ABSTRACT. A \( \pi \)-complex is a finite, connected 2-dimensional CW complex with fundamental group \( \pi \). The tree \( HT(\pi) \) of homotopy types of \( \pi \)-complexes has width \( < N \) if there is a root \( Y \) of the tree such that, for any \( \pi \)-complex \( X \), \( X \vee (\vee_{i=1}^{N} S^2_i) \) lies on the stalk generated by \( Y \). Let \( \pi \) be a finite abelian group with torsion coefficients \( \tau_1, \ldots, \tau_n \). The main theorem of this paper asserts that \( \text{width } HT(\pi) < n(n-1)/2 \). This generalizes the results of [4].

1. Introduction. Let \( \pi \) be a finitely presentable group. A \( \pi \)-complex is a finite connected 2-dimensional CW complex with fundamental group \( \pi \). In [4], we gave a complete classification of the homotopy and simple homotopy types of \( Z_n \)-complexes, where \( Z_n \) is the finite cyclic group of order \( n \). In general, we may describe the set of (simple) homotopy types of \( \pi \)-complexes \( (S)HT(\pi) \) as a directed tree—a directed, connected graph which has no circuits. A vertex of \( (S)HT(\pi) \) is the (simple) homotopy type \( [X] \) of a \( \pi \)-complex \( X \). The vertices represented by \( X \) and \( Y \) are joined by an edge directed from \( [X] \) to \( [Y] \) if and only if \( Y \cong (S) X \vee S^2 \). A \( \pi \)-complex is called a root if \( [X] \) possesses no predecessor; the stalk generated by \( X \) is the linearly ordered subgraph of \( (S)HT(\pi) \) determined by the (simple) homotopy types of \( X, X \vee S^2, X \vee S^2 \vee S^2, \ldots \).

The main theorem of [4] states that \( (S)HT(Z_n) \) is a single stalk generated by the pseudo projective plane \( P_n = S^1 \cup_n e^2 \). We say that the width of \( (S)HT(\pi) \leq n \) if there is a root \( X \) such that, for any \( \pi \)-complex \( Y \), \( Y \vee (\vee_{i=1}^{n} S^2_i) \) is on the stalk generated by \( X \).

It is known by the simple homotopy theory of J. H. C. Whitehead [14] that given any \( \pi \)-complex \( Y \) and any root \( X \) there is an integer \( m(Y) \) such that \( Y \vee (\vee_{i=1}^{m(Y)} S^2_i) \) is on the stalk generated by \( X \). Width\( (S)HT(\pi) \leq n \) indicates that there is a root \( X \) such that \( m(Y) \) can be chosen \( \leq n \) for any \( \pi \)-complex \( Y \).

**Theorem A.** Let \( \pi \) be a finite abelian group, \( n = n(\pi) = \text{the number of} \).

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torsion coefficients of $\pi$, and $k = k(\pi) = n(n - 1)/2$. Then the width $HT(\pi) \leq k(\pi)$.

If $p$ is any positive integer, Theorem A implies that width $HT(Z_p)$ is zero, which is the result of [4]. If $\pi = Z_p \times Z_q$, where $p$ divides $q$, then width $HT(\pi) \leq 1$. In this case, the homotopy tree of $Z_p \times Z_q$-complexes looks at worst like:

$$
\begin{array}{c}
\vdots \\
\cdots [X \vee (\sqrt[4]{S^2})] \\
\cdots [X \vee (\sqrt[3]{S^2})] \\
\cdots [X \vee S^2 \vee S^2] \\
\cdots [X \vee S^2] \\
\cdots [X]
\end{array}
$$

where $X$ is the cellular model [4] of the presentation $(a, b: a^p, b^q, aba^{-1}b^{-1})$ and the horizontal levels represent the vertices with common Euler characteristic. At the present time it is unknown whether any of the other “branches” exist. However, at a given level $\chi \geq 3$, there are only finitely many branches. See Theorem B.

As a corollary to A, we obtain a theorem on the cancellation of “large” sums of 2-spheres with $\pi$-complexes. If $\pi$ is a finite abelian group and $X, Y$ are $\pi$-complexes, then $X \vee (\bigvee_{i=1}^{s} S^2_i) \cong Y \vee (\bigvee_{i=1}^{t} S^2_i)$ and $s \geq t > k(\pi)$ imply that $\bigvee_{i=1}^{t-k(\pi)} S^2_i$ can be cancelled from each side (up to homotopy type).

For a given finite group $\pi$ let $\chi_\pi = \min \{\chi(X) \mid X$ is a $\pi$-complex}, $|\pi|$ be the order of $\pi$, and $\varphi$ be the Euler $\varphi$-function.

**THEOREM B.** Let $\pi$ be a finite group other than $Z_2$. The number of homotopy types of $\pi$-complexes with fixed Euler characteristic $\chi \geq \chi_\pi + 1$ is less than or equal to $\varphi(|\pi|)/2$.

**EXAMPLES.** (a) If $\pi = Z_2 \times Z_2$, then Theorems A and B imply that the tree of (simple) homotopy types looks at worst like:
where $X$ is the complex modeled on $(a, b: a^2, b^2, [a, b])$.

(b) If $\pi = \Sigma_3$, the group (of order 6) of permutations on 3 letters, then $\text{HT}(\Sigma_3)$ looks at worst like the above tree, where $X$ is a root of $\text{HT}(\Sigma_3)$ of minimal Euler characteristic. The complex $X$ modeled on the presentation $\{a, b: b^2, bab = a^2\}$ is such a root, since $H_2X = 0$ [16].

2. The chain functor. In [4], we associated with each finite presentation $P = (a_1, \ldots, a_n : r_1, \ldots, r_m)$ of a group $\pi$, its cellular model

$$P = \left( \bigvee_{i=1}^n S_i^1 \right) \cup \left( \bigvee_{j=1}^m B_j^2 \right),$$

which has a single 0-cell, one 1-cell for each generator of $P$, and one 2-cell for each relator of $P$. The $j$th 2-cell is attached to the 1-skeleton $\bigvee_{i=1}^n S_i^1$ according to the instructions provided by the $j$th relator $r_j$.

Then we associated with the cellular model $P$ the cellular chain complex $C_*(\widetilde{P})$ of its universal covering $\widetilde{P}$. $C_*(\widetilde{P})$ is a chain complex of free $\pi$-modules with preferred bases

$$(*) \quad C: C_2(y_1, \ldots, y_m) \xrightarrow{\partial_2} C_1(x_1, \ldots, x_n) \xrightarrow{\partial_1} C_0 = Z[\pi] \xrightarrow{\epsilon} Z \rightarrow 0$$

in which

(a) $\epsilon$ is the augmentation homomorphism $Z[\pi] \rightarrow Z[1]$ induced by $\pi \rightarrow 1$.

(b) Exactness holds at $C_1, C_0, Z$. 

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We can combine these two processes \( P \rightarrow P \) and \( P \rightarrow C_*(\tilde{\pi}) \) as follows. If \( P = (a_1, \ldots, a_n : r_1, \ldots, r_m) \) is a presentation for \( \pi \), let

\[
1 \rightarrow R_\pi \rightarrow F(a_1, \ldots, a_n) \xrightarrow{\varphi_\pi} \pi \rightarrow 1
\]

be the short exact sequence in which \( F = F(a_1, \ldots, a_n) \) is the free group of rank \( n \) on generators \( \{a_1, \ldots, a_n\} \) and \( R_\pi \) is the normal closure of the relators \( \{r_1, \ldots, r_m\} \). The elements \( \bar{x}_i = \varphi_\pi(a_i) \) \((1 \leq i \leq n)\) serve as a set of generators for \( \pi \). We associate a chain complex \( C_*(P) \) as follows. Let \( C_2(P) = C_2(y_1, \ldots, y_m) \) and \( C_1(P) = C_1(x_1, \ldots, x_n) \) be free \( \pi \)-modules with preferred bases \( \{y_1, \ldots, y_m\} \) and \( \{x_1, \ldots, x_n\} \) in 1-1 correspondence with the relators and generators of \( P \), respectively. Let \( C_0(P) \) be the integral group ring \( \mathbb{Z}[[\pi]] \). Then \( C_*(P) \) is the chain complex

\[
C_*(P): C_2(y_1, \ldots, y_m) \xrightarrow{\partial_2(P)} C_1(x_1, \ldots, x_n) \xrightarrow{\partial_1(P)} \mathbb{Z}[[\pi]] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0
\]

whose boundary operators have the following matrix representations with respect to the preferred bases:

\[
\partial_1(P) = (\bar{x}_1 - 1, \ldots, \bar{x}_n - 1) \quad \text{and} \quad \partial_2(P) = (\mathbb{Z}[[\varphi_\pi]](\partial r_i / \partial a_i))
\]

where \( \partial / \partial a_i : \mathbb{Z}[F] \rightarrow \mathbb{Z}[F] \) is the derivative with respect to \( a_i \) in the free calculus of R. H. Fox [5] and \( \mathbb{Z}[\varphi_\pi] : \mathbb{Z}[F] \rightarrow \mathbb{Z}[[\pi]] \) is induced by \( \varphi_\pi : F \rightarrow \pi \).

For example, let \( P = (a_1, a_2 : a_1a_2a_1^{-1}a_2^{-1}) \) be a presentation for \( \pi = \mathbb{Z} \times \mathbb{Z} \) under the correspondence \( \varphi_\pi(a_1) = \bar{x}_1 = (1, 0) \) and \( \varphi_\pi(a_2) = \bar{x}_2 = (0, 1) \). Then the associated chain complex \( C_*(P) \) takes the form

\[
\begin{align*}
C_2(y_1) \xrightarrow{(1 - \bar{x}_2)} & C_1(x_1, x_2) \xrightarrow{(\bar{x}_1 - 1, \bar{x}_2 - 1)} \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0.
\end{align*}
\]

**Definition.** We say that a chain complex \( C \) as in (*) above is realized by a presentation \( P \) of \( \pi \) if \( C_*(P) = C \).

3. The homomorphism \( \rho \). Given a finite presentation \( P = (a_1, \ldots, a_n : r_1, \ldots, r_m) \) of \( \pi \) there is a surjective group homomorphism \( \rho \) from the relator subgroup \( R_\pi \) onto the free abelian group \( \ker \partial_1(P) \) (a \( \pi \)-module also) \( \subset C_1(P) \) which has kernel \( [R_\pi, R_\pi] \).

Following J. H. C. Whitehead in [13] we define the cross homomorphism

\[
\bar{\rho} : F(a_1, \ldots, a_n) \rightarrow C_1(P) \equiv C_1(x_1, \ldots, x_n),
\]

where \( F \) is the free group of rank \( n \), by

(a) \( \bar{\rho}(a_i) = x_i \),

(b) \( \bar{\rho}(a_i^{-1}) = -\varphi_\pi(a_i^{-1})x_i \),

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(c) if $W_1, W_2$ are any words in $F$, then $\overline{\rho}(W_1 \cdot W_2) = \overline{\rho}(W_1) + \varphi_p(W_1) \cdot \overline{\rho}(W_2)$.

Recall that $\varphi_p : F \rightarrow \pi$ is the surjection given by the presentation $P$. Note that by (c), $\overline{\rho}|_{R_p} \equiv \rho$ is a homomorphism. Also, if $r \in R_p$, then

$$\rho(r) = \sum_{i=1}^{n} \left( Z[\varphi_p] \frac{\partial r}{\partial a_i} \right) x_i.$$ 

**Lemma.** The following sequence is exact:

$$1 \rightarrow [R_p, R_p] \xrightarrow{i} R_p \xrightarrow{\rho} \ker \partial_1(P) \rightarrow 0.$$ 

**Proof.** This is really a restatement of Theorem 8 of [13]. Part (a) of Theorem 8 says that $\rho(R_p) = \ker \partial_1(P)$. Part (b) says that $\ker \rho = \ker \overline{\rho} = \text{image of the commutator subgroup of } \pi_2(P, \pi_1(P)) \text{ in } R_p \subset \pi_1(P^{(1)}) (= F(a_1, \cdots, a_n)) \text{ under the boundary operator } \partial : \pi_2(P, \pi_1(P)) \rightarrow \pi_1(P^{(1)}).$ Since $\text{im } \partial = R_p$, $\ker \rho \subset [R_p, R_p]$. But $\ker \rho \supset [R_p, R_p]$ follows because $\ker \partial_1(P)$ is abelian as a group. □

4. Proof of Theorem A. Let $n = n(\pi)$ be the number of torsion coefficients of the finite abelian group $\pi$. Let $\{\tau_1, \cdots, \tau_n\}$ be the torsion coefficients of $\pi$, where $\tau_1, \tau_{i+1}$ for $i = 1, 2, \cdots, n-1$, and $k = k(\pi) = n(n-1)/2$. Furthermore, let $P$ be the $\pi$-complex modeled on the standard presentation

$$P = (a_1, \cdots, a_n : a_1^{\tau_1}, \cdots, a_n^{\tau_n}, \{[a_i, a_j] \mid 1 < i < j \leq n\}).$$

Note that $k(\pi)$ is the number of commutators in $P$ and that $P$ is a root of $(S)HT(\pi)$ (see [15]). We will show that if $X$ is any $\pi$-complex, then $X \vee (\vee_{i=1}^{k(\pi)} S_i^2)$ is on the stalk generated by $P$; i.e.,

$$X \vee \left( \bigvee_{i=1}^{k(\pi)} S_i^2 \right) \simeq P \vee \left( \bigvee_{i=1}^{D(X)} S_i^2 \right)$$

where $D(X) = \text{rank } H_2(X)$.

The given $\pi$-complex $X$ has the simple homotopy type of a $\pi$-complex $R$ modeled on the "pre-abelian" presentation

$$R = (b_1, \cdots, b_l : b_1^{r_1} W_1, b_2^{r_2} W_2, \cdots, b_l^{r_l} W_m, b_{l+1} W_{l+1}, \cdots, b_m W_m)$$

where each $W_i (i = 1, \cdots, m)$ has zero exponent sum on each $b_j (j = 1, \cdots, l)$ [4, Proposition 3]. Notice that $R \vee (\bigvee_{i=1}^{k(\pi)} S_i^2)$ has the simple homotopy type of the $\pi$-complex $S$ modeled on the presentation

$$S = (b_1, \cdots, b_l : b_1^{r_1} W_1, \cdots, b_l^{r_l} W_m, b_{l+1} W_{l+1}, \cdots, b_m W_m; \{[b_i, b_j] \mid 1 < i < j \leq n\}).$$

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Observe that in passing from $R \rightarrow S$ we have added only those commutators corresponding to the nontrivial generators $b_1, \ldots, b_n$.

Let

$$1 \rightarrow R_S \rightarrow F(b_1, \ldots, b_l) \xrightarrow{\varphi_S} \pi \rightarrow 1$$

be the short exact sequence of groups and homomorphism determined by $S$. Denote $\varphi_S(b_i)$ by $\bar{x}_i$ ($i = 1, \ldots, n$) and note that, since $\pi$ is abelian, $\varphi_S(b_i) = 1$ ($n + 1 \leq i \leq l$). The chain complex $C_*(S)$ is given as follows:

$$C_2(S) \xrightarrow{\partial_2(S)} C_1(S)$$

$$C_*(S) : C_2(y_1, \ldots, y_m; z_{12}, z_{13}, \ldots, z_{n-1,n}) \xrightarrow{\partial_2(S)} C_1(x_1, \ldots, x_l) \xrightarrow{\partial_1(S)} Z[\pi] \xrightarrow{\varepsilon} Z \rightarrow 0$$

where $\{z_{ij} \mid 1 \leq i < j \leq n\}$ corresponds to the set of special relators $\{[b_i, b_j] \mid 1 \leq i < j \leq n\}$. Thus

$$\partial_2(z_{ij}) = (1 - \bar{x}_j)x_i + (\bar{x}_i - 1)x_j \quad (1 \leq i < j \leq n).$$

Let $\tilde{Z}$ denote the $n \times k$ matrix of $\partial_2$ restricted to $\langle z_{12}, z_{13}, \ldots, z_{n-1,n} \rangle$, the submodule of $C_2(S)$ generated by $\{z_{ij} \mid 1 \leq i < j \leq n\}$. By examining the chain complex $C_*(P)$, it follows that $\ker(\partial_1(P)) \cong \ker(\partial_1(S)) \oplus \langle x_{n+1}, \ldots, x_l \rangle$ (we will henceforth identify $\ker \partial_1(P)$ as a submodule of $\langle x_1, \ldots, x_n \rangle \subset C_1(S)$) and that $\ker \partial_1(P)$ is generated by $\{N_ix_i \mid i = 1, \ldots, n\} \cup \{\partial_2z_{ij} \mid 1 \leq i < j \leq n\}$, where $N_i = \sum_{j=0}^{r-1} x_j \in Z[\pi]$. Note also that, since $R$ is a presentation of $\pi$ with the same generators as $S$, $\{\partial_2y_i \mid i = 1, \ldots, m\}$ generates $\ker \partial_1(S) = \ker \partial_1(R)$.

As in [4, §3], we use H. Jacobinski's theorem on the cancellation of projective $\pi$-modules (see [7], [11, Theorem 19.8], or [12, p. 178]) to choose a new basis $\{y'_1, \ldots, y'_m\} \cup \{z_{ij}\}$ for $C_2(S)$ such that the set $\{\partial_2y'_i \mid i = 1, \ldots, n, l + 1, \ldots, m\}$ generates $\ker \partial_1(P)$ and $\partial_2y'_j = x_j$ for $j = n + 1, \ldots, l$. The matrix for $\partial_2(S)$ with respect to the new basis for $C_2(S)$ and the original basis for $C_1(S)$ becomes

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Let \( \psi: F(b_1, \cdots, b_n) \to \pi \) be the surjection \( \phi_S \mid F(b_1, \cdots, b_n) \) and let \( \overline{R} = \ker \psi \). Since the homomorphism \( \rho: \overline{R} \to \ker \partial_1(P) \) is surjective, we can choose relators \( \{ r_1, \cdots, r_n, r_{l+1}, \cdots, r_m \} \subset \overline{R} \) such that

\[
\rho(r_i) = \sum_{j=1}^{n} \left( \partial^2_i \right)_{\partial b_j} \cdot x_j = \partial_2 y_i \quad (i = 1, \cdots, n, l + 1, \cdots, m).
\]

Here it is crucial that \( \partial_2 y_i \in \langle x_1, \cdots, x_n \rangle \) \( (i = 1, \cdots, n, l + 1, \cdots, m) \).

Claim. Each \( r_i \) can be written as

\[
r_i = b_1^{\eta_1} b_2^{\eta_2} \cdots b_n^{\eta_n} W_i \quad (i = 1, \cdots, n, l + 1, \cdots, m)
\]

where \( W_i \) has zero exponent sum on each \( b_j, j = 1, \cdots, n \), and \( W_i \in \overline{R} \cap \langle \overline{F}, \overline{F} \rangle \).

**Proof.** Abelianize \( \overline{F} = F(b_1, \cdots, b_n) \) and obtain the following commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & \pi \\
\downarrow A & & \downarrow \psi' \\
\overline{F} & \xrightarrow{\rho} & \overline{F} \cap \langle \overline{F}, \overline{F} \rangle
\end{array}
\]

where \( \overline{F}(b_1, \cdots, b_n) \) is the free abelian group of rank \( n \) generated by \( b_1, \cdots, b_n \). Since \( \psi(r_i) = 1 = \psi'(A(r_i)) = \psi'(b_1^{\eta_1} \cdots b_n^{\eta_n}) = x_1^{\eta_1} \cdots x_n^{\eta_n} \) it follows that each \( \eta_{ij} \) is divisible by order \( \eta_{ij} = \tau_j \) and \( r_i = b_1^{\eta_1} b_n^{\eta_n} W_i \), where \( W_i \in \ker A = \langle \overline{F}, \overline{F} \rangle \). Define \( \beta_{ij} = \eta_{ij} \tau_j \).

Claim. We may change part of the basis of \( C_2(S) \), say to \( \{ y_1'', \cdots, y''_n, y''_{l+1}, \cdots, y''_m \} \cup \{ z_{ij} \} \cup \{ y'_j, | n + 1 \leq j \leq l \} \), so that we may alter each \( r_i \) to \( r'_i = \prod_{j=1}^{n} b_j^{y''_j} \) and preserve \( \rho(r'_i) = \partial_2 y''_i \) for \( i = 1, \cdots, n, l + 1, \cdots, m \).

**Proof.** This follows because \( \rho([\overline{F}, \overline{F}]) \subset \ker \partial_1(P) \subset \langle x_1, \cdots, x_n \rangle \) is
generated by \( \{ \partial_2 z_{ij} \mid 1 \leq i < j \leq n \} \). Consider
\[
\rho(r_i) = \rho(b_1^{n_1} \cdots b_n^{n_m} W_i) = \rho(r'_i) + \rho(W_i).
\]
But
\[
\rho(W_i) = \sum_{1 \leq i < k \leq n} \delta_{ijk} \partial_2 z_{jk} \quad (\delta_{ijk} \in \mathbb{Z}[[t]])
\]
Let
\[
y''_i = y'_i - \sum_{1 \leq i < k \leq n} \delta_{ijk} z_{jk} \quad (i = 1, \cdots, n, l + 1, \cdots, m).
\]
Clearly \( \partial_2 y''_i = \rho(r'_i) \) and \( \{ y''_1, \cdots, y''_n, y''_{l+1}, \cdots, y''_m \} \cup \{ z_{ij} \} \cup \{ y''_{n+1}, \cdots, y''_{m} \} \) is a basis for \( C_2(S) \).
Thus \( \partial_2 (y''_i) = \rho(r''_i) \) and \( \{ y''_1, \cdots, y''_n, y''_{l+1}, \cdots, y''_m \} \cup \{ z_{ij} \} \cup \{ y''_{n+1}, \cdots, y''_{m} \} \) generates \( \ker \partial_t(P) \). Thus for each \( s = 1, \cdots, n \)
\[
N_s x_s = \sum_{i=1, l+1}^{n, m} \alpha_{si} \rho(r'_i) + \sum_{1 \leq i < j \leq n} r_{sij} \partial_2 z_{ij} \quad (\alpha_{si}, r_{sij} \in \mathbb{Z}[[t]])
\]
Denoting the second term by \( T_s (T_s \in \rho([F, F])) \) we have
\[
N_s x_s = \sum_i \alpha_{si} \left( \sum_j \beta_{ij} N_j x_j \right) + T_s = \sum_j \left( \sum_i \alpha_{si} \beta_{ij} \right) N_j x_j + T_s.
\]
By augmenting the above equation, and observing that \( e(T_s) = 0 \) and \( e(N_s) = \tau_j \), we have \( (\Sigma_i e(\alpha_{si}) \beta_{ij}) \tau_j x_j = \delta_{sj} \tau_j x_s \). Thus we deduce
\[
\sum_{i=1, l+1}^{n, m} e(\alpha_{si}) \beta_{ij} = \delta_{sj}, \quad \left\{ \begin{array}{l} s = 1, \cdots, n \\ j = 1, \cdots, n \end{array} \right\}
\]
The above argument shows we can choose \( \alpha_{si} \in \mathbb{Z} \) (let \( \alpha_{si} = \sum e(\alpha_{si}) \)) such that
\[
N_s x_s = \sum_{i=1, l+1}^{n, m} \alpha_{si} \rho(r'_i) \quad (s = 1, \cdots, n).
\]
Let \( p = m + n - l \), the number of basis elements in the set \( \{ y''_i \} \). Let \( (\alpha_{si}) = A \) and \( (\beta_{ij}) = B \) denote respectively the \( n \times p \) and \( p \times n \) matrices with integer coefficients. Relation (4.1) implies that
\[
AB = I_n
\]
where \( I_n \) is the identity \( n \times n \) matrix. Using (4.3), an easy argument on free abelian groups shows that there exists a nonsingular \( p \times p \) matrix \( C \) with integer coefficients such that
\[
CB = (I_n | 0) \quad (n \times p \text{ matrix}).
\]
Apply the matrix \( C \) to the partial basis \( \{ y''_i \mid i = 1, \cdots, n, l + 1, \cdots, m \} \)
of $C_2(S)$ to obtain a new basis $\{w_i | i = 1, \ldots, n, l + 1, \ldots, m\} \cup \{z_{ij}\} \cup \{y'_j | n + 1 \leq j \leq l\}$ for $C_2$. Then

$$\partial_2(w_i) = \partial_2 \sum_j c_{ij} y_j'' = \sum_j c_{ij} \partial_2 y_j'' = \sum_s \left(\sum_j c_{ij} \beta_{js}\right) N_s x_s$$

by (4.4). The matrix of $\partial_2$ with respect to this new basis for $C_2(S)$ and the old basis for $C_1(S)$ is

$$\begin{pmatrix}
N_1 & 0 & \cdots & 0 \\
0 & N_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & N_n \\
\end{pmatrix}$$

The chain complex with this new preferred basis for $C_2$ can clearly be realized by the presentation

$$V = \langle b_1, \ldots, b_l : b_1^r, \ldots, b_m^r, b_{n+1}, \ldots, b_l, 1, \ldots, 1, \{[b_i, b_j] | 1 \leq i < j \leq n\} \rangle.$$

The cellular model $V$ has the same simple homotopy type as $P \vee (\vee_{i=1}^{m-l} S^2)$. The chain complexes $C_*(S)$ and $C_*(V)$ differ only by a change of basis in $C_2$. Proposition 4 of [4] shows that there is a homotopy equivalence $f : V \to S$ which is the identity on the 1-skeleton and such that the matrix of $f_2# : C_2(V) \to C_2(S)$ is exactly the matrix $N$ recording the basis change in $C_2$. Furthermore, this matrix $N$ represents the Whitehead torsion $\tau(f) \in \text{Wh}(\pi_1 V)$ of the equivalence $f$. This completes the proof of Theorem A. \hfill $\square$

5. Proof of Theorem B. In this section we will show that for a finite group $\pi \neq Z_2$, the number of homotopy types of $\pi$-complexes with a given Euler characteristic $\chi$ is less than or equal to $\varphi(|\pi|)/2$, provided $\chi \geq \chi_{\pi} + 1$.

**Definition.** Let $M$ be a finitely generated $\pi$-module. $M$ has the **cancellation property** if for any finitely generated $\pi$-module $N$ such that $M \oplus (\mathbb{Z} \pi)^i \cong N \oplus (\mathbb{Z} \pi)^j$, we have $N \cong M \oplus (\mathbb{Z} \pi)^{i-j}$.
The following lemma was shown to us by R. G. Swan [12].

**Lemma.** Let \( X \) be any \( \pi \)-complex. Then \( \pi_2(X) \oplus \mathbb{Z}\pi \) has the cancellation property.

**Proof.** We will show that \( \pi_2(X) \oplus \mathbb{Z}\pi \) satisfies the Eichler condition. That \( \pi_2(X) \oplus \mathbb{Z}\pi \) has the cancellation property follows from the theorem of H. Jacobinski ([7], [12, p. 178]). A finitely generated, torsion free \( \pi \)-module \( M \) satisfies the Eichler condition \( \iff \) the algebra \( \text{End}_{\mathbb{Q}\pi}(Q \otimes M) \) has no totally definite quaternion algebra as a direct summand (see [7] for a definition).

Consider the cellular chain complex \( C_*(\tilde{X}) \) of the universal cover \( \tilde{X} \) of \( X \). This gives an exact sequence of \( \pi \)-modules

\[
0 \rightarrow \pi_2(X) \rightarrow (\mathbb{Z}\pi)^r \rightarrow (\mathbb{Z}\pi)^s \rightarrow \mathbb{Z} \rightarrow 0.
\]

Tensoring with \( \mathbb{Q} \), the rationals. The resulting sequence splits and gives

\[
\pi_2(X) \otimes \mathbb{Q} \cong (\mathbb{Q}I)^{r+1} \oplus \mathbb{Q}^n
\]

where \( n = r - s \) and \( I \) is the augmentation ideal. Therefore \( \mathbb{Q} \otimes (\pi_2(X) \oplus \mathbb{Z}\pi) \cong (\mathbb{Q}I)^{n+2} \oplus \mathbb{Q}^{n+1} \) and

\[
\text{End}_{\mathbb{Q}\pi}(Q \otimes (\pi_2(X) \oplus \mathbb{Z}\pi)) \cong M_{n+2}(\text{End}_{\mathbb{Q}\pi}QI) \times M_{n+1}(\mathbb{Q}).
\]

Since \( n \geq 0 \), no totally definite quaternion algebras occur. \( \square \)

We appeal to the theory of 2-types (see [10]) and the cancellation theorem above. Let \( X \) be any \( \pi \)-complex with \( \chi(X) \geq \chi_\pi + 1 \). By a theorem of J. H. C. Whitehead [13],

\[
\pi_2(X) \oplus (\mathbb{Z}[\pi])^m \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^n
\]

where \( Y \) is a \( \pi \)-complex with \( \chi(Y) = \chi_\pi \), and \( n \geq m + 1 \). The cancellation theorem above guarantees that

\[
\pi_2(X) \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^{n-m}
\]

where \( n - m = \chi(X) - \chi_\pi \). Thus \( \pi \)-complexes with fixed Euler characteristic \( \chi \geq \chi_\pi + 1 \) have the same second homotopy module

\[
\pi_2 \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^{\chi-\chi_\pi}.
\]

We conclude that their algebraic 2-types \((\pi, \pi_2, k)\) differ only by the obstruction invariant \( k \in H^3(\pi, \pi_2) \cong Z_{\text{int}} \). But each \( k \in H^3(\pi, \pi_2) \) which is the obstruction invariant for a \( \pi \)-complex must be a generator of \( H^3(\pi, \pi_2) \) (see [3]). There are exactly \( \varphi(|\pi|) \) such generators. The sign changing automorphism

\[
\lambda: \pi_2 \rightarrow \pi_2 \quad (\lambda(x) = -x, x \in \pi_2)
\]

together with \( \text{id}: \pi \rightarrow \pi \) gives an isomorphism of the 2-types

\[
\lambda: \pi_2 \rightarrow \pi_2 \quad (\lambda(x) = -x, x \in \pi_2)
\]
and shows that the number of $k$-invariants representing distinct homotopy types of $\pi$-complexes with Euler characteristic $\chi$ is less than or equal to $\varphi(|\pi|)/2$, since $\pi \neq \mathbb{Z}_2$.

BIBLIOGRAPHY