ON SOME CLASSES OF MULTIVALENT STARLIKE FUNCTIONS

BY

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ABSTRACT. Classes of multivalent functions analogous to certain classes of univalent starlike functions are defined and studied. Estimates on coefficients and distortion are made, using a variety of techniques.

1. Let $S_t$ denote the class of all functions $f(z) = z + \ldots$ analytic, univalent and starlike in the unit disc $U$. Such functions satisfy the condition $\text{Re}(zf'(z)/f(z)) > 0$, $z \in U$.

The problem of defining a corresponding class of multivalent starlike functions has been studied by several authors. Hummel [5] distinguishes six commonly used definitions, a typical one being $f(z)$ belongs to the class $S(p)$ if $f$ has at most $p$ zeros in $U$ and

$$\lim_{r \to 1^-} \min_{|z|=r} \frac{zf'(z)}{f(z)} > 0.$$ (1.1)

In this note we will study three classes of multivalent starlike functions which are analogues of certain subclasses of $S_t$.

2. Let $S_1(p, \alpha)$, $p$ a positive integer, $0 < \alpha < 2p$, denote the class of all functions $f(z) = a_0 + a_1z + \ldots$ analytic in $U$ with precisely $p$ zeros there such that

$$\lim_{r \to 1^-} \min_{|z|=r} \text{Re} \frac{zf'(z)}{f(z)} > \alpha.$$ (2.1)

$S_1(p, \alpha)$ is the generalization of the class $S(\alpha)$ of starlike functions of order $\alpha$ introduced by Robertson [11].

**Theorem 2.1.** Let $f(z)$ belong to the class $S_1(p, \alpha)$ and suppose $f$ has zeros at $z_1, z_2, \ldots, z_p$. Then $f(z)$ is $p$-valent in $U$ and there is a function $g$ in the class $S(\alpha/p)$ and a constant $A$ such that

$$f(z) = A \prod_{j=1}^{p} \Psi(z, z_j) [g(z)]^p.$$ (2.2)
where 

\[ \Psi(z, z_j) = (z - z_j) (1 - z_j^p z)/z. \]

(The asterisk denotes conjugation.)

**Proof.** Since inequality (2.1) implies inequality (1.1), it follows from [5] that \( f(z) \) is \( p \)-valent and that there is a function \( g \in \text{St} \) such that (2.2) holds. We compute

\[ \frac{zf'(z)}{f(z)} = \sum_{j=1}^{p} \frac{z\Psi'(z, z_j)}{\Psi(z, z_j)} + p \frac{zg'(z)}{g(z)}. \]

Since \( \text{Re}(z\Psi'/\Psi) = 0 \) on \( |z| = 1 \), it follows from (2.3) and the definition of \( S_1(p, \alpha) \) that \( \lim \sup \min \text{Re}(zg'(z)/g(z)) \geq \alpha/p \) and the result follows from the maximum principle.

For any \( f \in S(p) \), let \( z_1, \ldots, z_p \) be the zeros of \( f(z) \), let \( r_i = |z_i| \), \( R_M = \max\{r_i\} \), \( R_m = \min\{r_i | r_i \neq 0\} \), and let \( r = |z| \). We also assume that the constant \( A \) of Theorem 2.1 equals 1.

**Theorem 2.2.** Let \( S_1(p, \alpha; z_1, \ldots, z_p) \) denote the subclass of \( S_1(p, \alpha) \) of functions with zeros at \( z_1, \ldots, z_p \). Then the extreme points of the closed convex hull of \( S_1(p, \alpha; z_1, \ldots, z_p) \) are precisely the functions of the form

\[ f(z) = \prod_{j=1}^{p} \Psi(z_1 z_j) z^p (1 - xz)^{\alpha - 2p}, \quad |x| = 1. \]

**Proof.** It follows from (2.2) and the compactness of \( S(\alpha/p) \) that \( S_1(p, \alpha; z_1, \ldots, z_p) \) is compact. For \( z_1, \ldots, z_p \) fixed the mapping

\[ T: \{g(z)\} \rightarrow \prod_{j=1}^{p} \Psi(z_1 z_j) [g(z)]^p \]

is a linear homomorphism; thus it suffices to find the extreme points of \( \{[g(z)]^p | g \in S(\alpha/p)\} \). Since \( p > 0 \), the argument in [3] applies and we are done.

**Corollary 2.3.** Let \( f(z) = a_0 + \ldots \in S_1(p, \alpha; z_1, \ldots, z_p) \). Then

(i) \( |f(z)| \leq \prod_{j=1}^{p} (r + r_j) (1 + r_j r) (1 - r)^{\alpha - 2p}, \quad |z| < 1, \)

(ii) \( |f(z)| \geq \prod_{j=1}^{p} (r - r_j) (1 - r_j r) (1 + r)^{\alpha - 2p}, \quad |z| > R_M, \)

(iii) \( |f(z)| \geq \prod_{j=1}^{p} (r_j - r) (1 - r_j r) (1 + r)^{\alpha - 2p}, \quad |z| < R_m, \)

(iv) \( |a_n| \leq A_n \), where \( A_n \) is the coefficient of \( z^n \) in

\[ F(z) = \prod_{j=1}^{p} (z + r_j)(1 + r_j z)(1 - z)^{\alpha - 2p}, \]

(v) \( |f^{(k)}(z)| \leq f^{(k)}(r), \quad k = 1, 2, \ldots. \)
We note that it is possible to obtain sharp upper and lower bounds for $\text{Re}(zf'(z)/f(z))$ using the estimates in [6], but we do not state them here.

The problem of determining $\max |a_n|$ when $|a_1|, \ldots, |a_p|$ are fixed was first studied for the class $S(p)$ by Goodman [4]. We are only able to obtain a partial result.

**Lemma 2.4.** Let $f(z) = (z - z_0)(1 - \overline{z}_0 z) \cdot z^{-1}(z + \sum_{n=2}^{\infty} b_n z^n)^p$. Then if $f(z) = \sum_{n=p-1}^{\infty} a_n z^n$,

\begin{equation}
(2.4) \quad a_{p+1} = \left[ pb_3 - \frac{p(p + 1)}{2} b_2^2 + \frac{\overline{z}_0}{z_0} \right] a_{p-1} + pb_2 a_p.
\end{equation}

**Proof.** Let $\sum_{n=1}^{\infty} c_n z^n = (\sum_{n=1}^{\infty} b_n z^n)^p$, $b_1 = 1$. Then

$$f(z) = -z_0 c_p z^{p-1} + [-z_0 c_{p+1} + (1 + |z_0|^2) c_p] z^p$$

$$+ [-z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \overline{z}_0 c_p] z^{p+1} + \ldots.$$ Comparing coefficients, we have

$$a_{p-1} = -z_0 c_p, \quad a_p = -z_0 c_{p+1} + (1 + |z_0|^2) c_p,$$

$$a_{p+1} = -z_0 c_{p+2} + (1 + |z_0|^2) c_{p+1} - \overline{z}_0 c_p.$$ An easy calculation yields

\begin{equation}
(2.5) \quad a_{p+1} = \frac{c_{p+1}}{c_p} a_p + \left[ \frac{c_{p+2}}{c_p} - \left( \frac{c_{p+1}}{c_p} \right)^2 + \frac{\overline{z}_0}{z_0} \right] a_{p-1}.
\end{equation}

Since $c_p = 1$, $c_{p+1} = p b_2$, and $c_{p+2} = p(p - 1) b_2^2/2 + pb_3$, substitution of (2.5) yields (2.4).

**Theorem 2.5.** Let $f(z) = a_{p-1} z^{p-1} + a_p z^p + \ldots \in S_1(p, \alpha)$ with $p \geq 2$, $(2p^2 - 3p - (5p^2 - 4p\alpha)/2(p - 1)) \geq \alpha \geq 0$, $a_p \text{ real}$. Then

$$|a_{p+1}| \leq [2(p - \alpha)^2 - (p - \alpha) - 1] |a_{p-1}| + (p - \alpha) |a_p|.$$

**Proof.** It follows from Theorem 2.1 that $f(z)$ has a single real zero $z_0$ not at the origin and that $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ has real coefficients. From (2.4) we have

$$|a_{p+1}| \leq \left( 1 + p \left| b_3 - \frac{(p + 1)}{2} b_2^2 \right| \right) |a_{p-1}| + p |b_2| |a_p|.$$
For $\alpha/p \leq 1 - 1/p$, it follows from a result of Keogh and Merkes [7] that

\[ |b_3 - (p + 1)b_2^2/2| \leq (1 - \alpha/p)(2p - 2\alpha - 1) \] and thus

\[ 1 + p \left( b_3 - \frac{p + 1}{2} b_2^2 \right) \geq 1 - (p - \alpha) (2p - 2\alpha - 1). \] (2.6)

It remains to find an upper bound for the left-hand side of (2.6). It suffices to prove

\[ (p - \alpha) (2p - 2\alpha - 1) - 1 + p \left( \frac{p + 1}{2} \right) b_2^2 \geq 1 + pb_3, \]

which is certainly true if

\[ (p - \alpha) (2p - 2\alpha - 1) \geq 2 + pb_3. \] (2.7)

Since $|b_3| \leq (1 - \alpha/p) (3 - 2\alpha/p)$ [3], [11], (2.7) is certainly true if

\[ (p - \alpha) (2p - 2\alpha - 1) \geq 2 + (p - \alpha) (3 - 2\alpha/p), \]

or

\[ \alpha \leq (2p^2 - 3p - (5p^2 - 4p)^{1/2})/(2p - 1). \]

This proves the theorem, since $|b_2| \leq (1 - \alpha/p)$ [3], [11].

3. Let $S_2(p, \alpha)$ denote the subclass of $S(p)$ consisting of all functions $f(z)$ for which

\[ \lim \sup \min \Re \left[ \frac{1 + z f''(z)}{f'(z)} + (1 - \alpha) \frac{z f'(z)}{f(z)} \right] \geq 0. \]

The class $S_2(p, \alpha)$ is the analog of the class $C(\alpha)$ of $\alpha$-convex functions defined by Mocanu [10]. A short calculation gives

**Theorem 3.1.** Let $f(z) \in S_2(p, \alpha)$. Then $f(z) = \prod_{j=1}^{p} \Psi(z, z_j) \Phi(g(z))^p$, where $g(z) \in C(\alpha/p)$.

We cannot state an analogue of Theorem 2.2 since as yet the extreme points of $C(\beta)$ are not known for all $\beta$. However, we can obtain the results of Corollary 2.3.

**Theorem 3.2.** Let $f(z) = a_0 + \ldots \in S_2(p, \alpha), \alpha > 0$. Then:

(i) \[ |f(z)| \leq \prod_{j=1}^{p} (r + r_j) (1 + r_j) r^{-p} \left[ \frac{p}{\alpha} \int_0^{r_j} (1 - t)^{-p/\alpha} t^{p/\alpha - 1} dt \right]^{\alpha}, \quad |z| < 1, \]

(ii) \[ |f(z)| \geq \prod_{j=1}^{p} (r - r_j) (1 - r_j) r^{-p} \left[ \frac{p}{\alpha} \int_0^{r_j} (1 + t)^{-p/\alpha} t^{p/\alpha - 1} dt \right]^{\alpha}, \quad |z| > R_M, \]

(iii) \[ |f(z)| \geq \prod_{j=1}^{p} (r - j) (1 - r_j) r^{-p} \left[ \frac{p}{\alpha} \int_0^{r_j} (1 + t)^{-p/\alpha} t^{p/\alpha - 1} dt \right]^{\alpha}, \quad |z| < R_m, \]
(iv) \(|a_n| \leq A_n\); where
\[ F(z) = \prod_{j=1}^{p} \Psi(z, -r_j) \left[ \frac{p}{\alpha} \int_0^z (1 - t)^{-p/\alpha} t^{(\alpha - 1) / \alpha} \, dt \right]^{\alpha} = \sum_{n=0}^{\infty} A_n z^n, \]

(v) \(|f^{(k)}(z)| \leq F^{(k)}(r)\), \(k = 1, 2, \ldots\).

Proof. Inequalities (i), (ii) and (iii) follow from Theorem 3.1 and the distortion theorem for \(\alpha\)-convex functions (see Miller [9]). Statements (iv) and (v) follow from the recent result of P. Kulshrestha [8].

We mention without proof that a result similar to Theorem 2.5 holds for \(S_2(p, \alpha)\) using the technique of Theorem 2.5 and a result of J. Syzmal [12].

4. Let \(S_3(p, \alpha)\), \(0 < \alpha \leq 1\), denote the subclass of \(S(p)\) consisting of all functions \(f\) for which
\[ \limsup \max \left| \arg \left( \frac{zf'(z)/f(z)}{0} \right) \right| < \frac{\alpha \pi}{2}. \]

This extends the class \(S^*(\alpha)\) of strongly starlike functions of order \(\alpha\) defined by D. Brannan and W. Kirwan [2]. Note that a single valued branch of \(\arg \left( \frac{zf'(z)/f(z)}{0} \right)\) can be defined in some annulus \(\rho < |z| < 1\).

**Theorem 4.1.** Let \(f(z) \in S_3(p, \alpha)\). Then there is a function \(g(z) \in S^*(\alpha)\) such that \(f(z) = \prod_{j=1}^{p} \Psi(z, z_j) g(z)^p\).

Proof. This follows from the equation (2.3) since \(\Re(z \Psi'(z, z_j)/\Psi(z, z_j)) = 0\) on \(|z| = 1\).

**Corollary 4.2.** If \(f \in S_3(p, \alpha)\), then \(f\) is bounded in \(U\).

Proof. This follows from [2, Theorem 2.1] and the previous theorem.

**Lemma 4.3.** Let \(g(z) = z + b_2 z^2 + \ldots \in S^*(\alpha)\). Then if either \(\lambda \geq 3/4\) or \(3/4 - 1/4\alpha \geq \lambda, |b_3 - \lambda b_2| \leq |3\lambda^2 - 4\lambda \alpha^2|, and this result is sharp.

Proof. Using the notation of [1, Theorem 2.1], we have
\[ b_3 - \lambda b_2^2 = \frac{\alpha}{2} \left[ p_2 + \frac{3\alpha - 1 - 4\lambda \alpha}{2} p_1^2 \right], \]
where \(P(z) = 1 + p_1 z + p_2 z^2 + \ldots\) has \(\Re P(z) > 0\) in \(U\). We have
\[ b_3 - \lambda b_2^2 = \frac{\alpha}{2} \cdot 2 \int_0^{2\pi} e^{-2i\theta} d\mu(\theta) \]
\[ + \frac{\alpha}{2} \left( \frac{3\alpha - 1 - 4\lambda \alpha}{2} \right) \left( 2 \int_0^{2\pi} e^{-i\theta} d\mu(\theta) \right)^2, \]
where \(\mu(\theta)\) is an increasing function on \([0, 2\pi]\) with \(\mu(2\pi) - \mu(0) = 1\). Hence
\[
\frac{2}{a} \text{Re}(b_3 - \lambda b_2^2) = 2 \int_0^{2\pi} \cos \theta \, d\mu(\theta) \\
+ (6\alpha - 2 - 8\lambda \alpha) \left[ \left( \int_0^{2\pi} \cos \theta \, d\mu(\theta) \right)^2 - \left( \int_0^{2\pi} \sin \theta \, d\mu(\theta) \right)^2 \right].
\]

Suppose first that \(6\alpha - 2 - 8\lambda \alpha \geq 0\). Then

\[
\frac{2}{a} \text{Re}(b_3 - \lambda b_2^2) \leq 2 \int_0^{2\pi} \cos 2\theta \, d\mu(\theta) + (6\alpha - 2 - 8\lambda \alpha) \left( \int_0^{2\pi} \cos \theta \, d\mu(\theta) \right)^2 \\
\leq 4 \int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) - 2 + (6\alpha - 2 - 8\lambda \alpha) \int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) \\
\leq 6\alpha - 8\lambda \alpha,
\]

where we have used Jensen's inequality and the estimate \(\int_0^{2\pi} \cos^2 \theta \, d\mu(\theta) \leq 1\).

The case \(6\alpha - 2 - 8\lambda \alpha < 0\) is treated in a similar manner.

To show that the inequality is sharp, we consider the function \(g(z)\) defined by

\[
g'(z)/g(z) = ((1+z)/(1-z))^{a_2} = (1-z)/(1-z)^3 = b_2^3
\]

for which \(a_2 = 2\alpha, b_3 = 3\alpha^2\).

**Theorem 4.4.** Let \(f(z) = a_{p-1}z^{p-1} + a_p z^p + in \in S_3(p, \alpha)\) and suppose each \(a_n\) is real. If \(p > 3\) and \(\alpha \geq \min(1/3, (p^2 - 2p)^{-1/2})\),

\[
|a_{p+1}| \leq (2p^2\alpha^2 - p\alpha^2 - 1) |a_{p-1}| + 2\alpha p |a_p|,
\]

and this result is sharp.

**Proof.** By Lemma 2.4, it is sufficient to show that

\[
\left| 1 + pb_3 - p \left( \frac{p+1}{2} \right) b_2^2 \right| \leq 2p^2\alpha^2 - p\alpha^2 - 1,
\]

since \(|b_2| \leq 2\alpha [1]\). By Lemma 2.3, with \(\lambda = (p + 1)/2\),

\[
1 + pb_3 - p((p+1)/2) b_2^2 \geq 1 + p\alpha^2 - 2p^2\alpha^2
\]

and hence it suffices to show

\[
1 + pb_3 \leq 2p^2\alpha^2 - p\alpha^2 - 1.
\]

Since \(|b_3| \leq 3\alpha^2\) if \(\alpha \geq 1/3\), (4.2) follows if \(1 + 3p\alpha^2 \leq 2p^2\alpha^2 - p\alpha^2 - 1\), which is certainly true if \(p \geq 3\).

The sharpness of the result follows from the fact that the function \(g(z)\) defined by (4.1) simultaneously maximizes \(|b_2|\) and \(|b_3 - (p + 1)b_2^2/2|\).
Notes. 1. If $p \geq 4$, the result holds for all $\alpha \geq 1/3$.
2. For $0 < \alpha \leq 1/3$, a similar (but not sharp) result holds for $p > \left[\alpha^2 + \alpha + (17\alpha^4 + 2\alpha^3 + \alpha^2)\right] (4\alpha^2)^{-1}$.

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