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NEARNESS STRUCTURES AND PROXIMITY EXTENSIONS

BY

M. S. GAGRAT AND W. J. THRON

ABSTRACT. Proximity, contiguity and nearness structures are here studied from a unified point of view. In the discussion the role that grills can play in the theory is emphasized. Nearness structures were recently introduced by Herrlich and Naimpally. Thron pointed out the importance of grills in proximity theory. Nearness structures ν are then used to generate proximity extensions $(\phi, (X^\nu, \Pi^\nu))$ of a given LO-proximity space (X, Π), where $\Pi_\nu = \Pi$. Finally, the relation of the extensions $(\phi, (X^\nu, \Pi^\nu))$ to arbitrary extensions $(i, (Y, \Pi^*))$ is investigated.

1. **Introduction.** It was shown by Smirnov [16] that EF-proximities can be used to generate all T_2-compactifications of a given Tychonov space. Somewhat later Ivanova and Ivanov [8] introduced contiguity structures and showed that they can be employed to obtain a large class of T_1-compactifications of a given T_1-space. The concept of contiguity was further investigated and slightly modified by Terwilliger [17]. Very recently Herrlich, Naimpally, and Bentley [6], [12], [2] have introduced nearness structures and have applied them, among others, to the study of extensions of spaces. Also recently Thron [19] brought out the importance of grills in proximity theory.

All of these ideas and concepts are brought to bear here on the study of proximity extensions of proximity spaces. We introduce a construction which may have been first suggested by Bentley (see [14]) and is similar to one employed by Herrlich to obtain completions of N-spaces. The construction associates with every nearness structure ν, compatible with the given proximity Π on X, proximity extensions (X^ν, Π^ν) of the original space (X, Π). This is done in §3. That the simple construction of proximity

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extensions, employed by Leader [10] for EF-proximities cannot be extended to LO-proximities was recently shown by Naimpally and Whitfield [14]. In §4 we investigate to what extent all proximity extensions can be obtained as (X^ν, Π^ν).

In §2 we take another look at the definitions of proximity, contiguity, and nearness. This is done partly to emphasize the importance of the concept of grill, which appears naturally in $\lambda(\mathfrak{A})$ as well as in maximal λ-compatible families (for λ a contiguity or a nearness). We are also able to bring out the similarities as well as the differences between the three types of structures.

A structure λ shall be called *clan* (*bunch*) *generated* if it satisfies the condition

$$\mathfrak{A} \in \lambda \Rightarrow \exists \text{ a } \lambda\text{-clan (bunch) } \mathfrak{G} \text{ such that } \mathfrak{A} \subset \mathfrak{G}.$$

It is known [19] that all basic proximities are clan generated. In §2 we show that the same is true for all basic contiguities. Very recently Naimpally and Whitfield [13] have given an example of a nearness which is not clan generated. It follows that in this important respect nearness structures are much more complicated than proximities or contiguities.

Bunch generated structures are exactly the ones which can be topologically induced. A structure λ on X is said to be *topologically induced* if (X, c_λ) can be embedded via a map ϕ in a topological space (Y, d) and $\mathfrak{A} = [A_i: i \in I] \in \lambda$ if $|\mathfrak{A}|$ is appropriately restricted and $\bigcap [d(\phi(A_i)): i \in I] \neq \varnothing$. Here c_λ is the closure operator induced by λ (see Definition 2.5). For details on this result see Bentley [3].

In what follows there is always an underlying nonempty set X and frequently also a set $Y \supset X$. It will be convenient to denote elements of X or Y by $x, y, \ldots$, subsets by $A, B, \ldots$. Families of subsets will be denoted by $\mathfrak{A}, \mathfrak{B}, \ldots$. In particular $\mathfrak{F}$ will be used for filters, $\mathfrak{U}, \mathfrak{V}$ for ultrafilters, and $\mathfrak{G}$ for grills. Letters $\alpha, \beta, \gamma, \ldots$ shall be used for collections of families of sets (i.e. $\alpha \subset \mathfrak{P}(\mathfrak{P}(X))$). For nearness structures we shall use $\nu, \mu, \ldots$, for contiguities $\xi, \zeta, \ldots$, a collection which may be any of the three structures shall be denoted by $\lambda, \ldots$. However, for proximities we shall continue to use Π.

In analogy to its use for relations we shall employ the notation $\lambda(\mathfrak{A})$ to mean $\lambda(\mathfrak{A}) = [A: [A] \cup \mathfrak{A} \in \lambda]$. In addition $\lambda([[x]])$ shall be simplified to $\lambda(x)$ and $\Pi([A])$ to $\Pi(A)$. Otherwise we shall refrain from using abbreviations. In particular we shall write $\mathfrak{A} \in \lambda$ or $\mathfrak{A} \notin \lambda$. The notation $|A|, |\mathfrak{A}|, \ldots$ refers to the cardinal number of the set under consideration.

Clusters were initially defined for proximities by Leader [10] as λ-closed λ-clans (see Definition 2.8). We use this definition also for contiguities and for nearness structures, for both of which one can prove (Theorems 2.4 and 2.5) that the λ-closed λ-clans are exactly the maximal λ-compatible families. Thus there is no real conflict between our definition and that of Herrlich, Naimpally, and Bentley [6], [12], [2] who, for near structures, define a cluster as a maximal λ-compatible family. In order to save space many straightforward proofs shall be omitted or given only in outline. The authors would also like to thank S. A. Naimpally and the referee for a large number of valuable comments.

2. Proximities, contiguities, and nearness structures. We begin by recalling the definition of a stack and a grill.

Definition 2.1. A family $\mathfrak{S}$ of subsets of X is called a *stack* on X if it satisfies $A \supset B \in \mathfrak{S} \Rightarrow A \in \mathfrak{S}$. A stack $\mathfrak{G}$ on X is called a *grill* on X if it satisfies the conditions $\emptyset \notin \mathfrak{G}$; $A \cup B \in \mathfrak{G} \Rightarrow A \in \mathfrak{G}$ or $B \in \mathfrak{G}$.

A proximity is usually considered as a relation on X, but since it is assumed to be a symmetric relation it can be taken to be a collection of two element families $[A, B]$. This enables us to make the following definition.

Definition 2.2. A collection Π of families of subsets of X is called a *basic* (or *Čech*) *proximity* on X if it satisfies the requirements:

$$P_0\colon \mathfrak{A} \in \Pi \Rightarrow |\mathfrak{A}| = 2,$$

$$P_1\colon |\mathfrak{A}| = 2,\ \bigcap \mathfrak{A} \neq \emptyset \Rightarrow \mathfrak{A} \in \Pi,$$

$$P_2\colon \Pi(A) \text{ is a grill on } X \text{ for all } A \subset X..$$

The equivalence of this definition to Čech's [4] is established in [19].

In Terwilliger's modification of Ivanova and Ivanov's definition of a contiguity the LO-condition and separatedness are still included. We remove these conditions in defining a basic contiguity.

Definition 2.3. A collection ξ of families of subsets of X is called a *basic contiguity* on X if it satisfies the conditions:

$$C_0\colon \mathfrak{A} \in \xi \Rightarrow |\mathfrak{A}| < \aleph_0,$$

$$C_1\colon |\mathfrak{A}| < \aleph_0,\ \bigcap \mathfrak{A} \neq \emptyset \Rightarrow \mathfrak{A} \in \xi,$$

$$C_2\colon \xi(\mathfrak{A}) \text{ is a grill on } X \text{ for all } \mathfrak{A} \subset \mathfrak{P}(X),$$

$$C_3\colon \mathfrak{B} \subset \mathfrak{A} \in \xi \Rightarrow \mathfrak{B} \in \xi.$$

For every infinite cardinal number C one can define a *C-contiguity* by

replacing the requirement $|\mathfrak{A}| < \aleph_0$ in C_0 and C_1 by $|\mathfrak{A}| \leq C$.

It is helpful to introduce an operation $\circledcirc$ as well as a relation $\succ$ on families of subsets of X. We have

$$\mathfrak{A} \circledcirc \mathfrak{B} = [A \cup B : A \in \mathfrak{A},\ B \in \mathfrak{B}]$$

and

$$\mathfrak{B} \succ \mathfrak{A} \Leftrightarrow \forall B \in \mathfrak{B}\ \exists A \in \mathfrak{A} \text{ such that } B \supset A.$$

In terms of this notation we can, following Naimpally [12], define a basic (or Čech) nearness as follows:

Definition 2.4. A collection ν of families of subsets of X shall be called a *basic* (or *Čech*) *nearness* on X if it satisfies:

$$B_1 : \bigcap \mathfrak{A} \neq \emptyset \Rightarrow \mathfrak{A} \in \nu,$$

$$B_2 : \mathfrak{A} \in \nu \Rightarrow \emptyset \notin \mathfrak{A},$$

$$B_3 : \mathfrak{B} \succ \mathfrak{A} \quad \text{and} \quad \mathfrak{A} \in \nu \Rightarrow \mathfrak{B} \in \nu,$$

$$B_4 : \mathfrak{A} \notin \nu,\quad \mathfrak{B} \notin \nu \Rightarrow \mathfrak{A} \circledcirc \mathfrak{B} \notin \nu.$$

In the sequel it will be convenient to omit the prefixes "basic" or "Čech". Thus a nearness is understood to be a basic nearness and similarly for proximities and contiguities.

With these definitions we are able to prove:

Theorem 2.1. (a) *Let* $|\mathfrak{A}| < \aleph_0$, $|\mathfrak{B}| < \aleph_0$ *and let* ξ *be a contiguity then*

(i) $\mathfrak{B} \succ \mathfrak{A}$ *and* $\mathfrak{A} \in \xi \Rightarrow \mathfrak{B} \in \xi$,

(ii) $\mathfrak{A} \notin \xi, \mathfrak{B} \notin \xi \Rightarrow \mathfrak{A} \circledcirc \mathfrak{B} \notin \xi$.

(b) $\mathfrak{A} \in \lambda \Rightarrow \mathfrak{A} \subset \lambda(\mathfrak{A})$, *where* λ *may be a proximity, contiguity or nearness.*

(c) *If* ν *is a nearness on* X *then* $\nu(\mathfrak{A})$ *is a grill on* X *for all* $\mathfrak{A} \subset \mathfrak{P}(X)$.

Proof of (a). Let $\mathfrak{B} = [B_1, \ldots, B_m]$. If $\mathfrak{B} \succ \mathfrak{A}$ then for every B_k there exists an $A_k \in \mathfrak{A}$, such that $B_k \supset A_k$. Here $A_k = A_j$, $k \neq j$, is possible. Set $\mathfrak{A}' = [A_1, \ldots, A_m]$, then $\mathfrak{A}' \subset \mathfrak{A}$. Hence by C_3 and the assumption $\mathfrak{A} \in \xi$ we have $\mathfrak{A}' \in \xi$. Set $\mathfrak{A}'_k = [B_1, \ldots, B_k, A_{k+1}, \ldots, A_m]$. Then $\mathfrak{A}'_1 \in \xi$ since from $A_1 \in \xi([A_2, \ldots, A_m])$ it follows by C_2 that $B_1 \in \xi([A_2, \ldots, A_m])$ and hence that $\mathfrak{A}'_1 \in \xi$. Now assume that $\mathfrak{A}'_k \in \xi$. Then $A_{k+1} \in \xi(\mathfrak{A}'_k \sim [A_{k+1}])$ and hence $B_{k+1} \in \xi(\mathfrak{A}'_k \sim [A_{k+1}])$, that is $\mathfrak{A}'_{k+1} \in \xi$. By induction we thus arrive at $\mathfrak{A}'_m = \mathfrak{B} \in \xi$. This establishes (i). We now turn to the proof of (ii).

Set $\mathfrak{A} \circledcirc \mathfrak{B} = \mathfrak{C}$. Further, let $\mathfrak{A} = [A_1, \ldots, A_n]$, $\mathfrak{B} = [B_1, \ldots, B_m]$ and define $\mathfrak{A}_r = [A_i : i < r]$, $\mathfrak{B}_s = [B_j : j < s]$. Then $\mathfrak{A} = \mathfrak{A}_{n+1}$ and $\mathfrak{B} = \mathfrak{B}_{m+1}$. Assume $\mathfrak{C} \in \xi$. Either $\mathfrak{B}_{m+1} \cup \mathfrak{C} = \mathfrak{B} \cup \mathfrak{C} \in \xi$, in which case $\mathfrak{B} \in \xi$ follows from C_3, or there exists a least k, $1 \leq k \leq m+1$ such that $\mathfrak{B}_k \cup \mathfrak{C} \notin \xi$. Since $\mathfrak{B}_1 = \emptyset$ and $\emptyset \cup \mathfrak{C} = \mathfrak{C} \in \xi$ we must have $k > 1$. Set $r = k - 1$, then $\mathfrak{B}_r \cup \mathfrak{C} \in \xi$. Further, if $B_r \in \xi(\mathfrak{B}_r \cup \mathfrak{C})$ then $[B_r] \cup \mathfrak{B}_r \cup \mathfrak{C} = \mathfrak{B}_{r+1} \cup \mathfrak{C} = \mathfrak{B}_k \cup \mathfrak{C} \in \xi$, which is a contradiction. Hence $\mathfrak{B}_r \cup \mathfrak{C} \in \xi$ and $B_r \notin \xi(\mathfrak{B}_r \cup \mathfrak{C})$. We next observe that since $\mathfrak{A}_1 = \emptyset$ it is true that $\mathfrak{A}_1 \cup \mathfrak{B}_r \cup \mathfrak{C} \in \xi$. Let $1 \leq t \leq m$. Assume that we know that $\mathfrak{A}_t \cup \mathfrak{B}_r \cup \mathfrak{C} \in \xi$. Now $A_t \cup B_r \in \mathfrak{A} \circledcirc \mathfrak{B} = \mathfrak{C} \subset \mathfrak{A}_t \cup \mathfrak{B}_r \cup \mathfrak{C}$. Hence $A_t \cup B_r \in \xi(\mathfrak{A}_t \cup \mathfrak{B}_r \cup \mathfrak{C})$. Since $B_r \notin \xi(\mathfrak{B}_r \cup \mathfrak{C})$ it is true, a fortiori, that $B_r \notin \xi(\mathfrak{A}_t \cup \mathfrak{B}_r \cup \mathfrak{C})$. This fact, together with C_2, yields $A_t \in \xi(\mathfrak{A}_t \cup \mathfrak{B}_r \cup \mathfrak{C})$ and hence $\mathfrak{A}_{t+1} \cup \mathfrak{B}_r \cup \mathfrak{C} \in \xi$. By induction and recalling that $\mathfrak{A} = \mathfrak{A}_{n+1}$, we conclude that $\mathfrak{A} \cup \mathfrak{B}_r \cup \mathfrak{C} \in \xi$. Using C_3 it then follows that $\mathfrak{A} \in \xi$.

Proof of (b). The result follows from the observation that $A \in \mathfrak{A} \Rightarrow [A] \cup \mathfrak{A} = \mathfrak{A}$.

Proof of (c). Set $\mathfrak{C} = [C] \cup \mathfrak{A}$ and $\mathfrak{D} = [D] \cup \mathfrak{A}$. Then $\mathfrak{C} \circledcirc \mathfrak{D} \succ [C \cup D] \cup \mathfrak{A}$. Assume $C \cup D \in \nu(\mathfrak{A})$. This is equivalent to $[C \cup D] \cup \mathfrak{A} \in \nu$. Hence by B_3, $\mathfrak{C} \circledcirc \mathfrak{D} \in \nu$. An application of the contrapositive of B_4 yields $\mathfrak{C} \in \nu$ or $\mathfrak{D} \in \nu$. Thus $C \in \nu(\mathfrak{A})$ or $D \in \nu(\mathfrak{A})$. Now assume that $E \supset F \in \nu(\mathfrak{A})$. Then $[F] \cup \mathfrak{A} \in \nu$ and hence, by B_3, $[E] \cup \mathfrak{A} \in \nu$, that is $E \in \nu(\mathfrak{A})$. Clearly $\emptyset \notin \nu(\mathfrak{A})$ and hence $\nu(\mathfrak{A})$ is a grill.

The analogues of B_3 and B_4 thus are also valid for contiguities. It is also clear that C_2 follows from (a)(i) and (ii). The proof is completely analogous to that for (c). The analogue of C_2 holds for nearness structures. However we are not able to derive B_3 and B_4 from it, since in those axioms we may be dealing with infinite families $\mathfrak{A}$ and $\mathfrak{B}$. The axioms B_3 and B_4 can be derived from the following:

I: $\mathfrak{B} \subset \mathfrak{A}$ and $\mathfrak{A} \in \nu \Rightarrow \mathfrak{B} \in \nu$.

II: Let $\mathfrak{B}$ be well ordered by indexing $\mathfrak{B} = [B_j]$ by means of ordinal numbers and set $\mathfrak{B}_j = [B_i : i < j]$. Then $\mathfrak{B}_i \cup \mathfrak{C} \in \nu \ \forall i < j \Rightarrow \mathfrak{B}_j \cup \mathfrak{C} \in \nu$ provided either $\mathfrak{B} \succ \mathfrak{C}$ or $\mathfrak{C} = \mathfrak{A} \circledcirc \mathfrak{B}$ for some $\mathfrak{A} \notin \nu$.

The proof resembles the proof of (a) but is by transfinite induction.

Finally, note that if Π is a proximity and $[A, B] \in \Pi$ then $\Pi([A, B]) = [A, B]$. It follows that $\Pi(\mathfrak{A})$ is not always a grill.

Definition 2.5. For a nearness ν on X we define

$$\xi_\nu = [\mathfrak{A}: \mathfrak{A} \in \nu,\ |\mathfrak{A}| < \aleph_0], \qquad \Pi_\nu = [\mathfrak{A}: \mathfrak{A} \in \nu,\ |\mathfrak{A}| = 2],$$

$$c_\nu(A) = [x: [[x], A] \in \nu].$$

For a contiguity ξ on X we have

$$\Pi_\xi = [\mathfrak{A}: \mathfrak{A} \in \xi,\ |\mathfrak{A}| = 2], \qquad c_\xi(A) = [x: [[x], A] \in \xi].$$

If Π is a proximity on X we define $c_\Pi(A) = [x: [[x], A] \in \Pi]$.

The following results are immediate.

Theorem 2.2. *The family ξ_ν is a contiguity on X. Π_ν and Π_ξ are proximities on X. The functions c_ν, c_ξ, and c_Π are Čech closure operators on X. Finally, $\Pi_\nu = \Pi_{\xi_\nu}$, $c_\nu = c_{\xi_\nu} = c_{\Pi_{\xi_\nu}}$.*

Definition 2.6. A proximity (contiguity, nearness) λ is called a LO-*proximity* (*contiguity*, *nearness*) if it satisfies the additional condition: $[c_\lambda(A_i): i \in I] \in \lambda \Rightarrow [A_i: i \in I] \in \lambda$.

Definition 2.7. A proximity (contiguity, nearness) λ is called *separated* if it satisfies the additional condition $[[x], [y]] \in \lambda \Rightarrow x = y$.

A LO-nearness induces a LO-contiguity, which in turn induces a LO-proximity. However a non-LO-nearness may induce a LO-contiguity, and a non-LO-contiguity may induce a LO-proximity. This is illustrated by Examples 2.2 and 2.3 below.

It is well known that the closure operator induced by a LO-proximity (and hence by any LO-structure) is a Kuratowski closure operator.

Definition 2.8. Let λ be a proximity, or contiguity, or nearness on X, then a family $\mathfrak{A} \subset \mathfrak{P}(X)$ is called λ-*compatible* if $\mathfrak{B} \subset \mathfrak{A}$ (and $|\mathfrak{B}| = 2$, or $|\mathfrak{B}| < \aleph_0$, if appropriate) implies $\mathfrak{B} \in \lambda$. The family $\mathfrak{A}$ is called λ-*closed* if $[A] \cup \mathfrak{B} \in \lambda$, for all $\mathfrak{B} \subset \mathfrak{A}$ (and $|\mathfrak{B}| = 1$, or $|\mathfrak{B}| < \aleph_0$, as appropriate), implies $A \in \mathfrak{A}$. If λ is a nearness then a family $\mathfrak{A}$ is λ-closed iff $\lambda(\mathfrak{A}) \subset \mathfrak{A}$.

A λ-compatible grill is called a λ-*clan*. Finally, a λ-closed λ-clan is called a λ-*cluster*.

The next result was proved for separated LO-contiguities by Terwilliger [17]; it is also, in a disguised form, asserted by Herrlich [6] for LO-nearness structures.

Theorem 2.3. *Let λ be a contiguity or a nearness on X. Then every maximal λ-compatible family $\mathfrak{A}$ is a grill and hence a maximal λ-clan on X.*

Proof. Except for the "union property" of $\mathfrak{A}$ the proof is straightforward. We shall consider the case where λ is a contiguity. For nearness

structures the argument is somewhat simpler. Set $\alpha = [\mathfrak{B}: \mathfrak{B} \subset \mathfrak{A}, |\mathfrak{B}| < \aleph_0]$. Let $A \cup B \in \mathfrak{A}$. Then for all $\mathfrak{B} \in \alpha$ we have $[A \cup B] \cup \mathfrak{B} \in \lambda$. Now either $[A] \cup \mathfrak{B} \in \lambda$ for all $\mathfrak{B} \in \alpha$, or $[B] \cup \mathfrak{B} \in \lambda$ for all $\mathfrak{B} \in \alpha$, or there exist families $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in α such that $[A] \cup \mathfrak{B}_1 \notin \lambda$ and $[B] \cup \mathfrak{B}_2 \notin \lambda$. Then by (a)(ii) of Theorem 2.1 $([A] \cup \mathfrak{B}_1) \otimes ([B] \cup \mathfrak{B}_2) = \mathfrak{C} \notin \lambda$. However $\mathfrak{C} \succ [A \cup B] \cup (\mathfrak{B}_1 \cup \mathfrak{B}_2)$ and $\mathfrak{B}_1 \cup \mathfrak{B}_2 \in \alpha$. Thus $[A \cup B] \cup (\mathfrak{B}_1 \cup \mathfrak{B}_2) \in \lambda$. This contradicts (a)(i) of Theorem 2.1. Hence either $[A] \cup \mathfrak{A}$ or $[B] \cup \mathfrak{A}$ is a λ-compatible family. It follows from the maximality of $\mathfrak{A}$ that $A \in \mathfrak{A}$ or $B \in \mathfrak{A}$.

Theorem 2.4. *Let λ be a proximity, or a contiguity, or a nearness. Then every λ-cluster is a maximal λ-clan and a maximal λ-compatible family.*

Theorem 2.5. *Let λ be a contiguity, or a nearness. Then every maximal λ-compatible family is a λ-cluster.*

Proof. Since every maximal λ-compatible family $\mathfrak{A}$ is a λ-clan by Theorem 2.3, it suffices to show that $[A] \cup \mathfrak{B} \in \lambda$ for all $\mathfrak{B} \subset \mathfrak{A}$ with the appropriate cardinality restriction implies $A \in \mathfrak{A}$. This clearly follows from the maximality of $\mathfrak{A}$ as a λ-compatible family.

Theorem 2.6. *If λ is a proximity or contiguity then every λ-compatible family is contained in a maximal λ-compatible family.*

For a contiguity ξ it now follows from Theorems 2.3 and 2.6 that every ξ-compatible family is contained in a maximal ξ-clan. Hence all contiguities are clan generated.

Using Theorem 2.7 it is easy to construct examples of nearness structures ν, where not all ν-compatible families are contained in maximal families.

Example 2.1. Let $X = A \cup B$, $A \cap B = \emptyset$, $|A| \geq \aleph_0$, $|B| \geq \aleph_0$. Let a closure operator c be defined on X by requiring that X, A, B and all finite sets form a subbase for the closed sets of the space. Define a proximity Π^* on X by: $[C, D] \in \Pi^*$ iff $c(C) \cap c(D) \neq \emptyset$ or both C and D are infinite. Then

$$\mathfrak{B} = [C: C \subset X, |C| \geq \aleph_0]$$
$$= \bigcup [\mathfrak{U}: \text{is a nonprincipal ultrafilter on } X]$$

is a maximal Π^*-clan. The maximal Π^*-compatible families containing $\mathfrak{B}$ are $\mathfrak{F}_a = [D: a \in D, D \cap B \neq \emptyset] \cup \mathfrak{B}$ and $\mathfrak{F}_b = [D: b \in D, D \cap A \neq \emptyset] \cup \mathfrak{B}$ where $a \in A$, $b \in B$. It is also clear that $\mathfrak{B}$ is not a Π^*-cluster. Even for an EF-proximity Π there may be maximal Π-compatible families which are not clusters. An example can be constructed by considering the three sides of a triangle in the plane.

We can summarize the results obtained above, by considering the following statements:

(A) λ clusters are maximal λ-clans and maximal λ-compatible families,

(B) maximal λ-compatible families are grills,

(C) maximal λ-compatible families are λ-clusters,

(D) λ-compatible families are contained in maximal λ-compatible families,

(E) maximal λ-clans are maximal λ-compatible families.

The following table now gives the desired information:

	A	B	C	D	E
proximity	yes	no Ex. 2.1	no Ex. 2.1	yes	no Ex. 2.1
contiguity	yes	yes	yes	yes	yes
nearness	yes	yes	open	no Th. 2.7	open

Theorem 2.7. *Let $[\mathfrak{G}_i: i \in I]$ be a family of grills on X with the property that for every $x \in X$ there exists an i such that $[x] \in \mathfrak{G}_i$. Then ν defined by $\mathfrak{A} \in \nu$ iff $\mathfrak{A} \subset \mathfrak{G}_i$, for some $i \in I$, is a basic nearness on X.*

Proof. We show that B_4 holds, the other properties are easily seen to be true. If $\mathfrak{A} \notin \nu$ then for every $i \in I$ there exists a set $A_i \in \mathfrak{A}$ such that $A_i \notin \mathfrak{G}_i$. Similarly $\mathfrak{B} \notin \nu$ implies the existence of sets $B_i \in \mathfrak{B}$ such that $B_i \notin \mathfrak{G}_i$. Now assume that $\mathfrak{A} \otimes \mathfrak{B} \in \nu$. Then there exists a $j \in I$ such that $\mathfrak{A} \otimes \mathfrak{B} \subset \mathfrak{G}_j$. In particular $A_j \cup B_j$ must be in $\mathfrak{G}_j$ but this is a contradiction since neither A_j nor B_j belongs to $\mathfrak{G}_j$.

The following stronger theorem holds for contiguities.

Theorem 2.8. *Let $[\mathfrak{G}_i: i \in I]$ be a family of grills on X satisfying the two conditions*

(a) *for every $x \in X$ there exists a $\mathfrak{G}_i$ such that $[x] \in \mathfrak{G}_i$,*

(b) $i \neq j \Rightarrow \mathfrak{G}_i \not\subset \mathfrak{G}_j$.

Then ξ defined by $\mathfrak{A} \in \xi$ iff $\mathfrak{A} \subset \mathfrak{G}_i$ for some i and $|\mathfrak{A}| < \aleph_0$ is a basic contiguity on X. Every contiguity ξ on X is generated by the family of all its maximal ξ-clans.

Proof. The proof of the first part is completely analogous to the proof of Theorem 2.7. The second part follows from Theorems 2.3 and 2.6.

Theorem 2.8 can be thought of as a representation theorem for contiguity structures since it asserts that all such structures are of the simple type described there.

Theorem 2.9. *Let λ be a proximity, or contiguity, or nearness on X. Let $S \subset X$. Define $\lambda_S = [\mathfrak{A}: \mathfrak{A} \in \lambda \cap \mathfrak{P}(\mathfrak{P}(S))]$, then λ_S is a proximity, or contiguity, or nearness on S. λ_S is called the structure induced by λ on S. Finally, if λ is a* LO*-structure then so is λ_S.*

Definition 2.9. If λ is a proximity, or contiguity, or nearness on X then the pair (X, λ) is called a *proximity space*, or a *contiguity space* or a *nearness space*.

Definition 2.10. A mapping $f: (X, \lambda) \to (Y, \eta)$ is called a *proximity map*, or a *contiguity map*, or a *near map* if $\mathfrak{A} \in \lambda \Rightarrow [f(A): A \in \mathfrak{A}] \in \eta$. The expressions *p-continuous*, *c-continuous* and *n-continuous* are also used.

Definition 2.11. A structure λ will be said to be *larger* that a structure λ' if λ, considered as a collection of families of sets, contains λ'.

We now turn to the discussion of some special nearness and contiguity structures and to some examples.

Definition 2.12. Let ξ be a contiguity on X and let $[\mathfrak{G}_i^\xi: i \in I]$ be the family of all maximal ξ-clans. By $\nu(\xi)$ we shall denote the collection $[\mathfrak{A}: \mathfrak{A} \subset \mathfrak{G}_i^\xi$ for some $i \in I]$.

$[\mathfrak{G}_i^\xi]$ is a family of grills and satisfies the conditions of Theorem 2.7, hence $\nu(\xi)$ is a nearness. Since every $\mathfrak{A} \in \xi$ is contained in a maximal ξ-clan it is true that $\xi_{\nu(\xi)} = \xi$. From this equality it also follows that for every contiguity ξ there is at least one nearness, namely $\nu(\xi)$, which induces it.

Theorem 2.10. *Let ν be a nearness which satisfies the condition: $\mathfrak{A} \notin \nu$ iff $\exists \mathfrak{B} \subset \mathfrak{A}, |\mathfrak{B}| < \aleph_0, \mathfrak{B} \notin \nu$. (That is: ν is a "contigual nearness" as defined by Herrlich* [6].*) Then $\nu = \nu(\xi_\nu)$, where ξ_ν is the contiguity defined in Definition* 2.5. *Moreover, ν is clan generated. Finally, for any contiguity ξ the nearness $\nu(\xi)$ is a contigual nearness.*

Proof. That $\nu = \nu(\xi_\nu)$ can be seen as follows: $\mathfrak{A} \in \nu$ iff $\forall \mathfrak{B} \subset \mathfrak{A}, |\mathfrak{B}| < \aleph_0, \mathfrak{B} \in \nu$, iff $\mathfrak{A}$ is a ξ_ν-compatible family, iff $\mathfrak{A} \subset \mathfrak{G}_i$, where $\mathfrak{G}_i$ is a maximal ξ_ν-clan, iff $\mathfrak{A} \in \nu(\xi_\nu)$. Clearly all $\nu(\xi)$ are clan generated. The last assertion can be proved by substituting $\nu(\xi)$ for ν in the first argument and recalling that $\xi_{\nu(\xi)} = \xi$.

Definition 2.13. Let λ be a proximity or a contiguity or a nearness on X. A λ-clan $\mathfrak{G}$ on X will be called a *λ-bunch* iff $b(\mathfrak{G}) = [A: c_\lambda(A) \in \mathfrak{G}] = \mathfrak{G}$.

Theorem 2.11. *If λ is a* LO*-structure on X then every maximal λ-clan on X is a λ-bunch.*

Proof. This is an extension of a theorem proved for proximities by Thron [19].

Now let (X, Π) be given and consider any nearness structure ν such that $\Pi_\nu = \Pi$. Let $[\mathfrak{C}_i^\Pi: i \in I]$ be the collection of all maximal Π-clans. If $\mathfrak{A} \in \nu$ then $\mathfrak{A}$ is ξ_ν-compatible and hence there exists a maximal ξ_ν-compatible family $\mathfrak{H}$, which is a ξ_ν-clan, such that $\mathfrak{A} \subset \mathfrak{H}$. $\mathfrak{H}$ is a Π-clan and hence $\mathfrak{H}$ is contained in one of the maximal Π-clans $\mathfrak{C}_i^\Pi$. It is thus easy to see that there is a largest nearness ν^Π, which induces Π. It is defined as follows: $\nu^\Pi = [\mathfrak{A}: \mathfrak{A} \subset \mathfrak{C}_i^\Pi$ for some $i \in I]$. It is slightly easier to show that $\xi^\Pi = [\mathfrak{A}: |\mathfrak{A}| < \aleph_0, \mathfrak{A} \subset \mathfrak{C}_i^\Pi$ for some $i \in I]$ is the largest contiguity which induces Π.

If $[A, B] \in \Pi$ there are in general several ways (but always at least one) of choosing ultrafilters $\mathfrak{U}_A^B$ and $\mathfrak{U}_B^A$ so that $A \in \mathfrak{U}_A^B$, $B \in \mathfrak{U}_B^A$ and $\mathfrak{G}_{A,B} = \mathfrak{U}_A^B \cup \mathfrak{U}_B^A$ is a Π-clan. (This is shown in [19].) Any collection $[\mathfrak{A}: \mathfrak{A} \subset \mathfrak{G}_{A,B}$ for some $[A, B] \in \Pi]$ is a clan generated nearness. That these nearness structures are "small" in a very real sense follows from the fact that they are generated by grills which are as small as possible, at least for $A \cap B = \emptyset$. If $A \cap B \supset [x]$ one could agree to choose $\mathfrak{U}_A^B = \mathfrak{U}_B^A = \mathfrak{U}(x) = [C: x \in C]$ and thus, in that case, also have $\mathfrak{G}_{A,B} = \mathfrak{U}(x)$ as small as possible. Nevertheless, it will not in general be the case that these near structures are minimal with respect to inducing Π and being clan generated. It is conceivable that certain grills $\mathfrak{G}_{A,B}$ could be deleted from the family of grills defining one of these structures without affecting the proximity induced by the structure. This is always the case if $[A, B] \subset \mathfrak{G}_{C,D}$ for a pair $[C, D] \in \Pi$ and $\mathfrak{G}_{A,B} \neq \mathfrak{G}_{C,D}$. If minimal clan generated nearness structures compatible with a given proximity Π do indeed exist, they must be of the type discussed here.

That for certain proximities Π there does not exist a least near structure compatible with Π has just been shown by Njåstad [15].

The situation becomes much simpler for contiguities. For them we have the theorem:

Theorem 2.12. *Let Π be a given proximity then*

$$\xi_\Pi = [\mathfrak{B}: |\mathfrak{B}| < \aleph_0, \quad \mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2, \quad [\bigcap \mathfrak{B}_1, \bigcap \mathfrak{B}_2] \in \Pi]$$

is the least contiguity compatible with Π. ξ_Π *can also be characterized as*

$$\xi_\Pi = [\mathfrak{B}: |\mathfrak{B}| < \aleph_0, \mathfrak{B} \subset \mathfrak{G}_{A,B} \text{ for some } [A, B] \in \Pi].$$

ξ_Π *is independent of the choice of the family* $[\mathfrak{G}_{A,B}: [A, B] \in \Pi]$, *provided that for each* $[A, B] \in \Pi$ *there is at least one* $\mathfrak{G}_{C,D}$ *containing* $[A, B]$.

Proof. That the two characterizations define the same collection is easy to check. That the collection is a contiguity compatible with Π can be deduced from the second characterization. Clearly all $\mathfrak{B}$ with $[\bigcap \mathfrak{B}_1, \bigcap \mathfrak{B}_2] \in \Pi$ must be in every ξ compatible with Π. It follows that ξ_Π is the least contiguity compatible with Π.

If Π is a LO-proximity then, by Theorem 2.11, the $\mathfrak{G}_i^\Pi$ are Π bunches so that we have in particular $[c_\Pi(A_j): j \in J] \subset \mathfrak{G}_i^\Pi \Rightarrow [A_j: j \in J] \subset \mathfrak{G}_i^\Pi$. It follows that ν^Π and ξ^Π are LO-structures and hence the largest LO-structures inducing Π.

Example 2.2. Let X be the Euclidean plane and Π be the proximity induced by the usual metric on X. Then (X, Π) is a LO-proximity space, but ξ_Π (as defined above) is not a LO-contiguity. To see this let A and B be two disjoint closed sets with $A \in \Pi(B)$. Let

$$c_\Pi(A_1) = c_\Pi(A_2) = A, \qquad A_1 \cap A_2 = \emptyset,$$

$$c_\Pi(B_1) = c_\Pi(B_2) = B, \qquad B_1 \cap B_2 = \emptyset.$$

No union of two ultrafilters can contain the four disjoint sets A_1, A_2, B_1, B_2 hence $[A_1, A_2, B_1, B_2] \notin \xi_\Pi$; however $[c_\Pi(A_1), c_\Pi(A_2), c_\Pi(B_1), c_\Pi(B_2)] \in \xi_\Pi$.

"Small" LO-structures compatible with a given LO-proximity Π can be induced by bunches of the form $b(\mathfrak{G}_{A,B})$. In particular

$$\xi_\Pi^L = [\mathfrak{B}: |\mathfrak{B}| < \aleph_0, \mathfrak{B} \subset b(\mathfrak{G}_{A,B}) \text{ for some } [A, B] \in \Pi]$$

is the least LO-contiguity compatible with Π. As before, it can be shown that ξ_Π^L is independent of the choice of the $\mathfrak{G}_{A,B}$. ξ_Π^L can also be described as

$$\xi_\Pi^L = [\mathfrak{B}: \mathfrak{B} = [B_1^{(1)}, \ldots, B_n^{(1)}] \cup [B_1^{(2)}, \ldots, B_m^{(2)}],$$
$$[\bigcap [c_\Pi(B_i^{(1)})], \bigcap [c_\Pi(B_i^{(2)})]] \in \Pi].$$

The existence of ξ^{Π} and ξ^{L}_{Π} was known to Terwilliger [17].

Example 2.3. Let X be an infinite set and define $\mathfrak{U}(x) = [A: x \in A]$, $\mathfrak{B} = \bigcup[\mathfrak{U}: \mathfrak{U}$ is a nonprincipal ultrafilter$]$. Define the contiguity ξ on X as follows: $\mathfrak{A} \in \xi$ iff $|\mathfrak{A}| < \aleph_0$ and $\mathfrak{A} \subset \mathfrak{U}(x) \bigcup \mathfrak{B}$ for some $x \in X$. This contiguity induces the minimum T_1-topology on X and is a LO-contiguity. Now define a nearness ν by $\mathfrak{A} \in \nu$ iff $\mathfrak{A} \subset \mathfrak{U}(x) \cup \mathfrak{U}_1 \cup \ldots \cup \mathfrak{U}_k$ where $\mathfrak{U}_1, \ldots, \mathfrak{U}_k$ are arbitrary nonprincipal ultrafilters on X. Clearly, $\xi_\nu = \xi$. ν is not a LO-nearness, since for a LO-nearness every family of infinite sets would have to be near. This is so because in a minimum T_1-space the closures of all these sets would be equal to X.

Example 2.4. Let $X = \bigcup[A_k: k = 1, 2, \ldots]$, where $|A_k|$ is infinite for each k and all A_k are disjoint. Define a contiguity ζ on X by $\mathfrak{A} \in \zeta$ iff $|\mathfrak{A}| < \aleph_0$ and $\mathfrak{A} \subset \mathfrak{U}(x)$, for some x, or $\mathfrak{A} \subset \mathfrak{B}$. ($\mathfrak{U}(x)$ and $\mathfrak{B}$ are as defined in the preceding example.) Then Π_ζ is the proximity in which two sets are close iff they intersect or they are both infinite. This is the largest LO-proximity on X compatible with the discrete topology. Denote by $\mathfrak{G}$ any finite union of nonprincipal ultrafilters and by γ the collection of all of these grills. Then ζ can also be characterized by $\mathfrak{A} \in \zeta$ iff $|\mathfrak{A}| < \aleph_0$ and $\mathfrak{A} \subset \mathfrak{U}(x)$, for some x, or $\mathfrak{A} \subset \mathfrak{G}$ for some $\mathfrak{G} \in \gamma$.

For each k let $\mathfrak{B}_k$ be a nonprincipal ultrafilter containing A_k. Now define a nearness μ on X as follows: $\mathfrak{A} \in \mu$ iff $\mathfrak{A} \subset \mathfrak{U}(x)$ for some $x \in X$, or $\mathfrak{A} \subset \mathfrak{G}$, for some $\mathfrak{G} \in \gamma$, or $\mathfrak{A} \subset \bigcup[\mathfrak{B}_{2k}] = \mathfrak{H}_2$, or $\mathfrak{A} \subset \bigcup[\mathfrak{B}_{2k-1}] = \mathfrak{H}_1$. Clearly $\xi_\mu = \zeta$. For further reference note that $\mathfrak{A}_1 = [A_{2k-1}, k = 1, 2, \ldots] \subset \mathfrak{H}_1$, $\mathfrak{A}_2 = [A_{2k}, k = 1, 2, \ldots] \subset \mathfrak{H}_2$, that $\mathfrak{H}_1 \cup \mathfrak{H}_2 \notin \mu$ but that every finite subset of it is in ζ. We shall use this example in §3 to show that not all Π^ν can be generated by contigual nearness structures $\nu(\xi)$.

The final three results are of importance in §3.

Theorem 2.13. *Let ν be a nearness on X and let $\mathfrak{A} \in \nu$. If $\nu(\mathfrak{A}) \in \nu$ then $\nu(\mathfrak{A})$ is a ν-cluster.*

Theorem 2.14. *If λ is a* LO-*proximity or a* LO-*contiguity or a* LO-*nearness on X then, for every $x \in X$, $\lambda(x)$ is a λ-cluster.*

Theorem 2.15. *Let ν be a nearness on X and let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be families of subsets of X such that $\mathfrak{A} \cup \mathfrak{C} \notin \nu$, $\mathfrak{B} \cup \mathfrak{C} \notin \nu$. Then $(\mathfrak{A} \bigcirc \mathfrak{B}) \cup \mathfrak{C} \notin \nu$.*

Proof. Define $\mathfrak{C}^* = [A: \exists B \in \mathfrak{C}, A \supset B]$. Then $\mathfrak{A} \cup \mathfrak{C}^* \notin \nu$ and $\mathfrak{B} \cup \mathfrak{C}^* \notin \nu$. It follows from B_4 that

$$\mathfrak{D} = (\mathfrak{A} \cup \mathfrak{C}^*) \circledcirc (\mathfrak{B} \cup \mathfrak{C}^*) \notin \nu.$$

Now

$$\mathfrak{D} = (\mathfrak{A} \circledcirc \mathfrak{B}) \cup (\mathfrak{A} \circledcirc \mathfrak{C}^*) \cup (\mathfrak{B} \circledcirc \mathfrak{C}^*) \cup (\mathfrak{C}^* \circledcirc \mathfrak{C}^*).$$

The last three families are all contained in $\mathfrak{C}^*$ since $\mathfrak{C}^*$ is a stack. We thus obtain $\mathfrak{D} \subset (\mathfrak{A} \circledcirc \mathfrak{B}) \cup \mathfrak{C}^*$ and hence $(\mathfrak{A} \circledcirc \mathfrak{B}) \cup \mathfrak{C}^* \notin \nu$. In view of condition B_3 this is true iff $(\mathfrak{A} \circledcirc \mathfrak{B}) \cup \mathfrak{C} \notin \nu$.

3. **Proximities defined in terms of near structures.** We begin this section by reviewing some facts about extensions of topological spaces. We make certain modifications, such as the transition to dual traces, and extend concepts, where feasible, to closure spaces (see Čech [4]).

The triple $(\phi, (Y, d))$ is an *extension* of the closure space (X, c) if (Y, d) is a closure space and ϕ is a homeomorphism from (X, c) to $(\phi(X), d')$, where $d'(A) = d(A) \cap \phi(X)$, $A \subset \phi(X)$, and if $d(\phi(X)) = Y$. That is (X, c) is densely embedded in (Y, d). Two extensions $(\phi, (Y, d))$ and $(\phi^\dagger, (Y^\dagger, d^\dagger))$ are *equivalent* if there exists a homeomorphism ψ from (Y, d) onto $(Y^\dagger, d^\dagger)$ such that $\psi \circ \phi = \phi^\dagger$ on X. If there is no danger of confusion we may sometimes refer to (Y, d) as an extension of (X, c).

For each $y \in Y$ define

$$\tau(y) = \tau(y, (Y, d)) = [A\colon\ A \subset X,\ y \in d(\phi(A))],$$

the *dual trace* of the point y with respect to the extension $(\phi, (Y, d))$. The set $[\tau(y)\colon y \in Y]$ is called the *dual trace system* of the extension.

We note that $\tau(y)$ is a grill on X and hence its dual

$$D(\tau(y)) = [B\colon\ X \sim B \notin \tau(y)] = [B\colon\ B \cap A \neq \varnothing\ \forall A \in \tau(y)]$$

is a filter on X. If d is a Kuratowski closure operator then $\tau(y)$ is a c-grill (a grill $\mathfrak{G}$ is a *c-grill* iff $c(A) \in \mathfrak{G} \Longrightarrow A \in \mathfrak{G}$) and $D(\tau(y))$ is an open filter for all $y \in Y$ and $[D(\tau(y))\colon y \in Y]$ is the trace system of the extension $(\phi, (Y, d))$. A simple translation of the usual statement in terms of trace systems gives the following:

Let (X, c) be a T_0-topological space (this insures that $\tau(\phi(x_1)) \neq \tau(\phi(x_2))$ iff $x_1 \neq x_2$, $x_1, x_2 \in X$) and let X^* be a collection of c-grills on X containing all $\Gamma_c(x) = [A\colon x \in c(A)]$. Define

$$A^* = [\mathfrak{G}\colon\ \mathfrak{G} \in X^*,\ A \in \mathfrak{G}],$$

$\phi(x) = \Gamma_c(x)$, and

$$d^*(\alpha) = \bigcap [A^*\colon\ \alpha \subset A^*] \quad \text{for all } \alpha \subset X^*.$$

Then d^* is a Kuratowski closure operator and $(\phi, (X^*, d^*))$ is equivalent to the *principal extension* of (X, c) with respect to the dual trace system X^* (see Thron [18]). The dual trace system of this extension is indeed X^*. More specifically it is true, for every $\mathfrak{G} \in X^*$, that $r(\mathfrak{G}) = \mathfrak{G}$.

These (or equivalent) extensions have a number of other names. Banaschewski [1] calls them the *strict extensions* with respect to X^*. Lodato [11] in a more special context obtains the same extensions and Gagrat and Naimpally [5] refer to the topology generated by d^* as the *absorption topology* on X^*. Wagner [21] calls these extensions *filter spaces*.

We next observe that

$$d^*(\phi(A)) = \bigcap [B^*: \phi(A) \subset B^*] = A^* = (c(A))^* = d^*(\phi(c(A))),$$

and that in view of the definition of d^* the family $[A^*: A^* = d^*(\phi(A)), A \subset X]$ forms a base for the closed sets of (X^*, d^*).

An extension $(\phi, (Y, d))$, where the closure operator d is determined by $d(B) = \bigcap [d(\phi(A)): A \subset X, d(\phi(A)) \supset B]$ (this is equivalent to saying that the sets $d(\phi(A))$ form a base for the closed sets of the space) is called by Ivanova and Ivanov [8] a *regular extension*. Thus the principal extension is a regular extension.

Moreover, since the dual trace system (and hence the trace system) of an extension determines the family $[d(\phi(A)): A \subset X]$, and since knowledge of this family determines the dual trace system, there is exactly one (up to equivalent ones) regular extension of a given space for a given dual trace system, namely the principal extension. It is also clear that for any T_0-topological extension $(\phi, (X^*, d))$ of (X, c) with dual trace system X^*, consisting of c-grills, the relation $d(\alpha) \subset d^*(\alpha)$, $\alpha \subset X^*$, holds.

We now turn to the description of a method to define proximity extensions of proximity spaces. Let (X, Π) be a LO-proximity space and let ν be a LO-nearness on X which satisfies $\Pi_\nu = \Pi$. Define

$$X^\nu = [\mathfrak{A}: \mathfrak{A} \text{ is a } \nu\text{-clan}] \supset [\nu(x): x \in X],$$
$$A^\nu = [\mathfrak{A}: \mathfrak{A} \in X^\nu, A \in \mathfrak{A}] \quad \text{for } A \subset X,$$
$$\mathfrak{A}^\nu = [A^\nu: A \in \mathfrak{A}] \quad \text{for } \mathfrak{A} \subset \mathfrak{P}(X).$$

We then let Π^ν be the collection of sets $[\alpha, \beta]$, $\alpha \subset X^\nu$, $\beta \subset X^\nu$ determined as follows: $[\alpha, \beta] \in \Pi^\nu$ iff $(\bigcap \alpha) \cup (\bigcap \beta) \in \nu$. Note that we do not require X^ν to contain all ν-clans. We do however require that it contain all ν-clans of the form $\nu(x)$.

Theorem 3.1. *(X^ν, Π^ν) as defined above is a proximity space.*

Proof. If $\alpha \cap \beta \supset [\mathfrak{A}]$ then $(\bigcap \alpha) \cup (\bigcap \beta) \subset \mathfrak{A}$. Since $\mathfrak{A} \in \nu$, $[\alpha, \beta] \in \Pi^\nu$ follows. If $\alpha \supset \beta \in \Pi^\nu(\gamma)$ then $(\bigcap \beta) \cup (\bigcap \gamma) \in \nu$ and $\bigcap \alpha \subset \bigcap \beta$. Hence $(\bigcap \alpha) \cup (\bigcap \gamma) \in \nu$ and $\alpha \in \Pi^\nu(\gamma)$. Finally, $\alpha \notin \Pi^\nu(\gamma)$ and $\beta \notin \Pi^\nu(\gamma)$ implies $(\bigcap \alpha) \cup (\bigcap \gamma) \notin \nu$ and $(\bigcap \beta) \cup (\bigcap \gamma) \notin \nu$. An application of Theorem 2.15 yields $((\bigcap \alpha) \circledcirc (\bigcap \beta)) \cup (\bigcap \gamma) \notin \nu$. Now $\bigcap \alpha$ and $\bigcap \beta$ are stacks. It follows that $(\bigcap \alpha) \circledcirc (\bigcap \beta) \subset \bigcap (\alpha \cup \beta)$. Thus we obtain $(\bigcap (\alpha \cup \beta)) \cup (\bigcap (\gamma)) \notin \nu$.

Theorem 3.2. *Define* $\phi: (X, \Pi) \to (X^\nu, \Pi^\nu)$ *by* $\phi(x) = \nu(x)$, *for all* $x \in X$, *and let* $c^\nu = c_\Pi \nu$. *Then*

$$\phi(c_\Pi(A)) = A^\nu \cap \phi(X) = c^\nu(A^\nu) \cap \phi(X)$$

for all $A \subset X$, *and*

$$c_\Pi(A) \subset c_\Pi(B) \quad \textit{iff} \quad A^\nu \cap \phi(X) \subset B^\nu \cap \phi(X).$$

Proof. $\nu(x) \in \phi(c_\Pi(A))$ iff $x \in c_\Pi(A)$ that is iff $[[x], A] \in \Pi \subset \nu$, since $\Pi_\nu = \Pi$. Hence $\nu(x) \in \phi(c_\Pi(A))$ iff $A \in \nu(x)$ iff $\nu(x) \in A^\nu \cap \phi(X)$. Next, $\nu(x) \in c^\nu(A^\nu)$ iff $\nu(x) \cup [A] \in \nu$. This is the case, since $\nu(x)$ is a ν-cluster by Theorem 2.14, iff $A \in \nu(x) \in A^\nu \cap \phi(X)$. Though ϕ may not be one-to-one, it is true that $\phi(x) = \phi(x')$ iff $\nu(x) = \nu(x')$ iff x and x' are contained in the same closures.

Theorem 3.3. $\phi: (X, \Pi) \to (X^\nu, \Pi^\nu)$ *is a proximity mapping and* $\phi(X)$ *is dense in* (X^ν, c^ν). *Moreover,* $\phi: (X, c_\Pi) \to (\phi(X), (c^\nu)_{\phi(X)})$ *is a closed mapping. If* Π *is a separated proximity then* ϕ *is one-to-one and provides a proximal embedding of* (X, Π) *into* (X^ν, Π^ν). *Thus* $(\phi, (X^\nu, \Pi^\nu))$ *is a proximity extension of* (X, Π).

Proof. We first observe that for $A \subset X$, $C \in \bigcap \phi(A)$ if $[C] \cup [[a]] \in \nu$ for all $a \in A$ iff $A \subset c_\nu(C)$. Now $[\phi(A), \phi(B)] \notin \Pi^\nu$ iff $(\bigcap \phi(A)) \cup (\bigcap \phi(B)) \notin \nu$. Recalling that ν is a LO-nearness, we then have $[A] \cup [B] \notin \nu$ and thus finally $[A, B] \notin \Pi$. Hence ϕ is a proximity mapping. The remaining assertions of the theorem are easy to verify.

The properties of the space (X^ν, Π^ν) depend very much on the choice of X^ν. Π will continue to be a LO-proximity and ν a LO-nearness on X. Our first result is:

Theorem 3.4. *The proximity* Π^ν *is separated iff for* $\mathfrak{G}_1, \mathfrak{G}_2 \in X^\nu$, $\mathfrak{G}_1 \neq \mathfrak{G}_2$ *it is true that* $\mathfrak{G}_1 \cup \mathfrak{G}_2 \notin \nu$.

Proof. It follows from the definition of Π^ν that $[[\mathfrak{G}_1], [\mathfrak{G}_2]] \in \Pi^\nu$ iff $\mathfrak{G}_1 \cup \mathfrak{G}_2 \in \nu$.

Theorem 3.5. *The trace of the point $\mathfrak{G} \in X^\nu$ with respect to the extension $(\phi, (X^\nu, c^\nu))$ of (X, c_Π) is $\tau(\mathfrak{G}) = \nu(\mathfrak{G})$. If X^ν is such that all $\nu(\mathfrak{G})$ are ν-clans then Π^ν is separated iff $\nu(\mathfrak{G}_1) = \nu(\mathfrak{G}_2) \Rightarrow \mathfrak{G}_1 = \mathfrak{G}_2$. If one thinks of τ as a function from X^ν to $[\nu(\mathfrak{G}): \mathfrak{G} \in X^\nu]$ defined by $\tau(\mathfrak{G}) = \nu(\mathfrak{G})$ then the condition for Π^ν to be separated (provided all $\nu(\mathfrak{G})$ are ν-clans) becomes the one-to-one behavior of τ.*

Proof. $\tau(\mathfrak{G}) = [A: \mathfrak{G} \in c^\nu(\phi(A))] = [A: [A] \cup \mathfrak{G} \in \nu] = \nu(\mathfrak{G})$. If there exist $\mathfrak{G}_1, \mathfrak{G}_2 \in X^\nu$ such that $\mathfrak{G}_1 \neq \mathfrak{G}_2$ and $\nu(\mathfrak{G}_1) = \nu(\mathfrak{G}_2)$, then $\mathfrak{G}_1 \in \nu(\mathfrak{G}_2)$ and, since all $\nu(\mathfrak{G})$ are assumed to be ν-clans, $\mathfrak{G}_1 \cup \mathfrak{G}_2 \in \nu$ so that Π^ν is not separated. If Π^ν is not separated there exist $\mathfrak{G}_1 \neq \mathfrak{G}_2$ such that $\mathfrak{G}_1 \cup \mathfrak{G}_2 \in \nu$. It follows that $\mathfrak{G}_2 \subset \nu(\mathfrak{G}_1)$ which if $\nu(\mathfrak{G}_1) \in \nu$ implies $\nu(\mathfrak{G}_2) \supset \nu(\mathfrak{G}_1)$. By an analogous argument we obtain $\nu(\mathfrak{G}_1) \supset \nu(\mathfrak{G}_2)$ and hence $\nu(\mathfrak{G}_1) = \nu(\mathfrak{G}_2)$.

Theorem 3.6. *If all $\mathfrak{G} \in X^\nu$ are maximal ν-compatible families, then $\mathfrak{G} = \tau(\mathfrak{G})$ for all $\mathfrak{G} \in X^\nu$. Further, Π^ν is a* LO*-proximity on X^ν, the closure operator c^ν is a Kuratowski operator, and $(\phi, (X^\nu, c^\nu))$ is the principal extension of (X, c_Π) with respect to the dual trace system X^ν.*

Proof. If $\mathfrak{G}$ is maximal ν-compatible then $\mathfrak{G} = \nu(\mathfrak{G}) = \tau(\mathfrak{G})$. Let $\alpha \subset X^\nu$ then

$$c^\nu(\alpha) = [\mathfrak{G}: \bigcap \alpha = [A_i: i \in I] \subset \mathfrak{G}] = \bigcap [A_i^\nu].$$

Assume $[\alpha, \beta] \notin \Pi^\nu$ then $\mathfrak{A} = [A_i] = \bigcap \alpha$ and $\mathfrak{B} = [B_j] = \bigcap \beta$ is such that $\mathfrak{A} \cup \mathfrak{B} \notin \nu$. However $c^\nu(\alpha) \subset \bigcap [A_i^\nu]$ and $c^\nu(\beta) \subset \bigcap [B_j^\nu]$ and hence $[c^\nu(\alpha), c^\nu(\beta)] \notin \Pi^\nu$. Thus Π^ν is a LO-proximity and the closure operator induced by it is a Kuratowski operator.

Theorem 3.7. *Let (X^ν, Π^ν) be such that*

(a) *$\nu(\mathfrak{G}) \in \nu$, for all $\mathfrak{G} \in X^\nu$,*

(b) *$\mathfrak{G}_1 \neq \mathfrak{G}_2 \Rightarrow \nu(\mathfrak{G}_1) \neq \nu(\mathfrak{G}_2)$, $\mathfrak{G}_1, \mathfrak{G}_2 \in X^\nu$,*

(c) *c^ν is a Kuratowski closure operator on X^ν.*

Then $\tau: (X^\nu, c^\nu) \to (X^*, d^*)$, where $\tau(\mathfrak{G}) = \nu(\mathfrak{G})$, is a homeomorphism, and $\tau(\nu(x)) = \nu(x)$. Thus $(\phi, (X^\nu, c^\nu))$ is equivalent to the principal extension with respect to its dual trace system $X^* = [\nu(\mathfrak{G}): \mathfrak{G} \in X^\nu]$. If only (a) *and* (b) *hold one has $\tau(c^\nu(\alpha)) \supset d^*(\tau(\alpha))$.*

Proof. $\nu(\mathfrak{G}) \in \nu$ implies that $\nu(\mathfrak{G})$ are maximal ν-compatible hence the $\nu(\mathfrak{G})$ are also c-grills. The maximality of the $\nu(\mathfrak{G})$ together with the fact that $\nu(\nu(x)) = \nu(x)$ and condition (b) insures that (X, c_Π) is a T_0-space. (As a matter of fact it is a T_1-space since it is generated by a proximity.) This suffices for the existence of the principal extension $(\phi, (X^*, d^*))$ with dual trace system $X^* = [\nu(\mathfrak{G}): \mathfrak{G} \in X^\nu]$. We recall that

$$d^*(\beta) = \bigcap [A^*: \beta \subset A^*], \ \beta \subset X^*,$$

where $A^* = [\nu(\mathfrak{G}): A \in \nu(\mathfrak{G})]$. Thus for $\alpha \subset X^\nu$

$$\begin{aligned}
\tau(c^\nu(\alpha)) &= \tau[\mathfrak{G}: \bigcap \alpha = [A_i] \subset \nu(\mathfrak{G})] \\
&= [\nu(\mathfrak{G}): \bigcap \alpha = [A_i] \subset \nu(\mathfrak{G})] \\
&= \bigcap [[\nu(\mathfrak{G}): A_i \in \nu(\mathfrak{G})]: A_i \in \bigcap [\mathfrak{G}: \mathfrak{G} \in \alpha]] \\
&= \bigcap [A_i^*: A_i \in \bigcap [\mathfrak{G}: \mathfrak{G} \in \alpha]] \\
&\supset \bigcap [A_j^*: A_j \in \bigcap [\nu(\mathfrak{G}): \mathfrak{G} \in \alpha]] \\
&= \bigcap [A_j^*: A_j \in \bigcap [\nu(\mathfrak{G}): \nu(\mathfrak{G}) \in \tau(\alpha)]] \\
&= \bigcap [A_j^*: \tau(\alpha) \subset A_j^*] = d^*(\tau(\alpha)).
\end{aligned}$$

If c^ν is Kuratowski then we must have $\tau(c^\nu(\alpha)) \subset d^*(\tau(\alpha))$. Combining these two inclusion relationships we conclude that τ is a homeomorphism. That $\tau(\nu(x)) = \nu(x)$ follows from the fact, already noted before, that $\nu(\nu(x)) = \nu(x)$.

It would be desirable to have a better sufficient condition for c^ν to be a Kuratowski closure operator than that contained in Theorem 3.6. This however appears to be a difficult problem.

Partly motivated by Lodato's [11] constructions and partly by that employed in the construction of the principal extension we are led to the following definition.

Let (X, Π) be a LO-proximity space and let $X^\dagger$ be a family of Π-bunches on X such that $X^\dagger \supset [\Pi(x): x \in X]$. Define $\phi(x) = \Pi(x)$, $A^\dagger = [\mathfrak{H}: \mathfrak{H} \in X^\dagger, A \in \mathfrak{H}]$, and $d^\dagger(\alpha) = \bigcap [A^\dagger: \alpha \subset A^\dagger]$, $\alpha \subset X^\dagger$. We call $d^\dagger$ the absorption closure operator on $X^\dagger$.

By standard arguments, using the fact that the $\mathfrak{H}$ are grills, one shows that $d^\dagger$ is a Kuratowski closure operator on $X^\dagger$. Moreover, $\phi: (X, c_\Pi) \to (\phi(X), d^\dagger_{\phi(X)})$ is a homeomorphism, since the $\mathfrak{H}$ are c_Π-grills, provided Π is separated.

The fact that the $\mathfrak{H}$ are Π-compatible begins to play an important role only if $X^\dagger$ contains enough $\mathfrak{H}$ so that $[A, B] \in \Pi$ implies the existence of an $\mathfrak{H} \in X^\dagger$ such that $A, B \in \mathfrak{H}$. In this case

$$[A, B] \in \Pi \quad \text{iff} \quad d^\dagger(\phi(A)) \cap d^\dagger(\phi(B)) \neq \emptyset.$$

It is of interest to compare $d^\dagger$ with c^ν in the case $X^\nu = X^\dagger$. We have the following result.

Theorem 3.8. *Let (X, Π) be given and let $X^\nu = X^\dagger$ be a family of ν-bunches (and hence Π-bunches). Then $c^\nu = d^\dagger$ iff all $\mathfrak{G} \in X^\nu$ are maximal ν-compatible families.*

Proof. If all $\mathfrak{G}$ are maximal then by Theorem 3.6 $c^\nu = d^\dagger$. If $c^\nu = d^\dagger$, then in particular $c^\nu(A^\nu) = d^\dagger(A^\nu) = d^\dagger(A^\dagger) = A^\dagger = A^\nu$. However, if $\mathfrak{G}$ is not a maximal ν-compatible family then there exists an $A \subset X$ such that $A \notin \mathfrak{G}$ but $[A] \cup \mathfrak{G} \in \nu$. It follows that $\mathfrak{G} \in c^\nu(A^\nu)$ but $\mathfrak{G} \neq A^\nu$ and hence $c^\nu \neq d^\dagger$.

Let ξ be a LO-contiguity on X. For a family X^ξ of ξ-clans one can, in analogy to the definition of Π^ν, define Π^ξ by $[\alpha, \beta] \in \Pi^\xi$ iff $(\bigcap \alpha) \cup (\bigcap \beta)$ is a ξ-compatible family. However, $(X^\xi, \Pi^\xi) = (X^{\nu(\xi)}, \Pi^{\nu(\xi)})$, where $\nu(\xi)$ is the nearness defined in Definition 2.12. First, observe that the $\nu(\xi)$-clans are exactly the ξ-clans, hence any X^ξ is an $X^{\nu(\xi)}$, and conversely. Next, let $[\alpha, \beta] \in \Pi^{\nu(\xi)}$. This is equivalent to $\mathfrak{C} = (\bigcap \alpha) \cup (\bigcap \beta) \in \nu(\xi)$, which is true iff $\mathfrak{C}$ is $\nu(\xi)$-compatible. This is the same as $\mathfrak{C}$ is ξ-compatible, which is the same as $[\alpha, \beta] \in \Pi^\xi$. Thus nothing is lost by not considering separately extension spaces of the form (X^ξ, Π^ξ).

The question remains whether anything is gained by considering extension spaces generated by nearness structures. The answer is in the affirmative as the following example shows:

Let X, ζ, μ, etc. be defined as in Example 2.4. Set

$$X^\mu = X^\zeta = [\mu(x) : x \in X] \cup \gamma \cup [\mathfrak{H}_1, \mathfrak{H}_2].$$

Π^μ and Π^ζ are distinct, since $[[\mathfrak{H}_1], [\mathfrak{H}_2]] \notin \Pi^\mu$ but $[[\mathfrak{H}_1], [\mathfrak{H}_2]] \in \Pi^\zeta$. Π^μ could not be obtained from a smaller contiguity ζ' since for no smaller contiguity will all grills in X^ζ be ζ'-clans.

4. Comparison of extensions (X^ν, Π^ν) with arbitrary extensions. Let (X, Π) be a LO-proximity space and let $(i, (Y, \Pi^*))$, where i is the identity mapping, be an arbitrary proximity extension of (X, Π). The question we propose to investigate in this section is: how close can we come to (Y, Π^*) by a suitable choice of ν and X^ν? It will be convenient to impose on Π^*

a restriction which is slightly weaker then the LO-restriction.

Definition 4.1. Let (X, λ) be a nearness or contiguity or proximity space and let $S \subset X$. Then λ is said to be a LO/*S-structure* iff for all $A_i \subset S$

$$[c_\lambda(A_i): i \in I] \in \lambda \Rightarrow [A_i: i \in I] \in \lambda.$$

Theorem 4.1. *Let $(\phi, (Y, \Pi^*))$ be a proximity extension of a proximity space (X, Π) and let λ^* be a LO/$\phi(X)$ nearness or contiguity or proximity on Y such that $\Pi_{\lambda^*} = \Pi^*$. Define λ on X by $\mathfrak{A} \in \lambda$ iff $\mathfrak{A} \subset \mathfrak{P}(X)$ and $[\phi(A): A \in \mathfrak{A}] \in \lambda^*$. Then every $\tau(y)$ is a λ-clan on X. If in addition c_{λ^*} is a Kuratowski closure operator then the $\tau(y)$ are λ-bunches (and hence a fortiori c_λ-grills).*

Proof. It was already noted in §3 that the $\tau(y)$ are always grills on X. Since all elements of $[c_{\lambda^*}(\phi(A)): A \in \tau(y)]$ have the point y in common, the family $[c_{\lambda^*}(\phi(A)): A \in \tau(y)] \in \lambda^*$. Since $\phi(A) \subset \phi(X)$ and λ^* is LO/$\phi(X)$ it follows that $[\phi(A): A \in \tau(y)] \in \lambda^*$ and hence $\tau(y) \in \lambda$.

The function $\phi: (X, c_\lambda) \to (Y, c_{\lambda^*})$ is continuous; hence $\phi(c_\lambda(A)) \subset c_{\lambda^*}(\phi(A))$. If we now assume that c_{λ^*} is a Kuratowski operator then

$$c_{\lambda^*}(\phi(c_\lambda(A))) \subset c_{\lambda^*}(c_{\lambda^*}(\phi(A))) = c_{\lambda^*}(\phi(A)).$$

Thus $y \in c_{\lambda^*}(\phi(c_\lambda(A)))$ implies $y \in c_{\lambda^*}(\phi(A))$ so that from $c_\lambda(A) \in \tau(y)$ follows $A \in \tau(y)$. Hence if c_{λ^*} is a Kuratowski operator, then each $\tau(y)$ is a λ-bunch.

An immediate consequence of Theorem 4.1 is

Theorem 4.2. *If in $(\phi, (X^\nu, \Pi^\nu))$ Π^ν is a LO/$\phi(X)$-proximity then for all $\mathfrak{G} \in X^\nu$, $\tau(\mathfrak{G}) = \nu(\mathfrak{G})$ are Π-clans.*

Unfortunately, the behavior of the dual trace system of a proximity extension with respect to maximality of its members is not as nice as one might wish. This is illustrated by the following two examples.

Example 4.1. Let $Y = A_1 \supset A_2 \supset A_3 \supset \ldots$ where $\bigcap [A_k: k = 1, 2, \ldots] = \emptyset$. Further let $X \subset Y$ be such that $|(A_k \sim A_{k+1}) \cap X| = \infty$ and that $(A_k \sim A_{k+1}) \cap (Y \sim X) \neq \emptyset$ for all $k \geq 1$. Now let $k(B)$ be the smallest natural number k such that for $|B| = \infty$, $B \sim A_{k(B)}$ is infinite. The proximity Π on Y is then defined as follows: $[A, B] \in \Pi$ iff $c_\Pi(A) \cap C_\Pi(B) \neq \emptyset$, where for finite sets B, $c_\Pi(B) = B$. If $|B| = \infty$ then $c_\Pi(B) = B \cup A_{k(B)-1}$.

Then Π is a separated LO-proximity on Y and $(i, (Y, \Pi))$ has as dual traces, with respect to (X, Π_X), the following: for $y_k \in (A_k \sim A_{k+1}) \cap (Y \sim X)$

$$\tau(y_k) = [B: B \subset X, \ |B \sim A_{k+1}| = \infty].$$

Thus, clearly, $\tau(y_{k+1}) \supsetneq \tau(y_k)$ so that no $\tau(y_k)$ is maximal.

Example 4.2. Let $Y = A_1 \cup A_2 \cup A_3$, $A_k \cap A_m = \emptyset$, $k \neq m$, $k = 1, 2, 3$; $m = 1, 2, 3$. $X = A_1' \cup A_2' \cup A_3'$, $A_k' = A_k \cap X$, $|A_k'| = \infty$, $A_k \sim A_k' \neq \emptyset$. Let $\mathfrak{B}_A$ be a nonprincipal ultrafilter on Y containing A and define

$$\mathfrak{D}_C^Z = [B: B \subset Z, \ |B \cap C| = \infty].$$

It is a grill on Z. Now define c^* on Y by

$$c^*(B) = B \cup \bigcup [A_k: \ |A_k \cap B| = \infty]$$

and let Π^* be the least separated LO-proximity on Y for which $c_{\Pi^*} = c^*$. Then ν^*, is the least LO/X-nearness on Y which induces Π^*, where ν^* is defined by $\mathfrak{A} \subset \mathfrak{P}(Y)$ is in ν^* iff $\mathfrak{A} \subset \mathfrak{U}(y) \cup \mathfrak{D}_{A_k}^Y$, $y \in A_k$, or $\mathfrak{A} \subset \mathfrak{B}_A \cup \mathfrak{B}_B$, $A \subset Y$, $B \subset Y \sim X$ both infinite, or $\mathfrak{A} \subset \mathfrak{D}_{A_k'}^Y \cup \mathfrak{D}_{A_m'}^Y$. The proximity ν induced by ν^* on X can then be described as follows: $\mathfrak{A} \in \nu$ iff $\mathfrak{A} \subset \mathfrak{P}(X)$ and $\mathfrak{A} \subset \tau(y)$ some $y \in Y$ or $\mathfrak{A} \subset \mathfrak{D}_{A_k'}^X \cup \mathfrak{D}_{A_m'}^Y$. For $y \in A_k \sim A_k'$ we have $\tau(y) = \mathfrak{D}_{A_k'}^X$. These $\tau(y)$ are thus not maximal as ν-clans. We also observe that

$$\nu(\tau(y)) = \mathfrak{D}_{A_1'}^X \cup \mathfrak{D}_{A_2'}^X \cup \mathfrak{D}_{A_3'}^X,$$

which is a Π_ν-clan, but is not ν-compatible.

For purposes of comparison it seems desirable to choose X^ν to be the dual trace system of (Y, Π^*). However, since the dual trace system of (X^ν, Π^ν) is $[\nu(\mathfrak{G}): \mathfrak{G} \in X^\nu]$ it becomes clear that our choice can be completely successful only if the $\tau(y) = \mathfrak{G} = \nu(\mathfrak{G})$, that is only if the $\tau(y)$ are all maximal ν-clans. The two preceding examples show that this is not always attainable.

We now show that with the above choice of X, and with a suitable selection of ν, the natural mapping from (Y, Π^*) to (X^ν, Π^ν) is at least a proximity mapping.

Theorem 4.3. *Let $(i, (Y, \Pi^*))$ be an arbitrary proximity extension of the* LO-*proximity space (X, Π). Let ν^* be a* LO/X-*nearness on Y such that $\Pi_{\nu^*} = \Pi^*$. Finally, let $\nu = \nu^* \cap \mathfrak{P}(\mathfrak{P}(X))$. Set $X^\nu = [\tau(y): y \in Y]$ and define $f: (Y, \Pi^*) \to (X^\nu, \Pi^\nu)$ by $f(y) = \tau(y)$ for all $y \in Y$. Then f is a proximity mapping.*

Proof. Clearly $\Pi_\nu = \Pi$ and $\tau(y)$ is a ν-clan on X (Theorem 4.1). We also have $\tau(x) = \nu(x)$, for all $x \in X$, and hence $X^\nu = [\tau(y): y \in Y]$ is a permissible choice.

Let $D \subset X$; then

$$f^{-1}(D^\nu) = [y: y \in Y,\ D \in \Pi^*(y)] = [y: y \in Y,\ y \in c_{\Pi^*}(D)] = c_{\Pi^*}(D).$$

Now let $\alpha, \beta \subset X^\nu$ be such that $[\alpha, \beta] \notin \Pi^\nu$. Then $(\bigcap \alpha) \cup (\bigcap \beta) \notin \nu$. Set $A = f^{-1}(\alpha)$, $B = f^{-1}(\beta)$, $\bigcap \alpha = \mathfrak{A}$, and $\bigcap \beta = \mathfrak{B}$. Then $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{P}(X)$ and

$$A \subset f^{-1}\left(\bigcap [A_i^\nu: A_i \in \mathfrak{A}]\right) = \bigcap [f^{-1}(A_i^\nu): A_i \in \mathfrak{A}] = \bigcap [c_{\Pi^*}(A_i): A_i \in \mathfrak{A}].$$

Similarly $B \subset \bigcap [c_{\Pi^*}(B_j): B_j \in \mathfrak{B}]$. If $[A, B] \in \Pi^*$ then $[A, B] \in \nu^*$ and hence

$$[c_{\Pi^*}(A_i): A_i \in \mathfrak{A}] \cup [c_{\Pi^*}(B_j): B_j \in \mathfrak{B}] \in \nu^*.$$

Since ν^* is a LO/X-nearness and $A_i \subset X$, $B_j \subset X$, it follows that $\mathfrak{A} \cup \mathfrak{B} \in \nu^*$. Since $\mathfrak{A} \cup \mathfrak{B} \subset \mathfrak{P}(X)$ it is also true that $\mathfrak{A} \cup \mathfrak{B} \in \nu$. This is a contradiction and hence $[A, B] \notin \Pi^*$.

The content of the preceding theorem is meaningful only if for any proximity Π^* on Y, which induces a LO-proximity Π on X, there exists a LO/X-nearness ν^* on Y such that $\Pi_{\nu^*} = \Pi^*$. This, though not quite trivial, is indeed the case.

Theorem 4.4. *Let $X \subset Y$ and let (Y, Π^*) be a proximity space such that the proximity induced by Π^* on X is a* LO*-proximity. Then there exists a* LO/X*-nearness ν^* such that $\Pi_{\nu^*} = \Pi^*$.*

Proof. For every pair of sets $[C, D] \in \Pi^*$ we determine a pair $\mathfrak{B}_C^D$, $\mathfrak{B}_D^C$ of ultrafilters on Y such that $C \in \mathfrak{B}_C^D$, $D \in \mathfrak{B}_D^C$, and $\mathfrak{B}_C^D \cup \mathfrak{B}_D^C$ is a Π^*-clan. Next, for any grill $\mathfrak{H}$ on Y define

$$b_X(\mathfrak{H}) = [B: \exists A \subset X,\ B \supset A,\ c_{\Pi^*}(A) \in \mathfrak{H}].$$

It is then easy to verify that $b_X(\mathfrak{H})$ is a grill on Y, and that ν^* defined by $\mathfrak{A} \in \nu^*$ iff

$$\mathfrak{A} \subset \Pi^*(y), \text{ some } y \in Y,$$

or

$$\mathfrak{A} \subset \mathfrak{B}_C^D \cup \mathfrak{B}_D^C, \text{ some } [C, D] \in \Pi^*,$$

or

$$\mathfrak{A} \subset b_X(\mathfrak{B}_C^D \cup \mathfrak{B}_D^C), \text{ some } [C, D] \in \Pi^*,$$

is a LO/X-nearness on Y and satisfies the condition $\Pi_{\nu *} = \Pi^*$.

Theorem 4.5. *The map f, as defined in Theorem 4.3, is one-to-one iff* (I) $y_1 \neq y_2 \in Y$, *implies the existence of $A \subset X$ such that either $y_1 \in c_{\Pi^*}(A)$, $y_2 \notin c_{\Pi^*}(A)$ or $y_2 \in c_{\Pi^*}(A)$, $y_1 \notin c_{\Pi^*}(A)$, holds.*

Proof. Condition (I) is exactly what is needed to insure that $\tau(y_1) = \tau(y_2)$ and hence that $f(y) = \tau(y)$ is one-to-one. The condition is mentioned by Ivanov [9] as a possible additional requirement for an extension to be regular. Note that (I) implies but is stronger than the condition that Π^* is separated.

Theorem 4.6. *The function f: $(Y, \Pi^*) \to (X^\nu, \Pi^\nu)$ provides a proximally isomorphic mapping iff condition* (I) *of Theorem 4.5 and*: (J) *For A, $B \subset Y$, $[A, B] \notin \Pi^* \Rightarrow$ the existence of collections $\mathfrak{A}$, $\mathfrak{B} \subset \mathfrak{P}(X)$ such that $A \subset \bigcap[c_{\Pi^*}(A_i)$: $A_i \in \mathfrak{A}]$, $B \subset \bigcap[c_{\Pi^*}(B_j$: $B_j \in \mathfrak{B}]$ and $\mathfrak{A} \cup \mathfrak{B} \notin \nu$, are satisfied. Here f, ν and X^ν are defined as in Theorem 4.3. Condition* (J) *implies that all $\tau(y)$ are maximal ν-compatible families.*

Proof. Condition (I) is necessary and sufficient for f^{-1} to exist and (J) is necessary and sufficient for f^{-1} to be a proximity map. The conditions of Theorem 4.3, which are assumed to be satisfied, insure that f is a proximity map. Finally, $A \notin \tau(y)$ implies $[A, [y]] \notin \Pi^*$ and hence by (J), and if we assume that $A \subset X$, it follows that there exists a family $\mathfrak{B} \subset \tau(y) \subset \mathfrak{P}(X)$ such that $[A] \cup \mathfrak{B} \notin \nu$, that is $\tau(y)$ is a maximal ν-compatible family.

Theorem 4.7. *A sufficient condition for all $\tau(y)$ of $(i, (Y, \Pi^*))$ to be maximal ν-compatible families is that for $y \in Y$ and $E \subset X$ $E \notin \tau(y) \Rightarrow \exists N_y$ such that $[E, N_y] \notin \Pi^*$.*

Proof. Since $N_y \cap X = D \neq \varnothing$ we have that $E \notin \tau(y)$ implies the existence of $D \subset X$ such that $[E, D] \notin \Pi$. Hence every $\tau(y)$ is a maximal Π-compatible family and, a fortiori, maximal ν-compatible. A sufficient condition for these conditions to be satisfied is if Π^* is an RH-proximity (see [19]) and hence certainly if Π^* is an EF-proximity.

The conditions appearing in the results above all depend on Π^* as well as on the "position" (a sort of "super density") of X in Y. The stronger the assumptions on Π^* the less important will be the requirements on the "position" of X in Y. Conversely, with relatively weak assumptions on Π^* the requirements on X become critical.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302