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POLAR SETS AND PALM MEASURES IN THE THEORY OF FLOWS(1)

BY

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ABSTRACT. Given a flow (θ_t), t real, over a probability space Ω, we prove that certain measures on Ω (viewed as the state space of the flow) decompose uniquely into a Palm measure Q which charges no "polar set" and a measure supported by a polar set. Considering the continuous and discrete parts of the additive functional corresponding to Q, we find that Q further decomposes into a measure charging no "semipolar set" and a measure supported by one. As a consequence, Palm measures are exactly those which neglect sets which the flow neglects, and polar sets are exactly those neglected by every Palm measure. Finally, we characterize various properties, such as predictability and continuity, of an additive functional in terms of its Palm measure. These results further illuminate the role played by supermartingales in the theory of flows, as pointed by J. de Sam Lazaro and P. A. Meyer.

0. Introduction. Let $(\Omega, \mathcal{F}_t^0, P, \theta_t)$, $t \in \mathbf{R}$ (the real line), be a filtered dynamical system (all terminology will be explained below). In §1 we prove that a finite measure Q on $\mathcal{F}_{0-}^0$ which is "progressively absolutely continuous" decomposes uniquely into the sum of two measures $Q = P_\alpha^- + \mu$, where P_α^- is the restriction to $\mathcal{F}_{0-}^0$ of the Palm measure P_α of a predictable additive functional α, and μ is supported by a "polar" set in $\mathcal{F}_{0-}^0$. Decomposing α into its continuous and discrete parts, say α_c and α_d, we will see (in §2) that P_α^- splits into a measure P_c^- which charges no "semipolar" set and a measure P_d^- which is carried by a semipolar, but charges no polar, set. Thus we have a decomposition

$$Q = P_c^- + P_d^- + \mu \tag{1}$$

analogous to that of a measure on the state space of a Markov process [1,

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p. 283]; this is not entirely surprising in view of the Markovian nature of the flow θ_t: $(\Omega, \mathcal{F}_t^0) \to (\Omega, \mathcal{F}_0^0)$ (see [9]). The decomposition (1) requires the Doob-Meyer decomposition of supermartingales and is related to Föllmer's [3] correspondence between supermartingales and certain measures on $\mathbf{R}_+ \times \Omega$. As a corollary, we find that a finite measure Q on $\bigvee_{t \in \mathbf{R}} \mathcal{F}_t^0$ is a Palm measure iff it charges no polar set. The section concludes with a characterization of polar sets, and several applications of these ideas, particularly to local times.

In §2 we characterize various properties of a given additive functional α, such as well-measurability, predictability, and continuity, in terms of its Palm measure. Papangelou [14] has recently given some results on stationary point processes (which we construe as additive functionals which increase only by unit jumps) of the type we are considering. (Similar questions for Markov additive functionals have been treated by Revuz [16].) In the present article we will generalize several of Papangelou's results and obtain flow theory analogues of some of the Markovian ones. Some of our material will be recognized as a specialization of results in the "general theory of processes" [2], with more detail made possible by additional structure. In fact, our intention throughout is to solidify further the bridge built between the general theory and flow theory by J. de Sam Lazaro and P. A. Meyer [8], [9].

The remainder of this section is devoted to an explanation of the terminology and background material. Our notation is largely that of [1, Chapter 0], with this exception: if $(E, \mathcal{E})$ is a measurable space, we write (ambiguously) $f \in (\mathcal{E})$ to mean that f is an $\mathcal{E}$-measurable function on E, the range being clear from context; $f \in (\mathcal{E})_+$ indicates the range is $\mathbf{R}_+ = [0, \infty)$.

A *flow* $\theta = (\theta_t)$, $t \in \mathbf{R}$, on a probability space $(\Omega, \mathcal{F}^0, P)$ is a one-parameter group (under composition) of bimeasurable, measure-preserving bijections θ_t: $\Omega \to \Omega$ such that $\theta_0 =$ identity and the mapping $(t, \omega) \to \theta_t(\omega)$ is $\mathcal{B} \otimes \mathcal{F}^0/\mathcal{F}^0$-measurable. We further assume the existence of a *filtration*, i.e. an increasing family of σ-fields $\{\mathcal{F}_t^0\}$, $t \in \mathbf{R}$, on Ω whose generated σ-field $\bigvee_{t \in \mathbf{R}} \mathcal{F}_t^0$ is $\mathcal{F}^0$, and which is compatible with the flow θ in that $\theta_t^{-1}\mathcal{F}_s^0 = \mathcal{F}_{s+t}^0$, $s, t \in \mathbf{R}$. As usual we write $\mathcal{F}_{t+}^0 = \bigcap_{s>t} \mathcal{F}_s^0$, $\mathcal{F}_{t-}^0 = \bigvee_{s<t} \mathcal{F}_s^0$. Each $\mathcal{F}_t^0$ (and thus $\mathcal{F}^0$) is assumed separable. The P-completion of $\mathcal{F}^0$ is denoted $\mathcal{F}$, and then $\mathcal{F}_t$ is obtained by adjoining to $\mathcal{F}_t^0$ all sets in $\mathcal{F}$ of measure zero. The family $\{\mathcal{F}_t\}$ is then right-continuous [2] and compatible with θ, but θ: $(t, \omega) \to \theta_t(\omega)$ need not be $\mathcal{B} \otimes \mathcal{F}/\mathcal{F}$-measurable. The entity $(\Omega, \mathcal{F}_t^0, P, \theta_t)$ is a *filtered dynamical system*. Concepts from the general theory of processes, such as predictability, are in reference to the family $\{\mathcal{F}_t\}$ unless otherwise indicated.

An *additive functional* (AF) is a real-valued process $\alpha = \alpha(t, \omega)$ (or $\alpha_t(\omega)$), $t \in \mathbf{R}$, $\omega \in \Omega$, such that (i) $\alpha(0) = 0$; (ii) almost every path is right-continuous, nondecreasing; (iii) for each $s, t \in \mathbf{R}$ there is a set $N_{st} \in \mathcal{F}$ of measure zero such that

(2) $$\alpha(t + s, \omega) = \alpha(t, \omega) + \alpha(s, \theta_t\omega)$$

for $\omega \notin N_{st}$. By (ii) we may consider α as a measure on $\mathcal{B}$. Notice that α need not be adapted; sometimes we will use the phrase (due to Getoor and Sharpe) "raw additive functional" (RAF) to emphasize this point. We call α *adapted* if $\alpha(t) \in (\mathcal{F}_t)$ for $t \geq 0$, and this implies $\alpha(t) \in \mathcal{F}_0$ if $t \leq 0$. As shown in [9], there exists an AF $\overline{\alpha}$ indistinguishable from α (i.e. such that $\overline{\alpha}(t, \omega) = \alpha(t, \omega)$ for all $t \in \mathbf{R}$ a.s.) which is *perfect* in that the set N_{st} in (iii) may be chosen independently of s, t, $\mathcal{F}^0$-measurable, and such that (iv) $\overline{\alpha}(\pm\infty, \omega) = \pm\infty$ or $\overline{\alpha}(t, \omega) \equiv 0$ for every $\omega \in \Omega$.

Given an AF α, its *Palm measure* is

(3) $$P_\alpha(A) = E\int_0^1 I_A \circ \theta_t \, d\alpha_t \quad \left(= E\int_0^\infty e^{-t} I_A \circ \theta_t \, d\alpha_t\right), \qquad A \in \mathcal{F}^0,$$

where I_A is the indicator of A. Palm measures arise naturally in the study of "flows under a function" [7], local times [4], "time-changes" of flows [17], point processes ("Palm-Khinhin formulae" etc.), and level crossings ("horizontal-window" probabilities). They are exactly the measures which neglect sets in Ω which the flow neglects (Theorem (10)).

Finally, we will need these facts (see [5], [9]). P_α is always σ-finite and finite iff $E\alpha(1) < \infty$, in which case α is called *integrable* and $E\alpha(t) = tE\alpha(1)$. Two AF's α, β are indistinguishable iff their Palm measures are identical. (In particular, $P_\alpha = P_{\overline{\alpha}}$.) In addition, if α and β are both adapted (resp. predictable), then it is enough for the Palm measures to agree on $\mathcal{F}^0_{0+}$ (resp. $\mathcal{F}^0_{0-}$).

1. Decomposition theorems and characterizations of Palm measures. A function $\xi \in (\mathcal{F}^0)$ for which $\xi \circ \theta_t(\omega) \to \xi(\omega)$ as $t \downarrow 0$ for all $\omega \in \Omega$ will be called *translation continuous*. It is shown in [9] that there exists another filtration $\{\mathcal{G}^0_t\}$ such that $\mathcal{G}^0_t \subset \mathcal{F}^0_{t-}$, $\mathcal{G}_t = \mathcal{F}_t$ for all $t \in \mathbf{R}$, where $\{\mathcal{G}_t\}$ is the completed family corresponding to $\{\mathcal{G}^0_t\}$ (see §0), and $\mathcal{G}^0 = \bigvee_{t \in \mathbf{R}} \mathcal{G}^0_t$ is generated by the translation continuous functions. Moreover, the mapping $(t, \omega) \to \theta_t(\omega)$ is $\mathcal{B} \otimes \mathcal{G}^0/\mathcal{G}^0$-measurable. Since the two filtrations differ by sets of measure zero only, there is no essential loss of generality in assuming that $\mathcal{F}^0$ is itself generated by the translation continuous functions. For reasons (in addition to those above) which will soon be apparent, *we assume from now on*

(4) (I) $\mathcal{F}^0$ is generated by the translation continuous functions,
(II) $(\Omega, \mathcal{F}^0)$ is a Blackwell space.

The meaning of (II) is: (i) $\mathcal{F}^0$ is separable, and (ii) for every real-valued $\xi \in (\mathcal{F}^0)$ and $A \in \mathcal{F}^0$, the image $\xi(A)$ is analytic in $\mathbf{R}$. The basic fact we will require is this: let $(E, \mathcal{E})$ be a Blackwell space and $\mathcal{G}$ a separable sub σ-field of $\mathcal{E}$. A function $f \in (\mathcal{E})$ which is constant on the atoms of $\mathcal{G}$ is then $\mathcal{G}$-measurable. (See [13] and [9, Appendix Chapter I].)

We hasten to add that many of the results below do not depend on (I), though they are more complicated without it, and that most of the standard spaces arising in flow theory satisfy (I) and (II). Two examples are the function spaces $\mathfrak{C}$ and $\mathfrak{D}$ of all continuous functions (resp. right-continuous functions having left limits) from $\mathbf{R}$ to $\mathbf{R}$; for $f \in \mathfrak{C}$ (or $\mathfrak{D}$) we let $\theta_t f(s) = f(s+t)$, $X_s f = f(s)$, and $\mathcal{F}_t^0 = \sigma\{X_s\colon s \leq t\}$. Another example, arising in connection with point processes, is the space $\mathfrak{W}$ of all locally finite (i.e. having no finite accumlation point) nonempty subsets of $\mathbf{R}$. Here, for $w \in \mathfrak{W}$, we let $\theta_t w = w - t$ and take $\mathfrak{W}_t^0$ as the σ-field generated by the functions $N(A, w) =$ cardinality of $A \cap w$ for Borel sets $A \subset (-\infty, t]$. These examples will be discussed below.

Let $Z \in (\mathcal{F}_0)_+$ be such that $Z_t = e^{-t} Z \circ \theta_t$, $t \in \mathbf{R}_+$, is a supermartingale relative to $\{\mathcal{F}_t\}$; Z is then called *excessive*. We need the following result, which may be derived from [8] or [9].

(5) **Theorem.** *Let Z be excessive. Then there exists an excessive $Z^0 = Z$ a.s. for which $Z^0 \in (\mathcal{F}_{0+}^0)_+$ and the mapping $t \to Z^0 \circ \theta_t(\omega)$ is right-continuous and has left limits at every $t \in \mathbf{R}$, for almost every $\omega \in \Omega$.*

In particular, let $Z = (Z_t)$, $t \in \mathbf{R}_+$, be a potential [13] such that $Z_t = e^{-t} Z_0 \circ \theta_t$ a.s. for each t (Z is "almost homogeneous"). Then there exists a *homogeneous potential* $Z_t^0 = e^{-t} Z^0 \circ \theta_t$, with Z^0 as described in (5), such that Z_t^0 and Z_t are indistinguishable.

Let $\mathcal{B}_+$ denote the Borel σ-field in $\mathbf{R}_+$, and define, for any process $u \in (\mathcal{B}_+ \otimes \mathcal{F}^0)$, two new processes, $\theta^+ u$ and $\theta^- u$, by

$$\theta^+ u(s, \omega) = u(s, \theta_s \omega), \qquad \theta^- u(s, \omega) = u(s, \theta_{-s} \omega).$$

These are measurable, and θ^+ and θ^- are obviously inverses of one another. Now define $\mathcal{P}^0$ to be the σ-field on $\mathbf{R}_+ \times \Omega$ generated by all sets of the form $[t, \infty) \times A$, $t \in \mathbf{R}_+$, $A \in \mathcal{F}_{t-}^0$ (equivalently: all sets of the form $\{0\} \times A$, $A \in \mathcal{F}_{0-}^0$, and $(t, \infty) \times A$, $t \in \mathbf{R}_+$, $A \in \mathcal{F}_{t-}^0$). This is similar to the usual predictable σ-field [2], but more appropriate in the present context.

(6) **Lemma.** $\mathcal{P}^0 = \theta^+(\mathcal{B}_+ \otimes \mathcal{F}^0_{0-})$, *i.e.* $u \in (\mathcal{P}^0)$ *iff* $\theta^- u \in (\mathcal{B}_+ \otimes \mathcal{F}^0_{0-})$.

First note that $(\mathrm{R}_+ \times \Omega, \mathcal{B}_+ \otimes \mathcal{F}^0)$ is a Blackwell space, and $\mathcal{B}_+ \otimes \mathcal{F}^0_{0-}$ is a separable sub σ-field of $\mathcal{B}_+ \otimes \mathcal{F}^0$. Consider a generator of $\mathcal{P}^0$, say $[t, \infty) \times A$, with $A \in \mathcal{F}^0_{t-}$, and let $u(s, \omega) = I_{[t,\infty)}(s) I_A(\omega)$. We wish to show $\theta^- u$ is constant on the atoms of $\mathcal{B}_+ \otimes \mathcal{F}^0_{0-}$. Suppose $(s, \omega), (s', \omega')$ are contained in such an atom. Then, for every $B \in \mathcal{B}_+$, $C \in \mathcal{F}^0_{0-}$, we have $I_B(s) I_C(\omega) = I_B(s') I_C(\omega')$, so $s = s'$ and ω, ω' lie in the same atom of $\mathcal{F}^0_{0-}$. Hence $\theta^- u(s, \omega) = I_{[t,\infty)}(s) I_A(\theta_{-s}\omega)$ and $\theta^- u(s', \omega') = I_{[t,\infty)}(s) I_A(\theta_{-s}\omega')$ both vanish if $s \le t$, and are equal if $s \ge t$, because then $\theta_s A \in \mathcal{F}^0_{(t-s)-} \subset \mathcal{F}^0_{0-}$. We have shown $\mathcal{P}^0 \subset \theta^+(\mathcal{B}_+ \otimes \mathcal{F}^0_{0-})$. Next, let $v \in (\mathcal{B}_+ \otimes \mathcal{F}^0_{0-})$ be of the form $v(s, \omega) = I_B(s) I_C(\omega)$, $B \in \mathcal{B}_+$, $C \in \mathcal{F}^0_{0-}$. Noting that $\mathcal{P}^0$ is a separable sub σ-field of $\mathcal{B}_+ \otimes \mathcal{F}^0$, we will show that $\theta^+ v$ is constant on atoms of $\mathcal{P}^0$. Let $(s, \omega), (s', \omega')$ be in an atom of $\mathcal{P}^0$. Then $s = s'$, and $I_A(\omega) = I_A(\omega')$ for every $A \in \mathcal{F}^0_{s-}$. Thus $v(s, \theta_s\omega) = I_B(s) I_C(\theta_s\omega) = I_B(s') I_C(\theta_s\omega)$ since $\theta_s^{-1} C \in \mathcal{F}^0_{s-}$. The proof is complete.

(7) **Corollary.** (a) *If* $t \in \mathrm{R}_+$ *and* $A \in \mathcal{F}^0_{t+}$, *then* $(t, \infty) \times A \in \mathcal{P}^0$.
(b) *If* $\xi \in (\mathcal{F}^0_{0-})$, *then the process* $\xi \circ \theta = (\xi \circ \theta_t(\omega))$ *is* $\mathcal{P}^0$*-measurable.*

Part (a) results from $(t, \infty) \times A = \bigcup_{n=1}^\infty (t + n^{-1}, \infty) \times A$; (b) is trivial.

To each measure Q on $\mathcal{F}^0_{0-}$ we now associate a measure $\tilde{Q}$ on $\mathcal{P}^0$ as follows. Writing $\tilde{Q}(u)$ for $\int u\, d\tilde{Q}$,

$$\tilde{Q}(u) = \int_0^\infty e^{-s} \int_\Omega \theta^- u(s, \omega) Q(d\omega)\, ds, \qquad u \in (\mathcal{P}^0)_+. \tag{8}$$

We further write $\tilde{Q}_t(A) = \tilde{Q}[(t, \infty) \times A]$, $A \in \mathcal{F}^0_{t+}$, which is possible by (7)(a), and call $\tilde{Q}$ (or Q) *progressively absolutely continuous* (relative to P) if, for each $t \in \mathrm{R}_+$, $\tilde{Q}_t \ll P$ on $\mathcal{F}^0_{t+}$, or, equivalently, $\tilde{Q}_0 \ll P$ on $\mathcal{F}^0_{0+}$.

(9) **Theorem.** *A finite measure* Q *on* $\mathcal{F}^0_{0-}$ *which is progressively absolutely continuous may be written uniquely as* $Q = P^-_\alpha + \mu$ *where* P^-_α *is the restriction to* $\mathcal{F}^0_{0-}$ *of the Palm measure of an (integrable) predictable AF* α, *and the measure* μ *is concentrated on a polar set.*

The meaning of "polar" is this. Let ξ be a random variable on Ω; the random set $S_\xi(\omega) = \{t\colon \xi \circ \theta_t(\omega) \neq 0\}$ is called the *spoor* of ξ. If $\xi = I_A$, we speak of the "spoor of A". Then ξ (or A if $\xi = I_A$) is called polar (*thin*) if its spoor is a.s. empty (a.s. locally finite). Thus ξ is *polar* iff $\xi \circ \theta = (\xi \circ \theta_t(\omega))$ is an "evanescent" process [2]. Further, A is *semipolar* if it is contained in a countable union of thin sets. As indicated earlier, P^-_α further decomposes into a measure which charges no semipolar and a measure which lives on a semipolar, but charges no polar, set (see §2). We note that if Q charges no polar set in $\mathcal{F}^0_{0-}$, then (using a Fubini argument) it is progressively absolutely continuous.

As an immediate consequence of (9) (using the trivial filtration $\mathcal{F}^0_t \equiv \mathcal{F}^0$ for (b)) we have

(10) **Theorem.** (a) *A finite measure Q on $\mathcal{F}^0_{0-}$ is the restriction to $\mathcal{F}^0_{0-}$ of the Palm measure of an integrable, predictable AF iff Q charges no polar set in $\mathcal{F}^0_{0-}$.*

(b) *A finite measure Q on $\mathcal{F}^0$ is a Palm measure iff Q charges no polar set in $\mathcal{F}^0$, in which case the corresponding AF (α_t) is the right-continuous regularization of (dQ^t/dP) where $Q^t(A) = \int_0^t Q(\theta_s A)\, ds$, $A \in \mathcal{F}^0$.*

Proof of (9). The uniqueness is clear, since (3) implies that a Palm measure charges no polar set. Let $Z_t \in (\mathcal{F}^0_{t+})_+$ be the Radon-Nikodym derivative $d\tilde{Q}_t/dP$ on $\mathcal{F}^0_{t+}$. Since $t \to EZ_t$ is right-continuous one may choose an a.s. right-continuous version $Z = (Z_t)$ and it is easy to see that Z is a potential, though not necessarily of class (D) (see [13] for terminology).

Now a change of variables yields $\tilde{Q}_t(A) = e^{-t}\tilde{Q}_0(\theta_t A)$, $A \in \mathcal{F}^0_{t+}$, from which follows immediately that, for each $t \in \mathbf{R}_+$, $Z_t = e^{-t}Z_0 \circ \theta_t$ a.s. By the remark following (5), we may assume Z_t is homogeneous: $Z_t \equiv e^{-t}Z_0 \circ \theta_t$.

It is well known that any potential Z has a unique decomposition $Z = N + Y$, where N is a local martingale (and also a potential) and Y is a potential of class (D). We show now that N and Y may be chosen homogeneous. Notice that $Z_{t+s}(\omega) = e^{-t}Z_s \circ \theta_t(\omega)$ for all ω, s, t. For t fixed, consider the following two decompositions of Z_{t+s}, $s \geq 0$.

$$(11) \qquad Z_{t+s} = \begin{cases} N_{t+s} + Y_{t+s}, \\ e^{-t}N_s \circ \theta_t + e^{-t}Y_s \circ \theta_t. \end{cases}$$

It is tedious, but straightforward, to check that both N_{t+s} and $e^{-t}N_s \circ \theta_t$, $s \geq 0$, are local martingales and that both Y_{t+s} and $e^{-t}Y_s \circ \theta_t$, $s \geq 0$, are class (D) potentials, all relative to the σ-fields $\mathcal{F}_{t+s}$, $s \geq 0$. By uniqueness, the two expressions "match" correctly, and we conclude $N_{t+s} = e^{-t}N_s \circ \theta_t$ for all s, a.s., and similarly for Y_{t+s}. Putting $s = 0$, we find N, Y are "almost homogeneous" and may be replaced by homogeneous modifications, which we again denote by N and Y.

By the Doob-Meyer decomposition theorem [13, p. 119] we may write $Y = M - A$, where $M = (M_t)$ is a uniformly integrable martingale, and $A = (A_t)$

is a predictable (= natural) integrable increasing process. Since Y is homogeneous, we have two decompositions of Y_{t+s} for $s \geq 0$ (t fixed) analogous to (11):

$$Y_{t+s} = \begin{cases} M_{t+s} - A_{t+s} = M_{t+s} - A_t - (A_{t+s} - A_t), \\ e^{-t}M_s \circ \theta_t - e^{-t}A_s \circ \theta_t. \end{cases}$$

Noting that $e^{-t}M_s \circ \theta_t$ and $M_{t+s} - A_t$ are uniformly integrable martingales, and $e^{-t}A_s \circ \theta_t$, $A_{t+s} - A_t$ are predictable increasing processes (all relative to $\{\mathcal{F}_{t+s}\}$, $s \geq 0$, t fixed), we have by the uniqueness of the decomposition,

$$(12) \qquad A_{t+s} - A_t = e^{-t}A_s \circ \theta_t.$$

Now let $\alpha_t = \int_0^t e^s \, dA_s$. (By $\int_a^b$ we will always mean $\int_{(a,b]}$.) From (12) it follows easily that α is a predictable AF. (This argument was inspired by Maisonneuve [10].)

Now for any Palm measure, say P_β, we have (see [5])

$$(13)\ E\int_{\mathbf{R}} u(s, \omega)\beta(ds, \omega) = \int_\Omega \int_{\mathbf{R}} u(s, \theta_{-s}\omega)\, ds P_\beta(d\omega), \qquad u \in (\mathcal{B} \otimes \mathcal{F}^0)_+.$$

Hence for any $A \in \mathcal{F}^0_{t+}$,

$$E(Y_t; A) = E\left(\int_t^\infty e^{-s}\, d\alpha_s; A\right) = \int_t^\infty e^{-s} P_\alpha(\theta_s A)\, ds.$$

So we may write

$$(14)\ \int_t^\infty e^{-s} Q \circ \theta_s(A)\, ds = E(N_t; A) + \int_t^\infty e^{-s} P_\alpha \circ \theta_s(A)\, ds, \qquad A \in \mathcal{F}^0_{t+}.$$

Now define a measure $\tilde{\mu}$ on $\mathcal{P}^0$ by equation (8) with Q replaced by $\mu = Q - P^-_\alpha$, P^-_α being the restriction of P_α to $\mathcal{F}^0_{0-}$. Using (14), we see that $\tilde{\mu}$ is positive on sets of the form $(t, \infty) \times A$, $A \in \mathcal{F}^0_{t+}$, and hence on all of $\mathcal{P}^0$; moreover

$$(15) \qquad E(N_t; A) = \tilde{\mu}[(t, \infty) \times A] = \int_t^\infty e^{-s}\mu \circ \theta_s(A)\, ds, \qquad A \in \mathcal{F}^0_{t+}.$$

(The existence of a measure $\tilde{\mu}$ on $\mathbf{R}_+ \times \Omega$ satisfying the first equality in (15) is established by Föllmer [3] in a different situation.) It is easy to see that $\mu \geq 0$ so it only reamins to prove that μ lives on a polar set.

Having chosen N homogeneous, i.e. $N_t \equiv e^{-t}\overline{n} \circ \theta_t$ for an excessive function $\overline{n} \in (\mathcal{F}^0_{0+})_+$, note first that we may extend N_t to all $t \in \mathbf{R}$ and still have a supermartingale. Moreover, N_t will be right-continuous with finite left limits for all $t \in \mathbf{R}$ a.s. since $t \to \overline{n} \circ \theta_t$ has these properties.

Define, for each $n \geq 1$, $R_n = \inf\{r > 0, \text{ rational: } N_r > n\}$. Each $R_n \in \mathcal{S}^0_+$, the family of stopping times of $\{\mathcal{F}^0_{t+}\}$, $t \in \mathbf{R}_+$, and on right-continuous paths coincides with $\inf\{r > 0: N_r > n\}$. Starting with *discrete* $T \in \mathcal{S}^0_+$, one easily shows that the stochastic interval $]]T, \infty[[\in \mathcal{P}^0$ for *any* $T \in \mathcal{S}^0_+$, and hence $K = \bigcap_n]]R_n, \infty[[\in \mathcal{P}^0$. The argument in [3] shows that $\tilde{\mu}$ is supported by K and that K is evanescent. (The set K in [3] is slightly different, but, since $\tilde{\mu}$ puts no mass on $\{\infty\} \times \Omega$ because N_t is a potential, the argument given there goes through.)

Define $\xi(\omega) = \int_0^\infty e^{-r} I_K(r, \theta_{-r}\omega)\, dr$; since $K \in \mathcal{P}^0$, (6) implies $\xi \in (\mathcal{F}^0_{0-})$. Clearly $\mu(\xi) = \tilde{\mu}(K)$, and $\mu(1 - \xi) = \tilde{\mu}(K^c) = 0$. Hence $\xi = 1$ μ-a.e. and $\mu\{\xi = 0\} = 0$, i.e. μ lives on the set $\{\xi > 0\}$. To show this set is polar, let G be the set of $\omega \in \Omega$ on which the trajectory $N_t(\omega)$ has finite left limits and is right-continuous at all $t \in \mathbf{R}$. Then G, and so G^c, is invariant, and $P(G^c) = 0$. Suppose $\xi(\omega) > 0$. Then $(r, \theta_{-r}\omega) \in K$ for some $r > 0$, i.e. $R_n(\theta_{-r}\omega) < r$ for all $n \geq 1$, and this puts ω in G^c. But $\{\xi > 0\} \subset G^c$ implies $\{\xi > 0\}$ is polar since G^c is an invariant null set. This completes the proof of (9).

We now sketch the proof of another decomposition for an arbitrary (finite) Q on $\mathcal{F}^0_{0-}$, which is valid under the additional assumption that $\{\mathcal{F}^0_t\}$ is a *standard system* [3], [15], which means

(a) each $\mathcal{F}^0_t$ is σ-isomorphic to the Borel σ-field of a Polish space;

(b) for any increasing sequence t_n, and decreasing sequence of sets A_n, such that A_n is an atom of $\mathcal{F}^0_{t_n}$, we have $\bigcap_n A_n \neq \emptyset$.

Unfortunately, the usual filtrations on the standard spaces of flow theory, such as $\mathfrak{C}$ and $\mathfrak{B}$ previously mentioned, are not standard in the above sense. We will indicate later how to circumvent this difficulty for those two cases.

Let Q be a finite measure on $\mathcal{F}^0_{0-}$ and define $\tilde{Q}$ as before (see (8)). For each $t \in \mathbf{R}_+$, the measure $\tilde{Q}_t$ has a Lebesgue decomposition on $\mathcal{F}^0_{t+}$, namely $\tilde{Q}_t(A) = Q'_t(A) + Q''_t(A)$, with $Q'_t \ll P$ on $\mathcal{F}^0_{t+}$, and $Q''_t \perp P$ on $\mathcal{F}^0_{t+}$ ($\perp$ means "singular"). An easy argument using the uniqueness of the Lebesgue decomposition shows that $Q'_t = e^{-t} Q'_0 \circ \theta_t$ and $Q''_t = e^{-t} Q''_0 \circ \theta_t$. Let $Z_t = dQ'_t/dP$ on $\mathcal{F}^0_{t+}$. We may choose a homogeneous version of the potential $Z = (Z_t)$ just as in the proof of (9), and this splits into a local martingale N plus a class (D) potential Y, both of which are homogeneous. Now let $\tilde{\mu}$ be the *Föllmer measure* of the local martingale N, i.e. the unique measure on $\mathcal{P}^0$ such that the first equality in (15) holds. We thus have, α being as in the proof of (9),

$$\tilde{Q} = \tilde{Q}_\alpha + \tilde{\mu} + \tilde{\nu} \tag{16}$$

where $\tilde{Q}_\alpha$ is defined by the right-hand side of (8) with P_α in place of Q, $\tilde{\nu} \equiv \tilde{Q} - \tilde{Q}_\alpha - \tilde{\mu}$. In this way, $\tilde{\nu}[(t, \infty) \times A] = Q_t''(A)$, $A \in \mathcal{F}_{t+}^0$.

Equation (16) exhibits $\tilde{Q}$ as the sum of the progressively absolutely continuous measure $\tilde{M} = \tilde{Q}_\alpha + \tilde{\mu}$ and the "progressively singular" measure $\tilde{\nu}$. One establishes easily that such a decomposition is unique.

Define measures μ, ν on $\mathcal{F}_{0-}^0$ by $\mu(A) = \tilde{\mu}(I_A \circ \theta)$, $\nu(A) = \tilde{\nu}(I_A \circ \theta)$.

(17) **Lemma.** *For every* $u \in (\mathcal{P}^0)_+$,

(18) $$\tilde{\mu}(u) = \int_\Omega \int_0^\infty e^{-s} \theta^- u(s, \omega)\, ds\, \mu(d\omega)$$

and similarly for $\tilde{\nu}$ *and* ν.

Proof. Define a transformation T_t on $\mathcal{B}_+ \otimes \mathcal{F}^0$ for each $t \in \mathbf{R}_+$ by $T_t u(s, \omega) = u(s + t, \theta_{-t}\omega)$. A monotone class argument shows that T_t: $(\mathcal{P}^0)_+ \to (\mathcal{P}^0)_+$. From (8) we find that

(19) $$\tilde{Q}(T_t u) = e^t \tilde{Q}(I_{[[t,\infty[[} u), \qquad t \in \mathbf{R}_+, \quad u \in (\mathcal{P}^0)_+.$$

By looking at the generators of $\mathcal{P}^0$, we also find that $\tilde{M}(T_t u)$ defines a progressively absolutely continuous measure, while $\tilde{\nu}(T_t u)$ is progressively singular (t fixed). The uniqueness of the decomposition (16) shows that (19) holds for $\tilde{M}$ (resp. $\tilde{\nu}$) in place of $\tilde{Q}$; since $\tilde{Q}_\alpha$ also satisfies (19), so does $\tilde{\mu}$.

In view of (6), it will suffice to verify (18) for $u = \theta^+(I_{[t,\infty)}\xi)$, $t \geq 0$, $\xi \in (\mathcal{F}_{0-}^0)_+$, in which case the right-hand side of (18) reduces to $e^{-t}\tilde{\mu}(\xi \circ \theta)$. The left-hand side is

$$\tilde{\mu}(I_{[[t,\infty[[}\, \theta^+ \xi) = e^{-t} \tilde{\mu}(T_t(\xi \circ \theta)) \quad \text{(by (19))}$$
$$= e^{-t} \tilde{\mu}(\xi \circ \theta).$$

The fact that μ is carried by a polar set is proven just as before and we can state, *given that* $\{\mathcal{F}_t^0\}$ *is standard*:

(20) **Theorem.** *A finite measure* Q *on* $\mathcal{F}_{0-}^0$ *may be written as* $Q = P_\alpha^- + \mu + \nu$, *where* P_α^- *and* μ *are as described in* (9), *and* ν *is such that*

$$\tilde{\nu}(u) = \int_\Omega \int_0^\infty e^{-s} \theta^- u(s, \omega)\, ds\, \nu(d\omega), \qquad u \in (\mathcal{P}^0)_+,$$

is progressively singular.

As we indicated, the spaces $\mathfrak{C}$, $\mathfrak{D}$ and $\mathfrak{W}$ are *not* standard. To illustrate how to overcome this difficulty, introduce the space $\mathfrak{C}'$ consisting of all functions f: $\mathbf{R} \to \mathbf{R} \cup \{\Delta\}$, where $\Delta \notin \mathbf{R}$ is an adjoined "death point" such that f is continuous on $\mathbf{R}$ for all $t < \zeta$ and $f(t) = \Delta$ for all $t \geq \zeta$. The "lifetime" $\zeta \leq \infty$ depends on f. Clearly $\mathfrak{C} \subset \mathfrak{C}'$, and $f \in \mathfrak{C}$ iff $\zeta = +\infty$. Again let

θ_t "shift" f by t, $X_t f = f(t)$, and define $\mathcal{F}'_t = \sigma\{X_s, s \leq t\}$, all relative to $\mathfrak{C}'$. In this way, $\mathcal{F}^0_t = \mathcal{F}'_t \cap \mathfrak{C}$, $\mathfrak{C}$ is an invariant subset of $\mathfrak{C}'$, and a stationary measure P on $\mathfrak{C}$ extends naturally to $\mathfrak{C}'$ by $P'(A) = P(A \cap \mathfrak{C})$, $A \in \mathcal{F}' = \bigvee_t \mathcal{F}'_t$. The filtration $\{\mathcal{F}'_t\}$ is standard. (To "standardize" $\mathfrak{W}$, one introduces $\mathfrak{W}'$ consisting of all nonempty subsets $w \in \mathbf{R}$ which are locally finite before $\zeta = \sup w \leq \infty$. We only pursue the case $\mathfrak{C}'$; the argument for $\mathfrak{W}$ is entirely similar.)

Let Q be a finite measure on $\mathcal{F}^0_{0-} = \mathfrak{C} \cap \mathcal{F}'_{0-}$, and extend it to $\mathcal{F}'_{0-}$ by $Q'(A) = Q(A'\mathfrak{C})$, $A \in \mathcal{F}'_{0-}$. By (20)

(21) $$Q' = P^-_\alpha + \mu + \nu,$$

all measures on $\mathcal{F}'_{0-}$. Restricting each of these to $\mathcal{F}^0_{0-} \subset \mathcal{F}'_{0-}$ we obtain a decomposition of Q; it only remains to show it is of the desired form. Now P^-_α kills every polar set in $\mathcal{F}^0_{0-}$ since every polar set in $\mathcal{F}^0$ relative to P is also polar relative to P'; indeed, the restriction to $\mathfrak{C}$ of P_α is the Palm measure of the restriction of α to $\mathfrak{C}$. As for the restriction to $\mathcal{F}^0_{0-}$ of μ, it is obvious that it lives on $N\mathfrak{C}$, where $N \in \mathcal{F}'_{0-}$ is polar and $\mu(N^c) = 0$, and that $N\mathfrak{C} \in \mathcal{F}^0_{0-}$ is polar. One also can show that ν restricted to $\mathcal{F}^0_{0-}$ is as in (20).

Finally we remark that, if we write the Lebesgue and $\tilde{Q}'_t(A')$ (with the obvious notation),

$$\tilde{Q}_t(A) = E(Z_t; A) + K_t(A), \qquad A \in \mathcal{F}^0_{t+},$$
$$\tilde{Q}'_t(A') = E'(Z'_t; A') + K'_t(A'), \qquad A' \in \mathcal{F}'_{t+},$$

where K_t, K'_t are P-singular (resp. P'-singular), and Z_t, Z'_t are the Radon-Nikodym derivatives of the P- (P'-) absloutely continuous pieces, then Z, Z' are potentials and the pieces match properly. Indeed, $K'_t(A') = K_t(A'\mathfrak{C})$, and Z'_t may be taken as the "canonical extension" of Z_t to $\mathfrak{C}'$: first choose the homogeneous version of Z, and then let $Z'_t(f) = 0$ if $t \geq \zeta$ and $Z'_t(f) = Z_t(\tilde{f})$ if $t < \zeta$, where $\tilde{f} \in \mathfrak{C}$ is any function agreeing with $f \in \mathfrak{C}'$ for all $s \leq t + \epsilon$ for sufficiently small $\epsilon > 0$. The $\mathcal{F}^0_t$-measurability of Z_t guarantees the definition to be independent of the choice of $\tilde{f}$, and Z' is also homogeneous.

It is an open question whether every filtered dynamical system can be embedded in a standard system as in the above examples. We should also mention that the measure ν in (20) is something of a mystery to us.

We conclude this section by pointing out two applications of our results. Let $A = (A_t)$, $t \in \mathbf{R}_+$, be an integrable, increasing process relative to $\{\mathcal{F}_t\}$, that is (see [2]), $A_0 \equiv 0$, A is right-continuous, nondecreasing and $EA_\infty < \infty$.

Define a measure μ_A on $\mathbf{R}_+ \times \Omega$ by

$$\mu_A(u) = E\int_0^\infty u(s, \omega)\, dA_s(\omega), \qquad u \in (\mathcal{B}_+ \otimes \mathcal{F}^0)_+,$$

and a (finite) measure Q on $\mathcal{F}^0$ by $Q(\xi) = \mu_A(\xi \circ \theta)$. Obviously, Q kills every polar set in $\mathcal{F}^0$, and hence by (10) is the Palm measure of an (integrable) AF α. One easily checks that, in fact,

$$\alpha_t(\omega) = \int_{\mathbf{R}}\int_0^\infty I_{(0,t]}(s + r)\, dA_r(\theta_s\omega)\, ds, \qquad t \in \mathbf{R}_+.$$

We call α the *additive projection* of A. A special case of this is used in [6], and Mecke [12] has noted a similar idea.

The following result has a Markovian analogue [16].

(22) **Theorem.** *A set $N \in \mathcal{F}^0$ is polar if and only if it is charged by no Palm measure.*

Proof. If N is polar, we have already observed that $P_\alpha(N) = 0$ for every AF α. Suppose N is not polar. Its spoor $S_N(\omega)$ is then nonempty on a set of ω having positive probability; in fact, for almost every $\omega \in \Omega$ for which $S_N(\omega) \neq \emptyset$, we shall see that $S_N(\omega)$ is unbounded above. Observe, to begin with, that $S_N(\omega)$ is a "homogeneous set" in the sense that for every $t \in \mathbf{R}$, $S_N(\theta_t\omega) = S_N(\omega) - t$. Let $B_N = \{\sup S_N > n\}$. Since the projection on Ω of sets in $\mathcal{B} \otimes \mathcal{F}$ is in $\mathcal{F}$ (see [2, I., T32]), $B_n \in \mathcal{F}$ for each $n \geq 1$. The ergodic theorem now yields

$$\lim_{t\to\infty} t^{-1}\int_0^t I_{B_n}(\theta_{-s}\omega)\, ds = P(B_n \mid \mathcal{C}) \quad \text{a.s.}$$

where $\mathcal{C}$ denotes the θ-invariant σ-field in Ω. Suppose $S_N(\omega) \neq \emptyset$. Then $S_N(\theta_{-s}\omega) = S_N(\omega) + s$ has supremum greater than n when s is sufficiently large. Hence if $B = \{S_N \neq \emptyset\}$, we have $P(B_n | \mathcal{C}) = 1$ a.s. on B. Since B is invariant, $P(B \cap B_n^c) = 0$, which proves our point.

Now set $D(\omega) = S_N(\omega) \cap (0, \infty) \in \mathcal{B}_+$; clearly $P(D \neq \emptyset) = P(B) > 0$. According to [2, I., T37], there exists a nonnegative random variable $\tau \in (\mathcal{F})_+$ such that $\tau(\omega) \in D(\omega)$ for $D(\omega) \neq \emptyset$ and otherwise $\tau = \infty$. Define an integrable, increasing process $A = (A_t)$ by $A_t(\omega) = I_{\{\tau \leq t\}}(\omega)$; A is flat, except for a unit jump at τ if $\tau < \infty$. We then have

$$\mu_A(I_N \circ \theta) = E\int_0^\infty I_N \circ \theta_s(\omega)\, dA_s(\omega) = E(I_N \circ \theta_\tau;\, D \neq \emptyset) = P(B) > 0.$$

By our work above, $P_\alpha(N) = \mu_A(I_N \circ \theta) > 0$, where α is the additive projection of A, and (22) is proven.

Note. A similar argument, using the material in [2, Chapter VI (esp. §3 and T37)], shows that *a set $N \in \mathcal{F}^0$ has an a.s. countable spoor $S_N(\omega)$ iff $P_\alpha(N) = 0$ for every continuous AF α.*

For our second application, let $X = (X_t)$, $t \in \mathbf{R}$, be a strictly stationary measurable process such that $X_t = X_0 \circ \theta_t$, and assume X is adapted to a filtration $\{\mathcal{F}_t^0\}$. Denote by $(E, \mathcal{E})$ the state space of X, with $\mathcal{E}$ assumed separable, and let $\pi(\Gamma) = P\{X_t \in \Gamma\}$ (independent of t) be the one-dimensional distribution of the process. We say that X has a *local time* if there exist AF's $\alpha^x = (\alpha_t^x)$, $x \in E$, such that, for almost every $\omega \in \Omega$,

$$(23) \qquad \int_\Gamma \alpha_t^x(\omega)\pi(dx) = \int_0^t I_\Gamma(X_s(\omega))\,ds \quad \text{for all } t \in \mathbf{R}_+,\ \Gamma \in \mathcal{E}.$$

(Suitable measurability restrictions must be imposed; we omit the details.) Let $\{P^x\}$, $x \in E$, be a regular conditional probability given X_0, i.e. a family of measures on $\mathcal{F}^0$ such that, for every $A \in \mathcal{F}^0$, $\Gamma \in \mathcal{E}$, $P(A, X_0 \in \Gamma) = \int_\Gamma P^x(A)\pi(dx)$. We know [4] a local time exists iff P^x is a Palm measure for π-a.e. $x \in E$ (in which case the AF's are predictable for a.e. x).

Suppose only that there exist AF's α^x which are *predictable* and such that $P^x = P_{\alpha^x}$ on $\mathcal{F}_{0-}^0$ for a.e. x. Set

$$\beta_t^\Gamma(\omega) = \int_0^t I_\Gamma(X_s(\omega))\,ds, \quad \alpha_t^\Gamma(\omega) = \int_\Gamma \alpha_t^x(\omega)\pi(dx),$$

for $t \in \mathbf{R}_+$, $\Gamma \in \mathcal{E}$. An easy computation shows that β^Γ and α^Γ have the same Palm measure on $\mathcal{F}_{0-}^0$, hence on all of $\mathcal{F}^0$ since β^Γ and α^Γ are both predictable. Consequently $\alpha_t^\Gamma(\omega) = \beta_t^\Gamma(\omega)$ for all t, a.s., which yields (using separability of $\mathcal{E}$) $\alpha_t^\Gamma(\omega) = \beta_t^\Gamma(\omega)$ for all t and all Γ, a.s. Thus we have proven

(24) **Theorem.** *A local time exists iff, for almost every $x \in E$, the measure P^x charges no polar set in $\mathcal{F}_{0-}^0$.*

Notice, however, that "polar" is defined in terms of sets in Ω which are avoided by the *flow* θ_t rather than those in E which are avoided by the *process* X_t.

We conclude with an example, based on a construction due to Maisonneuve [10], of the "local time" of an arbitrary random set. Suppose, for each $\omega \in \Omega$, we are given a Borel set $M(\omega)$ of $\mathbf{R}$, homogeneous in that $M(\theta_t\omega) = M(\omega) - t$ for all $t \in \mathbf{R}$, $\omega \in \Omega$. (For example, $M(\omega) = \{t: X_t(\omega) = 0\}$ with (X_t) as above.) Define $\tau = \inf(M \cap (0, \infty))$ (or $\tau = \infty$ if $M \cap (0, \infty)$ is empty), and $\tau_t(\omega) = t + \tau \circ \theta_t(\omega)$. The random variable τ is "terminal": $\tau = \tau_t$ whenever $\tau > t$.

Let $Z = E(e^{-\tau} \mid \mathcal{F}_0)$. Then Z is excessive, and by (5) we may choose a nice version of Z so that $Z_t = e^{-t}Z \circ \theta_t$ is a homogeneous potential, obviously of class (D). Proceeding as in the proof of (9) we write the decomposition $Z_t = M_t - A_t$, and let $\alpha_t = \int_0^t e^s\,dA_s$. Then α is a predictable AF. Under further conditions on M, $\alpha(dt, \omega)$ is a.s. carried by $M(\omega)$. The Palm measure of α is

$$E_\alpha(\xi) = -E\int_0^\infty \xi^* \circ \theta_s \, de^{-\tau_s}, \qquad \xi \in (\mathcal{F}^0)_+,$$

where ξ^* is the "predictable projection" of ξ. Put another way, α is just the predictable projection of $-e^t de^{-\tau_t}$ (see §2). If $X = (X_t)$ has a local time, say α^x, we do not know the connection, if any, between α^x and the local time of $M^x(\omega) = \{t: X_t(\omega) = x\}$, $x \in E$, at least outside the Markov case.

2. Characterization of additive functionals. With assumptions (I), (II) of §1 still in force, we will need the following basic fact, borrowed from [8] (see also [9]).

(25) **Theorem.** *For every $\xi \in (\mathcal{F}^0)$ which is either bounded or nonnegative, there exists $\xi^\# \in (\mathcal{F}^0_{0+})$ (resp. $\xi^* \in (\mathcal{F}^0_{0-})$) such that $(\xi^\# \circ \theta_t)$, $t \in \mathbf{R}$ (resp. $(\xi^* \circ \theta_t)$) is the well-measurable (resp. predictable) projection of the process $(\xi \circ \theta_t)$, $t \in \mathbf{R}$.*

Moreover, $\xi^\#$ (resp. ξ^*) is bounded or nonnegative with ξ and is unique up to a polar function. We note that the process $\xi^* \circ \theta$ is actually in $(\mathcal{P}^0)$ by (7)(b) while the notions of well-measurability, etc. refer to the family $\{\mathcal{F}_t\}$.

Our results in this section will be of two kinds: the first type classifies an AF α in accordance with the behavior of P_α under projection, while, in the second type, we give conditions under which, for example, the dual predictable projection of α is a.s. absolutely continuous. These latter results are generalizations of some of the work of Papangelou [14].

Before going on, we recall some material from the general theory of processes [2]. Let $u = (u(t, \omega))$ be a process and $A = (A_t(\omega))$ an increasing process, $EA_t < \infty$, $t \in \mathbf{R}_+$, and $\{\mathcal{F}_t\}$ an increasing family of σ-fields on Ω which is right-continuous and with each $\mathcal{F}_t$ completed by all P-null sets. We write $\mathcal{W}$, $\mathcal{P}$ for the well-measurable (resp. predictable) σ-fields on $\mathbf{R}_+ \times \Omega$, and note that $\mathcal{P} \subset \mathcal{W}$. The accessible σ-field falls between $\mathcal{P}$ and $\mathcal{W}$, but will be omitted from our discussion.

Writing $w(u)$ and $p(u)$ for the well-measurable and predictable projections of the process u, the dual well-measurable (resp. predictable) projection of the increasing process A is defined as follows: A^w (resp. A^p) is the unique well-measurable (resp. predictable) increasing process such that

$$(26) \quad E\int_0^\infty w(u)(s, \omega)\, dA_s(\omega) = E\int_0^\infty u(s, \omega)\, dA^w_s(\omega), \qquad u \in (\mathcal{B}_+ \otimes \mathcal{F})_+,$$

and similarly for A^p. For an increasing process, we note that well-measurability is equivalent to being adapted.

For an integrable RAF α, we now denote by $\alpha^\#$ (resp. α^*) the dual well-measurable (resp. predictable) projection as defined above.

(27) **Theorem.** *The increasing processes* $\alpha^{\#}$, α^* *are AF's whose Palm measures are*

$$E_{\alpha^{\#}}(\xi) = E_\alpha(\xi^{\#}), \quad E_{\alpha^*}(\xi) = E_\alpha(\xi^*), \qquad \xi \in (\mathcal{F}^0)_+. \tag{28}$$

Proof. It suffices to treat the predictable case, the other being entirely analogous, even somewhat easier. Suppose, for the moment, that α^* is an AF. Let $\xi \in (\mathcal{F}^0)_+$. Then $p(\xi \circ \theta) = \xi^* \circ \theta$, and the predictable version of (26) gives

$$E\int_0^\infty e^{-s}\xi \circ \theta_s \alpha^*(ds) = E\int_0^\infty e^{-s}\xi^* \circ \theta_s \alpha(ds),$$

i.e. (28) holds. To show α^* is an AF, it is enough to establish

$$E[\alpha^*_{t+s} - \alpha^*_t;\, A] = E[\alpha^*_s \circ \theta_t;\, A], \qquad A \in \mathcal{F}. \tag{29}$$

The left side of (29) can be written as

$$\begin{aligned} E\int_0^\infty I_{(t,t+s]}(r) I_A(\omega)\alpha^*(dr,\omega) &= E\int_0^\infty p(I_{(t,t+s]} I_A)\, d\alpha \\ &= E\int_0^\infty I_{(t,t+s]}(r) P(A \mid \mathcal{F}_{r-})\alpha(dr) \\ &= E\int_0^s P(A \mid \mathcal{F}_{(r+t)-})\alpha(dr, \theta_t\omega), \end{aligned}$$

where $P(A|\mathcal{F}_{r-})$ denotes the left-continuous modification of the martingale $P(A|\mathcal{F}_r)$. Using stationarity of the flow, it is easy to prove that, for each r, t, $P(A|\mathcal{F}_{(t+r)-}) = P(\theta_t A|\mathcal{F}_{r-}) \circ \theta_t$ a.s., and so, formally, the last displayed expression becomes

$$= E\int_0^s P(\theta_t A|\mathcal{F}_{r-})\alpha(dr) = E\int_0^\infty I_{\theta_t A}\alpha^*(dr) = E[\alpha^*_s \circ \theta_t;\, A].$$

The problem is to show that $P(\theta_t A|\mathcal{F}_{r-}) \circ \theta_t$ may be chosen *indistinguishable* from $P(A|\mathcal{F}_{(t+r)-})$, as r varies. However, both processes are a.s. left-continuous in r, hence are indistinguishable. Q.E.D.

Since Palm measures determine AF's (up to indistinguishability):

(30) **Corollary.** *An AF* α *is adapted (resp. predictable) iff* $E_\alpha(\xi) = E_\alpha(\xi^{\#})$ *(resp.* $E_\alpha(\xi) = E_\alpha(\xi^*)$*) for every* $\xi \in (\mathcal{F}^0)_+$.

Notes. (1) If α is adapted, then α will be predictable iff $E_\alpha(\xi) = E_\alpha(\xi^*)$ for all $\xi \in (\mathcal{F}^0_{0+})_+$ since the Palm measure of an adapted AF is completely determined by its action on $\mathcal{F}^0_{0+}$.

(2) Since P^-_α kills polar sets in $\mathcal{F}^0_{0-}$, there exists (by (10)(a)) a *predictable* AF β such that $P^-_\beta = P^-_\alpha$; in fact, $\beta = \alpha^*$ since $E_\beta(\xi) = E_\beta(\xi^*) = E_\alpha(\xi^*) = E_{\alpha^*}(\xi)$ for $\xi \in (\mathcal{F}^0)_+$.

(31) **Corollary.** *Additive projection (see §1) preserves well-measurability (resp. predictability).*

We consider next the splitting of an AF α into the sum of a continuous AF α_c and a purely discrete AF α_d, i.e. the measure $\alpha_c(dt, \omega)$ has no atoms for each $\omega \in \Omega$, whereas $\alpha_d(dt, \omega)$ is the sum of countably many point masses depending on $\omega \in \Omega$. The corresponding Palm measures are denoted P_c, P_d. Given α, α_c and α_d are obtained from the usual decomposition of a measure into a continuous plus a discrete piece. If α is adapted, α_c and α_d will be likewise, and α_c will be predictable; if α is predictable, α_d will be also. Let $A \in \mathcal{F}^0$ have an a.s. countable spoor S_A. Clearly $P_c(A) = 0$. In particular, this is the case when A is semipolar.

Now consider the discrete part α_d. Denote the mass on $\{0\}$ by Δ: $\Delta(\omega) = \alpha_d(0, \omega) - \alpha_d(0-, \omega) = -\alpha_d(0-, \omega)$ (there is no restriction (see §0) in assuming $\Delta \in (\mathcal{F}^0)_+$); also let $\Delta_t(\omega) = \Delta \circ \theta_t(\omega)$ be the mass on $\{t\}$. It is well known that P_d is supported on the set $\Omega_d = \{\Delta > 0\}$. Writing $\Omega_d = \bigcup_{n=1}^{\infty}\{\Delta > 1/n\}$ and recalling that α_d is finite on compacts, we see that Ω_d is semipolar.

When α is predictable, we get a finer decomposition: we can then show that P_d is supported by a semipolar set *in* $\mathcal{F}^0_{0-}$, and this in turn will lead to the decomposition promised in §1.

(32) **Lemma.** *The Palm measure of a purely discrete, predictable AF α is carried by a semipolar set in $\mathcal{F}^0_{0-}$.*

Proof. We choose for α the "perfect version" described in §0. The process $\Delta \circ \theta_t = \alpha(t) - \alpha(t-)$ is then predictable, and is therefore indistinguishable from its predictable projection $\Delta^* \circ \theta_t$. It follows that the set $N^* = \{\Delta^* > 0\}$ (which is in $\mathcal{F}^0_{0-}$) is semipolar since its spoor is a.s. the same as the spoor of Ω_d, and $P_\alpha(\Delta^* = 0) = 0$, since Palm measures fail to distinguish indistinguishable processes. The set N^* is thus the one required by the theorem.

The following is now immediate upon recalling our remarks in §1:

(33) **Theorem.** *A finite measure Q as in (9) has a decomposition $Q = P_c^- + P_d^- + \mu$ where P_c^- charges no semipolar set in $\mathcal{F}^0_{0-}$, P_d^- lives on a semipolar, but charges no polar set in $\mathcal{F}^0_{0-}$, and μ lives on a polar set in $\mathcal{F}^0_{0-}$.*

As a consequence of (33), we obtain the following refinements of the results in §1:

(34) **Corollary.** (a) *A finite measure Q on $\mathcal{F}^0$ is the Palm measure of*

a continuous AF iff Q charges no semipolar set.

(b) *For an AF α, α^* is continuous iff P_α charges no semipolar set in $\mathcal{F}^0_{0-}$.*

Part (a) follows by taking the trivial filtration $\mathcal{F}^0_t \equiv \mathcal{F}^0$ in (33), and part (b) from (33) and the note following (30).

A similar situation obtains for discrete AF's:

(35) **Corollary.** *An AF α is purely discrete iff its Palm measure is carried by a semipolar set.*

If N is semipolar, it is contained in the union of thin sets B_n, so we may consider N thin. By the argument in the proof of (22), the spoor $S_N(\omega)$ will be unbounded in both directions a.s. Since N is thin, we may enumerate the points of $S_N(\omega)$: $\ldots R_{-1}(\omega) < R_0(\omega) \leq 0 < R_1(\omega) < \ldots$, and we have $R_{n+1} = R_n + R_1 \circ \theta_{R_n}$ for each integer n. Define $\nu(dt, \omega)$ as the measure which puts unit mass on each $R_n(\omega)$. We leave it to the reader to check that ν is an AF and that $\alpha(dt, \omega) \ll \nu(dt, \omega)$ for almost every $\omega \in \Omega$ (to show this it suffices to check $Q \ll P_\nu$). The general semipolar case is an easy extension of this method.

We conclude with a characterization of the absolute continuity of α^* in terms of P_α similar to that given in [14] for point processes.

(36) **Theorem.** *The following three statements are equivalent:*

(a) *α^* is a.s. absolutely continuous (i.e. $\alpha^*(dt, \omega) \ll dt$);*

(b) *$P_\alpha \ll P$ on $\mathcal{F}^0_{0-}$;*

(c) *$t^{-1}E(\alpha(t)|\mathcal{F}_0)$ converges in L^1-norm as $t \downarrow 0$.*

Proof. Suppose $\alpha^*(dt, \omega) = \xi(t, \omega)\,dt$. Then, for any $A \in \mathcal{F}^0_{0-}$, $P_\alpha(A) = P_{\alpha^*}(A) = E[\int_0^1 \xi(t, \theta_{-t}\omega)\,dt;\, A]$ which proves (a) $\Rightarrow$ (b). Conversely, assuming (b), we can write $dP^-_\alpha = \xi dP$ for some $\xi \in (\mathcal{F}^0_{0-})_+$. Now for any $\eta \in (\mathcal{F}^0)_+$, $E_{\alpha^*}(\eta) = E_\alpha(\eta^*) = E(\xi\eta^*)$ since $\eta^* \in (\mathcal{F}^0_{0-})_+$. If $\eta = 0$ a.s., the same is true of η^*, hence $P_{\alpha^*} \ll P$ and we can conclude that $\alpha^*(dt, \omega) = \xi \circ \theta_t(\omega)\,dt$, since both are predictable and have the same Palm measure on $\mathcal{F}^0_{0-}$. Thus (b) $\Rightarrow$ (a).

We next show that (a) is equivalent to (c). Recall the *local ergodic theorem* [9] which states that, if $\xi \in L^1(\Omega, \mathcal{F}^0, P)$, $t^{-1}\int_0^t \xi \circ \theta_s\,ds \to \xi$ (as $t \downarrow 0$) a.s. and in L^1. Now from the definition of α^* we have $E(\alpha(t)|\mathcal{F}_0) = E(\alpha^*(t)|\mathcal{F}_0)$ (see [2, VT 37]), and, assuming (a), we have

$$t^{-1}E(\alpha(t)\,|\,\mathcal{F}_0) = t^{-1}E\left(\int_0^t \xi \circ \theta_s\,ds\,|\mathcal{F}_0\right) \tag{37}$$

for some $\xi \in L^1$. But $t^{-1}\int_0^t \xi \circ \theta_s\, ds \to \xi(L^1)$, so the right member of (37) converges in L^1 to ξ as well.

Conversely, assume that $t^{-1}E(\alpha(t)|\mathcal{F}_0)$ converges in L^1 to some ξ, which we may take in $\mathcal{F}^0_{0-}$. Again using $E[\alpha(t) - \alpha(s)|\mathcal{F}_s] = E[\alpha^*(t) - \alpha^*(s)|\mathcal{F}_s]$, $s \leq t$, we will show

$$E\int_0^1 Y_s \alpha^*(ds) = E\int_0^1 Y_s \xi \circ \theta_s\, ds \tag{38}$$

for every continuous, adapted, bounded process Y_s. Equation (38) then extends to all predictable processes Y; since $\alpha^*(t)$ and $\int_0^t \xi \circ \theta_s\, ds$ are each predictable, (38) implies $\alpha^*(t) = \int_0^t \xi \circ \theta_s\, ds$ for all t, a.s.

Before proving (38), we require

(39) **Lemma.** *For $0 \leq \xi \in L^1$ and Y as described in* (38),

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} Y_{k/n}\, \xi \circ \theta_{k/n} = \int_0^1 Y_s \xi \circ \theta_s\, ds \quad (in\ L^1). \tag{40}$$

The L^1-norm of the difference of the two members of (40) may be written

$$\left\| n^{-1} \sum_0^{n-1} \left(Y_{k/n} \xi \circ \theta_{k/n} - n\int_{k/n}^{(k+1)/n} Y_s \xi \circ \theta_s\, ds \right) \right\| \leq C_n + D_n,$$

where

$$C_n = \left\| n^{-1} \sum_0^{n-1} Y_{k/n} \left(\xi \circ \theta_{k/n} - n\int_{k/n}^{(k+1)/n} \xi \circ \theta_s\, ds \right) \right\|,$$

$$D_n = \left\| \sum_0^{n-1} \int_{k/n}^{(k+1)/n} (Y_{k/n} - Y_s) \xi \circ \theta_s\, ds \right\|.$$

Let $|Y_s(\omega)| \leq M < \infty$ for all s, ω. Then

$$C_n \leq \frac{M}{n} \sum_0^{n-1} \left\| \xi - n\int_0^{1/n} \xi \circ \theta_s\, ds \right\| \to 0$$

by the local ergodic theorem. Also,

$$D_n' \equiv \left| \sum_0^{n-1} \int_0^{1/n} (Y_{k/n} - Y_{k/n+s}) \xi \circ \theta_s \circ \theta_{k/n}\, ds \right|$$

$$\leq 2M \sum_0^{n-1} \int_0^{1/n} \xi \circ \theta_{s+k/n}\, ds = 2M\int_0^1 \xi \circ \theta_s\, ds$$

since $\xi \geq 0$; hence D_n' is dominated by an L^1 function. Now, for $\omega \in \Omega$ fixed, $Y(\omega)$ is uniformly continuous on $[0, 1]$; hence, given $\epsilon > 0$, $|Y_{k/n}(\omega) -$

$Y_{k/n+s}(\omega)| \leq \epsilon$ for all sufficiently large n, all $k \leq n-1$, and all s in $[0, 1/n]$. Thus $D_n' \leq \epsilon \int_0^1 \xi \circ \theta_s \, ds$ for n large; by dominated convergence, $D_n \to 0$ and (39) is proven.

Returning to the proof of (38), we have only to show

$$E\int_0^1 Y_s \alpha^*(ds) = \lim_{n\to\infty} En^{-1} \sum_0^{n-1} Y_{k/n} \xi \circ \theta_{k/n}. \tag{41}$$

Let

$$A_n' = \left| \int_0^1 Y_s \alpha^*(ds) - \sum_0^{n-1} Y_{k/n}(\alpha^*(k+1)/n - \alpha^*(k/n)) \right|$$

$$= \left| \sum_0^{n-1} \int_{k/n}^{(k+1)/n} (Y_s - Y_{k/n}) \alpha^*(ds) \right|.$$

Since $|Y| \leq M$, we find $A_n' \leq 2M\alpha^*(1) \in L^1$; on the other hand, given $\epsilon > 0$, $A_n' \leq \epsilon\alpha^*(1)$ for all sufficiently large n, by a uniform continuity argument such as the one above. Hence by dominated convergence $EA_n' \to 0$ and we conclude

$$\int_0^1 Y_s \alpha^*(ds) = \lim_{n\to\infty} \sum_0^{n-1} Y_{k/n}(\alpha^*(k+1)/n - \alpha^*(k/n)) \quad (\text{in } L^1). \tag{42}$$

Now $\alpha^*(k+1)/n - \alpha^*(k/n) = \alpha^*_{1/n} \circ \theta_{k/n}$. Also

$$\left| E\left[\sum_0^{n-1} Y_{k/n} \alpha^*_{1/n} \circ \theta_{k/n} - n^{-1} \sum_0^{n-1} Y_{k/n} \xi \circ \theta_{k/n} \right] \right|$$

$$= \left| E\left[\sum_0^{n-1} Y_{k/n} E(\alpha^*_{1/n} | \mathcal{F}_0) \circ \theta_{k/n} - n^{-1} \sum_0^{n-1} Y_{k/n} \xi \circ \theta_{k/n} \right] \right|$$

$$= \left| \sum_0^{n-1} EY_{k/n} \circ \theta_{-k/n} (E(\alpha^*_{1/n} | \mathcal{F}_0) - \xi/n) \right|$$

$$\leq \frac{M}{n} \sum_0^{n-1} E| nE(\alpha^*_{1/n} | \mathcal{F}_0) - \xi|$$

$$= M \| nE(\alpha^*_{1/n} | \mathcal{F}_0) - \xi \| \to 0 \quad \text{by assumption.}$$

Putting (40), (41) and (42) together, we finally obtain (38).

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