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HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS

BY

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ABSTRACT. Let P be a $p \times q$ matrix of polynomials in n complex variables. If Ω is a domain of holomorphy in $\mathbf{C}^n$ and u is a q-tuple of holomorphic functions we show that the equation $Pv = Pu$ has a solution v which is a holomorphic q-tuple in Ω and which satisfies an L^2 estimate in terms of Pu. Similar results have been obtained by Y.-T. Siu and R. Narasimhan for bounded domains and by L. Hörmander for the case $\Omega = \mathbf{C}^n$.

Let Ω be a domain of holomorphy in $\mathbf{C}^n$ and let $A(\Omega)$ be the space of holomorphic functions in Ω. Let P be a $p \times q$ matrix of polynomials in $\mathbf{C}[z_1, \cdots, z_n]$ and let $f = (f_1, \cdots, f_p)$ be an element of $A(\Omega)^p$. Our main theorem states that if f satisfies certain growth conditions and if the system of linear equations $Pu = f$ has a solution u in $A(\Omega)^q$ then it has a solution which satisfies certain growth conditions at the boundary of Ω and at infinity. In the case where Ω is a *bounded* domain of holomorphy results of this type have been given by Y.-T. Siu [6] and R. Narasimhan [4], and in the case $\Omega = \mathbf{C}^n$ by L. Hörmander [2, Theorem 7.6.11]. Our method of proof in outline is the same as in [2], and we use the notation of [1] and [2]. As an illustration of the theorem we give a simple application to partial differential equations.

For each $z \in \Omega$ let $d(z) = \min(1, \operatorname{dist}(z, \partial\Omega))$ and define $\psi(z) = -\log d(z)$, $\theta(z) = \log(1 + |z|^2)$. Let φ be a plurisubharmonic function in Ω such that there exists a constant a such that

$$|\varphi(z) - \varphi(w)| \leqslant a \quad \text{if } z \in \Omega \text{ and } |z - w| \leqslant \tfrac{1}{2} d(z). \tag{1}$$

We note that if m and N are nonnegative integers then $\varphi + m\psi + N\theta$ is a plurisubharmonic function in Ω which satisfies (1). If $u = (u_1, \cdots, u_q) \in A(\Omega)^q$ we let $|u|^2 = |u_1|^2 + \cdots + |u_q|^2$.

THEOREM 1. *If $u \in A(\Omega)^q$ there exists $v \in A(\Omega)^q$ such that $Pu = Pv$ in Ω and*

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$$\int_\Omega |v|^2 e^{-\varphi-m\psi-N\theta} dV \leq C \int_\Omega |Pu|^2 e^{-\varphi} dV. \tag{2}$$

Here m and N are nonnegative integers which depend only on P, C is a constant which depends only on a and on P, and dV is Lebesgue measure.

For emphasis we note that m, N and C do not depend on Ω. If $z \in \Omega$ and D is the polydisk with center z and radii all equal to $\delta = \frac{1}{2} n^{-\frac{1}{2}} d(z)$, then $|v_j(z)| \leq \pi^{-n}\delta^{-2n} \int_D |v_j| dV$; and hence by the Cauchy-Schwarz inequality and (2) we obtain the estimate

$$|v(z)|^2 \leq C'(1 + |z|^2)^N d(z)^{-m-2n} e^{\varphi(z)}, \qquad z \in \Omega, \tag{3}$$

where $C' = \pi^{-n} n^n 2^{2n+N+m} e^a C \int_\Omega |Pu|^2 e^{-\varphi} dV$.

COROLLARY. *Let Γ be a closed convex salient cone in $\mathbf{R}^n$ with vertex at the origin and let Γ_0^+ be the interior of the dual cone. Let $P_1, \cdots, P_q$ be polynomials in $\mathbf{C}[z_1, \cdots, z_n]$.*

Then $P_1, \cdots, P_q$ have no common zero with real part in Γ_0^+ if and only if there exist temperate distributions $S_1, \cdots, S_q$ with supports in Γ such that

$$P_1(D)S_1 + \cdots + P_q(D)S_q = \delta. \tag{4}$$

Here $D_j = \partial/\partial x_j$ and δ is the unit mass at the origin.

PROOF OF COROLLARY. We first note if the temperate distributions S_j with supports in Γ satisfy (4) and if $u_j \in A(\Omega)$ is the Laplace transform of S_j, where $\Omega = \Gamma_0^+ + i\mathbf{R}^n$, then

$$P_1(z)u_1(z) + \cdots + P_q(z)u_q(z) = 1, \qquad z \in \Omega. \tag{5}$$

Conversely if $P_1, \cdots, P_q$ have no common zero in Ω then by Theorem 7.2.9 in [2] we can find $u_j \in A(\Omega)$ so that (5) is satisfied. Moreover by Theorem 1 and (3) we may choose u_j so that

$$|u_j(z)|^2 \leq C'(1 + |z|^2)^{N+n+1} d(z)^{-m-2n}.$$

By Theorem 1 of [5] u_j is the Laplace transform of a temperate distribution with support in Γ.

The proof of Theorem 1 will be postponed until we have established some preliminary results. We first need a theorem on trivial cohomology with bounds for the sheaf of germs of holomorphic functions in Ω.

If $\varphi \in C(\Omega)$ is real-valued we denote by $L^2_{(p,q)}(\Omega, \varphi)$ the Hilbert space of forms of type (p, q) on Ω whose coefficients are square integrable with respect to the measure $e^{-\varphi} dV$. Let $(\Omega_\nu)_{\nu \geq 1}$ be an open covering of Ω. If $\sigma \geq 0$ is an integer we denote by $C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi)$ the set of all alternating cochains $c = (c_\alpha)$

where $\alpha = (\alpha_0, \cdots, \alpha_\sigma)$ is a $(\sigma + 1)$-tuple of positive integers and $c_\alpha \in L^2_{(p,q)}(\Omega_\alpha, \varphi)$ where $\Omega_\alpha = \Omega_{\alpha_0} \cap \cdots \cap \Omega_{\alpha_\sigma}$ and $\bar\partial c_\alpha = 0$. We define

$$\|c\|_\varphi^2 = \sum_\alpha{}' \int_{\Omega_\alpha} |c_\alpha|^2 e^{-\varphi} dV$$

where $|c_\alpha|^2$ is the sum of the squares of the coefficients of the form c_α and where Σ' means we sum only over α such that $\alpha_0 < \cdots < \alpha_\sigma$.

We assume that the covering $(\Omega_\nu)_{\nu \geqslant 1}$ satisfies the following conditions.

(a) *There exist positive constants A and B such that if $z \in \Omega_\nu$ then*

$$B^{-1} \operatorname{diam}(\Omega_\nu) \leqslant d(z) \leqslant A \operatorname{diam}(\Omega_\nu).$$

In particular B is an upper bound for the diameters of the sets Ω_ν.

(b) *There exists an integer M such that more than M different sets Ω_ν always have an empty intersection.*

(c) *There exists a partition of unity $(\chi_\nu)_{\nu \geqslant 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu)$ and*

$$\sum_\nu |\bar\partial \chi_\nu(z)| \leqslant A\, d(z)^{-1}, \qquad z \in \Omega.$$

(d) *Each set Ω_ν is a domain of holomorphy.*

Open coverings with these properties may be constructed for any open proper nonempty subset Ω of $\mathbf{C}^n$ with constants A, B and M independent of Ω. Constructions of this type were introduced by Whitney (see E. M. Stein [7, p. 127]). We will construct open coverings with additional properties in Lemma 2 below.

We define the coboundary operator δ in the usual way: if $c \in C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi)$ then

$$(\delta c)_{\alpha_0 \cdots \alpha_{\sigma+1}} = \sum_{j=0}^{\sigma+1} (-1)^j c_{\alpha_0 \cdots \hat\alpha_j \cdots \alpha_{\sigma+1}}.$$

Then hypothesis (b) implies $\delta c \in C^{\sigma+1}((\Omega_\nu), Z_{(p,q)}, \varphi)$ and $\|\delta c\|_\varphi \leqslant M \|c\|_\varphi$.

THEOREM 2. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$, let φ be a plurisubharmonic function in Ω and let κ be a continuous function in Ω with $\kappa \leqslant 0$ such that e^κ is a lower bound for the plurisubharmonicity of φ.*

If $\sigma \geqslant 1$ then for each $c \in C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi + \kappa)$ with $\delta c = 0$ we can find a cochain $c' \in C^{\sigma-1}((\Omega_\nu), Z_{(p,q)}, \varphi + 2\psi)$ such that $\delta c' = c$ and $\|c'\|_{\varphi+2\psi} \leqslant K_\sigma \|c\|_{\varphi+\kappa}$ where $K_\sigma = (1 + 2ABM)^{\sigma-1}(1 + AM)M^{\sigma/2}$.

PROOF. The proof of Theorem 2 is exactly the same as the proof of Theorem 2.4.1 in [1] except that the estimates used there are replaced by the corresponding estimates arising from hypotheses (a) and (c) above in place of (i) and (iii) in [1].

Note by condition (b) we only have to consider $\sigma < M$ and hence we may replace the K_σ by a constant K independent of σ, and by previous remarks independent of Ω. In [2] the proof that Theorem 4.4.1 implies Theorem 4.4.2 is accomplished by replacing φ by $\varphi(z) + 2\log(1 + |z|^2)$. If we use the same device here we obtain

COROLLARY. *If φ is any plurisubharmonic function in Ω, if $\sigma \geqslant 1$ and $c \in C^\sigma((\Omega_\nu), Z_{(p,q)}, \varphi)$ and $\delta c = 0$ then there exists $c' \in C^{\sigma-1}((\Omega_\nu), Z_{(p,q)}, \varphi + 2\psi + 2\theta)$ such that $\delta c' = c$ and*

$$\|c'\|_{\varphi+2\psi+2\theta} \leqslant K\,\|c\|_\varphi. \tag{6}$$

In order to prove Theorem 1 we first need a semilocal version (Lemma 1). This semilocal result together with Theorem 2 (with $p = q = 0$) then gives a theorem on trivial cohomology with bounds for the sheaf of relations $\mathcal{R}(P)$. This result together with Lemma 1 then gives Theorem 1 easily. In outline this procedure is the same as in [2, §7.6].

LEMMA 1. *Let P be a $p \times q$ matrix over $\mathbf{C}[z_1, \cdots, z_n]$, let U be the open unit ball centered at the origin in $\mathbf{C}^n$. Then there exist nonnegative integers m and N and a constant t, $0 < t < 1$, depending on P, and if $B > 0$, a constant $C > 0$ depending on P and B, such that if $z \in \mathbf{C}^n$, $0 < r \leqslant B$ and $u \in A(z + rU)^q$ then there exists $v \in A(z + trU)^q$ such that $Pu = Pv$ in $z + trU$ and if φ is any continuous real-valued function on $z + r\overline{U}$ then*

$$r^m \int_{z+trU} |v|^2 e^{-\varphi - N\theta}\, dV \leqslant Ce^a \int_{z+rU} |Pu|^2 e^{-\varphi}\, dV \tag{7}$$

where $a = \sup_{z+rU} |\varphi(z') - \varphi(z'')|$.

If $0 < t < 1$ satisfies the conclusion of Lemma 1 then we will say that t is *good* for P. We will postpone the proof of Lemma 1 until we have completed the proof of Theorem 1. The proof of the lemma is the same as the proof of Proposition 7.6.5 in [2] except that it is necessary to pay considerably more attention to the estimates.

The *module of relations* $\mathcal{M}(P)$ of the columns of P is defined to be the kernel of the homomorphism

$$P\colon \mathbf{C}[z_1, \cdots, z_n]^q \longrightarrow \mathbf{C}[z_1, \cdots, z_n]^p$$

and the *sheaf of relations* $\mathcal{R}(P)$ of the columns of P is defined to be the kernel of the homomorphism $P\colon \mathcal{A}^q \longrightarrow \mathcal{A}^p$ where $\mathcal{A}$ is the sheaf of germs of holomorphic functions in $\mathbf{C}^n$. Since $\mathcal{M}(P)$ is finitely generated there is a $q \times s_0$ matrix of polynomials Q^0 whose columns generate $\mathcal{M}(P)$. Inductively we obtain an $s_{k-1} \times s_k$ matrix of polynomials Q^k whose columns generate $\mathcal{M}(Q^{k-1})$, $k \geqslant 1$. By homological dimension theory [3] $\mathcal{M}(Q^{n-1})$ is a projective module. By adding a sufficient number of columns of zeros to Q^{n-1} we can arrange for $\mathcal{M}(Q^{n-1})$ to be a free $\mathbf{C}[z_1, \cdots, z_n]$ module. (For this curious fact see R. G. Swan [8, Remark, p. 121].) With the obvious choice of Q^n we now have $Q^k = 0$ for $k > n$. By Lemma 7.6.3 in [2] for each z in $\mathbf{C}^n$ the stalk $\mathcal{R}(P)_z$ is generated as an $\mathcal{A}_z$ module by the germs at z of the polynomials in $\mathcal{M}(P)$. In particular by Theorem 7.2.9 in [2] if Ω is any domain of holomorphy in $\mathbf{C}^n$ we have a long exact sequence

$$(8) \qquad 0 \longrightarrow A(\Omega)^{s_n} \xrightarrow{Q^n} A(\Omega)^{s_{n-1}} \xrightarrow{Q^{n-1}} \cdots \xrightarrow{Q^0} A(\Omega)^q \xrightarrow{P} A(\Omega)^p.$$

Now let $\Omega \neq \mathbf{C}^n$ be a domain of holomorphy in $\mathbf{C}^n$ and assume for each $k = -1, 0, 1, \cdots, n+1$ we have an open covering $(\Omega_\nu^k)_{\nu \geq 1}$ of Ω such that the following conditions are satisfied.

(a)$'$ *There exists a positive constant A such that if $z \in \Omega_\nu^k$ then*

$$2 \operatorname{diam}(\Omega_\nu^k) \leq d(z) \leq A \operatorname{diam}(\Omega_\nu^k).$$

(b)$'$ *There exists an integer M such that for each k more than M different sets Ω_ν^k always have an empty intersection.*

(c)$'$ *There exists a partition of unity $(\chi_\nu)_{\nu \geq 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu^{n+1})$ such that $\Sigma_\nu |\bar\partial \chi_\nu(z)| \leq A\, d(z)^{-1}$, $z \in \Omega$.*

(d)$'$ *Each set Ω_ν^k is a domain of holomorphy.*

(e)$'$ *For each ν, $\Omega_\nu^{k+1} \subset \Omega_\nu^k$, $k = -1, 0, \cdots, n$, and* $\operatorname{dist}(\Omega_\nu^{k+1}, \partial\Omega_\nu^k) \geq$ $t^{-1} \operatorname{diam}(\Omega_\nu^{k+1})$ *where $0 < t < 1$ will be specified later.*

The proof that open covers satisfying all of these hypotheses exist will be given below in Lemma 2. M and A may be chosen so as not to depend on Ω. We note that $\Omega_\nu = \Omega_\nu^{n+1}$ gives an open cover which satisfies hypotheses (a)–(d) used earlier.

We define $C^\sigma((\Omega_\nu^k), \mathcal{R}(P), \varphi)$ as before except we now require $c_\alpha \in A(\Omega_\alpha^k)^q$ and $Pc_\alpha = 0$ in Ω_α^k. We note $Z_{(0,0)} = \mathcal{A}$. If $k < l$ and c is a cochain relative to $(\Omega_\nu^k)_{\nu \geq 1}$ we denote by ρc the restriction of c to $(\Omega_\nu^l)_{\nu \geq 1}$ defined by $(\rho c)_\alpha = c_\alpha|_{\Omega_\alpha^l}$.

THEOREM 3. *Let Ω be a domain of holomorphy in $\mathbf{C}^n$ and let φ be a plurisubharmonic function in Ω which satisfies* (1). *Let P and Q^k be as above and assume that t is good for $Q^0, \cdots, Q^n$.*

If $\sigma \geq 1$ and $c \in C^\sigma((\Omega_\nu^0), \mathcal{R}(P), \varphi)$ and $\delta c = 0$ then there exists $c' \in C^{\sigma-1}((\Omega_\nu^{n+1}), \mathcal{R}(P), \varphi + m\psi + N\theta)$ such that $\delta c' = \rho c$ and $\|c'\|_{\varphi + m\psi + N\theta} \leq K \|c\|_\varphi$. Here $m, N \geq 0$ are integers which depend on P and $K > 0$ is a constant which depends on a and P.

PROOF. In the proof K will denote various constants which may depend on

P, a and Q^k. We denote P by Q^{-1}. We will prove by induction if $-1 \leq k \leq n$ then:

(9) *If* $\sigma > 0$ *and* $c \in C^\sigma((\Omega_\nu^{k+1}), R(Q^k), \varphi)$ *and* $\delta c = 0$ *then there exists* $c' \in C^{\sigma-1}((\Omega_\nu^{n+1}), R(Q^k), \varphi + m\psi + N\theta)$ *such that* $\delta c' = \rho c$ *and* $\|c'\|_{\varphi+m\psi+N\theta} \leq K\|c\|_\varphi$.

When $k = -1$ we have the statement of the theorem. When $k = n$ then $R(Q^n) = 0$ implies $c = 0$ and so (9) holds. Now assume that $n > k \geq -1$ and that we have proved (9) for all larger values of k and for all plurisubharmonic functions φ which satisfy (1).

Let $\alpha = (\alpha_0, \cdots, \alpha_\sigma)$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_\sigma$ and assume that Ω_α^{k+2} is not empty. Let $r = t^{-1} \min_{\nu\in\alpha} \operatorname{diam}(\Omega_\nu^{k+2})$. Since $d(z) \leq 1$ hypothesis (a)$'$ implies $r \leq (2t)^{-1}$. If $z \in \Omega_\alpha^{k+2}$ then by (e)$'$

$$\operatorname{dist}(z, \partial\Omega_\alpha^{k+1}) = \min_{\nu\in\alpha} \operatorname{dist}(z, \partial\Omega_\nu^{k+1}) \geq r.$$

We also have

$$\operatorname{diam}(\Omega_\alpha^{k+2}) \leq \min_{\nu\in\alpha} \operatorname{diam}(\Omega_\nu^{k+2}) = tr.$$

It follows that if U is the open ball with center z and radius r and if U' is the open ball with center z and radius tr then $\Omega_\alpha^{k+2} \subseteq U' \subseteq U \subseteq \Omega_\alpha^{k+1}$. By the exactness of (8) with Ω replaced by Ω_α^{k+1} we have $c_\alpha = Q^{k+1}\widetilde{g}_\alpha$ where $\widetilde{g}_\alpha \in A(\Omega_\alpha^{k+1})^{s_{k+1}}$. Since $t > 0$ is good for Q^{k+1} Lemma 1 with $B = (2t)^{-1}$ implies we have $g_\alpha \in A(\Omega_\alpha^{k+2})^{s_{k+1}}$ such that $Q^{k+1}g_\alpha = c_\alpha$ in Ω_α^{k+2} and

$$r^{m'}\int_{\Omega_\alpha^{k+2}} |g_\alpha|^2 e^{-\varphi-N'\theta}\,dV \leq Ce^a\int_{\Omega_\alpha^{k+1}} |c_\alpha|^2 e^{-\varphi}dV \tag{10}$$

where C depends on t and Q^{k+1}. Here we have used (1) and the fact that if z, $w \in \Omega_\alpha^{k+1}$ then $|z-w| < \operatorname{diam}(\Omega_\alpha^{k+1}) \leq \frac{1}{2}d(z)$. Now if $z \in \Omega_\alpha^{k+2}$ then by (a)$'$ $d(z) \leq A \min_{\nu\in\alpha} \operatorname{diam}(\Omega_\nu^{k+2}) = Art$. Thus we have found $g \in C^\sigma((\Omega_\nu^{k+2}), A^{s_{k+1}}, \varphi + m'\psi + N'\theta)$ such that $Q^{k+1}g = \rho c$ and $\|g\|^2_{\varphi+m'\psi+N'\theta} \leq Ce^a(At)^{m'}\|c\|^2_\varphi$. Then $Q^{k+1}\delta g = \rho\delta c = 0$ implies that

$$\delta g \in C^{\sigma+1}((\Omega_\nu^{k+2}), R(Q^{k+1}), \varphi + m'\psi + N'\theta).$$

By the inductive hypothesis we can find

$$f \in C^\sigma((\Omega_\nu^{n+1}), R(Q^{k+1}), \varphi + m''\psi + N''\theta)$$

where $m'' \geq m'$ and $N'' \geq N'$ such that $\delta f = \rho\delta g$ and

$$\|f\|_{\varphi+m''\psi+N''\theta} \leq K\|\delta g\|_{\varphi+m'\psi+N'\theta}.$$

Then $\rho g - f \in C^\sigma((\Omega_\nu^{n+1}), A^{s_{k+1}}, \varphi + m''\psi + N''\theta)$ and $\delta(\rho g - f) = 0$. Therefore by the corollary to Theorem 2 (with $p = q = 0$) there exists

$$h \in C^{\sigma-1}((\Omega_\nu^{n+1}), A^{s_{k+1}}, \varphi + (m'' + 2)\psi + (N'' + 2)\theta)$$

such that $\delta h = \rho g - f$ and

$$\|h\|_{\varphi+(m''+2)\psi+(N''+2)\theta} \leqslant K\,\|\rho g - f\|_{\varphi+m''\psi+N''\theta}.$$

Now let $c' = Q^{k+1}h$, $m = m'' + 2$ and choose $N > N'' + 2$ such that $N - N'' - 2$ is an upper bound for the degrees of the polynomials which occur in Q^{k+1}. Then $\delta c' = Q^{k+1}\delta h = \rho Q^{k+1}g = \rho c$ and

$$\|c'\|_{\varphi+m\psi+N\theta} \leqslant K\,\|h\|_{\varphi+m\psi+(N''+2)\theta}.$$

Since $m'' \geqslant m'$ and $N'' \geqslant N'$ we also have

$$\|g\|_{\varphi+m''\psi+N''\theta} \leqslant \|g\|_{\varphi+m'\psi+N'\theta}$$

and hence the inequality in (9) follows.

Theorem 1 now follows immediately in the same way as Theorem 7.6.11 in [2].

PROOF OF THEOREM 1. We assume that t is good for $Q^0, \cdots, Q^n$ and also good for P. For each $\nu \geqslant 1$ let $h_\nu = Pu_{|\Omega_\nu^{-1}}$. Then $h \in C^0((\Omega_\nu^{-1}), A^p, \varphi)$ and $\|h\|_\varphi^2 \leqslant M\int_\Omega |Pu|^2 e^{-\varphi}dV$. Since t is good for P the same argument as in the proof of Theorem 3 shows that we can find $g \in C^0((\Omega_\nu^0), A^q, \varphi + m'\psi + N'\theta)$ such that $Pg = \rho h$ and $\|g\|_{\varphi+m'\psi+N'\theta} \leqslant K\,\|h\|_\varphi$. Then $P\delta g = \delta Pg = \rho\delta h = 0$ implies that $\delta g \in C^1((\Omega_\nu^0), R(P), \varphi + m'\psi + N'\theta)$. By Theorem 3 we can find $f \in C^0((\Omega_\nu^{n+1}), R(P), \varphi + m\psi + N\theta)$ where $m \geqslant m'$, $N \geqslant N'$ such that $\delta f = \rho\delta g$ and $\|f\|_{\varphi+m\psi+N\theta} \leqslant K\,\|\delta g\|_{\varphi+m'\psi+N'\theta}$. Then $\delta(\rho g - f) = 0$ implies there exists $v \in A(\Omega)^q$ such that $v = g_\nu - f_\nu$ in Ω_ν^{n+1}. Now $Pv = Pg_\nu = h_\nu = Pu$ in Ω_ν^{n+1} and therefore $Pv = Pu$. Moreover

$$\int_\Omega |v|^2 e^{-\varphi-m\psi-N\theta}dV \leqslant \|\rho g - f\|^2_{\varphi+m\psi+N\theta}$$

and again because $m \geqslant m'$ and $N \geqslant N'$ we obtain (2).

We now turn to the proof of Lemma 1. By an argument similar to the one which precedes the corollary to Theorem 1 it suffices to prove the following result.

LEMMA 1′. *Let P be a $p \times q$ matrix over $\mathbf{C}[z_1, \cdots, z_n]$ and let U be the open unit ball centered at the origin in $\mathbf{C}^n$. Then there exist constants $C > 0$, $0 < t < 1$ and integers $m, N \geqslant 0$ such that if $z \in \mathbf{C}^n$, $r > 0$ and $u \in A(z + rU)^q$ then there exists $v \in A(z + trU)^q$ such that $Pu = Pv$ in $z + trU$ and*

$$\sup_{z+trU} |v| \leqslant C(1 + r^{-m})(1 + |z| + r)^N \sup_{z+rU} |Pu|. \tag{11}$$

PROOF. This lemma is a refinement of Proposition 7.6.5 in [2]. The first part of the proof, induction on p, given there (p. 188) also works in the present case with obvious changes and we omit it. The second part of the proof, induction on the dimension n, is also the same [2, p. 192], but more care in the estimates is required, and therefore we give the proof here.

(12) *The lemma is true in dimension n for $p = 1$, if it is true in dimension $n - 1$ for all p.*

Since $p = 1$, $P = (P_1, \cdots, P_q)$. We may assume P_1 is not identically zero and $\mu = \deg P_1 = \max \deg P_j$. Define

$$\widetilde{P}_1(z; r) = \sup_{|w|<r} |P_1(z + w)|.$$

If we apply Lemma 7.6.6 in [2] to the polynomial $w \longrightarrow P_1(z + rw)$ we obtain

$$\widetilde{P}_1(z; r) \leqslant C \sup_{\theta\in\Theta} \inf_{|\tau|=1} |P_1(z + r\tau\theta)|$$

where Θ is a finite subset of $U \sim \{0\}$ and Θ and C may be chosen to depend only on μ and n. We let $E(\theta, r)$ be the set of all z in $\mathbf{C}^n$ such that

$$\widetilde{P}_1(z; r) \leqslant C \inf_{|\tau|=1} |P_1(z + r\tau\theta)|.$$

It is sufficient to prove that for each $\theta \in \Theta$ the lemma is true when $z \in E(\theta, r)$, since Θ is a finite set and so only a finite number of constants, all independent of r, will occur.

We now assume $\theta \in \Theta$ and $z \in E(\theta, r)$ are fixed. By a unitary transformation of coordinates we may assume $\theta = (0, 0, \cdots, s)$ where $0 < s < 1$. Let

$$f = P_1 u_1 + \cdots + P_q u_q \in A(z + rU).$$

If we apply Lemmas 7.6.7 and 7.6.8 in [2] to the polynomial $w \longrightarrow P_1(z + rw)$ we obtain a polydisk $\Delta = \{w \in \mathbf{C}^n\colon |w_j| < s', j = 1, \cdots, n - 1, |w_n| < s\}$ such that $\Delta \subseteq U$ and holomorphic functions $g, h \in A(z + r\Delta)$ such that

$$f = P_1 g + h \quad \text{in } z + r\Delta \tag{13}$$

and h is a polynomial in w_n of degree $<\mu^+$, where μ^+ is the number of roots of $\tau \longrightarrow P_1(z', \tau)$ in $|\tau - z_n| < rs$, where $z' = (z_1, \ldots, z_{n-1})$. Here s' may be chosen to depend only on s, μ and n, and g and h may be chosen so that

$$\widetilde{P}_1(z; r) \sup_{z+r\Delta} |g| + \sup_{z+r\Delta} |h| \leqslant C \sup_{z+r\Delta} |f| \tag{14}$$

where C is a constant which may be chosen to depend only on μ and n. By the same argument as in [2, p. 193] we obtain

$$P_1 f_1 + \cdots + P_q f_q = h \tag{15}$$

where $f_j \in A(z + r\Delta)$ and $f_2, \cdots, f_q$ are polynomials in w_n of degree $<\mu^+$. Then $P_1 f_1$ is a polynomial in w_n of degree $<\mu + \mu^+$. Now by Lemma 7.6.8 in [2] applied to the polynomial $w \longrightarrow P_1(z + rw)$ we have $P_1 = P^+P^-$ where P^+ and P^- are bounded functions in $A(z + r\Delta)$ and are polynomials in w_n. Moreover P^+ is monic, has degree μ^+ and if $\Delta' = \{w' \in \mathbf{C}^{n-1}: |w_j'| < s', j = 1, \cdots, n-1\}$ then for $w' \in z' + r\Delta'$ all the roots of $\tau \longrightarrow P^+(w', \tau)$ lie in $|\tau - z_n| < rs$. Moreover we have

$$(16) \qquad r^{-\mu^+}\widetilde{P}_1(z; r) \leqslant C \inf_{z+r\Delta} |P^-|, \qquad \sup_{z+r\Delta} |P^-| \leqslant C r^{-\mu^+}\widetilde{P}_1(z; r)$$

where C is a constant which may be chosen to depend only on μ and n. Since $P_1 f_1$ is a polynomial in w_n Lemma 7.6.9 in [2] implies $P^- f_1$ is a polynomial in w_n. Thus if we define $f_j' = P^- f_j$, $j = 1, \cdots, q$ and $h' = P^- h$ then these functions are holomorphic in $z + r\Delta$ and are polynomials in w_n of degree $< 2\mu$, and by (15)

$$(17) \qquad P_1 f_1' + \cdots + P_q f_q' = h'.$$

Moreover, from (14) and (16) we have the estimate

$$(18) \qquad \sup_{z+r\Delta} |h'| \leqslant C r^{-\mu^+}\widetilde{P}_1(z; r) \sup_{z+rU} |f|$$

where C is a constant which depends only on μ and n. Now f_j' and h' are polynomials in w_n with coefficients which are holomorphic in $z' + r\Delta'$ and therefore (17) is equivalent to a system of 3μ equations in the coefficients of the f_j' where the right-hand sides are the coefficients of h' and zeros. By the inductive hypothesis we can find constants $C > 0$, $0 < t' < 1$ and integers $m, N \geqslant 0$ such that if U' is the open unit ball in $\mathbf{C}^{n-1}$ with center at the origin then we have holomorphic functions $F_{jk} \in A(z' + t'rs'U')$ such that the polynomials $F_j(w) = \Sigma_k F_{jk}(w') w_n^k$ satisfy the equation $P_1 F_1 + \cdots + P_q F_q = h'$ and

$$(19) \qquad \sum \sup |F_{jk}| \leqslant C(1 + r^{-m})(1 + |z'| + r)^N \sum \sup |h_k|$$

where the first supremum is over $z' + t'rs'U'$ and the second is over $z' + rs'U'$ and where $h'(w) = \Sigma h_k(w') w_n^k$. Now we define

$$(20) \qquad v_1 = g + F_1/P^-, \qquad v_j = F_j/P^-, \quad j \geqslant 2.$$

Then $P_1 v_1 + \cdots + P_q v_q = P_1 g + h = f = Pu$ and it remains to check the estimate. Note if $D = \{\tau \in \mathbf{C}: |\tau| < s\}$ then the v_j are holomorphic in $z + r((t's'U') \times D)$.

We choose t with $0 < t \leqslant \min(s, t's')$. Since Θ is finite and s' depends only on s, μ and n only a finite number of values of s and s' have to be considered. Again since Θ is finite only a finite number of unitary transformations (at the beginning of the proof) have to be considered. For each of these P gives rise to a system of 3μ linear equations as above and hence only a finite number of t' arise.

Thus we may choose one value of t which satisfies the above conditions in all cases. Note we have $tU \subseteq (t's'U') \times D$. The same reasoning shows that the constants C, m and N in (19) may be chosen to depend only on P.

Now from (20), (16) and (14) we have

$$\sum \sup_{z+trU} |v_j| \leqslant \frac{C}{\widetilde{P}_1(z;r)} \left\{ \sup_{z+rU} |f| + r^{\mu^+} \sum \sup_{z+trU} |F_j| \right\}. \tag{21}$$

We now estimate the right-hand side of (21). For the first term we note if we apply the remark at the top of [2, p. 190] to the polynomial $w \longrightarrow P_1(z + rw)$ we obtain $\widetilde{P}_1(z;r) \geqslant Cr^{\mu}$ where the positive constant C depends only on the top order coefficients of P_1. Since $f = Pu$ we have

$$\frac{1}{\widetilde{P}_1(z;r)} \sup_{z+rU} |f| \leqslant Cr^{-\mu} \sup_{z+rU} |Pu| \tag{22}$$

where C depends only on P. Since $tU \subseteq (t's'U') \times D$ and $s < 1$ we have by (19)

$$\begin{aligned} \sum \sup_{z+trU} |F_j| &\leqslant C(1 + |z_n| + r)^{2\mu-1} \sum \sup |F_{jk}| \\ &\leqslant C(1 + r^{-m})(1 + |z| + r)^{N+2\mu-1} \sum \sup |h_k|. \end{aligned}$$

We digress for a moment. We can find a finite subset A of U and polynomials $Q_a \in \mathbf{C}[z_1, \cdots, z_n]$ of degree $\leqslant \mu$ such that if $Q \in \mathbf{C}[z_1, \cdots, z_n]$ has degree $\leqslant \mu$ then $Q(w) = \Sigma\, Q(a) Q_a(w)$ $(a \in A)$. If we apply this result to the polynomial $w \longrightarrow Q(z + rw)$ and then replace w by $r^{-1}(w - z)$ we obtain

$$Q(w) = \sum_{a \in A} Q(z + ra) Q_a(r^{-1}(w - z))$$

and therefore

$$\widetilde{Q}(0;1) \leqslant C(1 + r^{-\mu})(1 + |z|)^{\mu} \widetilde{Q}(z;r)$$

where the constant C depends only on μ and n. The norm $Q \longrightarrow \widetilde{Q}(0;1)$ is of course equivalent to any other norm on the space of polynomials of degree $\leqslant \mu$ and therefore we obtain if $Q(w) = \Sigma\, a_\beta w^\beta$ then

$$\sum |a_\beta| \leqslant C(1 + r^{-\mu})(1 + |z|)^{\mu} \widetilde{Q}(z;r) \tag{24}$$

where the constant C depends only on μ and n. If we apply (24) for the case of polynomials in one variable to h' then we obtain

$$\sum \sup_{z'+r\Delta'} |h_k| \leqslant C(1 + (rs)^{-2\mu+1})(1 + |z_n|)^{2\mu-1} \sup_{z+r\Delta} |h'|.$$

Since only a finite number of values of s occur we may incorporate s in C. Now by (23), (18) and since $f = Pu$

$$\sum \sup_{z+trU} |F_j| \leqslant C r^{-\mu^+} \widetilde{P}_1(z;r)(1+r^{-m-2\mu+1})(1+|z|+r)^{N+4\mu-2} \sup |Pu|.$$

If we combine this inequality with (21) and (22) we obtain

$$\sum \sup_{z+trU} |v_j| \leqslant C(1+r^{-m-2\mu+1})(1+|z|+r)^{N+4\mu-2} \sup_{z+rU} |Pu|$$

which completes the proof.

Finally to finish the proof of Theorem 1 and to see that Theorems 2 and 3 are not vacuous we need to exhibit open coverings of Ω with the required properties. We formulate the result which we need in terms of open subsets of $\mathbf{R}^n$.

LEMMA 2. *Let Ω be a proper nonempty open subset of $\mathbf{R}^n$. For each $x \in \Omega$ let $d(x) = \min(1, \operatorname{dist}(x, \partial\Omega))$. If $t > 0$ is a real number and $m \geqslant 0$ is an integer then there exist $m+1$ open coverings $(\Omega_\nu^k)_{\nu \geqslant 1}$ of Ω, $k = 0, 1, \cdots, m$, such that the following properties are all satisfied.*

(a) *There exists a constant A, depending on n, m and t, such that if $x \in \Omega_\nu^k$ then* $2\operatorname{diam}(\Omega_\nu^k) \leqslant d(x) \leqslant A \operatorname{diam}(\Omega_\nu^k)$.

(b) *There exists an integer M such that for each k more than M different sets Ω_ν^k always have an empty intersection. M depends on n, m and t.*

(c) *There exists a partition of unity $(\chi_\nu)_{\nu \geqslant 1}$ with $\chi_\nu \in C_c^\infty(\Omega_\nu^0)$ such that $\Sigma_\nu |D^\beta \chi_\nu(x)| \leqslant A_\beta d(x)^{-|\beta|}$ for certain constants A_β depending on β, m, n and t.*

(d) *Each Ω_ν^k is a cube with edges parallel to the coordinate axes.*

(e) *For each ν, $\Omega_\nu^k \subseteq \Omega_\nu^{k+1}$, $k = 0, 1, \cdots, m-1$, and* $\operatorname{dist}(\Omega_\nu^k, \partial\Omega_\nu^{k+1}) \geqslant t^{-1} \operatorname{diam}(\Omega_\nu^k)$.

PROOF. Let $\mathcal{M}_0$ be the mesh of all closed cubes in $\mathbf{R}^n$ with edges of unit length and with vertices in the lattice $\mathbf{Z}^n$. Assuming we have already defined $\mathcal{M}_{k-1}$ $(k \geqslant 1)$, let $\mathcal{M}_k$ be the mesh of all closed cubes in $\mathbf{R}^n$ obtained by bisecting the edges of each cube in $\mathcal{M}_{k-1}$. Let $s > 1$ be a real number to be specified later and for each integer $k \geqslant 1$ define

$$U_k = \{x \in \Omega: s n^{1/2} 2^{-k} < d(x) \leqslant s n^{1/2} 2^{-k+1}\}$$

and let

$$\mathcal{F}_0 = \bigcup_{k \geqslant 1} \{Q \in \mathcal{M}_k: Q \cap U_k \neq \varnothing\}.$$

Let $\mathcal{F}$ be the set of all maximal cubes in $\mathcal{F}_0$. By arguments similar to those in [7, p. 167] Ω is the union of the cubes in $\mathcal{F}$ and these cubes have disjoint interiors. Moreover if $Q \in \mathcal{F}$ and $x \in Q$ then

$$(s-1)\operatorname{diam}(Q) < d(x) \leqslant (2s+1)\operatorname{diam}(Q).$$

If $s \geqslant 5/2$ and if Q, Q' are cubes in $\mathcal{F}$ which intersect then

$$\tfrac{1}{2}\operatorname{diam}(Q) \leqslant \operatorname{diam}(Q') \leqslant 2 \operatorname{diam}(Q).$$

In particular there are at most $4^n + 3^n - 2$ cubes in F which meet a given cube in F.

If $\lambda \geqslant 1$ we let $Q(\lambda)$ be the cube obtained by enlarging the cube Q by a homothetic transformation from its center in the ratio λ to 1. A computation shows if $1 \leqslant \lambda \leqslant (1/5)(2s-1)$ then for each $Q \in F$ we have $Q(\lambda) \subseteq \Omega$ and

$$2 \operatorname{diam} Q(\lambda) < d(x) < 3s \operatorname{diam} Q(\lambda), \qquad x \in Q(\lambda). \tag{25}$$

Let $Q \in F$ and suppose that $Q(\lambda)$ meets $Q'(\lambda)$ for some other $Q' \in F$. Then (25) implies that $2 \operatorname{diam} Q(\lambda) \leqslant 3s \operatorname{diam} Q'(\lambda)$ and therefore

$$Q(\lambda) \subseteq Q'((3s+1)\lambda).$$

If $|Q|$ is the volume of Q we then have $|Q'| \geqslant (3s+1)^{-n}|Q|$. By symmetry we also have $Q' \subseteq Q'(\lambda) \subseteq Q((3s+1)\lambda)$. Since the cubes Q' in F have disjoint interiors we conclude that if $1 \leqslant \lambda \leqslant (1/5)(2s-1)$ and $Q \in F$ then there are at most $5^{-n}(3s+1)^{2n}(2s-1)^n$ cubes Q' in F such that $Q(\lambda) \cap Q'(\lambda) \neq \varnothing$.

Now choose s so large that $s \geqslant 1 + 5(1 + 2t^{-1}n^{1/2})^m$ and let $\lambda_k = 2(1 + 2t^{-1}n^{1/2})^k$ and define $\Omega_\nu^k =$ interior of $Q_\nu(\lambda_k)$, where $F = (Q_\nu)_{\nu \geqslant 1}$. Then conditions (a), (b) and (d) are certainly satisfied. Moreover $\Omega_\nu^{k+1} = \Omega_\nu^k(1 + 2t^{-1}n^{1/2})$ and so (e) is satisfied. Finally to obtain the partition of unity which satisfies (c) we argue as in [7].

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