SOME $H^\infty$-INTERPOLATING SEQUENCES AND THE BEHAVIOR OF CERTAIN OF THEIR BLASCHKE PRODUCTS

BY

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ABSTRACT. Let $f$ be a strictly increasing continuous real function defined near $0^+$ with $f(0) = f'(0) = 0$. Such a function is called a $K$-function if for every constant $k, f(\theta + kf(\theta))/f(\theta) \to 1$ as $\theta \to 0^+$. The curve in the open unit disc with corresponding representation $1 - r = f(\theta)$ is called a $K$-curve. Several analytic and geometric conditions are obtained for $K$-curves and $K$-functions. This provides a framework for some rather explicit results involving parts in the closure of $K$-curves, $H^\infty$-interpolating sequences lying on $K$-curves and the behavior of their Blaschke products. In addition, a sequence of points in the disc tending upper tangentially to 1 with moduli increasing strictly to 1 and arguments decreasing strictly to 0 is proved to be interpolating if and only if the hyperbolic distance between successive points remains bounded away from zero.

1. Introduction and preliminaries. Let $D$ be the open unit disc in the complex plane. We assume that the reader is somewhat familiar with the theory of $H^\infty(D)$ as a function algebra including such basic concepts (see [1]) as its maximal ideal space, the fiber $\mathcal{D}_1$ above 1, the pseudohyperbolic metric $\chi(z, w) = |z - w|/|1 - \overline{z}w|$, and the papers [2] and [3] on the parts of $H^\infty$.

Following through the proofs of [3] it is easy to see that all the results there hold for a wider class of curves than those hypothesized. In particular the only property of the convex curves in that paper which was used was that these curves were $K$-curves in the sense of the following definition.

Definition 1.1. Let $f$ be a strictly increasing continuous real function defined near $0^+$ with $f(0) = f'(0) = 0$. Then, $f$ is called a $K$-function if for every $k \in (-\infty, \infty),\n
\lim_{\theta \to 0^+} \frac{f(\theta + kf(\theta))}{f(\theta)} = 1.

If $\Gamma$ is an upper tangential curve in $D$ terminating at 1 whose polar...
representation is \(1 - r = f(\theta)\), then \(\Gamma\) is called a \(K\)-curve.

In §2 we will present a number of equivalent analytic and geometric criteria for \(K\)-curves and \(K\)-functions. This will lead to a new description of the various Wermer maps onto the parts in the closure of \(K\)-curves. We also show that the class of \(K\)-curves is definitely wider than the class of curves in [3].

A theorem of Wortman in [4] states that a sequence \(\{z_n\}\) of points lying on a convex upper tangential curve in \(D\) terminating at 1 is an interpolating sequence if and only if the numbers \(\chi(z_n, z_{n+1})\) are bounded away from zero. We call a sequence \(\{z_n = r_n e^{i\theta_n}\}\) in \(D\) tending to 1 a (strict) \(M\)-sequence if \(r_n\) increases (strictly) to 1, \(\theta_n\) decreases (strictly) to zero and \(\theta_n/(1 - r_n) \to \infty\) as \(n \to \infty\). In §3 we prove that an \(M\)-sequence is interpolating if and only if it satisfies Wortman’s condition, thus subsuming his result. In the special case that the points \(\{z_n\}\) are (asymptotically) equally \(\chi\)-spaced, we study the boundary behavior of the associated Blaschke product \(B\). We prove that the restriction of \(B\) to any part hit by the closure of its sequence of zeros is again a Blaschke product. In fact, we explicitly compute which Blaschke product it is. One interesting topological consequence is that for any \(K\)-curve \(\Gamma\) the set \(V_1 = (\Gamma^- - \Gamma)\) (the weak* closure) is disconnected.

2. Descriptions of \(K\)-curves. As in [3] we will often be interested in the quantities \((z = re^{i\theta}, w = pe^{i\varphi})\)

\[
a(z) = \frac{1 - r}{1 + 2r \cos \theta + r^2} = \Re \left( \frac{1 - z}{1 + z} \right),
\]

\[
b(z) = \frac{2r \sin \theta}{1 + 2r \cos \theta + r^2} = -\Im \left( \frac{1 - z}{1 + z} \right),
\]

\[
1 - \chi^2(z, w) = \frac{(1 - r^2)(1 - \rho^2)}{1 - 2r\rho \cos (\theta - \varphi) + r^2\rho^2}.
\]

One simple result we will soon refer to is

**Lemma 2.1.** Let \(\{re^{i\theta}\}\) and \(\{pe^{i\varphi}\}\) be two nets in \(D\) tending to 1 indexed by the same index set. If \(\delta, a\) and \(b\) are limits respectively of \(\chi(z, w)\), \((1 - \rho)/(1 - r)\) and \((\theta - \varphi)/(1 - r)\) then \(1 - \delta^2 = 4/(a^{-1} + 2 + a + b^2a^{-1})\).

**Proof.** Clear from the identity

\[
1 - \chi^2(z, w) = \frac{(1 + r)(1 + \rho)}{(1 - \rho^2 + 2r + r^2 \frac{1 - \rho}{1 - r} + 2r\rho \frac{1 - \cos(\theta - \varphi)}{(1 - r)(1 - \rho)}}.
\]

We next require some lemmas which later become part of the main theorem (Theorem 2.6).
Lemma 2.2. Suppose $f$ is a strictly increasing real function near $0^+$ with $f(0) = f'(0) = 0$. Suppose for some $k_0 \neq 0$, $f(\theta - k_0 f(\theta))/f(\theta) \to 1$ as $\theta \to 0^+$. Then $f$ is a $K$-function.

Proof. We will assume $k_0 > 0$; the other case is handled similarly. Because $f$ is monotone, the ratio in question tends to 1 if $k_0$ is replaced by $k \in (0, k_0)$. Let $k', \epsilon > 0$ be chosen such that $2\epsilon < k_0$ and $k' - 2\epsilon \equiv k_1 < k_0 < k'$. By our previous remark we may choose $0 < S < (k_0 - k_1)/k_1$ and $\delta$ so small that $f(\theta) < (1 + S)/(\theta - 2\epsilon f(\theta))$. Then, if $\varphi = \theta - 2\epsilon f(\theta)$,

$$k_1 f(\theta)/f(\varphi) < (1 + \delta)k_1 \equiv \alpha < k_0.$$

Thus,

$$1 \geq \frac{f(\theta - k' f(\theta))}{f(\theta)} \geq \frac{f(\varphi - k_1 f(\varphi))}{f(\varphi)} \geq \frac{f(\varphi - \alpha f(\varphi))}{f(\varphi)} \to 1.$$

Clearly, the ratio in question then tends to 1 for all $k > 0$.

Now, let $k > 0$, $\varphi = \theta + kf(\theta)$. Then, $k(f(\theta) - f(\varphi)) \leq 0$, so $\theta \geq \theta + kf(\theta) - kf(\varphi) = \varphi - kf(\varphi)$ and $f(\theta) \geq f(\varphi - kf(\varphi))$. Consequently,

$$1 \leq \frac{f(\theta + kf(\theta))}{f(\theta)} \leq \frac{f(\varphi)}{f(\varphi - kf(\varphi))} \to 1$$

by the first part of the proof. The result follows.

Lemma 2.3. Let $\{r_n e^{i\theta_n}\}$ be a strict $M$-sequence in $D$. Suppose that $k_n = (\theta_n - \theta_{n+1})/(1 - r_n)$ is bounded away from zero and $\gamma_n = (1 - r_{n+1})/(1 - r_n) \to 1$. If $f$ is any strictly increasing continuous real function near $0^+$ passing through the points $(\theta_n, 1 - r_n)$, then $f$ is a $K$-function, and for any two such functions $f, g$ we have $f(\theta)/g(\theta) \to 1$ as $\theta \to 0^+$.

Proof. Clearly, $f(0) = f'(0) = 0$. Choose $N$ so large that $\gamma_n > \frac{1}{2}$ for $n \geq N$. Select $k_0$ so that $0 < 2k_0 < k_n$ for all $n$. If $\theta_{n+1} \leq \theta \leq \theta_n$ and $n \geq N$, then $k_0 f(\theta) < k_0 f(\theta_n)$. Since $k_0/k_{n+1} < \frac{1}{2} < f(\theta_{n+1})/f(\theta_n)$, we have $\theta - k_0 f(\theta) > \theta_{n+1} - k_{n+1} f(\theta_{n+1}) = \theta_{n+2}$. Consequently,

$$1 \geq \frac{f(\theta - k_0 f(\theta))}{f(\theta)} \geq \frac{f(\theta_{n+2})}{f(\theta_n)} = \gamma_{n+1} \cdot \gamma_n \to 1.$$

By Lemma 2.2, $f$ is a $K$-function. The last assertion of the present lemma follows from the inequalities $f(\theta_{n+1})/f(\theta_n) \leq f(\theta)/g(\theta) \leq f(\theta_n)/f(\theta_{n+1})$ for $\theta_{n+1} \leq \theta \leq \theta_n$. 

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Since we are mainly interested in the homomorphisms in the closure of a K-curve the following definition and lemma are natural.

**Definition 2.4.** Two curves $\Gamma$ and $\Gamma'$ in $D$ terminating at 1 are equivalent if $(\Gamma^- - \Gamma) = (\Gamma'^- - \Gamma')$. Two real functions $f, g$ defined near $0^+$ are equivalent if $f(\theta)/g(\theta) \to 1$ as $\theta \to 0^+$.

**Lemma 2.5.** Suppose $\Gamma$ is a K-curve with polar representation $1 - r = f(\theta)$ and $\Gamma'$ is a curve in $D$ terminating at 1 which admits a polar representation $1 - r = g(\theta)$ with $g$ continuous, strictly increasing and $g(0) = g'(0) = 0$ for $\theta$ near $0^+$. If $g$ is equivalent to $f$, then $g$ is a K-function and $\Gamma'$ is equivalent to $\Gamma$. If $\Gamma'$ is equivalent to $\Gamma$, then $\Gamma'$ is a K-curve and $g$ is equivalent to $f$.

**Proof.** Suppose $g$ is equivalent to $f$. Then, for each $\epsilon > 0$, and small enough \( \theta, (1 - \epsilon)f(\theta) < g(\theta) < (1 + \epsilon)f(\theta) \). Thus,

\[
1 \geq \frac{g(\theta) - g(0)}{g(\theta)} \geq \frac{(1 - \epsilon)f(\theta) - (1 + \epsilon)f(\theta)}{(1 + \epsilon)f(\theta)}
\]

and by Lemma 2.2, since $f$ is a K-function, $g$ is also. Next, let $\{r_{\alpha}e^{i\theta_{\alpha}}\}$ be a net on $\Gamma$ tending to a homomorphism $h$. Choosing $\{\rho_{\alpha}e^{i\theta_{\alpha}}\}$ on $\Gamma'$ we have by direct computation, since $r_{\alpha} = 1 - f(\theta_{\alpha})$ and $\rho_{\alpha} = 1 - g(\theta_{\alpha})$,

\[
\chi(r_{\alpha}e^{i\theta_{\alpha}}, \rho_{\alpha}e^{i\theta_{\alpha}}) = |1 - g(\theta_{\alpha})/f(\theta_{\alpha})|/|1 + r_{\alpha}g(\theta_{\alpha})/f(\theta_{\alpha})| \to 0.
\]

Thus, $\rho_{\alpha}e^{i\theta_{\alpha}} \to h$. By symmetry, $(\Gamma^- - \Gamma) = (\Gamma'^- - \Gamma')$.

On the other hand, suppose that $\Gamma'$ is equivalent to $\Gamma$. Let $\{\varphi_{\alpha}\}$ be any universal net tending to $0^+$. Let $\rho_{\alpha}e^{i\varphi_{\alpha}}$ be on $\Gamma'$. Since $\Gamma$ is a K-curve all the homomorphisms in $\Gamma^-$ are nontrivial [3]. Since $\Gamma'$ is equivalent to $\Gamma$ it follows from [2, Theorem 6.1], that there are corresponding points $r_{\alpha}e^{i\theta_{\alpha}}$ on $\Gamma$ with $\chi(r_{\alpha}e^{i\theta_{\alpha}}, \rho_{\alpha}e^{i\varphi_{\alpha}}) \to 0$. By Lemma 2.1, $k_{\alpha} = (\theta_{\alpha} - \varphi_{\alpha})/(1 - r_{\alpha}) \to 0$ and $(1 - r_{\alpha})/(1 - \rho_{\alpha}) = f(\theta_{\alpha})/g(\varphi_{\alpha}) \to 1$. Thus,

\[
\frac{f(\varphi_{\alpha})}{g(\varphi_{\alpha})} = \frac{f(\theta_{\alpha} - k_{\alpha}f(\theta_{\alpha}))}{f(\theta_{\alpha})} \frac{f(\theta_{\alpha})}{g(\varphi_{\alpha})} \to 1
\]

since $f$ is a K-function. Thus $g$ is equivalent to $f$. Consequently, as above, $g$ is a K-function so $\Gamma'$ is a K-curve.

The next result underlies much of the explicitness obtained in the succeeding analysis of the parts in the closure of K-curves.

**Theorem 2.6.** Let $\Gamma$ be an upper tangential curve in $D$ terminating at 1 and let $1 - r = f(\theta)$ be its polar representation with $f$ strictly increasing,
continuous and \( f(0) = f'(0) = 0 \). Then, each of the following conditions is equivalent to each other condition and to the fact that \( \Gamma \) is equivalent to a K-curve and \( f \) is equivalent to a K-function.

1. There is a function \( g \) equivalent to \( f \) such that for each sequence \( \{\theta_n\} \), \( \theta_n \downarrow 0^+ \), \( [g(\theta_n) - g(\theta_{n+1})]/(\theta_n - \theta_{n+1}) \to 0 \).

2. \( f(\theta - f(\theta))/f(\theta) \to 1 \) as \( \theta \to 0^+ \).

3. For some (or for every) \( k \neq 0 \), \( f(\theta + kf(\theta))/f(\theta) \to 1 \) as \( \theta \to 0^+ \).

4. For some positive bounded measurable function \( h \) with essential limit zero as \( \theta \to 0^+ \), \((1/f(\theta)) \int_0^\theta h(t) dt \to 1 \) as \( \theta \to 0^+ \).

5. There is a curve \( \Gamma' \) equivalent to \( \Gamma \) such that for each strict \( M \)-sequence \( \{z_n\} \) on \( \Gamma' \), the slopes of the secant lines joining successive points \( z_n, z_{n+1} \), approach \(-\infty \) as \( n \to \infty \).

6. For some (or for every) \( k > 0 \) and for some (or for every) \( M \)-sequence \( \{r_n e^{i\theta n}\} \) on \( \Gamma \) such that \( (\theta_n - \theta_{n+1})/(1 - r_n) \to k \) one has \((1 - r_{n+1})/(1 - r_n) \to 1 \).

7. For some (or for every) sequence on \( \Gamma \) defined recursively by \( \theta_{n+1} = \theta_n - (1 - r_n) \) one has \((1 - r_{n+1})/(1 - r_n) \to 1 \).

8. For some (or for every) \( M \)-sequence \( \{z_n\} \) on \( \Gamma \) such that \( \chi(z_n, z_{n+1}) \to 0 \) in \((0, 1) \) one has \((1 - |z_{n+1}|)/(1 - |z_n|) \to 1 \).

**Proof.** We have already seen from Lemma 2.2 that (2) and (3) are equivalent. Suppose (1) holds. Since \( g(\theta)/\theta \to 0 \) we may define a sequence \( \{\theta_n\} \), \( \theta_n \downarrow 0^+ \), with \( \theta_{n+1} = \theta_n - g(\theta_n) \). (1) then implies that \( g(\theta_{n+1})/g(\theta_n) \to 1 \). It follows immediately from Lemma 2.2 (with \( 1 - r_n = g(\theta_n) \)) that (2) and (3) hold and also hold for the segmental function \( f_1 \) joining successive points \( (\theta_n, g(\theta_n)) \) linearly. Furthermore, \( f_1(\theta)/f(\theta) \to 1 \). Let \( h = f_1' \), a.e. Clearly, \( h \) is an (essentially) positive bounded measurable function with essential limit zero as \( \theta \to 0^+ \) and \( f_1 \) is recovered by integration. Thus (4) holds. On the other hand, (1) clearly follows from (4) with \( g(\theta) = \int_0^\theta h(t) dt \) and we see that the first four conditions are equivalent.

Next we show that (1) implies (5). Let \( \Gamma' \) correspond to \( g \) so \( \Gamma' \) is equivalent to \( \Gamma \) by Lemma 2.5. Let \( \{z_n = r_n e^{i\theta n}\} \) be a strict \( M \)-sequence on \( \Gamma' \). Suppressing subscripts by letting \( r = r_n, \rho = r_{n+1}, \theta = \theta_n, \varphi = \theta_{n+1} \), the slope, \( S_n \), of the secant joining \( z_n \) to \( z_{n+1} \) is

\[
S_n = \frac{r \sin \theta - \rho \sin \varphi}{r \cos \theta - \rho \cos \varphi}
= \frac{\sin \varphi - \sin \theta}{\sin(\theta - \varphi)/2} \frac{(r - \rho) \sin \theta/(\sin \theta - \sin \varphi) + \rho}{(\rho - r) \cos \varphi/(\sin(\theta - \varphi)/2) + 2r \sin(\theta + \varphi)/2}
\to -\infty \quad \text{as} \quad n \to \infty
\]
since our assumption implies that \((r - \rho)/(\theta - \varphi) \to 0\) (and, of course, \(\theta, \varphi \to 0\)).

Suppose that (5) holds and \(\theta_n\) is defined on \(\Gamma\) as in (6). If we take \(\rho_n e^{i\theta_n}\) on \(\Gamma'\), the fact that \(\Gamma\) is equivalent to \(\Gamma'\) gives \((1 + \rho_n)/(1 - r_n) \to 1\) and 
\[
(\theta_n - \theta_{n+1})/(1 - \rho_n) \to k.
\]
Then with notation similar to the last paragraph, and 
\[
\gamma = \gamma_n = (1 - r_{n+1})/(1 - r_n),
\]
\[
S_n = -\frac{2 \cos(\theta + \varphi)/2 \sin(\theta - \varphi)/2}{(1 - r) + \gamma \sin \varphi - \sin \theta}.
\]
If \(S_n \to -\infty\) as \(n \to \infty\), it is clear from the fact that \((\theta - \varphi)/(1 - r) \to k > 0\) that \(\gamma_n \to 1\) which gives the required conclusion of (6) for every such sequence.

A fortiori (7) holds.

Suppose, next that for some sequence \(\{\theta_n\}\) on \(\Gamma\) defined recursively by 
\[
\theta_{n+1} = \theta_n - (1 - r_n) \text{ that } (1 - r_{n+1})/(1 - r_n) \to 1.
\]
By Lemma 2.2, \(f\) satisfies (3), thus conditions (1) through (7) are equivalent.

Finally, we show that (6) and (8) are equivalent. Suppose (6) is true and 
\[
\chi(z_n, z_{n+1}) \to \delta \in (0, 1) \text{ as in (8). Choose any universal net, } n(\alpha), \text{ of indices, } n(\alpha) \to \infty.
\]
Let \(h_0 = \lim z_n(\alpha), h_1 = \lim z_n(\alpha)+1\). Because \(\Gamma\) is a \(K\)-curve, we have all the facts listed in [3]. In particular, there is a unique \(\xi \in D\) such that 
\[
L_{\alpha}(\xi) = (z_n(\alpha) + \xi)/(1 + \bar{z}_n(\alpha)\xi) \to h_1.
\]
Thus, \(\chi(L_{\alpha}(\xi), z_{n(\alpha)+1}) \to 0\) which implies that 
\[
(1 - |L_{\alpha}(\xi)|)/(1 - |z_{n(\alpha)+1}|) \to 1.
\]
By Lemma 1 of [3], 
\[
(1 - |L_{\alpha}(\xi)|)/(1 - |z_{n(\alpha)}|) \to \Re((1 - \xi)/(1 + \xi)) = a(\xi).
\]
But \(a(\xi) = 1\) since \(h_1\) is in the closure of \(\Gamma\). Thus, 
\[
(1 - |z_{n(\alpha)+1}|)/(1 - |z_{n(\alpha)}|) \to 1.
\]
This being true for every universal net, we have the required result.

Conversely, suppose (8) holds for some \(M\)-sequence \(\{z_n\}\). Since 
\[
\chi(z_n, z_{n+1}) \to \delta \in (0, 1) \text{ and } 1 - |z_{n+1}|/1 - |z_n| \to 1, \text{ we must have, by Lemma 2.1, that } (\theta_n - \theta_{n+1})/(1 - r_n) \to 2\delta \sqrt{1 - \delta^2} \text{ and (6) holds.}
\]

The next result gives very "explicit" descriptions of the Wermer maps onto parts in the closure of a \(K\)-curve.

**Theorem 2.7.** Let \(\Gamma\) be a \(K\)-curve with polar representation \(1 - r = f(\theta)\) and let \(b_0 > 0\). Choose \(0 < \theta_1 < \pi/2\) and define 
\[
z_n = r_n e^{i\theta_n} \text{ recursively on } \Gamma
\]
by \(\theta_{n+1} = \theta_n - b_0 f(\theta_n)\). Let \(h_0\) be a homomorphism in the closure in \(D_1\) of 
\[
\{z_n\}, \text{ say, } h_0 = \lim z_n(\alpha).
\]
Let \(\tau\) be the Wermer map of \(D\) onto the part of \(h_0\) such that \(\tau(0) = h_0\). Given \(z_0 \in D, \text{ let } N \text{ be the greatest integer in } b(z_0)/b_0\)
and let \(\lambda = b(z_0) - Nb_0\). Then 
\[
\tau(z_0) = \lim \rho_n(\alpha) e^{i\varphi_n(\alpha)} \text{ where } \varphi_n(\alpha) = \theta_n(\alpha) + N - \lambda f(\theta_n(\alpha) + N) \text{ and } \rho_n(\alpha) = 1 - f(\varphi_n(\alpha)) a(z_0).
\]

**Proof.** We will give the proof for \(b(z_0) > 0\). The proof for negative values is similar.
Let \( \varphi'_n = \theta_n - b(z_0)f(\theta_n) \), \( \rho'_n = 1 - f(\varphi'_n)a(z_0) \). Then, from [3], \( \tau(z_0) = \lim_{n \to \infty} \rho_n(\alpha) e^{i\varphi_n(\alpha)} \) so the theorem will follow if we can prove that
\[
\chi(\rho_n(\alpha) e^{i\varphi_n(\alpha)}, \rho'_n(\alpha) e^{i\varphi'_n(\alpha)}) \to 0.
\]

By Lemma 2.1 this will follow from proving that \( (\varphi_n - \varphi'_n)/(1 - \rho_n) \to 0 \) and \( (1 - \rho_n)/(1 - \rho'_n) \to 1 \).

We estimate first that
\[
\varphi'_n = \theta_n - \sum_{k=0}^{N-1} b_0 f(\theta_n) + \lambda f(\theta_{n+1})
\]
\[
\leq \theta_n - \sum_{k=0}^{N-1} b_0 f(\theta_{n+k}) + \lambda f(\theta_{n+N})
\]
\[
= \theta_{n+N} - \lambda f(\theta_{n+N}) = \varphi_n.
\]

Let \( \epsilon > 0 \). By Theorem 2.6, condition (7) for large \( n \), \( f(\theta_{n+j}) \leq (1 + \epsilon)f(\theta_{n+k}) \), \( j, k = 0, \ldots, N \). We then estimate that
\[
\varphi'_n \geq \theta_n - \sum_{k=0}^{N-1} b_0 (1 + \epsilon) f(\theta_{n+k}) - \lambda(1 + \epsilon) f(\theta_{n+N})
\]
\[
= \varphi_n - \epsilon b_0 \sum_{k=0}^{N-1} f(\theta_{n+k}) - \epsilon \lambda f(\theta_{n+N})
\]
\[
\geq \varphi_n - \epsilon [b_0 (1 + \epsilon) N + \lambda] f(\theta_{n+N}).
\]

Therefore,
\[
0 \leq \frac{\varphi_n - \varphi'_n}{1 - \rho_n} \leq \frac{\epsilon(b(z_0) + \epsilon b_0 N) f(\theta_{n+N})}{a(z_0) f(\theta_n)}. 
\]

It is clear that \( (\varphi_n - \varphi'_n)/(1 - \rho_n) \to 0 \). Similarly,
\[
(1 - \rho_n)/(1 - \rho'_n) = f(\theta_n)/f(\varphi'_n) \leq f(\theta_n)/f(\theta_{n+N+1}) \to 1.
\]

Using [3] it is not hard to see that another way of describing geometrically what we have done in Theorem 2.7 is the following: For each \( \gamma \in (0, \infty) \) and each \( \delta \in (|\gamma - 1|/|\gamma + 1|, 1) \), the circle \( \chi(z, z_n) = \delta \) eventually intersects the curve \( \Gamma(f, \gamma) \) (given by \( f(\theta) = \gamma(1 - r) \)) in two points which are asymptotically equivalent to the points on \( \Gamma(f, \gamma) \) with arguments \( \theta_{n+N} - \lambda f(\theta_{n+N}) \) and \( \theta_{n-N} + \lambda f(\theta_{n-N}) \) where \( N \) and \( \lambda \) are computed by \( |b| = Nb_0 + \lambda \), \( 0 \leq \lambda < b_0 \), and
\[
b^2 = \left[ \delta^2 - ((\gamma - 1)/(\gamma + 1))^2 \right] \left[ (\gamma + 1)^2/\gamma^2(1 - \delta^2) \right].
\]
(This evaluation of \( b \) follows from Lemma 2.1 and the relation \( \gamma = 1/a \) from [3].)
We close this section with an example which demonstrates that the class of $K$-functions is definitely wider than the class considered in [3].

**Example 2.8.** There exists a $K$-function which is equivalent to no convex function.

**Proof.** First, suppose $h$ is any $K$-function whose graph is piecewise linear. Suppose $g$ is convex and for some $K > 0$, $K^{-1} \leq h(\theta)/g(\theta) \leq K$ for $\theta$ near $0^+$. Let $f$ be the convex function whose graph forms the lower boundary of the convex hull of the graph of $h$. Let $\{\theta_n\}$ be the sequence with $\theta_n \to 0$ such that $(\theta_n, h(\theta_n))$ is a vertex lying on the graph of $f$. Let $\theta_{n+1} \leq \theta \leq \theta_n$, $\theta = \lambda \theta_n + (1 - \lambda)\theta_{n+1}$. Then

$$K^{-1}f(\theta) \leq f(\theta)g(\theta)/h(\theta) \leq g(\theta) \leq \lambda g(\theta_n) + (1 - \lambda)g(\theta_{n+1})$$

$$\leq K[\lambda h(\theta_n) + (1 - \lambda)h(\theta_{n+1})] = Kf(\theta).$$

Thus, if there is such a $g$, the function $f$ has the same properties.

Define $f$ to be the piecewise linear function passing through the points $(1/n, 1/n!)$ and $(n^2 + 1)/n^2(n + 1)$, $1/2 \cdot n!$. It is not hard to calculate that $f$ is the convex function bearing the same relationship to $h$ as above. Furthermore, the secant slopes of $h$ tend to zero so that by Theorem 2.6, condition (1), $h$ is a $K$-function. But, if $\theta_n = (n^2 + 1)/n^2(n + 1)$, then $h(\theta_n)/f(\theta_n) \to \infty$. Combining this with the above initial remarks verifies the example.

In fact, it is not hard to see that for each given convex function $g$, the $K$-curve corresponding to the function $h$ of the example contains whole parts in its closure which are not in the closure of the curve defined by $g$.

3. Interpolating sequences and Blaschke products. We begin with some facts about $M$-sequences. The first result can be computed by brute force and was noted in [4], while the second follows at once.

**Lemma 3.1.** If $z = re^{i\theta}$, $w_1 = \rho_1 e^{i\varphi_1}$, $w_2 = \rho_2 e^{i\varphi_2}$ and $0 < r \leq \rho_1 \leq \rho_2 < 1$, $0 < \varphi_2 \leq \varphi_1 < \theta < \pi/2$, then $\chi(z, w_1) \leq \chi(z, w_2)$.

**Corollary 3.2.** Suppose $\{z_n\}$ is an $M$-sequence and for some $\delta > 0$, $\chi(z_n, z_{n+1}) \geq \delta$. Then $\chi(z_n, z_m) \geq \delta$ for all $n, m$.

The next result is basic to the section and allows us to use some of the techniques of [4] to improve Wortman’s theorem on interpolating sequences.

**Lemma 3.3.** Let $\{z_n = r_n e^{i\theta_n}\}$ be an $M$-sequence such that $\chi(z_n, z_{n+1}) \geq \delta > 0$ for some $\delta$. Then, $\{z_n\}$ can be partitioned into two $M$-sequences $\{\alpha_k\}$ and $\{\beta_k\}$ such that

$$(1) \{\alpha_k\} \text{ is geometric}, i.e., \left|1 - \frac{\alpha_{k+1}}{\alpha_k}\right| < c < 1 \text{ for some } c.$$
(2) The numbers \((\varphi_k - \varphi_{k+1})/(1 - \rho_k)\) are bounded away from zero.

**Proof.** Let
\[
a_n = (1 - r_{n+1})/(1 - r_n), \quad b_n^2 = 2(1 - \cos(\theta_n - \theta_{n+1}))/((1 - r_n)^2),
\]
and for \(0 < c < 1\), let \(N_c = \{n: a_n > c\}\). If \(N_c\) is always eventually empty we may take \(\alpha_k = z_k\) and \(\{\beta_k\}\) to be empty. Otherwise, we compute, suppressing indices with \(r_n = r, r_{n+1} = \rho\),
\[
\frac{(1 + r)(1 + \rho)}{a^{-1} + 2r + r^2a + \rho \beta^2 a^{-1}} = 1 - \chi^2(z_n, z_{n+1}) \leq 1 - \delta^2 < 1.
\]
Solving for \(b_n^2\) and taking \(a\) as any limit point of \(\{a_n\}\) through \(N_c\),
\[
\liminf b_n^2 \geq \frac{(1 + a)^2 \delta^2 - (1 - a)^2}{(1 - \delta^2)} \geq \frac{(1 + c)^2 \delta^2 - (1 - c)^2}{(1 - \delta^2)},
\]
since for \(n \in N_c, a_n > c\) so \(a \geq c\). Therefore, we may choose \(b_0 > 0\) and \(c\) so close to 1 that \(b_n > b_0\) for \(n \in N_c\).

Fix such a \(c\). Let \(\{\alpha_k\}\) be the sequence (in natural order) \(\{z_n: n \in N_c\}\).

Given \(k\), for some \(n \in N_c\) and \(m > n\), \(\alpha_k = z_n, \alpha_{k+1} = z_m\). Then
\[
1 - |\alpha_{k+1}|/1 - |\alpha_k| = (1 - r_m)/(1 - r_n) \leq (1 - r_{n+1})/(1 - r_n) \leq c.
\]
Thus \(\{\alpha_k\}\) is geometric.

Consider the remaining sequence \(\{z_n: n \in N_c\} = \{\beta_k\}\) in natural order.

Given \(k\), for some \(n \in N_c\) and \(m > n\), \(\beta_k = z_n\) and \(\beta_{k+1} = z_m\). Then, because of the choice of \(c\),
\[
(\varphi_k - \varphi_{k+1})/(1 - \rho_k) = (\theta_n - \theta_m)/(1 - r_n)
\geq (\theta_n - \theta_{n+1})/(1 - r_n) = b_n > b_0 > 0.
\]

A crucial observation of Wortman [4] for us is that one can make an excellent lower estimate of an infinite real product \(\prod a_n\) provided the numbers \(\{a_n\}\) are bounded away from zero on the left.

**Lemma 3.4.** Let \(\{a_n\}\) be a real sequence such that for some \(\delta > 0, \delta \leq a_n < 1\) for all \(n\). Then,
\[
\exp \left( -\delta^{-1} \sum_{k=1}^{\infty} (1 - a_k) \right) \leq \prod_{k=1}^{\infty} a_k \leq \exp \left( -\sum_{k=1}^{\infty} (1 - a_k) \right).
\]

**Proof.** The proof is standard beginning with the inequality \((1 - x) < -\log x \leq \delta^{-1}(1 - x)\) for \(\delta \leq x \leq 1\).
We now isolate as a lemma the computations required for both of the main results, Theorems 3.6 and 3.7, of this section.

**Lemma 3.5.** Let \( \{z_k = r_k e^{i\theta_k}\} \) be an M-sequence in \( D \) such that
\[
(\theta_k - \theta_{k+1})/(1 - r_k) > b_0 > 0 \quad \text{for all} \quad k.
\]
Assume further that for all \( k, r_k > 1/2 \) and \( 0 < \theta_k < \pi/3 \). Let \( w = pe^{i\varphi} \in D \) and let \( n \) be the index for which \( \theta_{n+1} < \varphi < \theta_n \). Then

\[
\sum_{k \neq (n-1), \ldots, (n+2)} 1 - \chi^2(w, z_k) \leq \frac{8}{b_0 \sqrt{\rho}} \left[ \tan^{-1} \left( \frac{2(1 - \rho)}{b_0 \sqrt{\rho} (1 - r_{n-1})} \right) + \tan^{-1} \left( \frac{2(1 - \rho)}{b_0 \sqrt{\rho} (1 - r_{n+1})} \right) \right].
\]

**Proof.** The condition \( 0 < \theta_k < \pi/3 \) implies that \( 1 - \cos (\theta_k - \varphi) > (\theta_k - \varphi)^2/4 \). Thus, using \( r_k > \frac{1}{2} \) and beginning with the expression for \( 1 - \chi^2 \) listed at the beginning of §2, one has

\[
1 - \chi^2(w, z_k) \leq 4 \frac{(1 - r_k)(1 - \rho)}{(1 - \rho)^2 + \rho(\theta_k - \varphi)^2/4}.
\]

Replacing each term of \( \Sigma \) by this estimate and using the facts that for \( k = 1, \ldots, n - 2 \) one has \( 1 - r_k \leq (\theta_k - \theta_{k+1})/b_0 \), and for \( k = n + 3, \ldots \) one has \( 1 - r_k \leq 1 - r_{k-1} \leq (\theta_{k-1} - \theta_k)/b_0 \), we estimate

\[
\sum \leq \frac{4}{b_0} \left[ \sum_{k = 1}^{n-2} \frac{(1 - \rho)(\theta_k - \theta_{k+1})}{(1 - \rho)^2 + \rho(\theta_k - \varphi)^2/4} + \sum_{k = n+3}^{\infty} \frac{(1 - \rho)(\theta_{k-1} - \theta_k)}{(1 - \rho)^2 + \rho(\varphi - \theta_k)^2/4} \right]
\leq \frac{8}{b_0 \sqrt{\rho}} \left[ (\int_A^B + \int_C^D) \frac{d\theta}{1 + \theta^2} \right]
\]

where \( A = \sqrt{\rho} (\theta_{n-1} - \varphi)/2(1 - \rho) \), \( B = \sqrt{\rho} (\theta_1 - \varphi)/2(1 - \rho) \), \( C = \sqrt{\rho} (\varphi - \theta_{n+2})/2(1 - \rho) \), \( D = \sqrt{\rho} \varphi/2(1 - \rho) \). This last step follows from standard type estimates of sums by integrals. The inequality remains if one replaces \( B \) and \( D \) by infinity and, by the choice of \( n \), if one replaces \( A \) and \( C \), respectively, by the smaller values

\[
A \geq \sqrt{\rho} (\theta_{n-1} - \theta_n)/2(1 - \rho) \geq b_0 \sqrt{\rho} (1 - r_{n-1})/2(1 - \rho)
\]

and

\[
C \geq \sqrt{\rho} (\theta_{n+1} - \theta_{n+2})/2(1 - \rho) \geq b_0 \sqrt{\rho} (1 - r_{n+1})/2(1 - \rho).
\]

The result then follows.
We recall Carleson’s condition (see [1]) that a sequence \( \{z_n\} \) in \( D \) is an \( H^\infty \)-interpolating sequence if and only if \( \prod_{k \neq n} \chi(z_k, z_n) \geq \delta > 0 \) for some \( \delta \). Notwithstanding the elegance of this characterization it is generally not an easy matter to check the condition in special cases. In [4] Wortman verified Carleson’s condition under the hypothesis that the points \( \{z_n\} \) form a strict \( M \)-sequence lying on a convex curve terminating at 1 such that \( \chi(z_n, z_n + 1) \geq \delta_0 > 0 \) for all \( n \). We show next that the convex curve is superfluous.

**Theorem 3.6.** Let \( \{z_n\} \) be a strict \( M \)-sequence in \( D \). Then, a necessary and sufficient condition for \( \{z_n\} \) to be an \( H^\infty \)-interpolating sequence is that \( \chi(z_n, z_n + 1) \geq \delta_0 > 0 \) for some \( \delta_0 \).

**Proof.** That \( \chi(z_n, z_n + 1) \geq \delta_0 > 0 \) if \( \{z_n\} \) is interpolating follows at once from Carleson’s condition.

Suppose conversely that \( \chi(z_n, z_n + 1) \geq \delta_0 > 0 \). By Lemma 3.3, \( \{z_n\} \) can be partitioned into a geometric sequence \( \{\alpha_k\} \) and an \( M \)-sequence \( \{\beta_k = \rho_k e^{i\varphi_k}\} \) such that \( (\varphi_k - \varphi_{k+1})/(1 - \rho_k) \geq b_0 > 0 \) for some \( b_0 \). Since our result does not depend on the first few terms we may assume \( \rho_k > 1/2 \) and \( 0 < \theta_k < \pi/3 \). Then Lemma 3.5 applies with \( \{z_k\} = \{\beta_k: k \neq n\} \) and \( w = \beta_n \). Then certainly \( \sum_{k \neq n} 1 - \chi^2(\beta_n, \beta_k) \leq M \) for some constant \( M \). Since by Corollary 3.2 one has \( \chi(\beta_n, \beta_k) \geq \delta_0 \) for all \( n, k \), Carleson’s condition for \( \{\beta_k\} \) follows from Lemma 3.4. Thus \( \{\beta_k\} \) is interpolating. It is well known that any geometric sequence, and thus \( \{\alpha_k\} \), is interpolating.

Let \( B_1, B_2 \) be the Blaschke products with zeros \( \{\alpha_k\} \) and \( \{\beta_k\} \), respectively. Then \( B = B_1 B_2 \) is the Blaschke product with zeros \( \{z_n\} \). Referring to the very last material concerning interpolating sequences in [1] we have that the closure in \( D \) of each of \( \{\alpha_k\} \) and \( \{\beta_k\} \) is homeomorphic to the Stone-\( \check{\text{C}} \)ech compactification of the integers and \( B_1 \) and \( B_2 \) vanish only in the closures of \( \{\alpha_k\} \) and \( \{\beta_k\} \), respectively. But applying Corollary 3.2 to the sequence \( \{z_n\} \), we see that \( \{\alpha_k\} \) and \( \{\beta_k\} \) have disjoint closures. Thus \( \{z_n\} \) is homeomorphic to the Stone-\( \check{\text{C}} \)ech compactification of the integers and \( B \) vanishes only in the closure of \( \{z_n\} \). Consequently, \( \{z_n\} \) is interpolating.

In the special case that the sequence \( \{z_n\} \) is asymptotically equally spaced and lies on a \( K \)-curve one can estimate the boundary behavior of the modulus of the Blaschke product with zeros \( \{z_n\} \).

**Theorem 3.7.** Let \( \Gamma \) be a \( K \)-curve with polar representation \( 1 - r = f(\theta) \). Let \( \{z_k = r_k e^{i\theta_k}\} \) be points chosen on \( \Gamma \) recursively by \( \theta_{k+1} = \theta_k - b_0 f(\theta_k) \) for some \( b_0 > 0 \). Let \( B \) be the Blaschke product with zeros \( \{z_k\} \). Then:

1. For \( \gamma \in (0, \infty) \), \( \gamma \neq 1 \),
\[ \lim \inf \{ |B(w)| : w = re^{i\varphi} \rightarrow 1, f(\varphi) = \gamma(1 - \rho) \} \geq \exp\left( -\left(25^{-1}M(\gamma) \right) \right) \]

where \( \delta = (|\gamma - 1|/|\gamma + 1|)^2 \) and \( M(\gamma) = (16/b_0) \tan^{-1}(4/b_0 \gamma) + 4(1 - \delta) \).

(2) Let \( h_0 \) be in the closure in \( D_1 \) of \( \{z_k\} \) and let \( \tau \) be the Wermer map of \( D \) onto the part of \( h_0 \) such that \( \tau(0) = h_0 \). Then \( B \circ \tau \) is the Blaschke product whose zeros have coordinates \( a(z) = 1, b(z) = 0, \pm b_0, \pm 2b_0, \ldots \).

**Proof.** Let \( 1 \neq \gamma > 0 \). Every homomorphism in \( D_1 \) in the closure of \( \Gamma(f, \gamma) = \{re^{i\varphi} : f(\varphi) = \gamma(1 - \rho) \} \) is in the same part as some homomorphism in the closure in \( D_1 \) of \( \{z_k\} \) since the points \( \{z_k\} \) are asymptotically equally spaced (by \( b_0/\sqrt{4 + b_0^2} \)). Let \( h_0 \) be in the closure of \( \{z_k\} \), say, \( h_0 = \lim z_n \). Let \( \tau \) be the Wermer map of \( D \) onto the part of \( h_0 \) such that \( \tau(0) = h_0 \). Let \( z_0 \) be a point in \( D \) such that \( a(z_0) = 1/\gamma \) and let \( N \) be the greatest integer in \( b(z_0)/b_0 \) and \( \lambda = b(z_0) - N b_0 \). By Theorem 2.7, \( \tau(z_0) = \lim \rho_n(\alpha)e^{i\varphi_n(\alpha)} \) where \( \varphi_n(\alpha) = \theta_n(\alpha) + N - \lambda \gamma(\theta_n(\alpha) + N) \) and \( 1 - \rho_n(\alpha) = f(\varphi_n(\alpha))/\gamma \).

We may apply Lemma 3.5 to \( \{z_k\} \) and \( w = \rho_n(\alpha)e^{i\varphi_n(\alpha)} \). From the above we see that for \( b(z_0) \geq 0 \), \( \theta_n(\alpha) + N + 1 \leq \varphi_n(\alpha) \leq \theta_n(\alpha) + N \) while for \( b(z_0) < 0 \), \( \theta_n(\alpha) + N \leq \varphi_n(\alpha) \leq \theta_n(\alpha) + N - 1 \). We will handle explicitly only the case where \( b(z_0) \geq 0 \) since as we shall see the only important ingredient in our argument will be that the index of \( \theta_n(\alpha) \) is advanced or retarded by a fixed amount.

To apply Lemma 3.5, we first notice that

\[
\frac{1 - \rho_n(\alpha)}{1 - r_n(\alpha) + N + 1} = \gamma^{-1} \frac{f(\theta_n(\alpha) + N) - \lambda \gamma(\theta_n(\alpha) + N)}{f(\theta_n(\alpha) + N + 1)} \leq 2\gamma^{-1}
\]

for \( n(\alpha) \) large since \( f \) is a K-function and \( N \) is fixed. Similarly,

\[
\frac{1 - \rho_n(\alpha)}{1 - r_n(\alpha) + N} \leq 2\gamma^{-1}.
\]

Next, from [3] we have \( \chi(\gamma, z_0) \geq |\gamma - 1|/|\gamma + 1| \). Combining all of this we have

\[
\limsup_{\alpha} \sum_{k=1}^{\infty} (1 - \chi^2(w_n(\alpha), z_k)) \leq \frac{16}{b_0} \tan^{-1} \frac{4}{\gamma b_0} + 4 \left( 1 - \left( \frac{|\gamma - 1|}{|\gamma + 1|} \right)^2 \right).
\]

If we apply Lemma 3.4 to \( |B|^2 \) and recall the remark at the beginning of the proof we see that (1) follows.

In fact, the estimate in (1) holds for \( |(B \circ \tau)(z_0)| \) whenever \( a(z_0) = 1/\gamma \) regardless of the value of \( b(z_0) \). If we fix \( z \) to lie on any curve \( 0 \neq b(z) = b = \) constant, \( -\infty < b < \infty \), and let \( \gamma \rightarrow \infty \) we have \( |(B \circ \tau)(z)| \rightarrow 1 \). That is, the radial limits of \( |B \circ \tau| \) are 1 at all points of the unit circle except \(-1\). Taking \( b(z) = 0 \) and letting \( \gamma \rightarrow 0 \) we see that \( |B \circ \tau| \) does not tend to zero radially at \(-1\). Thus \( B \circ \tau \) is an inner function with no zero radial limits. Hence \( B \circ \tau \) is
a Blaschke product with a singularity only at \(-1\). Since from Theorem 3.6, \(\{z_k\}\) is interpolating, \(B \circ \tau\) vanishes exactly at those points in the part of \(h_0\) which are in the closure of \(\{z_k\}\). From Theorem 2.7 these points are precisely those given in (2).

**Corollary 3.8.** The closure in \(\mathcal{D}_1\) of any K-curve disconnects \(\mathcal{D}_1\).

**Proof.** Let \(\Gamma\) be a K-curve with polar representation \(1 - r = f(\theta)\). Let \(B\) be one of the Blaschke products discussed in Theorem 3.7. For \(0 \leq \gamma \leq \infty\), let \(A(f, \gamma) = \{h: \text{for some universal net } r_\alpha e^{i\theta_\alpha} \to h, f(\theta_\alpha)/(1 - r_\alpha) \to \gamma\}\). From [3] one sees that if \(\gamma' \in (0, \infty)\) and \(\gamma \neq \gamma'\), then \(A(f, \gamma) \cap A(f, \gamma') = \emptyset\). Since \(|B| = 1\) on \(A(f, \infty)\) and \(|B| < 1\) on \(A(f, 0)\) we see that \(A(f, 0) \cap A(f, \infty)\) is also empty. It is easily seen that \(\bigcup \{A(f, \gamma): \gamma < 1\}\) is closed and that its complement is \(A_1 = \bigcup \{A(f, \gamma): \gamma > 1\}\). Similarly, \(A_2 = \bigcup \{A(f, \gamma): \gamma < 1\}\) is open and \(A_1 \cap A_2 = \emptyset\). Since \(\mathcal{D}_1 - A(f, 1) = A_1 \cup A_2\) the result is clear.

**References**