\}, \qquad z \in B^n.$$

5. **N and M for Banach spaces.** Henceforth X will always be a complex Banach space with norm $\| \ \|$. Denote the unit ball in X as $B = \{x \in X\colon \|x\| < 1\}$. The dual of X is symbolized by X'. For each $x \in X$, we define $T(x) = \{x' \in X'\colon \|x'\| = 1, x'(x) = \|x\|\}$; the Hahn-Banach Theorem guarantees that $T(x)$ is nonempty.

If Ω is a region of X (a nonempty connected open subset of X) and Y another Banach space, then a function f defined on Ω with range in Y is said to be (F)-differentiable at $x_0 \in \Omega$ if

$$\lim_{\beta \to 0} \frac{1}{\beta}[f(x_0 + h) - f(x_0)] \equiv Df(x_0)h$$

exists for all $h \in X$ and $Df(x_0)$ is a bounded linear operator from X to Y, i.e. $Df(x_0) \in L(X, Y)$. The norm on $L(X, Y)$ will be $\|A\| = \sup\{\|Ax\|\colon \|x\| < 1\}$

for $A \in L(X, Y)$. We say that f is a holomorphic function on Ω if f is (F)-differentiable at each point of Ω. We denote the class of holomorphic functions f: $\Omega \longrightarrow X$ by $H(\Omega)$.

As in §2 a function $f \in H(\Omega)$ is said to be *locally biholomorphic* in Ω if $Df(x)$ is nonsingular at each point $x \in \Omega$. By the inverse function theorem for Banach spaces [12, p. 252] and the fact that $x \longrightarrow Df(x)$ is a holomorphic mapping of Ω into $L(X) \equiv L(X, X)$ [9, Proposition 3, p. 29], f being locally biholomorphic is equivalent to f having a local holomorphic inverse at each point in Ω.

We now generalize the classes N and M to Banach spaces as follows.

$$N = \{h \in H(B)\colon h(0) = 0, \operatorname{Re} x'(h(x)) > 0 \text{ for all nonzero } x \in B,\ x' \in T(x)\},$$

$$M = \{h \in N\colon Dh(0) = I\}.$$

For $X = C^n$ with the standard Euclidean norm, $T(x)$ for $x \neq 0$ consists of one element $x' = \langle\, , x\rangle/\langle x, x\rangle^{1/2}$; N and M as defined above then reduce to the definitions in §2. Analogous to Lemma 1, we have the following result.

LEMMA 3. *Let $h \in N$, then for all nonzero $z \in B$ and $x' \in T(x)$,* $\operatorname{Re} x'(Dh(0)x) > 0$ *and*

$$\frac{1 - \|x\|}{1 + \|x\|} \operatorname{Re} x'(Dh(0)x) \leqslant \operatorname{Re} x'(h(x)) \leqslant \frac{1 + \|x\|}{1 - \|x\|} \operatorname{Re} x'(Dh(0)x). \tag{23}$$

PROOF. Fix $x \in B$, $x \neq 0$ and also fix $x' \in T(x)$. For $\zeta \in C$, $|\zeta| < 1$, define

$$g(\zeta) = x'\left(h\left(\frac{x}{\|x\|}\zeta\right)\right).$$

Therefore $p(\zeta) = g(\zeta)/\zeta$ is holomorphic in ζ and

$$p(0) = x'\left(Dh(0)\frac{x}{\|x\|}\right).$$

Fix $\zeta \neq 0$ and define $\tilde{x} = (|\zeta|/\zeta)x'$. Thus since

$$\tilde{x}\left(\frac{x}{\|x\|}\zeta\right) = \frac{|\zeta|}{\zeta}x'\left(\frac{x}{\|x\|}\zeta\right) = |\zeta|,$$

we have that $\tilde{x} \in T((x/\|x\|)\zeta)$ and hence

$$\operatorname{Re}\left(\tilde{x}\left(h\left(\frac{x}{\|x\|}\zeta\right)\right)\right) > 0.$$

Consequently $\operatorname{Re} p(\zeta) > 0$ for all $|\zeta| < 1$ by the minimum principle for harmonic

functions. Using the classical inequalities in (1) applied to $p(\zeta)$ with $\zeta = \|x\|$, we obtain (23). This completes the proof of the lemma.

The analogy between Lemmas 3 and 1 is most clear if we consider the Banach space X with a semi-inner product structure. The notion of a semi-inner product space was introduced by Lumer [6] and Lumer and Phillips [7].

DEFINITION 4. A *semi-inner product* $[\ ,\ ]$ on a complex vector space V is a map from $V \times V$ into C such that:

(i) $[x + y, z] = [x, z] + [y, z]$, $[\lambda x, y] = \lambda[x, y]$ for $x, y \in V$, $\lambda \in C$,

(ii) $[x, x] > 0$ for $x \neq 0$,

(iii) $|[x, y]|^2 \leqslant [x, x]\,[y, y]$.

Such a space V is called a *semi-inner product space* (abbreviated s.i.p.s.).

It is easy to show that a s.i.p.s. is a normed linear space with the norm $[x, x]^{1/2}$. Conversely any normed linear space V can be made into a s.i.p.s.—in general in infinitely many ways—by choosing for each element $y \in V$ exactly one element $Jy \in V'$ such that $(Jy)(y) = \|y\|^2$ and $\|Jy\| = \|y\|$. Then $[x, y] \equiv (Jy)(x)$ defines a semi-inner product on V such that $[x, x] = \|x\|^2$. Note that $\|x\|^{-1}(Jx) \in T(x)$.

Following the lead of [6] and [7], we make the following definitions.

DEFINITION 5. A linear operator A with domain $D(A)$ in a s.i.p.s. V is called *strictly dissipative* if $\operatorname{Re}[Ay, y] < 0, y \in D(A), y \neq 0$.

DEFINITION 6. For any operator A with domain $D(A)$ in a s.i.p.s. V, the *numerical range* $W(A)$ is defined as

$$W(A) \equiv \{[Ay, y] : y \in D(A),\ \|y\| = 1\}.$$

We let $k(A) \equiv \inf\{\operatorname{Re} \lambda : \lambda \in W(A)\}$ and $m(A) \equiv \sup\{\operatorname{Re} \lambda : \lambda \in W(A)\}$.

Returning to the Banach space X, we see that for each function $J: X \longrightarrow X'$ such that $\|x\|^{-1}Jx \in T(x)$, we obtain a distinct semi-inner product $[\ ,\ ]$ which makes X into a s.i.p.s. so that $[x, x]^{1/2} = \|x\|$ for all $x \in X$. We let $(X, [\ ,\ ])$ denote the s.i.p.s. with semi-inner product $[\ ,\ ]$ obtained via J and thus consistent with the norm on X.

For $A \in L(X)$ and any s.i.p.s. structure $(X, [\ ,\ ])$ imposed on X, we know that $m(A)$ computed with respect to $[\ ,\ ]$ is equal to

$$\lim_{\beta \to 0} \frac{\|I + \beta A\| - 1}{\beta}$$

[6, Lemma 12, p. 36]. Therefore $m(A)$ and $k(A) = -m(-A)$ are quantities dependent only upon A and are independent of the choice of a semi-inner product on X. Lemma 3 can now be restated as the following.

LEMMA 4. *Given $h \in N$ and s.i.p.s. $(X, [\ ,\ ])$, then $-Dh(0)$ is a strictly dissipative operator and*

$$\frac{1-\|x\|}{1+\|x\|}\operatorname{Re}[Dh(0)x, x] \leqslant \operatorname{Re}[h(x), x] \tag{24}$$
$$\leqslant \frac{1+\|x\|}{1-\|x\|}\operatorname{Re}[Dh(0)x, x], \qquad x \in B.$$

The next lemma is a generalization of Lemma 2.

LEMMA 5. *Let $h \in N$. Then for each $x \in B$, the initial value problem*

$$\partial v/\partial t = -h(v), \qquad v(0) = x, \tag{25}$$

has a unique solution $v(t) = v(x, t)$ defined for all $t \geqslant 0$. Furthermore for fixed t, $v_t(x) = v(x, t)$ is a univalent-Schwarz function on B whose norm satisfies the following equality for each $(X, [\ ,\ ])$:

$$\|v(x, t)\| = \|x\| \exp\left(-\int_0^t \frac{\operatorname{Re}[h(v_x(s)), v_x(s)]}{\|v_x(s)\|^2}\, ds\right), \qquad x \neq 0. \tag{26}$$

PROOF. As noted in [10], Theorem 2.1 of [10] can be extended to a Banach space. Using this fact, we obtain the conclusion above (except for (26)) after noting the same comments mentioned in the beginning of the proof of Lemma 2.

To conclude Lemma 5, it suffices to mention that as in Lemma 2, $\partial\|v(x, t)\|/\partial t$ exists a.e. for $t \geqslant 0$ and that analogous to (9) we have the formula

$$\|v(x, t)\| \frac{\partial\|v(x, t)\|}{\partial t} = \operatorname{Re}\left[\frac{\partial v(x, t)}{\partial t}, v(x, t)\right], \qquad \text{a.e. } t \geqslant 0,$$

for $(X, [\ ,\ ])$ [4, p. 510]. Therefore

$$\frac{\partial \log\|v(x, t)\|}{\partial t} = -\frac{1}{\|v(x, t)\|^2}\operatorname{Re}[h(v(x, t)), v(x, t)], \qquad \text{a.e. } t \geqslant 0,$$

and

$$\log\frac{\|v(x, t)\|}{\|x\|} = -\int_0^t \frac{\operatorname{Re}[h(v(x, s)), v(x, s)]}{\|v(x, s)\|^2}\, ds, \qquad t \geqslant 0.$$

Exponentiation yields (26) and the proof is concluded.

REMARK. It is clear from (26) that the norm of the solution of (25) is strictly decreasing as a function of t since the integrand in (26) is always positive. In general we do not know if $\|v(x, t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$ when X is an arbitrary Banach space and h is an arbitrary element of N. However the question is settled for one important subset of N.

If $h \in M$, then by (24) we have

$$\frac{1-\|x\|}{1+\|x\|} \leqslant \frac{1-\|v(x,t)\|}{1+\|v(x,t)\|}$$

$$= \frac{1-\|v(x,t)\|}{1+\|v(x,t)\|} \; \frac{\operatorname{Re}[Dh(0)v(x,t), v(x,t)]}{\|v(x,t)\|^2}$$

$$\leqslant \frac{\operatorname{Re}[h(v(x,t)), v(x,t)]}{\|v(x,t)\|^2}.$$

In this case (26) yields

$$\|v(x,t)\| \leqslant \|x\| \exp\left(-\frac{1-\|x\|}{1+\|x\|}\, t\right)$$

and $\|v(x,t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$.

A more general approach involves picking a semi-inner product on X that satisfies the homogeneity property $[x, \lambda y] = \bar{\lambda}[x, y]$ for all $x, y \in X$ [2, Theorem. 1, p. 437]. This can be done by choosing exactly one nonzero $y \in X$ from each complex line through the origin and defining Jy. We then define J on the remainder of the line by $J\lambda y \equiv \bar{\lambda} Jy$. The semi-inner product associated with J then satisfies the condition $[x, \lambda y] = \bar{\lambda}[x, y]$. Any semi-inner product with this property will be denoted $[\ ,\ ]_1$.

If $A \in L(X)$, then $k(A)\|x\|^2 \leqslant \operatorname{Re}[Ax, x]_1 \leqslant m(A)\|x\|^2$. Using the same argument as for $h \in M$, we see that if $h \in N$ such that $0 < k(Dh(0))$, then (26) yields

$$\|v(x,t)\| \leqslant \|x\| \exp\left(-\frac{1-\|x\|}{1+\|x\|}\, k(Dh(0))t\right) \tag{27}$$

and $\|v(x,t)\| \longrightarrow 0$ as $t \longrightarrow +\infty$. This applies when X is a finite-dimensional Hilbert space; then $\langle\ ,\ \rangle = [\ ,\ ]$ and $k(Dh(0))$ is the usual lower bound for the numerical range of $(Dh(0) + Dh(0)^*)/2$.

We now define the notions of Φ-like functions and Φ-like regions in Banach spaces.

DEFINITION 7. Let $f \in H(B)$ such that $f(0) = 0$, $Df(0) = I$, and f is locally biholomorphic in B. Let $\Phi \in H(f(B))$. Then f is *Φ-like* if $(Df(x))^{-1}\Phi(f(x)) \in N$.

The normalization of $Df(0) = I$ forces $D\Phi(0) = Dh(0)$ for some $h \in N$. We see by Lemma 4 that $-D\Phi(0)$ is then a strictly dissipative operator on B with respect to any semi-inner product consistent with the norm on X.• This is the analogue of $(D\Phi(0) + D\Phi(0)^*)/2$ being positive definite in §2.

DEFINITION 8. Let Ω be a region in X containing 0, and let $\Phi \in H(\Omega)$ such that $\Phi(0) = 0$ and $\operatorname{Re} x'(D\Phi(0)x) > 0$ for all $x \in B$, $x \neq 0$, $x' \in T(x)$. Then Ω is *Φ-like* if for any $\alpha \in \Omega$, the initial value problem

$$dw/dt = -\Phi(w), \qquad w(0) = \alpha, \tag{28}$$

has a solution $w(t)$ defined for all $t \geqslant 0$ such that $w(t) \in \Omega$ for all $t \geqslant 0$ and $w(t) \longrightarrow 0$ as $t \longrightarrow +\infty$.

Just as the uniqueness and analyticity of the solution of (6) as a function of the initial value $\alpha \in C^n$ follows from the analyticity of Φ, one can show that for the Banach space X, the solution $w(\alpha, t)$ to (28) is unique—if it exists—and for fixed t holomorphic as a function of the initial value α. If the solution $w(\alpha, t)$ exists, then it may be found using the method of successive approximations [5, pp. 129–130]. For fixed t each of the approximations $w_n(\alpha, t)$ will be holomorphic in α and equilocally bounded in Ω. Thus since $\lim_{n\to\infty} w_n(\alpha, t) = w(\alpha, t)$ in Ω, we see that $w(\alpha, t) \in H(\Omega)$ for fixed t [3, Theorem 8.4.3, p. 272].

We note that requiring $\operatorname{Re} x'(D\Phi(0)x) > 0$ for all nonzero $x \in B$, $x' \in T(x)$ is equivalent to saying $-Dh(0)$ is a strictly dissipative operator on the unit ball in $(X, [\ ,\ ])$ where $[\ ,\ ]$ is consistent with the norm on X.

6. Φ-likeness and univalence criteria in a Banach space. In this section we give Banach space generalizations of the results in §3. Our characterization of univalence in terms of Φ-likeness for a general Banach space involves the additional requirement that $k(D\Phi(0))$ be positive. This insures that the solution to the initial value problem (25) tends to zero as $t \longrightarrow +\infty$.

THEOREM 5. *Let $f \in H(B)$ with $f(0) = 0$ and $Df(0) = I$. Then f is univalent in B if and only if f is Φ-like for some Φ with the property $k(D\Phi(0)) > 0$.*

PROOF. Let f be a univalent on B. Select any $h \in N$ and define Φ on $f(B)$ by

$$\Phi(f(x)) \equiv Df(x)h(x). \tag{29}$$

This defines a holomorphic function Φ on $f(B)$ such that f is Φ-like. In particular, we choose $h \in M$ and define Φ by (29). Then f is Φ-like and $k(D\Phi(0)) = 1$. Conversely, if f is Φ-like and $k(D\Phi(0)) > 0$, then $k(Dh(0)) > 0$ where $\Phi(f(x)) = Df(x)h(x)$. By (27) the solution to (25) tends to zero as $t \longrightarrow +\infty$. With the machinery developed in §5, the proof of the remainder of this theorem is a straightforward generalization of the proof of Theorem 1.

COROLLARY 2. *Let f be Φ-like in B. If the solution $v(x, t)$ of (25) tends to zero as $t \longrightarrow +\infty$, then f is univalent and $f(B)$ is Φ-like.*

THEOREM 6. *Let $f \in H(B)$ and univalent in B with $f(0) = 0$ and $Df(0) = I$; let $f(B)$ be Φ-like. Then f is Φ-like in B.*

PROOF. As in the proof of Theorem 2, we define $v_x(t) \equiv f^{-1}(w_x(t))$

where $w_x(t)$ is the solution of the initial value problem $dw_x(t)/dt = -\Phi(w_x(t))$, $w_x(0) = f(x)$. Setting $t = 0$, we obtain $\Phi(f(x)) = -Df(x)\partial v_x(0)/\partial t$. We now wish to show that $h(x) \equiv -\partial v_x(0)/\partial t$ is an element of N.

By the same argument used in Theorem 2, $v_x(t)$ is a Schwarz function for fixed t. Therefore for $\Psi \in T(x)$, $\operatorname{Re} \Psi(v_x(t)) \leqslant \|v_x(t)\| \leqslant \|x\|$ and consequently

$$\operatorname{Re} \Psi\left(\frac{\partial v_x(0)}{\partial t}\right) = \lim_{t \to 0^+} \operatorname{Re} \Psi\left(\frac{v_x(t) - x}{x}\right) \leqslant 0.$$

Hence $\operatorname{Re} \Psi(h(x)) \geqslant 0$ for all $x \in B$, $\Psi \in T(x)$. By Lemma 3 of [14], $\operatorname{Re} \Psi(h(x)) = 0$ if and only if $\operatorname{Re} \Psi(Dh(0)x) = 0$. Since $Dh(0) = D\Phi(0)$, the latter is impossible unless $x = 0$. Therefore $\operatorname{Re} \Psi(h(x)) > 0$ when $x \neq 0$ and $h \in N$. This concludes the proof of Theorem 6.

REMARK. In general we do not know whether or not f being Φ-like implies f is univalent and has a Φ-like image since Corollary 2 has an additional hypothesis about f. If for all $h \in N$, the solution of (25) can be shown to tend to zero as $t \longrightarrow +\infty$, then the question would be settled affirmatively.

7. Starlike maps in Banach spaces. Let X and Y be complex Banach spaces and let $B = \{x \in X\colon \|x\| < 1\}$. In this section we investigate locally biholomorphic maps $f\colon B \longrightarrow Y$ that map the origin of X to the origin of Y. We find necessary and sufficient conditions that f be univalent and map B onto a domain in Y which is starlike with respect to the origin. The definition of a starlike map in this context is the same as the definition given at the beginning of §4 except the domain is now B and the range is in Y. Our approach depends heavily upon the results concerning Φ-like functions obtained in §§5 and 6 of this paper even though Φ-like functions as defined have ranges contained in X rather than Y.

Suppose $I_Y \in L(Y)$ is the identity map on Y and we are given a locally biholomorphic map $f\colon B \longrightarrow Y$ such that

$$f(0) = 0 \quad \text{and} \quad I_Y f(x) = f(x) = Df(x)h(x), \qquad x \in B,$$

holds where $h \in N(X)$, the class N defined for the space X. Then $Df(0) = Df(0)Dh(0)$ or $Dh(0) = I_X$, the identity map on X.

As in §4 for the finite-dimensional case, it is easy to see that a region contained in Y is starlike with respect to the origin if and only if the region is I_Y-like (Definition 8). Therefore by slightly modifying the proofs in §6, we have obtained the following theorem.

THEOREM 7. *Let $f\colon B \longrightarrow Y$ be a locally biholomorphic map such that $f(0) = 0$. Then f is starlike if and only if $(Df(x))^{-1}f(x) \in N(X)$.*

This result was partially obtained by Suffridge in [14]. In proving $(Df(x))^{-1}f(x) \in N(X)$ implies starlikeness, Suffridge added to his hypothesis that for each r, $0 < r < 1$, there exists $M(r)$ such that $\|(Df(x))^{-1}\| \leqslant M(r)$ when $\|x\| \leqslant r$. We have shown that this is not necessary.

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