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POINTWISE BOUNDS ON EIGENFUNCTIONS AND WAVE PACKETS IN N-BODY QUANTUM SYSTEMS. III

BY

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ABSTRACT. We provide a number of bounds of the form $|\psi| \leq O(\exp(-a|x|^{\alpha}))$, $\alpha > 1$, for L^2-eigenfunctions ψ of $-\Delta + V$ with $V \longrightarrow \infty$ rapidly as $|x| \longrightarrow \infty$. Our strongest results assert that if $|V(x)| \geq cx^{2m}$ near infinity, then $|\psi(x)| \leq D_\epsilon \exp(-(c-\epsilon)^{1/2}(m+1)^{-1}x^{m+1})$, and if $|V(x)| \leq cx^{2m}$ near infinity, then for the ground state eigenfunction, Ω, $\Omega(x) \geq E_\epsilon \exp(-(c+\epsilon)^{1/2}(m+1)^{-1}x^{m+1})$.

1. Introduction. This is the last in our series of papers [19], [20] on pointwise bounds for L^2-eigenfunctions for Schrödinger operators $-\Delta + V$ on $L^2(\mathbf{R}^n)$. We have been partly motivated by a desire to extend and exploit the recent elegant techniques of O'Conner [15] and Combes-Thomas [3]. In (I) of this series, we considered the case $V = \sum V_{ij}(r_i - r_j)$ with $V_{ij}(x) \longrightarrow 0$ as $x \longrightarrow \infty$ and found exponential bounds $D_b \exp(-b|r|)$ but only for b smaller than some optimal b_0; in (II) of this series, we considered the case where V was bounded below and $V \longrightarrow \infty$ as $r \longrightarrow \infty$ and found exponential falloff for every b. In this paper, we wish to examine the case where V not only goes to infinity as $r \longrightarrow \infty$ but at least as fast as some power r^{2m}. Not surprisingly, we will find that there is then falloff of $O(\exp(-cr^{\alpha}))$ for some $\alpha > 1$.

The relation between α and n is simple and is "predicted" by the following heuristic argument of WKB type [14]: If $\Delta\psi = W\psi$ and we write $\psi = \exp(-h)$, we find that h obeys

$$(\operatorname{grad} h)^2 - (\Delta h) = W.$$

If the variations of h are primarily radial we have $(\partial h/\partial r)^2 - r^{-2}(\partial/\partial r) \cdot (r^2(\partial h/\partial r)) = W$. If $W \longrightarrow \infty$, then $\partial h/\partial r \longrightarrow \infty$ so that the second derivate makes a small contribution. Thus $h \sim \pm \int W^{1/2}\, dr$, i.e.

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$$\psi \sim \exp\left(-\int W^{1/2}\,dr\right).$$

If $W = r^{2m} - E$, we see that $\int W^{1/2} \sim r^{m+1}$, i.e. we expect to find that $\alpha = m + 1$.

For the case $n = 1$, it is often possible to use ordinary differential equation methods to control the falloff of eigenfunctions. For example, one has the following theorem of Hsieh-Sibuya [10] (see also the appendix by Dicke in [18]):

Theorem 1. *Let $\psi \in C^2(\mathrm{R})$ be a nonzero function obeying*

$$-\psi'' + V\psi = E\psi \tag{1}$$

with

$$V(x) = a_{2m}x^{2m} + \cdots + a_0; \qquad a_{2m} > 0.$$

Then, for suitable c_0, either:

(a) $c_0\psi(x) \to \infty$ *as* $x \to +\infty$, *in which case* $(m+1)(\ln c_0\psi(x))/a_{2m}^{1/2}x^{m+1} \to 1$ *as* $x \to \infty$, *or*

(b) $c_0\psi(x) \to 0$ *as* $x \to +\infty$, *in which case* $(m+1)(\ln c_0\psi(x))/(-a_{2m}^{1/2}x^{m+1}) \to 1$ *as* $x \to \infty$.

The proof of Theorem 1 depends on the explicit construction of two independent solutions of (1) and thereby of all solutions. When $n > 1$, we have a partial differential equation and, in general, one cannot use a method listing all solutions. For later reference, we do note that in the case where V on $\mathbf{R}^n$ is centrally symmetric, one can separate variables in spherical coordinates and employ Theorem 1 to give some information.

We attack the problem of bounds on eigenfunctions of

$$-\Delta\psi + V\psi = E\psi \tag{2}$$

by two methods. The first follows the approach of Combes-Thomas [3] and our earlier work [19], [20] and is discussed in §§2–4. We will be able to discuss fairly general V but our results will not always be as strong as might be hoped for. The second approach, found in §§5, 6 is completely independent of §§2–4 although it does depend on a result of Combes-Thomas type we proved in [20]. The V's we are able to discuss are somewhat restricted and so we restrict ourselves to multidimensional anharmonic oscillators, i.e. V will be a polynomial in $x_1, \ldots, x_n$ of degree $2m$ with the property that the leading term be strictly positive on the unit sphere (so that for x near ∞, $c_1|x|^{2m} \le V(x) \le c_2|x|^{2m}$). Our strongest result is (§6)

Theorem 2. *Let ψ be an L^2-eigenfunction for $-\Delta + V$. Suppose that V is C^∞ and for some $c > 0$, d:*

$$V(x) \ge c|x|^{2m} - d. \tag{3}$$

Then for any $\epsilon > 0$ *there is a* D_ϵ *with*

$$|\psi(x)| \le D_\epsilon \exp(-\sqrt{(c-\epsilon)}\,|x|^{m+1}(m+1)^{-1}). \tag{4}$$

Next suppose that ψ is the "ground state" eigenfunction, i.e. ψ is the eigenfunction associated to the lowest eigenvalue, E_0, of $-\Delta + V$. Then it is known (see, e.g. [22]) that E_0 is a nondegenerate eigenvalue and that ψ can be chosen to be a.e. strictly positive. For this ground state eigenfunction we have (§6)

Theorem 3. *Let* ψ *be the ground state eigenfunction for* $-\Delta + V$. *Suppose that* V *is* C^∞, $V \to \infty$ *at* ∞ *and for some* $e > 0$, f:

$$V(x) \le e|x|^{2m} + f. \tag{5}$$

Then, for any $\epsilon > 0$, *there is a* G_ϵ *with*

$$\psi(x) \ge G_\epsilon \exp(-\sqrt{(e+\epsilon)}\,|x|^{m+1}(m+1)^{-1}). \tag{6}$$

In particular, ψ *is strictly positive.*

We close this introduction with a series of remarks about Theorems 2 and 3.

1. The proofs of Theorems 2 and 3 rely on Theorem 1 and a simple comparison argument (§5). The comparison argument depends on certain methods from classical potential theory; we have borrowed the idea of using these potential theory methods from Lieb-Simon [11] who is turn were motivated in part by some remarks of Teller [23].

2. Our interest in Theorem 3 and in the more general problem of sharp bounds on eigenfunctions of multidimensional anharmonic oscillators comes in part from recent work of Eckmann [5] and J. Rosen [17] generalizing L. Gross' logarithmic Sobolev inequalities [8]. We discuss the use of Theorem 3 to generalizing Rosen's results in §7.

3. Still another method for controlling falloff of eigenfunctions for anharmonic oscillators is to look at the finite dimensional Lie algebra generated by $-\Delta$ and V and use Lie algebraic techniques on eigenfunctions treated as analytic vectors. This approach has been advocated and developed by Goodman [6], [7] and Gunderson [9].

2. L^2 bounds of WKB type.

Theorem 4. *Let* $V = V_+ - V_-$ *with* $V_+ \ge 0$, $V_+ \in (L^1)_{\text{loc}}$, $V_- \in L^q(\mathbf{R}^n) + L^\infty(\mathbf{R}^n)$ *with* $q = 1$ *if* $n = 1$, $q > 1$ *if* $n = 2$ *and* $q = n/2$ *if* $n \ge 3$ *(so that* V_- *is a form bounded perturbation of* $-\Delta$ *with form bound* 0). *Let* $H = -\Delta + V$ *defined as a sum of quadratic forms. Let* ψ *be an eigenfunction for* H *with eigenvalue* E *in the discrete spectrum for* H. *Suppose* W *is a real-valued* *absolutely continuous function on* $\mathbb{R}^n$ *with*

$$|\text{grad } W|^2 \leq c_1(H + c_2) \tag{7}$$

for suitable c_1, c_2. *Then, for some* $\alpha > 0$,

$$\exp(\alpha W(x))\psi(x) \in L^2(\mathbb{R}^n). \tag{8}$$

Remarks. 1. In most applications, $V_+ \to \infty$ at ∞ so H has compact resolvent by Rellich's criterion. In such situations, E is automatically in the discrete spectrum.

2. As a particular example, suppose $V_- = 0$ and let $\tilde{V}(r) = \inf_{|x|=r} V(x)$. Then we can take $W(r) = \int_{r_0}^{r} |\tilde{V}(r)|^{1/2}\, dr$, thereby obtaining L^2-bounds on ψ of the usual WKB form.

3. Our proof is a fairly direct modification of the idea of Combes-Thomas [3] which in turn is motivated by [1], [2] (see also [21]).

Proof. For real β, let $U(\beta)$ be the unitary operator of multiplication by $\exp(i\beta W(x))$. (8) is easily seen to be equivalent to the statement that ψ be an analytic vector for $U(\beta)$ in the sense of Nelson. For β real, let

$$H(\beta) = U(\beta) H U(\beta)^{-1}.$$

Then

$$H(\beta) = (p - \beta \text{ grad } W)^2 + V \tag{9}$$

where $p = i^{-1}$ grad. Thus

$$H(\beta) = H + \beta^2 (\text{grad } W)^2 - \beta[p(\text{grad } W) + (\text{grad } W)p]. \tag{9$'$}$$

Now, note the following estimates for $\phi \in Q(H) = Q(\Delta) \cap Q(V_+)$:

$$(\phi, (\text{grad } W)^2 \phi) \leq c_1(\phi, (H + c_2)\phi), \tag{10a}$$

$$2 \operatorname{Re}(p\phi, (\text{grad } W)\phi) \leq 2(p\phi, p\phi)^{1/2}(\phi, (\text{grad } W)^2\phi)^{1/2} \leq c_3(\phi, (H + c_4)\phi) \tag{10b}$$

where we have used (7) and the operator estimate $p^2 \leq p^2 + (p^2 - 2V_- + c_5) \leq 2(p^2 + V) + c_5$ which follows from the fact that V_- is a form perturbation of p^2 with relative bound 0.

Choose d with $H + d \geq 1$. It follows from (10(a)(b)) that for *complex* β sufficiently small, say $|\beta| \leq B$, ($9'$) defines a closed sectorial form on $Q(H)$. It follows that for $|\beta| < B$, $H(\beta)$ is an analytic family of type (B) [12].

By analytic perturbation theory, it follows that for $|\beta| < B_0$, $H(\beta)$ has only discrete eigenvalues $E_1(\beta), \ldots, E_n(\beta)$ in its spectrum near E and that the $E_i(\beta)$ are analytic. Since $H(\beta)$ is unitarily equivalent to H for β real, $E_i(\beta) = E$ for β real and thus, by analyticity for all β with $|\beta| < B_0$. Let

$$P(\beta) = \oint_{|\lambda - E| \le \epsilon} (-2\pi i)^{-1} (H(\beta) - \lambda)^{-1} \, d\lambda$$

so that $P(\beta)$ is the projection onto the eigenvectors for $H(\beta)$ with eigenvalue E. Since $U(\alpha)P(\beta)U(\alpha)^{-1} = P(\beta + \alpha)$ for α real with $|\beta|, |\beta + \alpha| < B_0$, a lemma of O'Connor [15] assures us that $\psi \in \operatorname{Ran} P(0)$ is an analytic vector for $U(\alpha)$. □

3. **Pointwise bounds, $m < 1$.** We now wish to turn the L^2-bounds, $\psi \in D(\exp(\alpha W(x)))$, into pointwise bounds of the form

$$|\psi(x)| \le A \exp(-\alpha' W(x)). \tag{11}$$

We consider the case $W(x) = |x|^{m+1}$. In this section, we will see how to use our method from [20] to obtain pointwise bounds in case $V_- = 0$ and $m \le 1$. We note that our method in [20] was motivated by an idea of Davies [4]. We exploit smoothing properties of $\exp(t\Delta)$:

Lemma 3.1. *Let $\psi \in D(\exp(a|x|^{m+1}))$ for some $a > 0$ and $0 < m \le 1$. Then for all t sufficiently small, there is an A and C (t dependent) so that*

$$|e^{t\Delta}\psi|(x) \le C \exp(-A|x|^{m+1}).$$

Proof. We first note that

$$1 + |x-y|^2 + |y|^{m+1} \ge |x-y|^{m+1} + |y|^{m+1}$$
$$\ge 2^{-m-1}(|x-y| + |y|)^{m+1} \ge 2^{-m-1}|x|^{m+1}$$

so that

$$\exp(-a|x-y|^2)\exp(-a|y|^{m+1}) \le \exp(1 - 2^{-m-1}a|x|^{m+1}).$$

Thus

$$\left| \int \exp[-(a+1)|x-y|^2]\psi(y)\,dy \right|$$
$$\le \int \exp(-(a+1)|x-y|^2 - a|y|^{m+1})|\exp(a|y|^{m+1})\psi(y)|\,dy$$
$$\le \exp(1 - 2^{-m-1}a|x|^{m+1}) \int dy\, e^{-(x-y)^2} |\exp(a|y|^{m+1})\psi(y)|\,dy$$
$$\le C\exp(-2^{m-1}a|x|^{m+1})$$

since both factors in the integral are L^2. On account of the explicit form of the kernel for $e^{t\Delta}$, the lemma is proven. □

Theorem 5. *Let $V \in (L^2)_{\mathrm{loc}}$ with*

$$\alpha|x|^{2m} \le V(x) + \beta \tag{12}$$

for suitable m, $0 < m \leq 1$, *and suitable* α, β. *Let* $H = -\Delta + V$ *defined as a selfadjoint operator sum* [13]. *Let* ψ *be an eigenfunction of* H. *Then, for some* $\gamma > 0$ *and* C:

$$|\psi(x)| \leq C \exp(-\gamma |x|^{m+1}). \tag{13}$$

Proof. By Rellich's criterion, (12) implies that H has only discrete spectrum. Letting $W(x) = |x|^{m+1}$ and using (12) and Theorem 4, we see that $\psi \in D(\exp(a|x|^{m+1}))$ for some $a > 0$.

Let V_k be a sequence of bounded functions with $V_k(x) - \beta$ converging montonically upward to V. Then using the fact that C_0^∞ is a common core [13], it is easy to see that $H_k \equiv -\Delta + V_k$ converges to H in strong resolvent sense [12], [16] as $k \to \infty$ so that $\exp(-tH)_k \to \exp(-tH)$ strongly as $k \to \infty$. Moreover, since $e^{t\Delta}$ is positivity preserving and $e^{t\beta} \geq e^{-tV_k} \geq 0$:

$$0 \leq (e^{-t\Delta/n} e^{-tV_k/n})^n |\phi| \leq e^{t\Delta} e^{t\beta} |\phi|$$

for all $\phi \in L^2$. By the Trotter product formula [16],

$$0 \leq e^{-tH_k} |\phi| \leq e^{t\beta} e^{t\Delta} |\phi|,$$

so by the convergence result:

$$0 \leq e^{-tH} |\phi| \leq e^{t\Delta} e^{t\beta} |\phi|.$$

Thus for any eigenfunctions ψ with $H\psi = E\psi$:

$$|\psi| = e^{tE} |e^{-tH}\psi| \leq e^{t(E+\beta)} e^{t\Delta} |\psi|.$$

By the lemma, and the fact noted above that $|\psi| \in D(\exp(a|x|^{m+1}))$ we obtain (13). □

4. Pointwise bounds, $m > 1$. When $m > 1$, we are not able to use the method of the last section to obtain pointwise bounds. Instead, we rely on Sobolev type estimates and therefore obtain results whose hypotheses depend on n, the dimension of space. We illustrate the ideas first in the special case $n \leq 3$ where only minimal additional hypotheses are needed.

Lemma 4.1. *Let* $f(x) = a(x^2 + 1)^{(m+1)/2}$ *on* $\mathbb{R}^n$. *If* $\psi \in L^2(\mathbb{R}^n)$ *and* ψ, $\Delta\psi \in D(e^f)$, *then for any multi-index* α *with* $\alpha \leq 2$, $D^\alpha \psi \in D(\exp[(1-\epsilon)f])$ *for all* $\epsilon > 0$. *In particular*, $\Delta(e^{(1-\epsilon)f}\psi) \in L^2$.

Proof. By a simple argument, we need only prove a priori estimates for $\psi \in C_0^\infty(\mathbb{R}^n)$. We note first that for any β:

$$\int e^{\beta f} |\nabla \psi|^2 = -\int \psi^* (\Delta\psi) e^{\beta f} - \int \psi^* (\beta e^{\beta f}) \nabla f \cdot \nabla \psi. \tag{14}$$

Let $\beta < 1$, then since $e^{\beta f}\psi^*$, $\Delta\psi \in L^2$ and $\nabla f e^{\beta f}\psi^* \in L^2$, the R.H.S. of (14) is finite and thus $\nabla\psi \in D(e^{\beta f/2})$. We can now apply (14) when $\beta < 3/2$ to conclude the R.H.S. is finite so that $\nabla\psi \in D((3/4 - \epsilon)f)$. Repeating the argument, we see that $\nabla\psi \in D(\exp((1-\epsilon)f))$. From

$$\Delta(e^{\beta f}\psi) = e^{\beta f}\Delta\psi + 2\beta(\nabla f)e^{\beta f}\nabla\psi + [\Delta(e^{\beta f})]\psi$$

we conclude that $e^{\beta f}\psi \in D(\Delta)$ for $\beta < 1$ so that $D^\alpha(e^{\beta f}\psi) \in L^2$ if $|\alpha| \le 2$. Since $\nabla\psi \in D(e^{\beta f})$, we see that $D^\alpha\psi \in D(\exp((1-\epsilon)f))$. □

Theorem 6. *Suppose that the hypotheses of Theorem 4 hold with $n \le 3$ and $W(x) = |x|^{m+1}$. Suppose in addition that*

$$(15)\qquad |V(x)| \le C_1 \exp(C_2|x|^\alpha)$$

with $\alpha < m+1$. Then any eigenfunction ψ of $-\Delta + V$ obeys

$$(16)\qquad |\psi(x)| \le C_3 \exp(-C_4|x|^{m+1})$$

for suitable $C_3, C_4 > 0$.

Proof. By Theorem 4, $\psi \in D(\exp(af))$ for suitable $a > 0$ with $f = (1+|x|^2)^{(m+1)/2}$. Since $\Delta\psi = V\psi - E\psi$, $\Delta\psi \in (\exp((a-\epsilon)f))$ on account of (15). Thus by Lemma 4.1, $e^{(a-\epsilon)f}\psi \in L^2(\mathbf{R}^n) \cap D(\Delta)$. By a Sobolev estimate, $e^{(a-\epsilon)f}\psi$ is a bounded continuous function, so (16) holds. □

For general n, we need

Lemma 4.2. *Let k be a positive integer and let $D^\alpha\psi, \Delta(D^\alpha\psi) \in D(\exp(f))$ with $f|x| = a(x^2+1)^{(m+1)/2}$ for $|\alpha| \le 2k$. Then $D^\alpha\psi \in D(\exp((1-\epsilon)f))$ for all $\epsilon > 0$ and $|\alpha| \le 2(k+1)$. In particular, $\Delta^{(k+1)}(e^{(1-\epsilon)f}\psi) \in L^2$.*

Proof. This follows immediately from Lemma 4.1. □

Theorem 7. *Fix n and m. Suppose the distributional derivatives $D^\alpha V$ for $|\alpha| \le 2[n/4 + 9/8]$ (where $[x] \equiv$ greatest integer less than or equal to x) are locally L^1 obeying*

$$|D^\alpha V| \le C_\alpha \exp(-D_\alpha|x|^{m+1-\epsilon}) \qquad (\epsilon > 0)$$

and that moreover

$$V(x) \ge C|x|^{2m} - D.$$

Then any eigenfunction ψ of $-\Delta + V$ obeys

$$|\psi(x)| \le A \exp(-B|x|^{m+1})$$

for suitable $A, B > 0$.

Proof. Similar to Theorem 6 but employing $-\Delta D^\alpha\psi + D^\alpha(V\psi) = ED^\alpha\psi$ as well as $-\Delta\psi + V\psi = E\psi$. □

5. A comparison argument. We now turn to a method of obtaining falloff information for eigenfunctions which is independent of and stronger than the results of §§2–4 but under stronger hypotheses. As we have already stated in the introduction, this method is motivated by [23], [11] although the basic idea is fairly standard. J. M. Combes (private communication) has informed me that T. Kato (unpublished) has used a not dissimilar idea in the one-dimensional case. The basic comparison theorem is

Theorem 8.(2) *Let S be a closed ball in $\mathbf{R}^n$. Suppose that f, g are functions C^∞ in a neighborhood of $\overline{\mathbf{R}^n\setminus S}$, and that*
(i) $\Delta|f| \le V|f|$ *all* $x \notin S$,
(ii) $\Delta|g| \ge W|g|$ *all* $x \notin S$,
(iii) $f, g \to 0$ *as* $x \to \infty$,
(iv) $W(x) \ge V(x) \ge 0$ *all* $x \notin S$,
(v) $|f(x)| \ge |g(x)|$ *all* $x \in \partial S$.
Then $|f(x)| \ge |g(x)|$ *all* $x \in S$.

Remark. (i), (ii) are intended in the sense of distributional inequalities.

Proof. Let $D = \{x \mid |f(x)| < |g(x)|\}$ and let $\psi = |g(x)| - |f(x)|$ on D, which is open. Then, on D,

$$\begin{aligned}\Delta\psi &\ge W|g| - V|f| && \text{(by (i), (iv))}\\ &\ge V(|g| - |f|) && \text{(by (iv))}\\ &\ge 0 && (\text{by } x \in D).\end{aligned}$$

Thus ψ is subharmonic on D and so takes its maximum value on $\partial D \cup \{\infty\}$. But $\psi \to 0$, at ∞ by (iii), at points $x \in \partial D \cap \partial S$ by (i) and at points $x \in \partial D\setminus\partial S$ by definition. Thus $\psi(x) \le 0$ on D. But, by definition, $\psi(x) > 0$ on D so D is empty. □

6. Eigenfunctions of anharmonic oscillators.

Lemma 6.1. *For any $m > 0$, $C > 0$, there exist an f and E so that $-\Delta f + C(x^2+1)^m f = Ef$ with*

$$0 < f(x) \le D_\epsilon \exp(-(C-\epsilon)^{+1/2}|x|^{m+1}/(m+1)^{-1}) \tag{17}$$

all x. Moreover, for suitable $D' > 0$

(2) Added in proof. H. Kalf has pointed out a similar result in P. Hartman and A. W. Winter, *Partial differential equations and a theorem of A. Kneser*, Rend. Cir. Mat. Palermo (2) 4 (1955), 237–255. MR 18, 214.

(17′) $$f(x) \geq D'_\epsilon \exp(-(C+\epsilon)^{1/2}(m+1)^{-1}(x)^{m+1}).$$

Proof. Let $H = -\Delta + (x^2+1)^m$. Choose f to be the ground state eigenfunction for H (which exists since H has purely discrete spectrum). Then f is a.e. nonnegative, so by the symmetry of H, f is spherically symmetric. Thus f obeys a suitable second order *ordinary* differential equation so that it is impossible that f and ∇f both vanish. But since $f \geq 0$ and C^∞ (elliptic regularity) $f = 0$ implies that $\nabla f = 0$ so f is strictly positive.

We claim that (17) holds near infinity and so everywhere. This follows either by appealing to a suitable generalization of Theorem 1 (since $|x|^{m-1/2}f$ obeys an equation similar to 1 but with an extra $C|x|^{-2}$ in the potential) or by appealing directly to Theorem 1, using Theorem 8 and an argument similar to that used in Theorem 2 below. □

We now repeat

Theorem 2. *Let V be a C^∞ function on $\mathbf{R}^n$ and let g be an eigenfunction of $-\Delta + V$. Suppose that $V(x) \geq c|x|^{2m} - d$ for some $c, d > 0$. Then, for any $\epsilon > 0$, there is a D_ϵ with*

(18) $$|g(x)| \leq D_\epsilon \exp(-(c-\epsilon)^{1/2}|x|^{m+1}/(m+1)^{-1}).$$

Remark. It is easy to replace C^∞ by C^p for suitable finite p.

Proof. Let $(-\Delta + V)g = Eg$. Given ϵ, find f with $[-\Delta + (c - \epsilon/2)|x|^{2m}]f = E_0 f$, $0 < f \leq D_\epsilon \exp(-(c-\epsilon)^{1/2}|x|^{m+1})$. Let $\widetilde{V} = (c-\epsilon/2)|x|^{2m} - E_0$; $\widetilde{W} = V - E$. Find a sphere S with $\widetilde{V} \geq \widetilde{W} \geq 0$ outside S. Since $f > 0$, f is bounded below on ∂S, so choose $\widetilde{f}$ a multiple of f with $|g| \leq \widetilde{f}$ on ∂S. By Kato's inequality [13]

$$\Delta|g| \geq \operatorname{Re}((\operatorname{sgn} g)\Delta g) = \operatorname{Re}(W|g|) = W|g|.$$

Finally, we note that by the exponential falloff inequalities on g [20], $g \to 0$ at ∞. Thus applying Theorem 8, $|g| \leq f$ outside S. (18) now follows. □

Now consider V which is C^∞ with $V \to \infty$ at ∞. By Rellich's criterion, $-\Delta + V$ has compact resolvent and so a lowest eigenvalue E_0. By a standard argument [22], E_0 is simple, and the corresponding eigenvector, ψ is a.e. positive. Following [23], [11] we first note

Lemma 6.2. *If ψ is a.e. positive, C^∞ with $-\Delta\psi = (-V + E)\psi$ with V C^∞, then ψ is everywhere strictly positive.*

Proof. Suppose that $\psi(0) = 0$. We will prove that ψ is identically zero near 0 violating the fact that ψ is a.e. positive. This will prove that $\psi(0) \neq 0$ and by similar argument that $\psi \neq 0$ for all x.

Thus, suppose $\psi(0) = 0$. Let $c(r) = \int_{|x|=r} \psi(x)\, d\Omega$. Then $c(r) \to 0$ as $r \to 0$ and

$$r^{n-1} \frac{dc}{dr} = \int_{|x|=r} \frac{\partial \psi}{\partial r}\, dS = \int_{|x| \le r} (\Delta\psi)\, dr$$

$$\le \max_{|x|\le r} (|V - E|) r^{n-1} \int_0^r c(x)\, dx.$$

Fix R_0 and let $D = \max_{|x| \le R_0}(|V - E|)$. Then for $0 < r \le R_0$

$$\frac{dc}{dr} \le D \int_0^r c(x)\, dx \le (Dr) \max_{0 \le x \le r} c(x).$$

Since $c(0) = 0$: $c(r) \le (\tfrac{1}{2}Dr^2) \max_{0\le x\le r} c(x)$ so for $0 < r \le R$, $\max_{0\le x\le r} c(x) \le (\tfrac{1}{2} Dr^2) \max_{0 \le x \le r} c(x)$.

Choosing r so small that $Dr^2 < 2$ and $0 < r < R$, we see that $\max_{0\le x\le r} c(x) = 0$ so that $\psi(x) = 0$ if $|x| < r$. □

We next repeat

Theorem 3. *Let ψ be the ground state eigenfunction for $-\Delta + V$ where V is C^∞ and $V \to \infty$. Suppose that $V(x) \le e|x|^{2m} + s$. Then for any $\epsilon > 0$, there is a G_ϵ with*

$$\psi(x) \ge G_\epsilon \exp(-\sqrt{(e+\epsilon)}\, |x|^{m+1}(m+1)^{-1}). \tag{19}$$

Proof. Let $f = \psi$, $\tilde{V} = V - E$ and let $W = (e + \epsilon/2)|x|^{2m} + s$. Let g be the ground state of $-\Delta + W$ with ground state energy $\tilde{E}$ and let $\tilde{W} = W - \tilde{E}$. Pick S so that $\tilde{W} \ge \tilde{V} \ge 0$ outside S. Since f is strictly positive and C^∞ by Lemma 6.2, choose a multiple $\tilde{g}$ of g with $f \ge \tilde{g}$ on ∂S. Then $f \ge \tilde{g}$ on $\mathbf{R}^n/S$ by Theorem 8. Thus, by Lemma 6.1, (19) follows. □

When V is a polynomial, we can say much more about the eigenfunctions.

Theorem 9. *Let V be a polynomial in n variables on $\mathbf{R}^n$ with $C(x^{2m} - 1) \le V(x) \le d(x^{2m} + 1)$ for $m \ge 1$. Let ψ be an L^2-eigenfunction for $-\Delta + V$. Then*:

(a) *ψ is a real-analytic function and has an analytic continuation to the entire space $\mathbf{C}^n$.*

(b) *For any $y \in \mathbf{R}^n$, $\epsilon > 0$,*

$$|\psi(x + iy)| \le C_{y,\epsilon} \exp[-(m+1)^{-1}(d - \epsilon)^{1/2}|x|^{m+1}]$$

for all $x \in \mathbf{R}^n$.

(c) *For any $\epsilon > 0$, there are constants E and F with*

$$|\psi(z)| \le E \exp(-F|z|^{m+1})$$

all $z \in \mathbf{C}^n$ *with* $\arg z_1 = \cdots = \arg z_n$ *and* $|\arg z_1| \le \pi/2(m+1) - \epsilon$.

(d) *For any* $\epsilon > 0$, *there are constants* G_1 *and* G_2 *with*

$$|\psi(z)| \le G_1 \exp(-G_2|z|^{m+1})$$

for all $z \in \mathbf{C}^n$ *with* $|\arg z_i| \le \pi/4m - \epsilon$, $i = 1, \ldots, n$.

Remark. With a minimal amount of extra work, one should be able to improve (d).

Proof. By the basic Combes-Thomas argument [3] we see that ψ is an entire analytic vector for the group $\{U(a) | a \in \mathbf{R}^n\}$ where $U(a)\psi(b) = \psi(b-a)$. Thus $\hat{\psi}$, the Fourier transform of ψ, has the property that $e^{ip\cdot a}\hat{\psi} \in L^2$ for all $a \in \mathbf{C}^n$. It follows that ψ is an entire function, proving (a). Moreover, $\psi(\cdot + iy)$ is an L^1-eigenfunction of $-\Delta + V(\cdot + iy)$ so the methods of §4 (or §5) allow one to prove (b). The bounds in (c), (d) follow by similar arguments (and a Phragmen-Lindelöf argument to get uniform constants) but using the group of dilations [1], [2], [21]. For (c) we note that $-\beta^{-2}\Delta + V(\beta x)$ is an analytic family of operators sectorial (in the sense of [16]) so long as $|\arg \beta| < \pi/2(m+1)$ and for (d) that

$$-\sum_{D=1}^{n} \beta_i^{-2} \frac{d^2}{dx_i} + V(\beta_1 x_1, \ldots, \beta_n x_n)$$

is accretive if $|\arg \beta_i| < \pi/4m$. □

Remark. Results related to Theorem 9 have been found by different methods in [9].

7. Supercontractive estimates à la J. Rosen. In [8], Gross considered the following situation. Let $H = -\Delta + V$ on $L^2(\mathbf{R}^n, dx)$ where V is a polynomial bounded from below. Let Ω be the ground state eigenfunction for H and let $\hat{H} = H - (\Omega, H\Omega)$. Let $d\mu$ be the probability measure $\Omega^2\, dx$. Then $\hat{H}$ on $L^2(\mathbf{R}^n, dx^n)$ is unitarily equivalent to $G = \Omega^{-1}\hat{H}\Omega$ on $L^2(\mathbf{R}^n, d\mu)$. G is a Dirichlet form in the sense that $(\psi, G\phi) = \int \overline{\operatorname{grad} \psi} \cdot \operatorname{grad} \phi\, d\mu$. Eckmann [5], following a suggestion of Gross [8], proved a variety of estimates which imply that G generates a hypercontractive semigroup [22] on $L^2(\mathbf{R}^n, d\mu)$ in case $n = 1$ or V is central and these estimates were improved by Rosen [17] who proved, in particular, that e^{-tG} is bounded from $L^p(\mathbf{R}^n, d\mu)$ to $L^q(\mathbf{R}^n, d\mu)$ for all $t > 0$, $p, q \ne 1, \infty$, again if $n = 1$. In Rosen's proof $n = 1$ enters in two places. First, he uses the fact that on $\mathbf{R}$, $f \le c(d^2/dx^2 + 1)$ if

$f \in L^1(\mathbf{R}, dx)$, but on $\mathbf{R}^n$ this can be replaced by $f \le c(-\Delta + 1)$ if $f \in L^p(\mathbf{R}^n, dx)$, $p > n/2$ $(n > 2)$. More critically, he requires that $\Omega = e^{-h}$ with $(h)^{2m/m+1} \le a(V + b)$ if $m = \frac{1}{2}\deg V$. This requires a lower bound on the falloff of Ω which was not available to him.

Our considerations in §6 were partially motivated by a desire to prove Rosen's estimates in case $n > 1$ and our results there allow us to mimic Rosen's proof [17] and conclude:

Theorem 10. *Let V be a polynomial on $\mathbf{R}^n$ with $a(x^{2m} - 1) \le V(x) \le b(x^{2m} + 1)$. Let $H = -\Delta + V$, Ω be its ground state, $d\mu = \Omega^2 d^nx$ and G be the Dirichlet form on $L^2(\mathbf{R}^n, d\mu)$. Then*:

(i) *For all $f \in C_0^\infty(\mathbf{R}^n)$*:

$$\int |f|^2 (\log_+ |f|)^{2km/m+1} d\mu \le C_k \sum_{|\alpha| \le k} \int |D^\alpha f|^2 d\mu + \|f\|_2^2 (\log \|f\|_2)^{2km/m+1}.$$

(ii) $D(G^{k/2}) \equiv \{f \in (C_0^\infty)' \,|\, D^\alpha f \in L^2;\ |\alpha| \le k\}$.

(iii) *For all $t > 0$, $p, q \ne 1, \infty$, e^{-tG} is bounded from $L^p(\mathbf{R}^n, d\mu)$ to $L^q(\mathbf{R}^n, d\mu)$.*

Remark. By using the upper bounds we have on Ω, we can show that the inequality in (i) fails if a factor of $\log_q(\log_q(\cdots \log_q(|f|)))$ (j times) is added to the integral for any $j > 0$. This follows by Rosen's arguments [17].

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