THE RADON-NIKODYM PROPERTY
IN CONJUGATE BANACH SPACES

BY

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ABSTRACT. We characterize conjugate Banach spaces $X^*$ having the
Radon-Nikodym Property as those spaces such that any separable subspace
of $X$ has a separable conjugate. Several applications are given.

Introduction. There are several equivalent formulations of the Radon-Nikodym
Property (RNP) in Banach spaces; we give perhaps the earliest definition: a
Banach space $X$ has RNP if given any finite measure space $(S, \Sigma, \mu)$ and any $X$
valued measure $m$ on $\Sigma$, with $m$ having finite total variation and being absolutely
continuous with respect to $\mu$, then $m$ is the indefinite integral with respect to $\mu$
of an $X$ valued Bochner integrable function on $S$. The first study of this property
was by Dunford and Pettis [4] and Phillips [11] (see also [5]).

It follows from the work of Dunford and Pettis and Phillips that reflexive
Banach spaces and separable conjugate spaces have RNP. More generally, the fol-
lowing is true:

**Theorem A.** If $X$ is a Banach space such that for any separable subspace
$Y$ of $X$, $Y^*$ is separable, then $X^*$ has RNP.

The above result was observed by Uhl [15] and also can be obtained from
a result of Grothendieck (Theorem B below).

The first characterizations of RNP were given by Grothendieck in [6].
Grothendieck’s approach, the one we shall use, is that of studying certain classes
of operators. An operator $T: X \to Y$ is a continuous linear function $T$ from the
Banach space $X$ to the Banach space $Y$. An operator $T: X \to Y$ is said to be an
integral operator if there exist a compact Hausdorff space $K$, a Radon measure $\mu$
on $K$, and operators $R$, and $S$, such that

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is commutative. The operator $Q$ from $Y$ to $Y^{**}$ is the canonical evaluation operator; the operator $J$ is the canonical operator from $C(K)$, the continuous real (or complex) valued functions on $K$, to $L_1(K, \mu)$, the equivalence classes of $\mu$-measurable, absolutely summable functions on $K$. An operator $T: X \to Y$ is nuclear if there exist sequences $\{x_n^*\} \subseteq X^*$, $\{y_n\} \subseteq Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \cdot \|y_n\| < +\infty$ and $Tx = \sum_{n=1}^{\infty} x_n^*(x)y_n$. Let $K$ be a compact Hausdorff space and $\mu$ a Radon measure on $K$. A bounded subset of $L_1(K, \mu)$ is said to be equi-measurable [6, p. 20] (with respect to $\mu$) if for each $\varepsilon > 0$ there exists a compact subset $K_0$ of $K$ such that $|\mu(K \setminus K_0)| < \varepsilon$ and $\{f|_{K_0}: f \in S\}$ is a relatively compact subset of $L^1(K_0, \mu)$. Grothendieck proved the following [6, Proposition 9, p. 64]:

**Theorem B.** Let $X$ be a Banach space, $\mu$ a Radon measure on the compact Hausdorff space $K$, and $T$ an operator from $X$ to $L_1(K, \mu)$; the operator $JT$ is nuclear if and only if $\{JTx: \|x\| \leq 1\}$ is an equi-measurable subset of $L_1(K, \mu)$.

From this theorem the following results can be obtained:

(B.1) $X^*$ has RNP if and only if every integral operator $T: X \to L_1(S, \Sigma, \mu)$ is nuclear ($L_1(S, \Sigma, \mu)$ any measure space). (This is implicit in [6], but see [3] for a development of this approach.)

(B.2) $X$ has RNP if and only if for any operator $T: L_1(S, \Sigma, \mu) \to X$ there exist a set $\Gamma$ and operators $S: I_1(\Gamma) \to X$, $R: I_1(S, \Sigma, \mu) \to I_1(\Gamma)$ such that $SR = T$. (This result was perhaps first obtained in [9] where several applications are given.)

We now give a geometrical characterization of RNP. The following definition is due to Rieffel [13] (who also proved a Radon-Nikodym theorem [14]). A subset $S$ of a Banach space will be called dentable if for every $\varepsilon > 0$ there is an $x \in S$ such that $x \notin \overline{B(S \setminus B(x, \varepsilon))}$. $B(X, \varepsilon)$ is the closed ball about $x$ of radius $\varepsilon$ and $\overline{\text{C}(M)}$ is the closed convex hull of the set $M$. Rieffel proved that if $X$ is a Banach space such that every bounded subset of $X$ is dentable then $X$ has RNP [13]. In [10] Maynard made the following definition: a subset $S$ of a Banach space will be called s-dentable if for every $\varepsilon > 0$ there is an $x \in S$ such that $x \notin s(S \setminus B(x, \varepsilon))$ ($s(M)$, the sequential hull of $M$, is the set of all converging series $\sum_{i=1}^{\infty} \lambda_i x_i$ such that $\lambda_i \geq 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$, and $x_i \in M$). Maynard proved that if a Banach space $X$ has a bounded, non-s-dentable subset then $X$ fails RNP.

Recently, R. Phelps and W. J. Davis [1] have shown that if a Banach space has a
bounded, nondentable subset then it has a bounded, non-s-dentable subset. These results may be combined to give the following result:

**Theorem C.** A Banach space $X$ has RNP if and only if every bounded subset of $X$ is dentable.

This is by no means a comprehensive discussion of the Radon-Nikodym Property. The reader is referred to the papers listed above as well as their bibliographies for more information.

Our purpose here is to prove the converse of Theorem A: if $X^*$ has RNP then for every separable subspace $Y$ of $X$, $Y^*$ is separable. We prove a more general result (Theorem 1 below) from which the above result follows. We give several applications of this result.

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**Results.** We begin with the following elementary observation.

**Lemma 1.** Let $Y$ be a nonseparable Banach space. Then for $e > 0$, there exists for every countable ordinal $\alpha$, $y_\alpha \in Y$, $y_\alpha^* \in Y^*$ such that $\|y_\alpha\| = 1$, $\|y_\alpha^*\| < 1 + e$ and

$$y_\beta^*(y_\alpha) = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha < \beta. \end{cases}$$

**Proof.** Choose $y_1 \in Y$ and $y_1^* \in Y^*$ such that $\|y_1\| = \|y_1^*\| = y_1^*(y_1) = 1$. Assume we have made the construction for all $\alpha$, $\alpha < \beta$, where $\beta$ is a countable ordinal. Since $\{y_\alpha\}_{\alpha < \beta}$, the closed linear span of $\{y_\alpha\}_{\alpha < \beta}$, is separable there exists a $z_\beta^* \in Y^*$ such that $z_\beta^*(y_\alpha) = 0$ for all $\alpha$, $\alpha < \beta$. Let $y_\beta^* = (1 + e/2)z_\beta^*/\|z_\beta^*\|$. Since $1 < \|y_\beta^*\| < 1 + e$ there exists $y_\beta$, $\|y_\beta\| = 1$ such that $y_\beta^*(y_\beta) = 1$.

If we let $\Delta$ denote the Cantor set, by a Haar system on $\Delta$ we mean a sequence of functions $\{h_{n,i}\} \subseteq C(\Delta)$, $n = 0, 1, 2, \cdots$, $i = 0, 1, \cdots, 2^n - 1$; $h_{n,i} = \chi_{A_{n,i}}$ (the characteristic function of the set $A_{n,i}$); $A_{0,0} = \Delta$; each $A_{n,i}$ is nonempty, open and closed; for each $n$, $\bigcup_{i=0}^{2^n-1} A_{n,i} = \Delta$ and $\{A_{n,i}\}$ is pairwise disjoint; $A_{n,i} = A_{n+1,2i} \cup A_{n+1,2i+1}$; and, for each choice of indices $i_n$, $0 \leq i_n \leq 2^n - 1$, $\bigcap_{n=0}^{\infty} A_{n,i_n}$ is either empty or a one point set.

**Theorem 1.** If $X$ is a separable Banach space such that $X^*$ is nonseparable, then for $e > 0$ there exist a subset $\Delta$ of the unit sphere of $X^*$ which is weak*
homeomorphic to the Cantor set, a Haar system \{h_{n,i}\} for \(\Delta\), and a sequence \(\{x_{n,i}\} \subseteq X\) with \(\|x_{n,i}\| < 1 + \varepsilon\) such that if \(T: X \to C(\Delta)\) is the canonical evaluation operator, then
\[
\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \varepsilon.
\]

**Proof.** Since \(X^*\) is nonseparable, apply Lemma 1 to obtain \(\{x^*_\alpha\} \subseteq X^*\), \(\{x^*_\alpha\} \subseteq X^{**}\), \(1 < \alpha < \omega_1\), \(\omega_1\) the first uncountable ordinal, such that \(\|x^*_\alpha\| = 1\), \(\|x^{**}_\alpha\| < 1 + \varepsilon\), and \(x^{**}_\alpha(x^*_\alpha) = 1\), \(x^{**}_\beta(x^*_\alpha) = 0\) if \(\alpha < \beta\). Since \(\{x^*_\alpha: \|x^*_\alpha\| < 1\}\), the unit ball of \(X^*\), is a compact metric space in the weak* topology and \(\{x^*_\alpha\}\) is an uncountable subset of the unit ball, the set \(A\) of condensation points of \(\{x^*_\alpha\}\) contains all but an at most countable subset of \(\{x^*_\alpha\}\). Thus there exists a countable ordinal \(\gamma\) such that for any \(\beta \geq \gamma\) and any weak* open set \(U\) containing \(x^*_\beta\), the set \(U \cap \{x^*_\alpha\}_{\alpha \geq \beta}\) is uncountable.

We shall construct for each \(n = 0, 1, 2, \cdots\) weak* open sets in the unit ball of \(X^*\), and a sequence \(\{x^*_n\}_{n=0}^{\infty} = 2^n - 1\) in \(X\) such that

1. weak* diameter \((U_{n,i}) < 1/(n + 2)\) and the weak* closure of \(U_{n,i}\), \(\overline{U}_{n,i}\), is disjoint from \(\{x^*_n: \|x^*_n\| < 1/(n + 2)\}\);
2. \(U_{n,i} \cap A \neq \emptyset\);
3. \(U_{n+1,2i} \cup U_{n+1,2i+1} \subseteq U_{n,i}\);
4. \(x_{n,i} \in X, \|x_{n,i}\| < 1 + \varepsilon\) and for each \(n\), \(|x^*(x_{n,i}) - \delta_{ij}| < \varepsilon/4^{n+1}\) for \(x^* \in U_{n,i}\).

For \(n = 0\), choose any \(x^*_0 \in A\). Since \(x^{**}_\beta(x^*_\alpha) = 1\) and \(\|x^{**}_\beta\| < 1 + \varepsilon\), we know by Helly’s Theorem [2, Theorem 3, p. 38] that there exists an \(x_{0,0} \in X\), \(\|x_{0,0}\| < 1 + \varepsilon\), such that \(x^*_\beta(x_{0,0}) = 1\). Let \(U_{0,0}\) be a weak* open neighborhood of \(x^*_\beta\) of weak* diameter less than \(1/2\), \(U_{0,0} \subseteq \{x^*_n: \|x^*_n\| < 1\} \cap \{x^*_n: \|x^*_n\| < 1 + \varepsilon/4\} \cap A\), and \(U_{0,0} \cap \{x^*_n: \|x^*_n\| < 1/2\} = \emptyset\).

Assume we have made the construction up to \(n\). Choose \(x^*_{\beta_{n,i}} \in U_{n,i} \cap A\) with \(\beta_{n,0} < \beta_{n,1} < \cdots < \beta_{n,2^n-1}\). Choose \(x^*_{\beta_{n+1,0}} \in U_{n,0} \cap A\) with \(\beta_{n+1,0} > \beta_{n,2^n-1}\). Since \(x^{**}_{\beta_{n+1,0}}(x^*_{\beta_{n,0}}) = 1\) and \(x^{**}_{\beta_{n+1,0}}(x^*_{\alpha}) = 0\) for all \(\alpha < \beta_{n+1,0}\), there exists by Helly’s theorem an \(x_{n+1,0} \in X, \|x_{n+1,0}\| < 1 + \varepsilon\), such that \(x^*_{\beta_{n,i}}(x_{n+1,0}) = 0\) for \(0 < i < 2^n\) and \(x^*_{\beta_{n+1,0}}(x_{n+1,0}) = 1\). Choose \(x^*_{\beta_{n+1,2n}} \in U_{n,1} \cap A\) such that \(\beta_{n+1,2} > \beta_{n+1,0}\) and \(x^*_{\beta_{n+1,2}}(x_{n+1,0}) < \varepsilon/4^{n+2}\). (This happens because \(x_{n+1,0}\) vanishes at some point of \(U_{n,1} \cap A\) so \(x_{n+1,0}\) is less than \(\varepsilon/4^{n+2}\) on some open, hence uncountable, subset of \(U_{n,1} \cap A\); in this uncountable set there must be a point of index larger than \(\beta_{n+1,0}\).) Now choose \(x_{n+1,2} \in X, \|x_{n+1,2}\| < 1 + \varepsilon\), \(x^*_{\beta_{n,i}}(x_{n+1,2}) = 0\) for \(0 < i < 2^n\), \(x^*_{\beta_{n+1,0}}(x_{n+1,2}) = 0\) and \(x^*_{\beta_{n+1,2}}(x_{n+1,2}) = 1\). In general, for \(0 < k < 2^n\), choose \(x^*_{\beta_{n+1,2k}} \in \)
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Choose \( x_{n+1,1}^* \in U_{n,0} \cap A \) such that \( |x_{n+1,1}^* (x_{n+1,2k})| < \varepsilon/4^{n+2} \) for \( 0 \leq k < 2^n \) and \( \beta_{n+1,1} = \beta_{n+1,2k+1} \). Choose \( x_{n+1,1}^* \in X, \|x_{n+1,1}^*\| < 1 + \varepsilon \) such that \( x_{n+1,1}^* (x_{n+1,2k}) = 0 \) for \( 0 \leq k < 2^n \), \( x_{n+1,2k}^* (x_{n+1,1}) = 0 \) for \( 0 \leq k < 2^n \) and \( x_{n+1,2k+1}^* (x_{n+1,1}) = 0 \). In general, for \( 0 \leq k < 2^n \), choose \( x_{n+1,2k+1}^* \in U_{n,k} \cap A, \beta_{n+1,2k+3} > \beta_{n+1,2k+1} \), and \( x_{n+1,2k+1}^* \in X, \|x_{n+1,2k+1}^*\| < 1 + \varepsilon \), such that

\[
\begin{align*}
(I) & \quad x_{n+1,2k}^* (x_{n+1,2k+1}) = 0, 0 \leq k < 2^n; \\
(ii) & \quad x_{n+1,2k+2}^* (x_{n+1,2k+1}) = 0, 0 \leq k < 2^n; \\
(iii) & \quad x_{n+1,2k+3}^* (x_{n+1,2k+1}) = 1, 0 \leq k < 2^n; \\
(iv) & \quad |x_{n+1,2k+1}^* (x_{n+1,2k+1})| < \varepsilon/4^{n+2}, 0 \leq k < l < 2^n.
\end{align*}
\]

Define, for \( 0 \leq j < 2^{n+1} \),

\[ U_{n+1,j} = \{ x^* \in U_{n,[j/2]} : |x^*(x_{n+1,k}) - \delta_{kj}| < \varepsilon/4^{n+2} \text{ for all } k, 0 \leq k < 2^{n+1} \}. \]

We have that \( x_{n+1,j}^* \in U_{n+1,j} \) for \( 0 \leq j < 2^{n+1} \). Choose \( U_{n+1,j} \) a weak* neighborhood of \( x_{n+1,j}^* \) with weak* diameter less than \( 1/(n+3) \), weak* closure of \( U_{n+1,j} \) is disjoint from \( \{ x^* : \|x^*\| \leq 1/(n+3) \} \), and \( U_{n+1,j} \subseteq U_{n+1,j}^* \). This completes the construction.

Let \( \Delta = \bigcap_{n=1}^{\infty} \bigcup_{j=0}^{n-1} \overline{U}_{n,j}^* \). As is well known \([2, p. 93]\), \( \Delta \) is homeomorphic to the Cantor set and we have that \( \Delta \subseteq \{ x^* : \|x^*\| = 1 \} \). If we let \( h_{n,i} = x_{A_{n,i}} A_{n,i} = \Delta \cap \overline{U}_{n,i}^* \), then \( \{ h_{n,i} \} \) is a Haar system and

\[
\sup\{ |x^*(x_{n,i}) - h_{n,i}(x^*)| : x^* \in \Delta \} = \sup\{ |x^*(x_{n,i}) - h_{n,i}(x^*)| : x^* \in \bigcup U_{n,j}, 0 \leq j < 2^n \} \leq \varepsilon/4^{n+1}.
\]

Let \( T : X \to C(\Delta) \) be the canonical evaluation operator \( (Tx)(x^*) = x^*(x) \); then we have that

\[
\sum_{b=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| \leq \sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \frac{\varepsilon}{4^{n+1}} = \sum_{n=0}^{\infty} 2^n \frac{\varepsilon}{4^{n+1}} = \frac{\varepsilon}{2}. \quad \text{Q.E.D.}
\]
COROLLARY 1. If $X$ is separable and $S$ is a nonseparable subset of $X^*$ in the norm topology and is a weak* $G_δ$ set, then for $e > 0$ there exist a subset $Δ ⊆ S$ which is weak* homeomorphic to the Cantor set, a Haar system $\{h_{n,i}\}$ on $Δ$, a sequence $\{x_{n,i}\} ⊆ X$, and a constant $C > 0$ such that $\|x_{n,i}\| ≤ C$ and $\sum_{n=0}^{∞} \sum_{i=0}^{n-1} \|Tx_{n,i} - h_{n,i}\| < ε$ where $T: X → C(Δ)$ is the canonical evaluation operator.

PROOF. The proof is essentially the same as that of Theorem 1 with the additional restriction that each $U^*_n ⊆ V_n$ where $V_n$ are weak* open sets such that $\cap_{n=0}^{∞} V_n = S$.

THEOREM 2. Suppose $X^*$ has RNP. Then for every separable subspace $Y$ of $X$, $Y^*$ is separable.

PROOF. Assume there exists a separable subspace $Y$ of $X$ such that $Y^*$ is not separable. By Theorem 1, there exist a Haar system $\{h_{n,i}\}$ on the Cantor set and an operator $T: Y → C(Δ)$ such that $\|T\| ≤ 1$, $\sum_{n=0}^{∞} \sum_{i=0}^{n-1} \|Ty_{n,i} - h_{n,i}\| < ε$, $\|y_{m,i}\| < 1 + ε$. For $v$ any measure on $Δ$ let $L^∞_{∞}(Δ, ν)$ denote the equivalence classes of $ν$ essentially bounded functions on $Δ$. We shall consider $T$ as an operator from $Y$ to $L^∞_{∞}(Δ, ν)$. We may extend $T$ to an operator $\tilde{T}: X → L^∞_{∞}(Δ, ν)$ since $L^∞_{∞}(Δ, ν)$ is an injective space [2, pp. 94–95]. We shall complete the proof in two different ways:

(1) Suppose $ν$ is not purely atomic. Let $K_0$ be a compact subset of $Δ$, $K_0$ has no atoms, $ν(K_0) > 0$. It is easy to see that $\{Ty_{n,i}|K_0: n = 0, 1, \cdots; 0 ≤ i < 2^n\}$ is not relatively compact in $L^∞_{∞}(K_0, ν)$. By Theorem B, $JT$ is not nuclear ($J: L^∞_{∞}(Δ, ν) → L^*_{1}(Δ, ν)$ the canonical operator). Thus by Theorem (B.1) $X^*$ does not have RNP.

(2) Suppose $ν$ is the measure on $Δ$ such that $\int h_{n,i}dν = 2^{-n}$.

Regarding $2^n h_{n,i}$ as elements of $L^∞_{∞}(Δ, ν)^*$, let $S = \{\tilde{T}^*(2^n h_{n,i})\}$. Suppose $n ≥ m$ and $i ≠ j$ if $n = m$; then

$$\|\tilde{T}^*(2^n h_{n,i}) - \tilde{T}^*(2^m h_{m,j})\| ≥ \frac{1}{1 + ε} \left| 2^n \int_{A_{n,i}} x^*(y_{n,i})dν - 2^m \int_{A_{m,j}} x^*(y_{n,i})dν \right|$$

$$= \frac{1}{1 + ε} \left| 2^n \int_{A_{n,i}} h_{n,i}(x^*)dν - 2^m \int_{A_{m,j}} h_{n,i}(x^*)dν \right|$$

$$+ \frac{1}{1 + ε} \left| 2^n \int_{A_{n,i}} [x^*(y_{n,i}) - h_{n,i}(x^*)]dν - 2^m \int_{A_{m,j}} [x^*(y_{n,i}) - h_{n,i}]dν \right|$$

$$≥ \frac{1}{1 + ε} \left( 1 - \frac{1}{2} - \frac{ε}{4^n+1} - \frac{ε}{4^n+1} \right) = \frac{1}{1 + ε} \left( \frac{1}{2} - \frac{2ε}{4^n+1} \right).$$
By choosing $0 < \epsilon < 1/4$, then the distance between any two distinct points of $S$ is greater than $1/4$; but $T^* \left(2^n h_{n,i} \right) = \frac{1}{2} \left( T^* \left( 2^{n+1} h_{n+1,1,i} \right) + T^* \left( 2^{n+1} h_{n+1,2,i+1} \right) \right)$. Thus $S$ is clearly not dentable (not even $s$-dentable); by Theorem C, $X^*$ does not have RNP.

**Corollary 2.** Let $X$ be a Banach space such that there exists a separable subspace $Y$ of $X$ such that $Y^*$ is nonseparable. (Equivalently, $X^*$ does not have RNP.) Then there exists a separable subspace $Z$ of $X^*$ such that $Z$ is not isomorphic to a subspace of a separable conjugate space.

**Proof.** It is proved in [15] that if every separable subspace $Z$ of $X^*$ is isomorphic to a subspace of a separable conjugate space, then $X^*$ has RNP.

It is not difficult to see that the arguments of Lemma 1 and Theorem 1 may be repeated for higher ordinals. In particular, this argument will give a proof of the complex version of a theorem proved by Leach and Whitfield [8] in the real case:

**Theorem 3.** Let $X$ be a Banach space such that $\dim X < \dim X^*$. ($\dim X$ is the smallest cardinal $m$ such that there exists a set $S$ of cardinality $m$ such that $|S| = X$.) Then there exists a separable subspace $Y$ of $X$ such that $Y^*$ is nonseparable.

Let $\{h_{n,i}\}$ be a Haar system for the Cantor set and let $\mu$ be the measure on $\Delta$ such that $\int h_{n,i} d\mu = \mu(A_{n,i}) = 2^{-n}$, $h_{n,i} = \chi_{A_{n,i}}$. Let $I_1 = \{(t_{n,i}) : n = 0, i = 0, 1, 2, \cdots, 0 \leq i < 2^{n-1}, t_{n,i}$ real (or complex), $|t_{0,0}| + \Sigma_{n=1}^{\infty} \Sigma_{i=0}^{2^{n-1}-1} |t_{n,i}| < +\infty \}$. Let $b_{0,0} = h_{0,0}$; $b_{n,i} = h_{n,2i} - h_{n,2i+1}, n = 1, 2, \cdots, 0 \leq i < 2^{n-1}$; and define $H: I_1 \rightarrow L_\infty(\Delta, \mu)$ by $H(t_{n,i}) = t_{0,0} b_{0,0} + \Sigma_{n=1}^{\infty} \Sigma_{i=0}^{2^{n-1}-1} t_{n,i} b_{n,i}$.

**Theorem 4.** Let $X$ be a separable Banach space and $Y$ a Banach space and $T: X \rightarrow Y$ such that $T^*(Y^*)$ is nonseparable. Then there exist operators $R: I_1 \rightarrow X$ and $S: Y \rightarrow L_\infty(\Delta, \mu)$ such that $STR = H$, where $H$ is the operator given above.

**Proof.** Since $T^*(Y^*)$ is nonseparable, $\{T^*y^* : \|y^*\| \leq 1\}$ is a non-norm-separable, weak* compact subset of $X^*$. By Corollary 1, there exist a subset $\Delta \subseteq \{T^*y^* : \|y^*\| \leq 1\}$, $\Delta$ weak* homeomorphic to the Cantor set, a Haar system $\{h_{n,i}\}$ on $\Delta$, a sequence $\{x_{n,i}\} \subseteq X$, a constant $C > 0$ such that $\|x_{n,i}\| \leq C$, and $\Sigma_{n=0}^{\infty} \Sigma_{i=0}^{2^{n-1}-1} \|Ux_{n,i} - h_{n,i}\| < \epsilon < 1$ where $U$ is the canonical evaluation operator. Define $\{b_{n,i}\}$ as above; $\{b_{n,i}\}$ is a Schauder basis for $C(\Delta)$. Define $g_{0,0} = Ux_{0,0}$, $g_{n,i} = U(x_{n,2i} - x_{n,2i+1}), n = 1, 2, \cdots, 0 \leq i < 2^{n-1}$. Since

$$\|g_{0,0} - b_{0,0}\| + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} \|g_{n,i} - b_{n,i}\| < \epsilon < 1$$
the Paley-Wiener stability theorem [12] shows the existence of an operator 
\[ A : C(\Delta) \to C(\Delta), \]
such that \( A \) is an onto isomorphism, \( \|A\| < 1 + \epsilon, \|A^{-1}\| < 1/(1 - \epsilon), \) and \( A g_{n,i} = b_{n,i} \). We have the following relations:

\[
\{A^n \eta : \|\eta\| \leq 1\} \subseteq \{\nu : \|\nu\| \leq (1 + \epsilon)/(1 - \epsilon)\}
\]

and

\[
\{U^*A^n \eta : \|\eta\| \leq 1\} \subseteq \{U^*\nu : \|\nu\| \leq (1 + \epsilon)/(1 - \epsilon)\}
\] \subseteq \{T^*y : \|y\| \leq (1 + \epsilon)/(1 - \epsilon)\}.
\]

Since \( AU \) has dense range, \( U^*A^* \) is one-to-one. Let \( \Delta_1 = \{U^*A^*\delta_k : \delta_k \) a positive point mass in \( C(\Delta)^*\} \) and let \( K = \{y^* \in Y^* : \|y^*\| \leq (1 + \epsilon)/(1 - \epsilon)\} \). Choose \( T^*y^* \) in \( \Delta_1 \). Since \( K \) is a weak* compact subset of \( Y^* \) the canonical evaluation operator \( V : Y \to C(K) \) is well defined. Let \( Q : C(\Delta) \to C(K) \) be the operator such that \( (Qf)y^* = f(k) \) where \( T^*y^* = U^*A^*(\delta_k) ; Q \) is an isometry of \( C(\Delta) \) into \( C(K) \) and \( QAU = VT \). Let \( I : C(\Delta) \to L_{\infty}(\Delta, \mu) \) denote the canonical operator. Since \( Q \) is an isometry and \( L_{\infty}(\Delta, \mu) \) is injective [2, pp. 94–95] there exists an operator \( \tilde{T} : C(K) \to L_{\infty}(\Delta, \mu) \) such that \( \tilde{T}Q = I \). Define \( R : l_1 \to X \) by

\[
R((t_{n,i})) = t_{0,0}x_{0,0} + \sum_{n=1}^{\infty} \sum_{i=0}^{2^{n-1}-1} t_{n,i}(x_{n,2i} - x_{n,2i+1}).
\]

Combining the facts above we have that \( \|R\| \leq 2C, IAUR = H, \) and \( (\tilde{T}V)TR = \tilde{T}(VT)R = \tilde{T}(QAU)R = IAUR = H. \) If we let \( S = \tilde{T}V \) then we have the desired result.

**Corollary 3.** Let \( X \) be a separable Banach space such that \( X^* \) is not separable. Then \( X \) has a bounded biorthogonal set of the cardinality of the continuum.

**Proof.** By Theorem 1, for \( \epsilon > 0 \), there exist a subset \( \Delta \) of the unit sphere of \( X^* \), \( \Delta \) weak* homeomorphic to the Cantor set, a Haar system \( \{h_{n,i}\} \) on the Cantor set, and a sequence \( \{x_{n,i}\} \subseteq X, \|x_{n,i}\| < 1 + \epsilon \) and

\[
\sum_{n=0}^{\infty} \sum_{i=0}^{2^n-1} \|Tx_{n,i} - h_{n,i}\| < \epsilon
\]

where \( T \) is the canonical evaluation operator. Let \( x^* \in \Delta \). Choose the unique sequence \( (n, i_n) \) such that \( h_{n,i_n}(x^*) = 1, n = 0, 1, \cdots \). Let \( x^{**} \) be any weak* cluster point in \( X^{**} \) of the sequence \( \{x_{n,i_n}\}, \|x^{**}\| < 1 + \epsilon. \) Since

\[
\sum_{n=0}^{\infty} |x^*(x_{n,i_n}) - 1| < \epsilon,
\]

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the sequence \( x^*(x_{n,i}) \) converges to 1, but also clusters at \( x^{**}(x^*) \) so \( x^{**}(x^*) = 1 \). For any \( y^* \in \Delta, y^* \neq x^* \),

\[
\sum_{n=0}^{\infty} |y^*(x_{n,i}) - h_{n,i}(y^*)| < \epsilon,
\]

but there exists a positive integer \( N \) such that for \( n \geq N, h_{n,i}(y^*) = 0 \), so \( \sum_{n=0}^{\infty} |y^*(x_{n,i})| < \epsilon \). Thus \( y^*(x_{n,i}) \) converges to 0, but also clusters at \( x^{**}(y^*) \) so \( x^{**}(y^*) = 0 \). Thus for each \( x^* \in \Delta \), there exists an \( x^{**} \in X^{**} \), \( \|x^{**}\| < 1 + \epsilon \), such that \( x^{**}(x^*) = 1 \) and \( x^{**}(y^*) = 0 \) for all \( y^* \in \Delta \), \( y^* \neq x^* \). Thus \( \{(x^*, x^{**}) \in \Delta \} \) is a biorthogonal system of the cardinality of the continuum.

**Corollary 4.** Let \( X \) be a separable Banach space. A necessary and sufficient condition that \( X^* \) be nonseparable is that there exists a bounded biorthogonal sequence \( \{(x_i^*, x_i^{**})\} \) in \( X^* \) such that \( \{x_i^*\} \) is dense in itself in the weak* topology.

**Proof.** If \( X^* \) is nonseparable, by Corollary 3, there exists a bounded biorthogonal system \( \{(x^*, x^{**})\} \) such that \( \{x^*\} \) is weak* homeomorphic to the Cantor set. Thus we have only to choose a sequence \( \{x_i^*\} \) in \( \{x^*: x^* \in \Delta \} \) that is weak* dense in \( \Delta \).

If \( \{(x_i^*, x_i^{**})\} \) is such a biorthogonal system then the construction in Theorem 1 can be repeated with slight modifications to construct an operator \( T: X \to C(\Delta) \) such that \( T^*(C(\Delta)*) \) has nonseparable range.

**Corollary 5.** Let \( X \) be a Banach space such that for any bounded sequence \( \{x_i^*\} \) in \( X^* \), the weak* closure of \( \{x_i^*\} \) is norm separable. Then \( X^* \) has RNP.

**Proof.** If \( X^* \) does not have RNP then from Theorem 1, we know there exist an operator \( S: L_1(\Delta, \nu) \to X^* \) (\( \nu \) some nonatomic measure on the Cantor set \( \Delta \)) and a Haar system \( \{h_{n,i}\} \) on \( \Delta, \int h_{n,i}d\nu = 2^{-n} \) such that \( S(2^n h_{n,i}) \) does not have separable weak* closure.

We point out that the converse of Corollary 5 is false. Precisely, there exists a compact Hausdorff space \( K \) such that \( K \) is separable, uncountable, and has no perfect subsets. Since \( C(K)^* \) is isometric to \( l_1(K) \), \( C(K)^* \) has RNP [4] but if \( \{k_i\} \) is a dense sequence in \( K \), then the weak* closure of \( \{\delta_{k_i} \} \) in \( C(K)^* \) contains all \( \delta_k \) which is not a norm separable set.

To obtain such a space \( K \), let \( (n, i) = k_{n,i} \) be the sequence of pairs of integers for \( n = 0, 1, 2, \cdots, 0 \leq i < 2^n \) and let \( k \) be any sequence of the form \( (n, i_n), n = 0, 1, \cdots, \), with \( 2i_n \leq i_{n+1} < 2(i_n + 1) \). The set of \( \{k\} \) is uncountable. Define the topology on \( \{k_{n,i}, k \} \) to be the following:
(1) each \( \{k_{n,i}\} \) is an open set;
(2) a neighborhood basis of each \( k = \{(n, i_n)\} \) is given by \( U_N = \{k, k_{n,i_n} : n \geq N\} \) for each \( N = 0, 1, 2, \cdots \).

It is easy to see that \( \{k, k_{n,i}\} \) is a locally compact Hausdorff space, so we let \( K \) be the one-point compactification of this space.

For reference we state the following result.

**Corollary 6.**

(1) If \( X^* \) has RNP and \( Y \) is isomorphic to a subspace of a quotient space of \( X \), then \( Y^* \) has RNP.

(2) If there is a subspace \( Y \) of \( X \) such that \( Y^* \) and \( [X/Y]^* \) have RNP then \( X^* \) has RNP.

**Proof.** Since (1) is obvious we shall prove only (2). Suppose \( Q : X \to X/Y \) is the canonical quotient operator. Let \( Z \) be a separable subspace of \( X \). Since \( Q \) is onto there exists a separable subspace \( W \) of \( X \), \( Z \subseteq W \) and \( Q(W) \) is closed in \( X/Y \). Let \( T : W \to Q(W) \), \( T = Q|_W \). The kernel of the operator \( T \) is \( W \cap Y \). Both \( Q(W) \) and \( W \cap Y \) are separable and their duals have RNP so their duals are separable. From this it is clear that \( W^* \) is separable so \( Z^* \) is separable. Thus \( X^* \) has RNP.

Finally, we state a tensor product formulation of Theorem 2 (see [6]).

**Corollary 7.** Let \( X \) be a Banach space. For \( X^* \) to have RNP it is necessary and sufficient that for every Banach space \( Y \), the natural operator from \( X^* \hat{\otimes} Y^* \) to \([X \hat{\otimes} Y]^*\) is onto.

**Questions.** Related to Theorem C and our Theorem 2 is the following question: if \( X \) does not have RNP do there exist a bounded sequence \( \{x_i\} \) in \( X \) and a \( \delta > 0 \) such that \( \|x_i - x_j\| \geq \delta \) for all \( i, j \) with \( i \neq j \) and for each \( i \) there exists \( j, k \neq i \neq k \), such that \( x_i = \frac{1}{2}(x_j + x_k) \)? By Theorem C, if such a sequence exists then \( X \) does not have RNP. Our Theorem 2 shows there is such a sequence in conjugate spaces not having RNP.

Related to Corollary 5 is the following question: if the set of extreme points of the unit ball of \( X^* \) is a norm separable set, is \( X^* \) separable?

If \( X \) has RNP does every separable subspace of \( X \) embed in a separable conjugate space? This is a problem posed by Uhl [15].

Probably the best known question about a separable Banach space \( X \) with \( X^* \) nonseparable is the following: Does \( X \) have a subspace isomorphic to \( l_1 \) (the space of all absolutely summing sequences)? Since the preparation of this paper R. C. James [7] has shown that the answer to this question is negative. In fact, James' example seems to indicate that the construction in our Theorem 1 is the best possible.
REFERENCES

7. R. C. James, A conjecture about $l_1$ subspaces (to appear).

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