I-RINGS

BY

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ABSTRACT. A ring $R$, possibly with no identity, is called an $I_0$-ring if each one-sided ideal not contained in the Jacobson radical $J(R)$ contains a non-zero idempotent. If, in addition, idempotents can be lifted modulo $J(R)$, $R$ is called an $I$-ring. A survey of when these properties are inherited by related rings is given. Maximal idempotents are examined and conditions when $I_0$-rings have an identity are given. It is shown that, in an $I_0$-ring $R$, primitive idempotents are local and primitive idempotents in $R/J(R)$ can always be lifted. This yields some characterizations of $I_0$-rings $R$ such that $R/J(R)$ is primitive with nonzero socle. A ring $R$ (possibly with no identity) is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents can be lifted modulo $J(R)$. These rings are characterized in several new ways: among them as $I_0$-rings with no infinite orthogonal family of idempotents, and as $I_0$-rings $R$ with $R/J(R)$ semisimple artinian. Several other properties are derived. The connection between $I_0$-rings and the notion of a regular module is explored. The rings $R$ which have a regular module $M$ such that $J(R) = \text{ann}(M)$ are studied. In particular they are $I_0$-rings. In addition, it is shown that, over an $I_0$-ring, the endomorphism ring of a regular module is an $I_0$-ring with zero radical.

1. Definitions and examples. Throughout this paper all rings are assumed to be associative but do not necessarily have an identity. When a ring has an identity, modules are assumed to be unital. Unless otherwise stated, all modules are left modules and homomorphisms are written on the right of their arguments. The Jacobson radical of a ring $R$ will be denoted by $J(R)$.

**Lemma 1.1.** If $R$ is a ring the following conditions are equivalent:

1. Every left ideal $L \subseteq J(R)$ contains a nonzero idempotent.

2. Every right ideal $T \subseteq J(R)$ contains a nonzero idempotent.

3. If $a \in J(R)$ then $xax = x$ for some $x \neq 0$.

**Proof.** Given (1), let $a \in J(R)$. If $e = ra \in Ra$ is a nonzero idempotent,
(3) follows with $x = \text{rar}$. The converse is clear and the proof that (1) $\iff$ (3) is analogous.

**Definition 1.2.** A ring $R$ is called an $I_0$-ring if it satisfies the conditions of Lemma 1.1. An $I_0$-ring in which idempotents can be lifted modulo $J(R)$ is called an $I$-ring.

An $I$-ring with a nil Jacobson radical will be called a Zorn ring. This terminology is used by Kaplansky [5, p. 19], and Bourbaki [3, p. 75]. However the reader is cautioned that Jacobson [4, p. 210] and Levitzky [8, p. 385] refer to our Zorn rings as $I$-rings. Many of the results below will be true for Zorn rings, but we shall not mention this fact in most instances. Our primary concern here is with $I$-rings. A deep study of Zorn rings can be found in Levitzky [8].

The class of $I$-rings is quite large. It obviously contains all division rings and, more generally, contains all local rings where, in this paper, a ring $R$ will be called local if it has an identity and $R/J(R)$ is a division ring. In a different direction, each primitive ring with a minimal left ideal is an $I$-ring. In fact every semiprime ring with essential socle is an $I$-ring. On the other hand any $I_0$-ring with identity and no divisors of zero is a local ring, so the integers are an example of a noetherian semiprime ring which is not an $I$-ring.

An element $a$ in a ring $R$ is called (von Neumann) regular if $aba = a$ for some $b \in R$. The ring $R$ is called regular if each of its elements is regular and $R$ is called $\pi$-regular if some power of each element is regular. It is easily verified that every $\pi$-regular ring is an $I$-ring (in fact a Zorn ring). In particular, every algebraic algebra is an $I$-ring [4, p. 210].

**Lemma 1.3.** Let $R$ be a ring, let $L$ be a left (right) ideal of $R$ and let $x \in L$. If there is an idempotent $f$ such that $f - x \in J(R)$, then there exists an idempotent $e \in L$ such that $e - x \in J(R)$.

**Proof.** If $f^2 = f$ and $f - x \in J(R)$, choose $a \in J(R)$ such that $a + (f - x) = a(f - x)$. Then $faf + f - fx = f(x - ax)f$. Take $e = f(x - ax)$. □

**Proposition 1.4.** Let $R$ be a ring in which idempotents can be lifted modulo $J(R)$. Then $R$ is an $I$-ring if and only if $R/J(R)$ is an $I$-ring.

A ring $R$ (possibly with no identity) will be called semiperfect if $R/J(R)$ is artinian and idempotents can be lifted modulo $J(R)$.

**Corollary 1.5.** Every semiperfect ring is an $I$-ring.

**Proposition 1.6.** If $R$ is an $I$-ring ($I_0$-ring) so is each one-sided ideal of $R$.

**Proof.** Let $L$ be a left ideal of $R$. Then $J(L) = \{a \in L \mid L a \subseteq J(R)\}$ [1,
If $M$ is a left ideal of $L$ and $M \trianglelefteq J(L)$ then $LM \trianglelefteq J(R)$ so there exists $0 \neq e^2 = e \in LM \subseteq M$. Suppose now that $x \in L$, $x^2 - x \in J(L)$. Then $x^4 - x^2 \in J(R)$ so, by Lemma 1.3, choose $e^2 = e \in L$ such that $e - x^2 \in J(R)$. Then $e - x \in J(L)$. □

**Corollary 1.7.** If $R$ is an $I$-ring ($I_0$-ring) so is each subring $aRb$ where $a, b \in R$.

**Proof.** $aRb$ is a left ideal of $aR$. □

Note that Proposition 1.6 and its corollary are true for Zorn rings as well. We remark here that the center of an $I$-ring need not be an $I$-ring (see Example 1.9).

The $n \times n$ matrix ring over a ring $R$ will be denoted by $M_n(R)$. If $r \in R$ let $E_{ij}(r)$ denote the matrix with $r$ in the $(i, j)$-position and zeros elsewhere.

**Proposition 1.8.** If $R$ is an $I_0$-ring so is the ring $M_n(R)$ of all $n \times n$ matrices over $R$.

**Proof.** Let $L$ be a left ideal of $M_n(R)$ with $L \trianglelefteq J[M_n(R)] = M_n[J(R)]$. There exists a matrix $A = (a_{ij}) \in L$ with $a_{pq} \in J(R)$ for some $p, q$. Then $L_0 = \{x \in R|x = x_{pq}$ for some $(x_{ij}) \in L\}$ is a left ideal of $R$ and $L_0 \trianglelefteq J(R)$. If $0 \neq e^2 = e \in L_0$, let $X = (x_{ij}) \in L$ with $x_{pq} = e$. One verifies that $E_{qp}(e)X = \Sigma_j E_{j}(e_{pj})$ is a nonzero idempotent in $L$. □

A natural question is the following: Is $M_n(R)$ an $I$-ring whenever $R$ is an $I$-ring? It is sufficient to answer this in the case $n = 2$. The answer is affirmative if $R$ is semiperfect or if $J(R)$ is locally nilpotent. More generally, one can ask: If a ring $R$ is such that idempotents can be lifted modulo $J(R)$, does $M_2(R)$ have the same property? We remark in this connection that it is an open question whether $M_2(R)$ is a nil ring whenever $R$ is a nil ring (see [7]).

It is obvious that a direct sum of rings is an $I$-ring (an $I_0$-ring) if and only if the same is true of each summand. It is also clear that $R/A$ is an $I$-ring ($I_0$-ring) if $R$ is an $I$-ring ($I_0$-ring) and $A \subseteq J(R)$ is an ideal. However the class of $I$-rings is not closed under homomorphic images.

**Example 1.9.** Let $\Delta$ be a division ring and let $S \subseteq \Delta$ be any subring. Let $R$ be the ring of all countably infinite square matrices of the form

$$
\begin{pmatrix}
A & 0 \\
\delta & \ddots
\end{pmatrix}
$$

where $A \in M_n(\Delta)$ for some $n \geqslant 1$ and $\delta \in S$. Denote this matrix by $(A, \delta)$. If
(A, δ) ≠ 0 we can assume A ≠ 0 so, by Proposition 1.8, choose a matrix B such
that BA is a nonzero idempotent. Then (B, 0)(A, δ) = (BA, 0) is an idempotent
in R, and it follows that R is an I₀-ring with J(R) = 0. The map R → S given
by (A, δ) → δ is a ring epimorphism so it is clear that a homomorphic image of
an I-ring need not be an I-ring. If S is in the center of A, the center of R is
isomorphic to S and so need not be an I₀-ring. □

2. Primitive idempotents. In any ring R there is a natural partial ordering
of the idempotents defined by f ≤ e if f ∈ eRe. A nonzero idempotent which is
minimal in this partial ordering is called primitive. It is easily verified that e is
primitive if and only if 0 ≠ f² = f ∈ Re implies Re = fRe.

We say that e = e² is a local idempotent if eRe is a local ring. It is well
known that every primitive idempotent in a semiperfect or regular ring is local.
We generalize this as follows:

PROPOSITION 2.1. The following conditions are equivalent for an idempotent e in an I₀-ring R:

1. e is primitive.
2. If L ⊆ Re is a left ideal and L ⊆ J(R) then L = Re.
3. e is local.

PROOF. (1) → (2) by the above remark (since R is an I₀-ring) and (3) ⇒
(1) is obvious. Assume (2). If a ∈ eRe and a ∈ J(eRe) = eRe ∩ J(R), then
Ra = Re by (2). Hence a has a left inverse in eRe, proving (3). □

An immediate consequence of this is that an I₀-ring has a unique nonzero
idempotent if and only if it has the form L ⊙ A where L is local and J(A) = A.

Two idempotents e and f in a ring R are said to be equivalent if there exist
x ∈ eRf and y ∈ fRe such that e = xy and f = yx.

COROLLARY 2.2. Two primitive idempotents e and f in an I₀-ring R are
equivalent if and only if eRf ⊆ J(R).

PROOF. If x ∈ eRf, x ∈ J(R), we have Rx = Rf by Proposition 2.1. Write
f = yx with y ∈ fRe. Then 0 ≠ (yx)² = xy ∈ eRe so xy = e. The converse is
trivial. □

We have the following result on the existence of primitive idempotents:

PROPOSITION 2.3. An I₀-ring with J(R) = 0 has a primitive idempotent
if and only if it has a maximal left (right) annihilator.

PROOF. Let L = {a ∈ R|aS = 0} be a maximal left annihilator. If 0 ≠
x ∈ S and 0 ≠ e² = e ∈ xR, we have L = {a|ae = 0} by maximality. We claim
e is primitive. If $0 \neq f^2 = f \in eRe$ we have $L = \{a|af = 0\}$ so $e - f \in L$. But then $0 = (e - f)e = e - f$. The converse follows from Proposition 2.1. □

We have immediately the following which retrieves a result of Koh [6] when $R$ is regular.

**Corollary 2.4.** An $I_0$-ring $R$ with $J(R) = 0$ is primitive with nonzero socle if and only if it is a prime ring with a maximal left (right) annihilator.

The next result shows that, in an $I_0$-ring, primitive idempotents in $R/J(R)$ can always be lifted to $R$. It will be referred to several times below.

**Lemma 2.5.** Let $R$ be an $I_0$-ring and suppose $x \in R$ is such that $x + J(R)$ is a primitive idempotent in $R/J(R)$. There exists an idempotent $e \in R$ such that $e - x \in J(R)$.

**Proof.** Choose $0 \neq f^2 = f \in Rx$. Then, in $\overline{R} = R/J(R)$, $f \in \overline{R}x$ so $\overline{R}f = \overline{R}x$ by Proposition 2.1. If we set $e = f + xf - fxf$, then $e^2 = e \neq 0$ and $\overline{e} = \overline{x}$. □

**Proposition 2.6.** The following are equivalent for a ring $R$:

1. $R$ is an $I_0$-ring and every nonzero idempotent contains a primitive idempotent.
2. Each left (right) ideal $L \subseteq J(R)$ contains a primitive idempotent.

**Proof.** If (2) holds and $e^2 = e \neq 0$ let $f^2 = f \in Re$ be primitive. Then $(ef)^2 = ef \in eRe$ and $ef$ is primitive by Proposition 2.1 since $Re = Rf$. Hence (2) ⇒ (1); the converse is clear. □

We say a ring $R$ has primitive idempotents if it satisfies these conditions. Lemmas 2.5 and 1.3 immediately yield

**Corollary 2.7.** An $I_0$-ring $R$ has primitive idempotents if and only if the same is true of $R/J(R)$.

Hence every semiperfect ring has primitive idempotents. It is well known that a primitive ring with nonzero socle has primitive idempotents (the socle is large as a left ideal). The ring in Example 1.9 is easily shown to be primitive with nonzero socle and so a homomorphic image of a ring with primitive idempotents need not have this property. On the other hand, the methods of §1 show that, if $R$ has primitive idempotents, the same is true of any one-sided ideal of $R$, any subring of the form $aRb$, and any matrix ring $M_n(R)$.

If $R$ is a ring with identity, a module $M \neq 0$ is called local if it is projective and $Rx = M$ for each $x \in M - \text{rad } M$. If $e^2 = e \in R$ then $\text{rad}(Re) = J(R)e$ and it follows easily from Proposition 2.1 that $Re$ is local if and only if $e$ is a local idempotent.

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Theorem 2.8. The following are equivalent for an $I_0$-ring $R$:

1. $R$ has primitive idempotents and any two primitive idempotents are equivalent.
2. $R$ contains a primitive idempotent and $J(R)$ is a prime ideal.
3. $R/J(R)$ is a primitive ring with nonzero socle.

If $R$ has an identity these are equivalent to

4. $R$ has a local module and $J(R)$ is a prime ideal.

Proof. If $R$ has an identity, (2) $\Leftrightarrow$ (4) by the remark above; (3) $\Rightarrow$ (1) by Corollary 2.7 and [4, p. 51]; and (2) $\Rightarrow$ (3) is clear. If (1) holds let $aRb \subseteq J(R)$ where $a, b \in R - J(R)$. Choose primitive idempotents $e \in Ra$ and $f \in bR$. By hypothesis, $e = xy$ where $x \in eRf$ and $y \in fRe$ so $e \in RaRbR \subseteq J(R)$, a contradiction. This means $J(R)$ is a prime ideal so (1) $\Rightarrow$ (2). \[\square\]

In fact, the equivalence of (2) and (4) can be proved under the weaker assumption that $J(R)$ is small in $R$ as a left ideal. In this form it generalizes the fact that a prime ring is primitive with nonzero socle if and only if it has an irreducible projective module.

3. Maximal idempotents. An idempotent $e$ in a ring $R$ is said to be the greatest idempotent if $e \geq f$ for every idempotent $f$ in $R$. Such an idempotent is central since $e \geq e + er - ere$ and $e \geq e + re - ere$ for each $r \in R$. Hence:

Lemma 3.1. An $I_0$-ring with a greatest idempotent has the form $S \oplus A$ where $S$ has an identity and $J(A) = A$.

An idempotent $e$ is called maximal if it is not the greatest idempotent and $e \leq f, f^2 = f$ implies $e = f$ or $f$ is the greatest idempotent.

Lemma 3.2. Let $R$ be a ring with no greatest idempotent. The following are equivalent for an idempotent $e \in R$:

1. $e$ is maximal.
2. If $Re \subseteq Rf$ where $f^2 = f$ then $Re = Rf$.
3. If $ef = 0$ where $f^2 = f$ then $f = 0$.

Moreover the left-right analogs of (2) and (3) are equivalent to (1).

Proof. (1) $\Rightarrow$ (2). If $Re \subseteq Rf$ let $g = e + f - fe$. Then $g^2 = g \geq e$ and (2) follows.

(2) $\Rightarrow$ (3). If $ef = 0$ let $g = e + f - fe$. Then $g^2 = g$ and $Re \subseteq Rg$ so $g = ge = e$ by (2). Hence $f = f^2 = (fe)f = 0$.

(3) $\Rightarrow$ (1). If $g^2 = g \geq e$ then $e(g - e) = 0$ so $g = e$. \[\square\]

Theorem 3.3. If an $I_0$-ring $R$ has a maximal idempotent then $R/J(R)$ has
an identity and the following are equivalent:

(1) \( x \in Rx \) for each \( x \in R \).

(2) \( J(R) \) is small as a right ideal.

(3) \( R \) has a left identity.

**Proof.** If \( R \) has a greatest idempotent the result follows from Lemma 3.1. Otherwise, let \( e^2 = e \) be maximal. The right ideal \( T = \{a \in R | ea = 0\} \) is contained in \( J(R) \) by Lemma 3.2 so \( r - er \in J(R) \) for every \( r \in R \). Similarly \( r - re \in J(R) \) so \( R/J(R) \) has an identity.

Now (3) \( \Rightarrow \) (2) is well known and (2) \( \Rightarrow \) (3) follows since \( R = eR + J(R) \).

If (1) holds let \( r \in R \) and write \( x = r - er \) where \( e^2 = e \) is maximal. If \( x = tx \), \( t \in R \) then \( x = (t - te)x \) so \( x = 0 \) (since \( t - te \in J(R) \)). Hence (1) \( \Rightarrow \) (3); the converse is obvious. \( \square \)

In particular a regular ring with a maximal idempotent has an identity. The next result generalizes the well-known fact that a semisimple artinian ring has an identity.

**Corollary 3.4.** Let \( R \) be an \( I_0 \)-ring with \( J(R) = 0 \). If \( R \) has a minimal right annihilator of the form \( r(e) = \{a \in R | ea = 0\}, e^2 = e \) then \( R \) has an identity.

**Proof.** It suffices to show \( R \) has a greatest idempotent by Lemma 3.1. If not, let \( r(e) \) be minimal. If \( e \leq f, f^2 = f \) then \( r(f) \subseteq r(e) \) so either \( r(f) = r(e) \) (so \( e = f \)) or \( r(f) = 0 \) (so \( fR = R \)). It follows that \( R \) has a maximal idempotent and so has an identity, a contradiction. \( \square \)

We remark finally that an \( I \)-ring \( R \) has a maximal idempotent if and only if \( R/J(R) \) has an identity and, in this case, the maximal idempotents are just the pre-images of the identity in \( R/J(R) \).

4. Finiteness conditions. In this section we study \( I_0 \)-rings which satisfy chain conditions on the set of idempotents and obtain some new characterizations of semiperfect rings.

**Lemma 4.1.** Let \( R \) be any ring. The following are equivalent:

(1) \( R \) has maximum condition on idempotents.

(2) \( R \) has maximum condition on left ideals \( Re, e^2 = e \) (on right ideals \( eR, e^2 = e \)).

(3) \( R \) has no infinite orthogonal family of idempotents.

**Proof.** The proofs of (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) are omitted. Given (1) let \( Re_1 \subseteq Re_2 \subseteq \cdots, e_i^2 = e_i \). Define \( f_1, f_2, \ldots \) as follows: \( f_1 = e_1, f_{k+1} = f_k + e_{k+1} - e_{k+1}f_k \) for \( k \geq 1 \). Then \( f_k \in Re_k \) for each \( k \) and consequently \( f_k^2 = f_k \).
and \( f_1 \leq f_2 \leq \cdots \). If \( f_n = f_{n+1} = \cdots \) for some \( n \) then \( R e_{n+1} = R e_{n+2} = \cdots \). Hence (1) \( \Rightarrow \) (2). \( \square \)

We remark that a ring \( R \) with minimum condition in left annihilators of idempotents has maximum condition on idempotents. Moreover, the converse is true if \( R \) has no total right annihilators; that is if \( R a = 0 \) implies \( a = 0 \).

It is surprising that the minimum condition on idempotents is, in general, weaker than the maximum condition. (Consider \( R = \Delta \oplus \Delta \oplus \cdots \) where \( \Delta \) is a division ring.) The two are equivalent for rings with identity as the next result shows.

**Lemma 4.2.** The following are equivalent in any ring \( R \):

1. \( R \) has minimum condition on idempotents.
2. \( R \) has minimum condition on left ideals \( Re, e^2 = e \) (on right ideals \( eR, e^2 = e \)).
3. Any bounded ascending chain of idempotents terminates.

**Proof.** Given (1) let \( Re_1 \supseteq Re_2 \supseteq \cdots, e^2_k = e_k \). Then \( e_1 e_2 \cdots e_k \) is an idempotent for each \( k \), \( e_1 e_2 \cdots e_k = Re_k \) and \( e_1 \geq e_2 \geq \cdots \). Hence (1) \( \Rightarrow \) (2). We leave (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1) to the reader. \( \square \)

The next result contains some new characterizations of semiperfect rings (defined following Proposition 1.4).

**Theorem 4.3.** The following are equivalent for any ring \( R \):

1. \( R \) is semiperfect.
2. \( R \) is an \( I_0 \)-ring with maximum condition on idempotents.
3. If \( L \subseteq R \) is a left (right) ideal there exist an idempotent \( e \in L \) and a left (right) ideal \( M \subseteq J(R) \) such that \( L = Re + M \).
4. \( R \) is an \( I_0 \)-ring and \( R/J(R) \) is semisimple artinian.

**Proof.** (1) \( \Rightarrow \) (2). If \( R \) is semiperfect it is an \( I_0 \)-ring by Corollary 1.5. If \( e_1 \leq e_2 \leq \cdots \) are idempotents in \( R \) then \( e_n = e_{n+1} = \cdots \) in \( R/J(R) \) for some \( n \). Hence \( (e_{k+1} - e_k)^2 = (e_{k+1} - e_k) \in J(R) \) for each \( k \geq n \) and so \( e_n = e_{n+1} = \cdots \).

(2) \( \Rightarrow \) (3). If \( L \subseteq J(R) \) take \( e = 0 \) and \( M = L \). Otherwise let \( 0 \neq e^2 = e \in L \) be maximal in \( L \). Then \( L = Re + M \) where \( M = \{x \in L | xe = 0 \} \). If \( M \not\subseteq J(R) \) choose \( 0 \neq f^2 = f \in M \). Then \( g = e + f - ef \) is an idempotent in \( L \) and \( e \leq g \). Hence \( e = g \) by the choice of \( e \) and so \( f = f^2 = f(ef) = 0 \), a contradiction. Hence \( M \subseteq J(R) \).

(3) \( \Rightarrow \) (4). This is obvious.

(4) \( \Rightarrow \) (1). Since \( R \) is an \( I_0 \)-ring, primitive idempotents in \( R/J(R) \) can be
lifted by Lemma 2.5. But every idempotent in $R/J(R)$ is a sum of orthogonal primitive idempotents and so can be lifted by standard techniques. □

If $R$ has an identity it is not hard to show (using Lemma 2.3 of [2]) that condition (3) is equivalent to the condition that every cyclic $R$-module has a projective cover. The following consequences of Theorem 4.3 give, in the case when $R$ has an identity, some new proofs of well-known facts.

**Corollary 4.4.** Let $R$ be a semiperfect ring. A subring of $R$ is semiperfect if and only if it is an $I_0$-ring. In particular, one-sided ideals of $R$ and subrings of the form $aRb$, $a, b \in R$, are semiperfect.

**Proof.** Simply observe that the maximum condition on idempotents is inherited by subrings and use Proposition 1.6. □

**Corollary 4.5.** A ring $R$ is semiperfect if and only if the ring $M_n(R)$ is semiperfect for all $n \geq 1$ (some $n \geq 1$).

**Proof.** If $R$ is semiperfect then $M_n(R)$ is an $I_0$-ring by Proposition 1.8 and $M_n(R)/J[M_n(R)] \cong M_n[R/J(R)]$ is semisimple artinian. Conversely, $R$ can be embedded in $M_n(R)$ as the subring of all matrices with zeros in all positions except $(1,1)$. Since this is the intersection of a left and a right ideal, Corollary 4.4 completes the proof. □

**Corollary 4.6.** Every homomorphic image of a semiperfect ring is semiperfect.

**Proof.** If $R$ is semiperfect and $A \subseteq R$ is an ideal, let $L \supseteq A$ be a left ideal. Choose an idempotent $e \in L$ and a left ideal $M \subseteq J(R)$ such that $L = Re + M$. Then, in $\widetilde{R} = R/A$, $\widetilde{L} = \widetilde{Re} + \widetilde{M}$ and $\widetilde{M} \subseteq J(\widetilde{R})$. Hence $R/A$ is semiperfect. □

We remark that if $R$ is a semiperfect ring and $A$ is an ideal of $R$ then $J(R/A) = (A + J)/A$ where $J$ denotes $J(R)$. Indeed, there is a ring epimorphism $R/A \rightarrow R/(A + J)$ with kernel $(A + J)/A$. But $R/(A + J)$ is a homomorphic image of $R/J$ so $J(R/A) \subseteq (A + J)/A$. The reverse inclusion always holds.

5. Regular idempotents. The notion of regularity has been extended to projective modules by Ware [9] and to a wider class of modules by Zelmanowitz [10]. In this section we shall study the connection between these modules and the $I_0$-rings. In particular we show that, over an $I_0$-ring, the endomorphism ring of a regular module is an $I_0$-ring with zero radical.

Let $M$ be a left $R$-module and let $M^* = \text{Hom}_R(M, R)$. Zelmanowitz calls $M$ regular if, for any $x \in M$, there exists $\alpha \in M^*$ such that $(\alpha x)x = x$. If $M$ is an $R$-module, $x \in M$ and $\alpha \in M^*$, define a map $[\alpha, x] \in \text{end}(M)$ by $y[\alpha, x] = (y\alpha)x$.
for every \( y \in M \). We say \( M \) is projective if it has a dual basis, that is if there exist subsets \( \{ x_\nu | \nu \in I \} \subseteq M \) and \( \{ \alpha_\nu | \nu \in I \} \subseteq M^* \) (indexed by the same set \( I \)) such that, for each \( x \in M \), \( x\alpha_\nu = 0 \) for all but a finite number of \( \nu \in I \) and \( x = \sum_\nu x[\alpha_\nu, x_\nu] \). The following easily verified facts from [10] will be needed.

**Lemma 5.1.** Let \( M \) be a regular module, let \( x \in M \), and suppose \( \alpha \in M^* \) satisfies \( (x\alpha)x = x \). Then:

1. \( e = x\alpha \) is an idempotent and \( ex = x \).
2. \( \alpha |_{Rx} : Rx \to Re \) is an isomorphism so \( Rx \) is projective.
3. \( M = Rx \oplus \ker[\alpha, x] \).

This shows that, if \( M \) is regular and \( x \in M \), then \( Rx \) is a projective summand of \( M \). Zelmanowitz proves the converse [10, Theorem 2.2], and also shows that \( Rx_1 + \cdots + Rx_n \) is a projective summand for any \( x_i \in M \).

**Lemma 5.2.** If \( e^2 = e \in R \), the following are equivalent:

1. \( Re \) is a regular \( R \)-module.
2. For each \( x \in Re \) there exists \( y \in R \) such that \( xyx = x \).
3. For each \( x \in Re \) there exists \( f^2 = f \in Re \) with \( Rx = Rf \).

The easy proof is left to the reader. An idempotent \( e \) in a ring \( R \) will be called left regular if \( Re \) is a regular \( R \)-module. Clearly, every idempotent in a regular ring is left regular. Also, \( eRe \) is a regular ring if \( e \) is a left regular idempotent (the converse is false as an example below will show).

The following is a strengthening of a remark in [10, p. 346].

**Lemma 5.3.** Let \( R \) be a ring and let \( M \) be a regular \( R \)-module. If \( a \notin \text{ann} M \) then \( aR \) contains a nonzero left regular idempotent. In particular \( J(R) \subseteq \text{ann} M \).

**Proof.** If \( ax \neq 0 \), \( x \in M \), choose \( \alpha \in M^* \) such that \( [(ax)\alpha]ax = ax \). Then \( (ax)\alpha \in aR \) is a nonzero left regular idempotent by Lemma 5.1 and the fact that submodules of a regular module are regular. \( \Box \)

Prompted by this, we say that a ring \( R \) has left regular idempotents if each right ideal \( T \not\subseteq J(R) \) contains a nonzero left regular idempotent.

**Theorem 5.4.** The following are equivalent for a ring \( R \):

1. \( J(R) \) is an intersection of annihilators of regular left modules.
2. \( J(R) = \text{ann} M \) for some regular left module \( M \).
3. \( R \) has left regular idempotents.

Moreover, when this is the case, \( J(R) \) is the intersection of the annihilators of the cyclic regular left ideals \( Re, e^2 = e \in R \).
Proof. (1) $\Rightarrow$ (2) since a direct sum of regular modules is again regular [10, Theorem 2.8]. (2) $\Rightarrow$ (3) by Lemma 5.3. Hence assume (3). If $a \notin J(R)$ choose $0 \neq e^2 = e \in aR$ such that $Re$ is regular. Then $a \notin \text{ann}(Re)$ so $J(R) \supseteq \{\text{ann}(Re) | e^2 = e \text{ left regular}\}$. This must be equality by Lemma 5.3 so the proof is complete. \[ \square \]

**Corollary 5.5.** If $M$ is a regular left $R$-module and $A = \text{ann} M$, then $R/A$ has left regular idempotents and $J(R/A) = 0$.

**Proposition 5.6.** Let $R$ be a ring, let $A \subseteq J(R)$ be an ideal, let $T \subseteq R$ be a right ideal and let $e^2 = e \in R$. If $R$ has left regular idempotents so do the rings $R/A$, $T$ and $eRe$.

Proof. Let $M$ be a regular $R$ module with $\text{ann} M = J(R)$. Then $M$ is a regular $R/A$-module and $\text{ann}_{R/A}(M) = J(R/A)$.

Turning to $eRe$, $eM$ is an $eRe$-module and $\text{ann}_{eRe}(eM) = J(R) \cap eRe = J(eRe)$. We show that $eM$ is regular. If $x \in eM$ let $\alpha: M \rightarrow R$ satisfy $(\alpha x) x = x$. Define $\bar{\alpha}: eM \rightarrow eRe$ by $y \bar{\alpha} = (y \alpha)e$. Then $\bar{\alpha} \in (eM)^*$ and $(\alpha x) x = (\alpha x) e x = x$.

Finally, $TM$ is a $T$-module and $\text{ann}_T(TM) = \{t \in T | tTM = 0\} = \{t \in T | tT \subseteq J(R)\} = J(T)$. Now let $x \in TM$ and suppose $\alpha: M \rightarrow R$ satisfies $(\alpha x) x = x$. Since $(TM) \alpha \subseteq T(M\alpha) \subseteq T$, the map $\alpha_{|TM}: TM \rightarrow T$ is a $T$-homomorphism and $(\alpha x)_{|TM} x = x$. This shows $TM$ is a regular $T$-module. \[ \square \]

**Proposition 5.7.** If a ring $R$ has left regular idempotents the same is true of the matrix ring $M_n(R)$.

Proof. Let $T \subseteq J[M_n(R)]$ be a right ideal. As in the proof of Proposition 1.8, there exists an idempotent $X$ in $T$ of the form $X = \sum_{i=1}^n E_{ip}(r_i e)$. It is easy to show that each element in $M_n(R)X$ has the form $Y = \sum_{i=1}^n E_{ip}(t_i)$ where each $t_i \in Re$. By Lemma 5.2, we must show each such $Y$ is regular in $M_n(R)$. By Lemma 5.2, let $t_is_it_j = t_j$. Then $YE_{ip}(s_1)y - y = \sum_{i=2}^n E_{ip}(t_is_1t_j - t_j)$ and it suffices to show this is regular in $M_n(R)$. In other words, we may assume $Y$ has the form $Y = \sum_{i=2}^n E_{ip}(t_i)$, $t_i \in Re$. This procedure can be continued to complete the proof. \[ \square \]

**Examples and remarks.**

1. Every regular ring has left (and right) regular idempotents.

2. Every ring $R$ with $J(R) = 0$ and which has primitive idempotents has left (and right) regular idempotents.

3. If a prime ring $R$ has a nonzero regular left module then it has left regular idempotents and $J(R) = 0$.

4. Any local ring $R$ with $J(R) \neq 0$ is an example of an $I$-ring which does not have regular left (or right) idempotents.
5. The ring $R$ in Example 1.9 is primitive with nonzero socle and so has left (and right) regular idempotents. Hence, if $R$ has left regular idempotents, the same need not be true of the center of $R$ or of a homomorphic image of $R$.

6. Let $\Delta$ be a division ring and let $R = \begin{pmatrix} \Delta & \Delta \\ \Delta & \Delta \end{pmatrix} \subseteq M_2(\Delta)$. If $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $Re$ is a regular module and $\text{ann}(Re) = J(R)$. On the other hand $fR = R$ for any $0 \neq f^2 = f \in R$ so $R$ has no regular right modules. In particular, $Re$ can be regular while $eR$ is not even when $eRe$ is a division ring.

We conclude this section with some remarks about the endomorphism ring $E(M)$ of a regular module $M$. We shall write $A(M) = \{ \alpha \in E(M) | \ker \alpha \text{ is large in } M \}$. It is well known that $A(M)$ is an ideal of $E(M)$.

**Theorem 5.8.** If $M$ is a regular $R$-module, each one-sided ideal of $E(M)$ which is not contained in $A(M)$ contains a nonzero idempotent.

**Proof.** If $\alpha \in E(M) - A(M)$, choose $0 \neq x \in M$ such that $Rx \cap \ker \alpha = 0$. Then $\alpha |_{Rx}: Rx \rightarrow Rx\alpha$ is an isomorphism. We have $M = Rx\alpha \oplus K$ by Lemma 5.1 so define $\beta: M \rightarrow M$ by setting $K\beta = 0$ and $\beta |_{Rx\alpha} = (\alpha |_{Rx})^{-1}$. Then $\beta \alpha \beta = \beta \neq 0$ and the result follows. □

**Corollary 5.9.** Let $M$ be a regular module such that every monomorphism in $E(M)$ has a right inverse. Then $E(M)$ is an $I_0$-ring and $J[E(M)] = A(M)$.

**Proof.** Since $\ker \alpha \cap \ker(1 - \alpha) = 0$ for every $\alpha \in E(M)$, our hypothesis implies $A(M) \subseteq J[E(M)]$. This must be equality since $J[E(M)]$ contains no nonzero idempotent. □

**Corollary 5.10.** Let $M$ be a regular $R$-module with the property that $My \subseteq J(R)$ for every $\gamma \in M^*$ with large kernel. Then $E(M)$ is an $I_0$-ring with $J[E(M)] = 0$.

**Proof.** Suppose $\alpha \in A(M)$ and let $y \in M\alpha$. There exists $\gamma \in M^*$ such that $(y\gamma)y = y$. But $\ker \alpha \subseteq \ker(\alpha \gamma)$ and consequently $M\gamma \subseteq J(R)$. Thus $y\gamma$ is an idempotent in $J(R)$ and so $y\gamma = 0$. It follows that $M\alpha = 0$ and hence that $A(M) = 0$. □

One situation where the condition in Corollary 5.10 is met is when $R$ is an $I_0$-ring.

**Corollary 5.11.** If $R$ is an $I_0$-ring and $M$ is a regular module then $E(M)$ is an $I_0$-ring with $J[E(M)] = 0$.

**Proof.** Suppose $\gamma \in M^*$ has large kernel and $M\gamma \subseteq J(R)$. Let $e = y\gamma$ be a nonzero idempotent and put $x = ey$. Then $x\gamma = e \neq 0$ and $x = ex$ and it follows that $Rx \cap \ker \gamma = 0$. This is a contradiction since $Rx \neq 0$ and so the result follows from Corollary 5.10. □
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