TREES OF HOMOTOPY TYPES OF
2-DIMENSIONAL CW COMPLEXES. II

BY

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ABSTRACT. A \( \pi \)-complex is a finite, connected 2-dimensional CW complex with fundamental group \( \pi \). The tree \( HT(\pi) \) of homotopy types of \( \pi \)-complexes has width \( \leq N \) if there is a root \( Y \) of the tree such that, for any \( \pi \)-complex \( X \), \( X \vee \bigvee_{i=1}^{N} S^2 \) lies on the stalk generated by \( Y \). Let \( \pi \) be a finite abelian group with torsion coefficients \( \tau_1, \ldots, \tau_n \). The main theorem of this paper asserts that width \( HT(\pi) \leq n(n-1)/2 \). This generalizes the results of [4].

1. Introduction. Let \( \pi \) be a finitely presentable group. A \( \pi \)-complex is a finite connected 2-dimensional CW complex with fundamental group \( \pi \). In [4], we gave a complete classification of the homotopy and simple homotopy types of \( \mathbb{Z}_n \)-complexes, where \( \mathbb{Z}_n \) is the finite cyclic group of order \( n \). In general, we may describe the set of (simple) homotopy types of \( \pi \)-complexes \( (S)HT(\pi) \) as a directed tree—a directed, connected graph which has no circuits. A vertex of \( (S)HT(\pi) \) is the (simple) homotopy type \([X]\) of a \( \pi \)-complex \( X \). The vertices represented by \( X \) and \( Y \) are joined by an edge directed from \([X]\) to \([Y]\) if and only if \( Y \cong_X X \vee S^2 \). A \( \pi \)-complex is called a root if \([X]\) possesses no predecessor; the stalk generated by \( X \) is the linearly ordered subgraph of \( (S)HT(\pi) \) determined by the (simple) homotopy types of \( X, X \vee S^2, X \vee S^2 \vee S^2, \ldots \).

The main theorem of [4] states that \((S)HT(\mathbb{Z}_n)\) is a single stalk generated by the pseudo projective plane \( P_n = S^1 \cup_n e^2 \). We say that the width of \((S)HT(\pi) \leq n \) if there is a root \( X \) such that, for any \( \pi \)-complex \( Y \), \( Y \vee \bigvee_{i=1}^{n} S^2 \) is on the stalk generated by \( X \).

It is known by the simple homotopy theory of J. H. C. Whitehead [14] that given any \( \pi \)-complex \( Y \) and any root \( X \) there is an integer \( m(Y) \) such that \( Y \vee \bigvee_{i=1}^{m(Y)} S^2 \) is on the stalk generated by \( X \). Width\((S)HT(\pi) \leq n \) indicates that there is a root \( X \) such that \( m(Y) \) can be chosen \( \leq n \) for any \( \pi \)-complex \( Y \).

Theorem A. Let \( \pi \) be a finite abelian group, \( n = n(\pi) = \) the number of

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torsion coefficients of $\pi$, and $k = k(\pi) = n(n - 1)/2$. Then the width $HT(\pi) \leq k(\pi)$.

If $p$ is any positive integer, Theorem A implies that width $HT(Z_p)$ is zero, which is the result of [4]. If $\pi = Z_p \times Z_q$, where $p$ divides $q$, then width $HT(\pi) \leq 1$. In this case, the homotopy tree of $Z_p \times Z_q$-complexes looks at worst like:

As a corollary to A, we obtain a theorem on the cancellation of "large" sums of 2-spheres with $\pi$-complexes. If $\pi$ is a finite abelian group and $X$, $Y$ are $\pi$-complexes, then $X \vee (\vee_{i=1}^s S^2_i) \simeq Y \vee (\vee_{i=1}^t S^2_i)$ and $s \geq t \geq k(\pi)$ imply that $\vee_{i=1}^{t-k(\pi)} S^2_i$ can be cancelled from each side (up to homotopy type).

For a given finite group $\pi$ let $\chi_{\pi} = \min \{\chi(X) | X \text{ is a } \pi\text{-complex} \}$, $|\pi|$ be the order of $\pi$, and $\varphi$ be the Euler $\varphi$-function.

**Theorem B.** Let $\pi$ be a finite group other than $Z_2$. The number of homotopy types of $\pi$-complexes with fixed Euler characteristic $\chi \geq \chi_{\pi} + 1$ is less than or equal to $\varphi(|\pi|)/2$.

**Examples.** (a) If $\pi = Z_2 \times Z_2$, then Theorems A and B imply that the tree of (simple) homotopy types looks at worst like:
where $X$ is the complex modeled on $(a, b: a^2, b^2, [a, b])$.

(b) If $\pi = \Sigma_3$, the group (of order 6) of permutations on 3 letters, then
$HT(\Sigma_3)$ looks at worst like the above tree, where $X$ is a root of $HT(\Sigma_3)$ of
minimal Euler characteristic. The complex $X$ modeled on the presentation \{a, b:
$b^2, bab = a^2$\} is such a root, since $H_2X = 0$ [16].

2. The chain functor. In [4], we associated with each finite presentation
$P = (a_1, \ldots, a_n : r_1, \ldots, r_m)$ of a group $\pi$, its cellular model

$$ P = \left( \bigvee_{i=1}^{n} S_i^1 \right) \cup_r \left( \bigvee_{j=1}^{m} B_j^2 \right), $$

which has a single 0-cell, one 1-cell for each generator of $P$, and one 2-cell for
each relator of $P$. The $j$th 2-cell is attached to the 1-skeleton $\bigvee_{i=1}^{n} S_i^1$ according
to the instructions provided by the $j$th relator $r_j$.

Then we associated with the cellular model $P$ the cellular chain complex
$C_*(\widetilde{P})$ of its universal covering $\widetilde{P}$. $C_*(\widetilde{P})$ is a chain complex of free $\pi$-modules
with preferred bases

$$(*) \quad C: C_2(y_1, \ldots, y_m) \xrightarrow{\partial_2} C_1(x_1, \ldots, x_n) \xrightarrow{\partial_1} C_0 = Z[\pi] \xrightarrow{\epsilon} Z \to 0$$
in which

(a) $\epsilon$ is the augmentation homomorphism $Z[\pi] \to Z[1]$ induced by $\pi \to 1$.

(b) Exactness holds at $C_1, C_0, Z$. 

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(c) \(\{y_1, \ldots, y_m\}\) and \(\{x_1, \ldots, x_n\}\) are the preferred bases for \(C_2\) and \(C_1\).

We can combine these two processes \(\mathcal{P} \rightarrow P\) and \(P \rightarrow C_*(\mathcal{P})\) as follows. If \(\mathcal{P} = (a_1, \ldots, a_n : r_1, \ldots, r_m)\) is a presentation for \(\pi\), let

\[
1 \rightarrow R_p \rightarrow F(a_1, \ldots, a_n) \xrightarrow{\varphi_p} \pi \rightarrow 1
\]

be the short exact sequence in which \(F = F(a_1, \ldots, a_n)\) is the free group of rank \(n\) on generators \(\{a_1, \ldots, a_n\}\) and \(R_p\) is the normal closure of the relators \(\{r_1, \ldots, r_m\}\). The elements \(\bar{x}_i = \varphi_p(a_i) (1 \leq i \leq n)\) serve as a set of generators for \(\pi\). We associate a chain complex \(C_*(\mathcal{P})\) as follows. Let \(C_2(\mathcal{P}) = C_2(y_1, \ldots, y_m)\) and \(C_1(\mathcal{P}) = C_1(x_1, \ldots, x_n)\) be free \(\pi\)-modules with preferred bases \(\{y_1, \ldots, y_m\}\) and \(\{x_1, \ldots, x_n\}\) in 1-1 correspondence with the relators and generators of \(\mathcal{P}\), respectively. Let \(C_0(\mathcal{P})\) be the integral group ring \(\mathbb{Z}[\pi]\). Then \(C_*(\mathcal{P})\) is the chain complex

\[
C_*(\mathcal{P}): C_2(y_1, \ldots, y_m) \xrightarrow{\partial_2(\mathcal{P})} C_1(x_1, \ldots, x_n) \xrightarrow{\partial_1(\mathcal{P})} \mathbb{Z}[\pi] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0
\]

whose boundary operators have the following matrix representations with respect to the preferred bases:

\[
\partial_1(\mathcal{P}) = (\bar{x}_1 - 1, \ldots, \bar{x}_n - 1) \quad \text{and} \quad \partial_2(\mathcal{P}) = (Z[\varphi_p](\partial r_i/\partial a_i))
\]

where \(\partial/\partial a_i: Z[F] \rightarrow Z[F]\) is the derivative with respect to \(a_i\) in the free calculus of R. H. Fox [5] and \(Z[\varphi_p]: Z[F] \rightarrow Z[\pi]\) is induced by \(\varphi_p: F \rightarrow \pi\).

For example, let \(\mathcal{P} = (a_1, a_2 : a_1 a_2 a_1^{-1} a_2^{-1})\) be a presentation for \(\pi = \mathbb{Z} \times \mathbb{Z}\) under the correspondence \(\varphi_p(a_1) = \bar{x}_1 = (1, 0)\) and \(\varphi_p(a_2) = \bar{x}_2 = (0, 1)\). Then the associated chain complex \(C_*(\mathcal{P})\) takes the form

\[
\begin{align*}
C_2(y_1) \xrightarrow{(\bar{x}_1 - 1)} C_1(x_1, x_2) \xrightarrow{(\bar{x}_1 - 1, \bar{x}_2 - 1)} \mathbb{Z}[\mathbb{Z} \times \mathbb{Z}] \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.
\end{align*}
\]

**Definition.** We say that a chain complex \(C\) as in (*) above is realized by a presentation \(\mathcal{P}\) of \(\pi\) if \(C_*(\mathcal{P}) = C\).

3. The homomorphism \(\rho\). Given a finite presentation \(\mathcal{P} = (a_1, \ldots, a_n : r_1, \ldots, r_m)\) of \(\pi\) there is a surjective group homomorphism \(\rho\) from the relator subgroup \(R_p\) onto the free abelian group \(\ker \partial_1(\mathcal{P})\) (a \(\pi\)-module also) \(\subset C_1(\mathcal{P})\) which has kernel \([R_p, R_p]\).

Following J. H. C. Whitehead in [13] we define the **crossed homomorphism**

\[
\overline{\rho}: F(a_1, \ldots, a_n) \rightarrow C_1(\mathcal{P}) \equiv C_1(x_1, \ldots, x_n),
\]

where \(F\) is the free group of rank \(n\), by

(a) \(\overline{\rho}(a_i) = x_i\),

(b) \(\overline{\rho}(a_i^{-1}) = -\varphi_p(a_i^{-1})x_i\).
(c) if \( W_1, W_2 \) are any words in \( F \), then \( \overline{\rho}(W_1 \cdot W_2) = \overline{\rho}(W_1) + \varphi_p(W_1) \cdot \overline{\rho}(W_2) \).

Recall that \( \varphi_p : F \to \pi \) is the surjection given by the presentation \( P \). Note that by (c), \( \overline{\rho}|_{\Gamma_p} = \rho \) is a homomorphism. Also, if \( r \in \Gamma_p \), then

\[
\rho(r) = \sum_{i=1}^{n} \left( \frac{\partial \varphi_p}{\partial a_i} \right) x_i.
\]

**Lemma.** The following sequence is exact:

\[
1 \to [\Gamma_p, \Gamma_p] \to \Gamma_p \to \ker \partial_1(\Gamma) \to 0.
\]

**Proof.** This is really a restatement of Theorem 8 of [13]. Part (a) of Theorem 8 says that \( \rho(\Gamma_p) = \ker \partial_1(\Gamma) \). Part (b) says that \( \ker \rho = \ker \overline{\rho} \) is the image of the commutator subgroup of \( \pi_2(P, P^{(1)}) \) in \( \Gamma_p \subset \pi_1(P^{(1)}) (= F(a_1, \cdots, a_n)) \) under the boundary operator \( \partial : \pi_2(P, P^{(1)}) \to \pi_1(P^{(1)}) \). Since \( \im \partial = \Gamma_p \), \( \ker \rho \subset [\Gamma_p, \Gamma_p] \). But \( \ker \rho \supset [\Gamma_p, \Gamma_p] \) follows because \( \ker \partial_1(\Gamma) \) is abelian as a group. \( \square \)

4. Proof of Theorem A. Let \( n = n(\pi) \) be the number of torsion coefficients of the finite abelian group \( \pi \). Let \( \{\tau_1, \cdots, \tau_n\} \) be the torsion coefficients of \( \pi \), where \( \tau_i |_{i+1} \) for \( i = 1, 2, \cdots, n-1 \), and \( k = k(\pi) = n(n-1)/2 \). Furthermore, let \( P \) be the \( \pi \)-complex modeled on the standard presentation

\[
P = (a_1, \cdots, a_n : a_1^{\tau_1}, \cdots, a_n^{\tau_n}, \left\{ [a_i, a_j] : 1 \leq i < j \leq n \right\}).
\]

Note that \( k(\pi) \) is the number of commutators in \( P \) and that \( P \) is a root of \( (S)\mathrm{HT}(\pi) \) (see [15]). We will show that if \( X \) is any \( \pi \)-complex, then \( X \vee (\bigvee_{i=1}^{k(\pi)} S_i^2) \) is on the stalk generated by \( P \); i.e.,

\[
X \vee \left( \bigvee_{i=1}^{k(\pi)} S_i^2 \right) \simeq P \vee \left( \bigvee_{i=1}^{\dim(X)} S_i^2 \right)
\]

where \( \dim(X) = \rank H_2(X) \).

The given \( \pi \)-complex \( X \) has the simple homotopy type of a \( \pi \)-complex \( R \) modeled on the “pre-abelian” presentation

\[
R = (b_1, \cdots, b_l : b_1^{\tau_1} W_1, b_2^{\tau_2} W_2, \cdots, b_l^{\tau_l} W_l, b_{l+1} W_{l+1}, \cdots, b_m W_m)
\]

where each \( W_i \) has zero exponent sum on each \( b_j \) (\( j = 1, \cdots, l \)). [4, Proposition 3]. Notice that \( R \vee (\bigvee_{i=1}^{k(\pi)} S_i^2) \) has the simple homotopy type of the \( \pi \)-complex \( S \) modeled on the presentation

\[
S = (b_1, \cdots, b_l : b_1^{\tau_1} W_1, \cdots, b_l^{\tau_l} W_l, b_{l+1} W_{l+1}, \cdots, b_m W_m ; \left\{ [b_i, b_j] : 1 \leq i < j \leq n \right\}).
\]
Observe that in passing from \( R \rightarrow S \) we have added only those commutators corresponding to the nontrivial generators \( b_1, \ldots, b_n \).

Let
\[
1 \rightarrow R_s \rightarrow F(b_1, \ldots, b_i) \xrightarrow{\varphi_S} \pi \rightarrow 1
\]
be the short exact sequence of groups and homomorphism determined by \( S \). Denote \( \varphi_S(b_i) \) by \( x_i \) \( (i = 1, \ldots, n) \) and note that, since \( \pi \) is abelian, \( \varphi_S(b_i) = 1 \) \((n + 1 \leq i \leq l)\). The chain complex \( C_*(S) \) is given as follows:

\[
\begin{array}{ccc}
C_2(S) & \xrightarrow{\partial_2(S)} & C_1(S) \\
C_*(S): C_2(y_1, \cdots, y_m; z_{12}, z_{13}, \cdots, z_{n-1,n}) & \xrightarrow{\partial_2(S)} & C_1(x_1, \cdots, x_l) \\
& \xrightarrow{\partial_1(S)} & Z[\pi] \xrightarrow{\varepsilon} Z \rightarrow 0 \\
& \xrightarrow{(x_1 - 1, \cdots, x_n - 1, 0, \cdots, 0)} & \\
\end{array}
\]

where \( \{z_{ij} | 1 \leq i < j \leq n \} \) corresponds to the set of special relators \( \{[b_i, b_j] | 1 \leq i < j \leq n \} \). Thus
\[
\partial_2(z_{ij}) = (1 - x_j)x_i + (\overline{x}_i - 1)x_j \quad (1 \leq i < j \leq n).
\]

Let \( \widetilde{Z} \) denote the \( n \times k \) matrix of \( \partial_2 \) restricted to \( \langle z_{12}, z_{13}, \cdots, z_{n-1,n} \rangle \), the submodule of \( C_2(S) \) generated by \( \{z_{ij} | 1 \leq i < j \leq n \} \).

By examining the chain complex \( C_*(P) \), it follows that \( \ker(\partial_1(P)) \cong \ker(\partial_1(S)) \oplus \langle x_{n+1}, \cdots, x_l \rangle \) (we will henceforth identify \( \ker \partial_1(P) \) as a submodule of \( \langle x_1, \cdots, x_n \rangle \subset C_1(S) \)) and that \( \ker(\partial_1(P)) \) is generated by \( \{N_i x_i | i = 1, \cdots, n \} \cup \{\partial_2 z_{ij} | 1 \leq i < j \leq n \} \), where \( N_i = \Sigma_{j=0}^{l-1} x_j \in Z[\pi] \). Note also that, since \( R \) is a presentation of \( \pi \) with the same generators as \( S \), \( \{\partial_2 y_i | i = 1, \cdots, m \} \) generates \( \ker \partial_1(S) = \ker \partial_1(R) \).

As in [4, §3], we use H. Jacobinski's theorem on the cancellation of projective \( \pi \)-modules (see [7], [11, Theorem 19.8], or [12, p. 178]) to choose a new basis \( \{y'_1, \cdots, y'_m \} \cup \{z_{ij} \} \) for \( C_2(S) \) such that the set \( \{\partial_2 y'_i | i = 1, \cdots, n, l + 1, \cdots, m \} \) generates \( \ker \partial_1(P) \) and \( \partial_2 y'_j = x_j \) for \( j = n + 1, \cdots, l \). The matrix for \( \partial_2(S) \) with respect to the new basis for \( C_2(S) \) and the original basis for \( C_1(S) \) becomes
Let \( \psi: \overline{F}(b_1, \cdots, b_n) \to \pi \) be the surjection \( \varphi_S|F(b_1, \cdots, b_n) \) and let \( \overline{R} = \ker \psi \). Since the homomorphism \( \rho: \overline{R} \to \ker \partial_1(P) \) is surjective, we can choose relators \( \{ r_1, \cdots, r_n, r_{l+1}, \cdots, r_m \} \subset \overline{R} \) such that

\[
\rho(r_i) = \sum_{j=1}^{n} \left( Z[\psi]\left( \frac{\partial r_i}{\partial b_j} \right) \right) x_j = \partial_2 y_i \quad (i = 1, \cdots, n, l+1, \cdots, m).
\]

Here it is crucial that \( \partial_2 y_i \in \langle x_1, \cdots, x_n \rangle \) (\( i = 1, \cdots, n, l+1, \cdots, m \)).

**Claim.** Each \( r_i \) can be written as

\[
r_i = b_1^{n_1 r_1} b_2^{n_2 r_2} \cdots b_n^{n_n r_n} W_i \quad (i = 1, \cdots, n, l+1, \cdots, m)
\]

where \( W_i \) has zero exponent sum on each \( b_j, j = 1, \cdots, n, \) and \( W_i \in \overline{R} \cap [\overline{F}, \overline{F}] \).

**Proof.** Abelianize \( \overline{F} = \overline{F}(b_1, \cdots, b_n) \) and obtain the following commutative diagram:

\[
\begin{array}{ccc}
\overline{F} & \xrightarrow{\psi} & \pi \\
\downarrow A & & \downarrow \psi' \\
\overline{FA}(b_1, \cdots, \overline{b}_n) & &
\end{array}
\]

where \( \overline{FA}(\overline{b}_1, \cdots, \overline{b}_n) \) is the free abelian group of rank \( n \) generated by \( \overline{b}_1, \cdots, \overline{b}_n \) \( (A(b_j) = \overline{b}_1) \). Since \( \psi(r_i) = 1 = \psi'(A(r_i)) = \psi'(\overline{b}_1^{|r_i|} \cdots \overline{b}_n^{|r_n|}) = \overline{x}_1^{|r_1|} \cdots \overline{x}_n^{|r_n|} \) it follows that each \( n_{ij} \) is divisible by order \( \overline{x}_j = r_j \) and \( r_i = b_1^{n_1} b_2^{n_2} \cdots b_n^{n_n} W_i \), where \( W_i \in \ker A = [\overline{F}, \overline{F}] \). Define \( \beta_{ij} = \eta_{ij} r_j \).

**Claim.** We may change part of the basis of \( C_2(S) \), say to \( \{ y''_1, \cdots, y''_n, y''_{l+1}, \cdots, y''_m \} \cup \{ z_i \} \cup \{ y'_i | n+1 < j < l \} \), so that we may alter each \( r_i \) to \( r'_i = \prod_{j=1}^{n} b_j^{\beta_{ij} y''_j} \) and preserve \( \rho(r'_i) = \partial_2 y''_i \) for \( i = 1, \cdots, n, l+1, \cdots, m \).

**Proof.** This follows because \( \rho(\overline{F}, \overline{F}) \subset \ker \partial_1(P) \subset \langle x_1, \cdots, x_n \rangle \) is
generated by \( \{ \partial_2 z_{ij} | 1 \leq i < j \leq n \} \). Consider 
\[
\rho(r_0) = \rho(b_0^{n_1} \cdots b_0^{n_m} w_i) = \rho(r'_0) + \rho(w_i). 
\]
But 
\[
\rho(w_i) = \sum_{1 \leq j < k \leq n} \delta_{ijk} \partial_2 z_{jk} \quad (\delta_{ijk} \in \mathbb{Z}[n]).
\]
Let 
\[
y''_i = y'_i - \sum_{1 \leq j < k \leq n} \delta_{ijk} z_{jk} \quad (i = 1, \cdots, n, l + 1, \cdots, m).
\]
Clearly \( \partial_2 y''_i = \rho(r'_i) \) and \( \{ y''_1, \cdots, y''_n, y''_{l+1}, \cdots, y''_m \} \cup \{ z_{ij} \} \cup \{ y'_{n+1}, \cdots, y'_m \} \) is a basis for \( C_2(S) \).

Thus \( \partial_2 (y''_i) = \rho(r''_i) = \sum_{j=1}^n \beta_{ij} N_j x_j \), where \( \beta_{ij} \in \mathbb{Z} \) \((i = 1, \cdots, n, l + 1, \cdots, m) \). Again notice that \( \{ \partial_2 (y''_i) \mid i = 1, \cdots, n, l + 1, \cdots, m \} \cup \{ \partial_2 z_{ij} \mid 1 \leq i < j \leq n \} \) generates \( \ker \partial_2(P) \). Thus for each \( s = 1, \cdots, n \)
\[
N_s x_s = \sum_{i=1, l+1}^{n, m} \alpha_{si} \rho(r'_i) + \sum_{1 \leq i < j \leq n} r_{sij} \partial_2 z_{ij} \quad (\alpha_{si}, r_{sij} \in \mathbb{Z}[n]).
\]
Denoting the second term by \( T_s \) \((T_s \in \rho(\{ F, F \})) \) we have
\[
N_s x_s = \sum_i \alpha_{si} \left( \sum_j \beta_{ij} N_j x_j \right) + T_s = \sum_j \left( \sum_i \alpha_{si} \beta_{ij} \right) N_j x_j + T_s.
\]
By augmenting the above equation, and observing that \( e(T_s) = 0 \) and \( e(N_s) = \tau_j \), we have \( (\sum_i e(\alpha_{si}) \beta_{ij} \tau_j x_j = \delta_{sj} \tau_s x_s \). Thus we deduce
\[
\sum_{i=1, l+1}^{n, m} e(\alpha_{si}) \beta_{ij} = \delta_{sj}, \quad \begin{cases} s = 1, \cdots, n, \\ j = 1, \cdots, n \end{cases}
\]
The above argument shows we can choose \( \alpha_{si} \in \mathbb{Z} \) \((\alpha_{si} \lneq e(\alpha_{si})) \) such that
\[
N_s x_s = \sum_{i=1, l+1}^{n, m} \alpha_{si} \rho(r'_i) \quad (s = 1, \cdots, n).
\]
Let \( p = m + n - l \), the number of basis elements in the set \( \{ y''_i \} \). Let \( \alpha_{si} = A \) and \( \beta_{ij} = B \) denote respectively the \( n \times p \) and \( p \times n \) matrices with integer coefficients. Relation (4.1) implies that
\[
AB = I_n
\]
where \( I_n \) is the identity \( n \times n \) matrix. Using (4.3), an easy argument on free abelian groups shows that there exists a nonsingular \( p \times p \) matrix \( C \) with integer coefficients such that
\[
CB = (I_n | 0) \quad (n \times p \text{ matrix}).
\]
Apply the matrix \( C \) to the partial basis \( \{ y''_i \mid i = 1, \cdots, n, l + 1, \cdots, m \} \)
of $C_2(S)$ to obtain a new basis $\{w_i | i = 1, \cdots, n, l + 1, \cdots, m\} \cup \{z_{ij}\} \cup \{y_j' | n + 1 \leq j \leq l\}$ for $C_2$. Then

$$\partial_2(w_i) = \partial_2 \sum_j c_{ij}y_j'' = \sum_j c_{ij}\partial_2 y_j'' = \sum_j (\sum_i c_{ij}\beta_{js}) N_s x_s$$

$$= \begin{cases} N_i x_i & \text{if } i = 1, \cdots, n, \\ 0 & \text{if } i = l + 1, \cdots, m \end{cases}$$

by (4.4). The matrix of $\partial_2$ with respect to this new basis for $C_2(S)$ and the old basis for $C_1(S)$ is

$$
\begin{pmatrix}
N_1 & 0 & \cdots & 0 \\
0 & N_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & N_n \\
0 & l_{(l-n)} & \cdots & 0 \\
\end{pmatrix}
$$

The chain complex with this new preferred basis for $C_2$ can clearly be realized by the presentation

$$V = \langle b_1, \cdots, b_l : b_1^{r_1}, \cdots, b_n^{r_n}, b_{n+1}, \cdots, b_l, 1, \cdots, 1, ([b_i, b_j] | 1 \leq i < j \leq n) \rangle.$$
The following lemma was shown to us by R. G. Swan [12].

**Lemma.** Let $X$ be any $\pi$-complex. Then $\pi_2(X) \oplus \mathbb{Z}\pi$ has the cancellation property.

**Proof.** We will show that $\pi_2(X) \oplus \mathbb{Z}\pi$ satisfies the Eichler condition. That $\pi_2(X) \oplus \mathbb{Z}\pi$ has the cancellation property follows from the theorem of H. Jacobinski ([7], [12, p. 178]). A finitely generated, torsion free $\pi$-module $M$ satisfies the Eichler condition $\iff$ the algebra $\text{End}_{\mathbb{Q}\pi}(Q \otimes M)$ has no totally definite quaternion algebra as a direct summand (see [7] for a definition).

Consider the cellular chain complex $C_*(\tilde{X})$ of the universal cover $\tilde{X}$ of $X$. This gives an exact sequence of $\pi$-modules

$$0 \to \pi_2(X) \to (\mathbb{Z}\pi)^r \to (\mathbb{Z}\pi)^s \to \mathbb{Z} \to 0.$$ 

Tensoring with $Q$, the rationals. The resulting sequence splits and gives

$$\pi_2(X) \otimes \mathbb{Q} \cong (Q/I)^{n+1} \oplus Q^n$$

where $n = r - s$ and $I$ is the augmentation ideal. Therefore $\mathbb{Q} \otimes (\pi_2(X) \oplus \mathbb{Z}\pi) \cong (Q/I)^{n+2} \oplus Q^{n+1}$ and

$$\text{End}_{\mathbb{Q}\pi}(Q \otimes (\pi_2(X) \oplus \mathbb{Z}\pi)) \cong M_{n+2}(\text{End}_{\mathbb{Q}\pi}(Q/I)) \times M_{n+1}(Q).$$

Since $n \geq 0$, no totally definite quaternion algebras occur. $\square$

We appeal to the theory of 2-types (see [10]) and the cancellation theorem above. Let $X$ be any $\pi$-complex with $\chi(X) \geq \chi_{\pi} + 1$. By a theorem of J. H. C. Whitehead [13],

$$\pi_2(X) \oplus (\mathbb{Z}[\pi])^m \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^n$$

where $Y$ is a $\pi$-complex with $\chi(Y) = \chi_{\pi}$, and $n \geq m + 1$. The cancellation theorem above guarantees that

$$\pi_2(X) \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^{n-m}$$

where $n - m = \chi(X) - \chi_{\pi}$. Thus $\pi$-complexes with fixed Euler characteristic $\chi \geq \chi_{\pi} + 1$ have the same second homotopy module

$$\pi_2 \cong \pi_2(Y) \oplus (\mathbb{Z}[\pi])^{\chi - \chi_{\pi}}.$$ 

We conclude that their algebraic 2-types $(\pi, \pi_2, k)$ differ only by the obstruction invariant $k \in H^3(\pi, \pi_2) \cong Z_{|\pi|}$. But each $k \in H^3(\pi, \pi_2)$ which is the obstruction invariant for a $\pi$-complex must be a generator of $H^3(\pi, \pi_2)$ (see [3]). There are exactly $\varphi(|\pi|)$ such generators. The sign changing automorphism

$$\lambda: \pi_2 \to \pi_2 \quad (\lambda(x) = -x, x \in \pi_2)$$

together with $\text{id}: \pi \to \pi$ gives an isomorphism of the 2-types.
and shows that the number of $k$-invariants representing distinct homotopy types of $\pi$-complexes with Euler characteristic $\chi$ is less than or equal to $\varphi(|\pi|)/2$, since $\pi \neq Z_2$.

BIBLIOGRAPHY


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