ON THE REPRESENTATION OF LATTICES BY MODULES

BY

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ABSTRACT. For a commutative ring \( R \) with unit, a lattice \( L \) is "representable by \( R \)-modules" if \( L \) is embeddable in the lattice of submodules of some unitary left \( R \)-module. A procedure is given for generating an infinite first-order axiomatization of the class of all lattices representable by \( R \)-modules. Each axiom is a universal Horn formula for lattices. The procedure for generating the axioms is closely related to the ring structure, and is "effective" in the sense that many nontrivial axioms can be obtained by moderate amounts of computation.

1. Introduction. Given a ring \( R \) with unit, let \( \Gamma(M; R) \) denote the lattice of submodules of a unitary left \( R \)-module \( M \). We may also regard \( \Gamma(M; R) \) as the lattice of congruences of \( M \) [3, Theorem 1, p. 159]. These lattices are among the most important examples of modular lattices. Let \( L(R) \) denote the class of all lattices representable by \( R \)-modules, that is, lattices \( L \) embeddable in \( \Gamma(M(L); R) \) for some \( R \)-module \( M(L) \). (All rings will have a unit and all modules will be unitary left modules hereafter.)

Several authors have considered \( L(R) \) from the model theory viewpoint. The classes \( L(R) \) are particular cases of a general class of models considered by B. M. Schein [18, Main Theorem, p. 15]. Schein's result implies that there exists a set of first-order universal lattice formulas axiomatizing \( L(R) \) for any ring \( R \), not necessarily commutative. Furthermore, it is known that a class of algebras is axiomatizable by universal Horn formulas if and only if it admits isomorphic images, subalgebras, products (including the one-element algebra) and ultraproducts [13], [14]. (For lattices as algebras, a universal Horn formula is the universal closure of either a lattice identity or a formula of the form:

\[
[(e_1 = e_2) \& (e_3 = e_4) \& \cdots \& (e_{2n-3} = e_{2n-2})] \Rightarrow (e_{2n-1} = e_{2n}),
\]

for lattice polynomials \( e_1, e_2, \ldots, e_{2n} \) in some denumerable set of variables.)

It is easily seen that \( L(R) \) admits isomorphic images, sublattices and the trivial lattice, for any \( R \). If \( L_i \) is embeddable in \( \Gamma(M_i; R) \) for all \( i \in I \), then \( \Pi_{i \in I} L_i \) is embeddable in \( \Gamma(\Pi_{i \in I} M_i; R) \) in an obvious way, and so \( L(R) \) admits products. Since Schein's result implies that \( L(R) \) admits ultraproducts, it follows that \( L(R) \)
is first-order axiomatizable by universal Horn formulas. C. Herrmann and W. Poguntke gave a general class of models admitting ultraproducts; their technique also shows that \( L(R) \) admits ultraproducts for any \( R \) [6, Theorem 6, p. 5]. Also, Herrmann and Poguntke [6, Theorem 3, p. 2] show that \( L(R) \) is not finitely first-order axiomatizable for various rings \( R \), in particular, for the unitary subrings of the field \( \mathbb{Q} \) of rationals. (They construct a lattice in \( L(\mathbb{Q}) \) which is an ultraproduct of certain lattices, none of which is representable by \( \mathbb{Z} \)-modules (abelian groups).) M. Makkai and G. McNulty [17] considered the method of Schein applied specifically to classes \( L(R) \), and obtained some further results. In [17, Corollary 2], they use the method of W. Craig [4] to assert the existence of a primitive recursive set of universal Horn formulas axiomatizing \( L(R) \) if \( R \) is defined on a recursive set and has recursive ring operations. (B. M. Schein's method of [18] can be used to prove that \( L(R) \) has a recursive axiomatization by universal formulas if \( R \) is recursive, and he observes that the result still holds if \( R \) is a finitely-presented ring.) In [17, Theorem 3], it is shown that \( L(R) \) depends only on the set of finite systems of equations satisfiable in \( R \). By a finite system of equations is meant a set of formal ring equations, each of form \( u_1 + u_2 = u_3 \) or \( u_1 u_2 = u_3 \), where each term \( u_i \) is either a variable or 0 or 1. The system is satisfiable in \( R \) if some assignment of elements of \( R \) to the variables causes every equation to hold in \( R \), with 0 and 1 interpreted as usual. In [6] and [17], there are also related axiomatization results for classes \( L(K) \) of lattices \( L \) embeddable in some \( \Gamma(M(L); R(L)) \), where \( R(L) \) belongs to a given first-order axiomatizable class \( K \) of rings and \( M(L) \) is some \( R(L) \)-module.

Our main theorem, giving a Horn formula axiomatization of \( L(R) \) for commutative \( R \), requires a much longer proof than the model theory proofs. However, the result is more than an existence proof: individual axioms with desired properties can often be generated by reasonable amounts of computation. More important, the procedure generating the axiomatization of \( L(R) \) is closely related to the ring structure of \( R \), and so may yield further insights into the connection between rings \( R \) and lattices representable by \( R \)-modules.

Our approach adapts techniques from the classical coordinatization theory for projective geometries [1, Chapter 2] and for complemented modular lattices [20]. However, the representations are not obtained by direct constructions as in the classical theory, but indirectly via the embedding theorem for small abelian categories [15], [5]. In [7], a small abelian category \( A_L \) is constructed from any modular lattice \( L \) having a smallest element 0 and satisfying the property that, for each \( x \) in \( L \), there exist \( y \) and \( z \) in \( L \) such that:

\[
x \land y = x \land z = y \land z = 0, \quad x \lor y = x \lor z = y \lor z.
\]

(If this is so, we say that every element of \( L \) "can be tripled" and that \( L \) is an...
"abelian" lattice. In [7], a slightly more general definition of "abelian" lattice was given.) Using the embedding theorem for $A_L$, we obtain [7, Theorem 4.3, p. 182]: A lattice is representable by abelian groups if and only if it is embeddable in an interval sublattice of some abelian lattice. The construction of $A_L$ resembles the regular ring construction in the proof of von Neumann's coordinatization theorem [20, Theorem 14.1, p. 208]: Every complemented modular lattice having a homogeneous basis of order $n \geq 4$ is isomorphic to the lattice of principal right ideals of some regular ring $R$. The sets $S(A, B)$ of [7, p. 163] used to construct abelian category morphisms are a "relativized" version of the sets $L_{ij}$ [20, p. 95] used by von Neumann to construct the ring elements. Negative graphs of homomorphisms are used in both theories (see [20, pp. 133, 148] and [7, p. 157]). This leads to the same formula (von Staudt's multiplication) for ring multiplication in [20] and abelian category composition in [7]. Then similar associativity results are obtained (compare [20, Lemmas 5.1, 5.2, pp. 117–118] with [7, 3.5, 3.6, pp. 164–166]). The transitivity of perspectivity in [20, Theorem 2.2, p. 265] is closely related to the correspondence between projectivities and category isomorphisms [7, 3.27, p. 179]. However, the complementation axiom is not required in [7], so the theory is not restricted to regular rings. The "tripling" property used to define "abelian" lattice may be regarded as a strong form of the von Neumann requirement for a homogeneous basis. (Unfortunately, these connections were not pointed out to the author in time to cite von Neumann's work in [7].)

Many other authors have considered coordinatization problems since [20], but we will only mention the work of B. Jónsson related to the representation of modular lattices. In [10], Jónsson gives a lattice identity equivalent to Desargues' Theorem for a projective plane. This "Arguesian" identity is satisfied in every lattice isomorphic to a lattice of commuting equivalence relations on some set [10, Lemma 2.1, p. 196]. It is not satisfied in every modular lattice [10, Theorem 2.3, p. 198]. As a corollary, it follows that not every modular lattice is representable by abelian groups. (Note that a lattice representable by R-modules is representable by abelian groups and by commuting equivalence relations.) In [11, Theorem 2.14, pp. 308–309], Jónsson shows that a complemented modular lattice satisfying the Arguesian identity is representable by abelian groups. In the same paper [11, Theorem 3.6, p. 313], there is an example of an Arguesian (modular) lattice that is not representable by abelian groups. In [12, Theorem 2, p. 457], he gives an infinite family of universal lattice Horn formulas which are satisfied in a lattice if and only if it is representable by commuting equivalence relations.

The proof of our main theorem rests on a complex construction and lengthy computations verifying its properties. We consider this construction separately, in the third section. In the second section, the procedure for generating the
axiomatization of $L(R)$ is described and motivated, the main theorem and related results are stated, and it is demonstrated that the main theorem follows from the asserted properties of the construction mentioned above.

This paper documents the main result announced in [8]. Other results announced in [8] and more recent results are given in [9]. In particular, we state a fundamental result [9, Theorem 1] below. Let $R$-Mod denote the abelian category of $R$-modules and $R$-linear maps. For any infinite cardinal $m$, let $R$-Mod$(m)$ denote the exact full subcategory of $R$-Mod of all $R$-modules having cardinality less than $m$. Then for any rings with unit $R$ and $S$, $L(R) \subset L(S)$ if and only if there exists an exact embedding functor $R$-Mod$(m) \rightarrow S$-Mod for every infinite cardinal number $m$.

It is an open question whether $L(R) \subset L(S)$ is equivalent to existence of an exact embedding functor $R$-Mod $\rightarrow S$-Mod in all cases.

2. The main theorem. Let $R$ be a commutative ring with unit. Our first goal is to define the procedure which generates an infinite axiomatization of $L(R)$. Essentially, we define certain configurations, called "$R$-frames", including a special one called the "initial" $R$-frame. Each $R$-frame determines a certain universal lattice Horn formula. Four operations are defined by which $R$-frames can be modified. A Horn formula is "$R$-constructible" if it is determined by an $R$-frame which can be obtained from the initial $R$-frame by a finite number of operations of these four types. The $R$-constructible Horn formulas are the axioms for $L(R)$. In the following, the passages labelled "procedure" contain the formal definitions, and the passages labelled "interpretation" explain the intended meanings of the definitions.

Procedure. Two denumerably infinite sets of variables are introduced: $B = \{b_0, b_1, b_2, \ldots \}$ and $X = \{x_0, x_1, x_2, \ldots \}$. Let $F_R(B)$ denote the free $R$-module with generating set $B$. Its elements are sums $\sum_{i \geq 0} r_ib_i$ for sequences $r_0, r_1, r_2, \ldots$ in $R$ such that at most finitely many terms $r_i$ are nonzero. Let $L_P(X)$ denote the set of lattice polynomials generated by the set of variables $X$.

Interpretation. Let $M$ be an $R$-module. The variables of $B$ will correspond to elements of $M$, and the variables of $X$ will correspond to submodules of $M$. The special variable $x_0$ will act like the trivial submodule $0$ of $M$.

Procedure. An "$R$-frame" is an ordered triple $(\Psi, G, \alpha)$ such that $\Psi$ is a formula

$$(e_1 = e_2) \& (e_3 = e_4) \& \cdots \& (e_{2n-1} = e_{2n}),$$

for lattice polynomials $e_1, e_2, \ldots, e_{2n}$ in $L_P(X)$, $G$ is a finite (possibly empty) subset of $F_R(B)$, and $\alpha$ is a function $B_0 \rightarrow L_P(X)$ for some finite $B_0$. $\{b_0\} \subset B_0 \subset B$, such that $b_k$ is in $B_0$ if $b_k$ has nonzero coefficient $r_k$ in any $\sum_{i \geq 0} r_ib_i$ in $G$. Let Domain$(\alpha)$ denote the domain of $\alpha$ hereafter. The Horn formula "determined by"
The "initial" $R$-frame is $\langle \Psi_0, \emptyset, \alpha_0 \rangle$, where $\Psi_0$ denotes the formula $x_0 = x_0$, $\text{Dom}(\alpha_0) = \{b_0\}$ and $\alpha_0(b_0) = x_0$. (The formula determined by the initial $R$-frame is $(x_0 = x_0) \implies (x_0 = x_0)$.)

**Interpretation.** Suppose $f: X \to \Gamma(M; R)$ is a function, assigning submodules of $M$ to variables in $X$. There is a unique function $\bar{f}: LP(X) \to \Gamma(M; R)$ extending $f$ and preserving meets and joins. Let $\langle \Psi, G, \alpha \rangle$ be an $R$-frame as described above, and say that $f$ "satisfies" $\Psi$ if $\bar{f}(e_{2k-1}) = \bar{f}(e_{2k})$ for every conjunct $e_{2k-1} = e_{2k}$ of $\Psi$. This is just the normal use of $\Psi$ as the hypothesis of a Horn formula.

The terms $G$ and $\alpha$ form a system of constraints, which determines a submodule of $M$ if $f$ is given. Let $M^B$ denote the $R$-module of all functions $B \to M$, regarded as assignments of elements of $M$ to variables of $B$. An element $g = \Sigma_{i>0} r_i b_i$ of $G$ is interpreted as the $R$-linear equation $g = 0$ in $M$. More precisely, the "$R$-linear set" $g^*$ is the submodule of $M^B$ given by

$$g^* = \left\{ h \in M^B : \sum_{i \geq 0} r_i h(b_i) = 0 \right\}.$$ Since only finitely many coefficients $r_i$ are nonzero, the sum above is well defined.

The "box" $f_*(\alpha)$ determined by $\alpha$ and $f: X \to \Gamma(M; R)$ is the submodule of $M^B$ given by

$$f_*(\alpha) = \{ h \in M^B : h(b_k) = \bar{f}(\alpha(b_k)) \text{ if } b_k \in \text{Dom}(\alpha) \}.$$ That is, $\alpha$ is regarded as formally specifying membership of $b_k$ in $\alpha(b_k)$. Given $f: X \to \Gamma(M; R)$ and $h: B \to M$, $\alpha$ becomes a conjunction of statements about membership of elements of $M$ in submodules of $M$.

Suppose $G = \{g_1, g_2, \ldots, g_n\}$. The "extended solution set" $\mu_0(G, \alpha, f)$ is the submodule of $M^B$ given by

$$\mu_0(G, \alpha, f) = f_*(\alpha) \land g_1^* \land g_2^* \land \cdots \land g_n^*,$$

the intersection of a box and finitely many $R$-linear sets. Let $\pi_0: M^B \to M$ be the projection given by $\pi_0(h) = h(b_0)$. The "solution set" $\mu(G, \alpha, f)$ is $\pi_0[\mu_0(G, \alpha, f)]$, a submodule of $M$. That is, the "extended solutions" are the assignments of elements of $M$ to variables in $B$ satisfying the constraints $G$ and $\alpha$, and the "solutions" are elements of $M$ which can be assigned to $b_0$ as part of an extended solution.
**Procedure.** Suppose \( \langle \Psi, G, \alpha \rangle \) is an \( R \)-frame, \( b_k \) is in \( \text{Dom}(\alpha) \), and \( b_p \) and \( b_q \) are distinct variables not in \( \text{Dom}(\alpha) \). Then \( \langle \Psi_1, G_1, \alpha_1 \rangle \) is a “union augmentation” of \( \langle \Psi, G, \alpha \rangle \) if

\[
\Psi_1 = \Psi \cup (\alpha(b_k) \subseteq x_p \lor x_q),
G_1 = G \cup \{b_k - b_p - b_q\}, \quad \text{Dom}(\alpha_1) = \text{Dom}(\alpha) \cup \{b_p, b_q\},
\alpha_1(b_p) = x_p, \quad \alpha_1(b_q) = x_q \quad \text{and} \quad \alpha_1(b_i) = \alpha(b_i) \quad \text{for} \quad b_i \in \text{Dom}(\alpha).
\]

A union augmentation of an \( R \)-frame is an \( R \)-frame.

**Interpretation.** Union augmentation abstracts the principle that join equals sum in \( \Gamma(M; R) \). (“If \( b_k \in x_p \lor x_q \), then there exist \( b_p \in x_p \) and \( b_q \in x_q \) such that \( b_k - b_p - b_q = 0 \).”) We need only the following:

2.1. Suppose \( \langle \Psi, G, \alpha \rangle \) is an \( R \)-frame and \( \langle \Psi_1, G_1, \alpha_1 \rangle \) is a union augmentation of it using \( b_k - b_p - b_q \). Let \( f: X \rightarrow \Gamma(M; R) \) satisfy \( \Psi_1 \), such that \( \bar{f}(x_0) = \bar{f}(\alpha(b_0)) = \mu(G, \alpha, f) \). Then \( \bar{f}(x_0) = \bar{f}(\alpha_1(b_0)) = \mu(G_1, \alpha_1, f) \).

**Proof.** Assume the hypotheses. Now \( p \) and \( q \) are nonzero, because \( b_p \) and \( b_q \) are not in \( \text{Dom}(\alpha) \), and so \( \alpha(b_0) = \alpha_1(b_0) \) and \( \mu(G_1, \alpha_1, f) \subset \pi_0[f(x_0)] = \bar{f}(\alpha_1(b_0)) = \bar{f}(x_0) \). Let \( v \in \bar{f}(x_0) = \mu(G, \alpha, f) \), so there exists \( h: B \rightarrow M \) in \( \mu_0(G, \alpha, f) \) such that \( h(b_0) = v \). Since \( h \in f_*(\alpha) \), \( h(b_k) \in \bar{f}(\alpha(b_k)) \). Now \( \alpha(b_k) \subseteq x_p \lor x_q \) is a conjunct of \( \Psi_1 \), so \( \bar{f}(\alpha(b_k)) \subset \bar{f}(x_p) \lor \bar{f}(x_q) \). Choose \( v_1 \in \bar{f}(x_p) \) and \( v_2 \in \bar{f}(x_q) \) such that \( h(b_k) = v_1 + v_2 \). Let \( h_1: B \rightarrow M \) be given by \( h_1(b_p) = v_1, \quad h_1(b_q) = v_2 \quad \text{and} \quad h_1(b_i) = h(b_i) \quad \text{for} \quad i \neq p, q \). If \( g = \Sigma_{j \geq 0} \, r_j b_j \) is in \( G \), then \( r_p = r_q = 0 \) by the definition of \( R \)-frame, since \( b_p \) and \( b_q \) are not in \( \text{Dom}(\alpha) \). So, \( \Sigma_{j \geq 0} \, r_j h_1(b_j) = \Sigma_{j \geq 0} \, r_j h(b_j) = 0 \), and \( h_1 \in g^* \). Using the assumptions on \( h, v_1 \) and \( v_2 \), we can show that \( h_1 \in f_*(\alpha_1) \land (b_k - b_p - b_q)^* \). So, \( h_1 \) is in \( \mu_0(G_1, \alpha_1, f) \), and \( v = h_1(b_0) \) is in \( \mu(G_1, \alpha_1, f) \). Therefore, \( \bar{f}(x_0) \subset \mu(G_1, \alpha_1, f) \), completing the proof.

**Procedure.** Suppose \( \langle \Psi, G, \alpha \rangle \) is an \( R \)-frame, \( b_k \) is not in \( \text{Dom}(\alpha) \), and \( g = \Sigma_{i \geq 0} \, r_i b_i \) is in \( F_R(B) \) such that \( r_k = 1 \) and \( r_i = 0 \) for all \( b_i \) not in \( \text{Dom}(\alpha) \cup \{b_k\} \). If \( r_i \neq 0 \) for some \( b_i \) in \( \text{Dom}(\alpha) \), let \( \rho_0(g, b_k, \alpha) \) denote the join in \( LP(X) \) of all terms \( \alpha(b_i) \) such that \( r_i \neq 0 \) and \( i \neq k \). Otherwise, let \( \rho_0(g, b_k, \alpha) \) denote the variable \( x_\omega \). Then \( \langle \Psi, G_2, \alpha_2 \rangle \) is a “defined variable augmentation” of \( \langle \Psi, G, \alpha \rangle \) if

\[
G_2 = G \cup \{g\}, \quad \text{Dom}(\alpha_2) = \text{Dom}(\alpha) \cup \{b_k\},
\alpha_2(b_k) = \rho_0(g, b_k, \alpha) \quad \text{and} \quad \alpha_2(b_i) = \alpha(b_i) \quad \text{for all} \quad b_i \in \text{Dom}(\alpha).
\]

A defined variable augmentation of an \( R \)-frame is an \( R \)-frame.

**Interpretation.** If certain elements of an \( R \)-module belong to specified submodules, then any \( R \)-linear combination of the elements belongs to the join of the submodules. Defined variable augmentation and the “constraint reduction” operation discussed later both abstract the above principle.
2.2. Suppose \( \langle \Psi, G, \omega \rangle \) is an R-frame and \( \langle \Psi, G_2, \alpha_2 \rangle \) is a defined variable augmentation of it using \( g = \sum_{i \geq 0} r_i b_i \) such that \( b_k \) is not in \( \text{Dom}(\alpha) \) and \( r_k = 1 \). Let \( f: X \rightarrow \Gamma(M; R) \) satisfy \( \Psi \), and suppose that \( \bar{f}(x_0) = \bar{f}(b_0) = \mu(G, \alpha, f) \). Then \( \bar{f}(x_0) = \bar{f}(b_0) = \mu(G_2, \alpha_2, f) \).

Proof. Assume the hypotheses. Then \( \mu(G_2, \alpha_2, f) \subset \bar{f}(b_0) = \bar{f}(x_0) \), as in 2.1. For \( v \in \bar{f}(x_0) \), choose \( h: B \rightarrow M \) in \( \mu_0(G, \alpha, f) \) such that \( h(b_0) = v \). Let \( h_2: B \rightarrow M \) be given by \( h_2(b_k) = h(b_k) - \sum_{i \neq k} r_i h(b_i) \) and \( h_2(b_i) = h(b_i) \) for \( i \neq k \). As before, \( h_2 \in g_0^* \) for \( g_0 = \sum_{j > 0} s_j b_j \) in \( G \), because \( b_k \) not in \( \text{Dom}(\alpha) \) implies \( s_k = 0 \), and \( h \in g_0^* \) was assumed. Also, \( h_2 \in g_2^* \) because

\[
\sum_{j > 0} r_j h_2(b_j) = \sum_{j = 0}^{k-1} r_j h(b_j) + h_2(b_k) + \sum_{j < k} r_j h(b_j) = 0.
\]

To prove \( h_2 \in f_*(\alpha_2) \), consider two cases. If \( g \neq b_k \), delete all terms with zero coefficient and let \( g = b_k + \sum_{j = 1}^{t} r_j b_j \) in \( F_R(B) \) (\( t \geq 1 \), and \( k, i_1, i_2, \ldots, i_t \) are distinct positive integers). Then \( h_2 \in g_2^* \) implies

\[
h_2(b_k) = -\sum_{j = 1}^{t} r_j h_2(b_j) = -\sum_{j = 1}^{t} r_j h(b_j).
\]

Since \( h(b_j) \in \bar{f}(b_j) \), \( h_2(b_k) \in \bigvee_{j=1}^{t} \bar{f}(b_j) = \bar{f}_0(g, b_k, \alpha) \). So, \( h_2 \in f_*(\alpha_2) \). If \( g = b_k \), \( h_2(b_k) = 0 \in \bar{f}(x_0) = \bar{f}_0(g, b_k, \alpha) \), and again \( h_2 \in f_*(\alpha_2) \). Therefore, \( h_2 \in \mu_0(G_2, \alpha_2, f) \), and so \( v \in \mu(G_2, \alpha_2, f) \). But then we have \( \bar{f}(x_0) \subset \mu(G_2, \alpha_2, f) \), completing the proof.

Procedure. Suppose \( \langle \Psi, G, \omega \rangle \) is an R-frame, \( G = \{g_1, g_2, \ldots, g_n\} \) and \( g = \sum_{i=1}^{n} r_i g_i \) is in \( F_R(B) \) for some \( r_1, r_2, \ldots, r_n \) in \( R \). Then \( \langle \Psi, G_3, \alpha \rangle \) is called a “linear combination augmentation” of \( \langle \Psi, G, \alpha \rangle \) if \( G_3 = G \cup \{g\} \). A linear combination augmentation of an R-frame is an R-frame.

Interpretation. This operation abstracts the usual principle that a solution of a system of R-linear equations also satisfies any R-linear combination of system equations.

2.3. Suppose \( \langle \Psi, G, \omega \rangle \) is an R-frame and \( \langle \Psi, G_3, \alpha \rangle \) is a linear combination augmentation of it such that \( G_3 = G \cup \{g\} \). Let \( f: X \rightarrow \Gamma(M; R) \) satisfy \( \Psi \), and suppose that \( \bar{f}(x_0) = \bar{f}(b_0) = \mu(G_3, \alpha, f) \). Then \( \bar{f}(x_0) = \bar{f}(b_0) = \mu(G_3, \alpha, f) \).

Proof. Given \( g_1, g_2 \) in \( F_R(B) \) and \( r \) in \( R \), we can check that \( (rg_1)^* \supset g_1^* \) and \( (g_1 + g_2)^* \supset g_1^* \land g_2^* \). If \( g = \sum_{i=1}^{n} r_i g_i \) for \( G = \{g_1, g_2, \ldots, g_n\} \), then \( g_1^* \land g_2^* \land \cdots \land g_n^* \subset g^* \). But then \( \mu_0(G_3, \alpha, f) = \mu_0(G, \alpha, f) \), and the result follows.

Procedure. Suppose \( \langle \Psi, G, \omega \rangle \) is an R-frame and \( g = \sum_{i>0} r_i b_i \) is in \( G \) such that \( r_k = 1 \). Define \( \rho_0(g, b_k, \alpha) \) in \( LP(X) \) as before: If \( g \neq b_k \), let \( \rho_0(g, b_k, \alpha) \)
denote the join in \(LP(X)\) of all terms \(\alpha(b_i)\) such that \(i \neq k\) and \(r_i \neq 0\). If \(g = b_k\), let \(\rho_0(g, b_k, \alpha)\) denote \(x_\omega\). Then \(\langle \Psi, G, \alpha_4 \rangle\) is a "constraint decrease" of \(\langle \Psi, G, \omega \rangle\) if

\[
\text{Dom}(\alpha_4) = \text{Dom}(\alpha), \quad \alpha_4(b_k) = \alpha(b_k) \land \rho_0(g, b_k, \alpha) \quad \text{and} \\
\alpha_4(b_i) = \alpha(b_i) \quad \text{for all } b_i \in \text{Dom}(\alpha), i \neq k.
\]

A constraint decrease of an R-frame is an R-frame.

**Interpretation.** Constraint decrease is the key operation, allowing modification of \(\alpha(b_0)\).

2.4. Suppose \(\langle \Psi, G, \omega \rangle\) is an R-frame and \(\langle \psi, G, \alpha_4 \rangle\) is a constraint decrease of it using \(g = \sum_{i \geq 0} r_i b_i \in G\) at \(b_k\). So, \(r_k = 1\). Let \(f: X \to L\) satisfy \(\Psi\), and \(\bar{f}(x_0) = \bar{f}(b_0) = \mu(G, \alpha, f)\). Then \(\bar{f}(x_0) = \bar{f}(b_0) = \mu(G, \alpha_4, f)\).

**Proof.** Assume the hypotheses. Since \(\alpha_4(b_0)\) is either \(\alpha(b_0)\) or the meet of \(\alpha(b_0)\) and some other lattice polynomial, we have \(\mu(G, \alpha_4, f) \subseteq \bar{f}(b_0) \subseteq \bar{f}(x_0)\). For \(v \in \bar{f}(x_0)\), choose \(h: B \to M\) in \(\rho_0(G, \alpha, f)\) such that \(h(b_0) = v\) as usual. To prove that \(h\) is in \(\mu_0(G, \alpha_4, f)\), it suffices to prove that \(h(b_k) \in \bar{f}(b_k) \land \bar{f}(b_k)\) is known, and \(h \in g^*\) implies \(h(b_k) \in \bar{f}(b_k)\) as in 2.2. This proves the proposition.

**Procedure.** A sequence \(u_1, u_2, \ldots, u_n\) \((n \geq 1)\) of R-frames is called "proper" if \(u_1\) is the initial R-frame and, for each \(i\) such that \(1 \leq i < n\), \(u_{i+1}\) is either a union augmentation, a defined variable augmentation, a linear combination augmentation or a constraint decrease of \(u_i\). The formula determined by the last term \(u_n\) of a proper sequence is called "R-constructible". That is, if \(u_n = \langle \Psi, G, \omega \rangle\), then the universal closure of \(\Psi \implies (x_0 = \alpha(b_0))\) is an R-constructible universal Horn formula.

2.5. If a lattice \(L\) is representable by R-modules, then every R-constructible Horn formula is satisfied in \(L\).

**Proof.** Let \(M\) be an R-module, and suppose that \(u_1, u_2, \ldots, u_n\) is a proper sequence of R-frames and \(u_n = \langle \Psi, G, \omega \rangle\). An induction on \(n\) proves that \(\bar{f}(x_0) = \bar{f}(b_0) = \mu(G, \alpha, f)\) if \(f: X \to \Gamma(M; R)\) satisfies \(\Psi\). (If \(n = 1\), so \(u_n\) is the initial R-frame, this is easily proved. The induction step follows from 2.1, 2.2, 2.3 and 2.4.) So, the universal closure of the formula \(\Psi \implies (x_0 = \alpha(b_0))\) is satisfied in \(\Gamma(M; R)\). It then follows that every R-constructible universal Horn formula is satisfied in a lattice \(L\) that is representable by R-modules.

We have proved the less difficult half of our result, which can now be fully stated.

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Main theorem. Let $R$ be a nontrivial commutative ring with unit. A lattice $L$ is representable by $R$-modules if and only if every $R$-constructible universal Horn formula is satisfied in $L$.

Two examples of Horn formula generation by our procedure are given below:

Example 1. Consider the "Fano" lattice identity of $R$. Wille [21, p. 134]:

\[(x_1 \vee x_2) \wedge (x_3 \vee x_4) \subseteq [(x_1 \vee x_3) \wedge (x_2 \vee x_4)] \vee [(x_1 \vee x_4) \wedge (x_2 \vee x_3)].\]

If the above identity is satisfied in the projective geometry $\Gamma(M; F)$ of subspaces of a vector space $M$ of dimension three or more over a division ring $F$, then the Fano postulate fails for every quadrangle in $\Gamma(M; F)$, and so $\text{char}(F) = 2$ ([21], [2, pp. 37–38]). If $R$ is a ring with unit, not necessarily commutative, and $\text{char}(R) \neq 2$, then there exists a lattice in $L(R)$ for which the above identity is not satisfied. For example, let $R^3$ denote the free $R$-module generated by $y_1, y_2$ and $y_3$, and observe that the above identity fails if $x_1 = Ry_1$, $x_2 = R(y_1 - y_2)$, $x_3 = R(y_2 - y_3)$ and $x_4 = Ry_3$ in $\Gamma(R^3; R)$. We now show that the above identity is satisfied in every lattice in $L(R)$ if $\text{char}(R) = 2$, for any ring $R$. Let $Z_2$ denote the ring of integers modulo 2. Beginning with the initial $Z_2$-frame, introduce two union augmentations using

\[g_1 = b_0 - b_1 - b_2 = b_0 + b_1 + b_2 \quad \text{and} \quad g_2 = b_0 - b_3 - b_4 = b_0 + b_3 + b_4.\]

Make defined variable augmentations at $b_5$ using $g_3 = b_1 + b_3 + b_5$ and at $b_6$ using $g_4 = b_1 + b_4 + b_6$. Then add $g_5 = g_1 + g_2 + g_3 = b_2 + b_4 + b_5$, $g_6 = g_1 + g_2 + g_4 = b_2 + b_3 + b_6$, and $g_7 = g_2 + g_3 + g_4 = b_0 + b_5 + b_6$ by linear combination augmentations. Finally, perform constraint decreases at $b_5$ using $g_5$, at $b_6$ using $g_4$ and at $b_0$ using $g_7$, in that order. The $Z_2$-constructible Horn formula determined by the last term of this proper sequence is

\[(x_0, x_1, x_2, x_3, x_4)([x_0 = x_0] \wedge [x_0 \subseteq x_1 \vee x_2] \wedge [x_0 \subseteq x_3 \vee x_4])
\Rightarrow (x_0 \subseteq [(x_1 \vee x_3) \wedge (x_2 \vee x_4)] \vee [(x_1 \vee x_4) \wedge (x_2 \vee x_3)]).\]

This Horn formula is equivalent to the Fano identity above in any lattice. If $\text{char}(R) = 2$, then $L(R) = L(Z_2)$ [8], [9, Theorem 5(6)], and so the Fano identity is satisfied in every lattice in $L(R)$ by 2.5.

Example 2. B. Jonsson’s Arguesian lattice identity ([10], cited in [3, p. 109, #7]) is lattice equivalent to a Horn formula which is $R$-constructible for any commutative $R$. Beginning with the initial $R$-frame, make three union augmentations using $g_1 = b_0 - b_1 - b_4$, $g_2 = b_0 - b_2 - b_5$ and $g_3 = b_0 - b_3 - b_6$. Then make three defined variable augmentations using $g_4 = -b_1 + b_2 + b_7$, $g_5 = -b_1 + b_3 + b_8$ and $g_6 = -b_2 + b_3 + b_9$. Now introduce five linear combina-
tion augmentations, obtaining five new equations

\[ g_7 = g_1 + g_2 + g_4, \quad g_8 = -g_1 + g_3 + g_5, \quad g_9 = -g_2 + g_3 + g_6, \]
\[ g_{10} = g_4 - g_5 + g_6 \quad \text{and} \quad g_{11} = -g_4. \]

Perform constraint decreases at \( b_7 \) using \( g_7 \), at \( b_8 \) using \( g_8 \), at \( b_9 \) using \( g_9 \), at \( b_7 \) using \( g_{10} \), at \( b_1 \) using \( g_{11} \), at \( b_4 \) using \( g_7 \) and at \( b_0 \) using \( g_1 \), in that order. The Horn formula determined by the last term of this proper sequence is equivalent to the Arguesian identity in any lattice.

To complete the main theorem proof, we must use the complex construction mentioned previously. At this point, we will outline this construction \( M(K; R) \) and describe its relevant properties. Consider first a generalization of \cite[4.2, 4.3, pp. 181-183]{}. Let \( M \) be an \( R \)-module, \( N \) the set of positive integers, and \( M^N \) the \( R \)-module of all functions \( N \rightarrow M \). Say that \( M_0 \) in \( \Gamma(M^N; R) \) has “finite support” if there exists \( n_0 \) in \( N \) such that \( h \in M_0 \) and \( n > n_0 \) imply \( h(n) = 0 \). That is, every element of \( M_0 \) has all coordinates zero outside a certain designated finite set of coordinates. Let \( \Gamma_f(M^N; R) \) denote the set of submodules of \( M^N \) having finite support. Just as in \cite[4.2]{}, we can verify that \( \Gamma_f(M^N; R) \) is an ideal of \( \Gamma(M^N; R) \) and is an abelian lattice. Define a lattice embedding \( \psi_M: \Gamma(M; R) \rightarrow \Gamma_f(M^N; R) \) by

\[ \psi_M(M_1) = \{ h \in M^N : h(1) \in M_1, h(n) = 0 \text{ for } n > 1 \}, \]

since \( \psi_M(M_1) \) as defined above has finite support.

Let \( K \) be a \((0, 1)\) lattice, that is, a lattice with a smallest element \( 0 \) and a largest element \( 1 \). We construct a lattice \( M(K; R) \) abstracting \( \Gamma_f(M^N; R) \) together with a map \( \psi: K \rightarrow M(K; R) \) which abstracts the embedding \( \psi_M: \Gamma(M; R) \rightarrow \Gamma_f(M^N; R) \). The points of \( M(K; R) \) are equivalence classes of constraint systems. These new constraint systems \((G, \alpha)\) are slightly different from those appearing in \( R \)-frames \( \langle \Psi, G, \alpha \rangle \). In particular, the new lattice constraint functions \( \alpha \) have values in \( K \) rather than in \( LP(X) \). Also, new variables are introduced so that the solution sets (given an embedding \( \iota: K \rightarrow \Gamma(M; R) \) such that \( \iota(0) = 0 \)) will be elements of \( \Gamma_f(M^N; R) \) rather than submodules of \( M \). The equivalence relation for the constraint systems is generated by seven rules, four of them resembling the \( R \)-frame operations. Under the given interpretation, equivalent constraint systems have the same solution set in \( \Gamma_f(M^N; R) \). We now assert the properties of \( M(K; R) \) and \( \psi \) that will be established in the final section.

2.6. For every \((0, 1)\) lattice \( K \), there exist an abelian lattice \( M(K; R) \) and a lattice homomorphism \( \psi: K \rightarrow M(K; R) \) such that \( \psi \) is an embedding if every \( R \)-constructible Horn formula is satisfied in \( K \). Furthermore, for every object \( A \) of \( \mathcal{A}(M(K; R)) \), there exists a unit-preserving ring homomorphism \( \xi_A \) from \( R \) into
the endomorphism ring \( \text{Hom}(A, A) \) in \( A_{M(K; R)} \). Finally, if \( f: A \to B \) in \( A_{M(K; R)} \) and \( r \in R \), then \( f\xi_A(r) = \xi_B(r)f \).

After two preparatory results, we can show that 2.6 suffices to prove the main theorem.

2.7. If a lattice \( L \) satisfies a set \( \Sigma \) of universal lattice Horn formulas, then there exists a \((0, 1)\) lattice \( K \) extending \( L \) such that every formula of \( \Sigma \) is satisfied in \( K \).

**Proof.** Assume the hypotheses. Adjoin a smallest element to \( L \) if it does not already have one. Dually, adjoin a largest element if necessary. The resulting \((0, 1)\) lattice \( K \) extends \( L \). It can be shown that every finitely-generated sub lattice of \( K \) can be embedded in \( L \). But then every formula of \( \Sigma \) must be satisfied in \( K \).

**Definition.** If \( C \) is \( R \)-Mod or is a small abelian category, let \( \Gamma(A; C) \) denote the lattice of subobjects of an object \( A \) of \( C \).

2.8. Let \( C \) be a small abelian category. Suppose that there exist ring homomorphisms \( \xi_A: R \to \text{Hom}_{C^0}(A, A) \) preserving the ring unit, for every object \( A \) of \( C \), and suppose that \( \xi_B(r)f = f\xi_A(r) \) for every \( f: A \to B \) in \( C \) and every \( r \) in \( R \). Then there exists an exact embedding functor \( G: C \to R\text{-Mod} \).

**Proof.** Assume the hypotheses. By the embedding theorem [15], [5], there exists an exact embedding functor \( F: C \to \text{Ab} \), where \( \text{Ab} \) is the category of abelian groups and homomorphisms. For each object \( A \) of \( C \), \( F(A) \) has an additive group structure. We make \( F(A) \) into an \( R \)-module, denoted \( G(A) \), by defining \( rv = (F\xi_A(r))(v) \) for \( r \) in \( R \) and \( v \) in \( F(A) \). That is, \( F\xi_A(r) = r1_{G(A)} \). If \( f: A \to B \) in \( C \), then \( Ff: G(A) \to G(B) \) is \( R \)-linear because \( \xi_B(r)f = f\xi_A(r) \) holds for all \( r \) in \( R \). So, \( G(A) \) and \( Gf = Ff \) determine a functor \( G: C \to R\text{-Mod} \). Since \( F \) is an exact embedding, so is \( G \).

**Outline of main theorem proof.** Assume 2.6. Suppose that every \( R \)-constructible Horn formula is satisfied in \( L \). By 2.7, there exists an embedding \( L \to K \) for a \((0, 1)\) lattice \( K \) such that every \( R \)-constructible Horn formula is satisfied in \( K \). Let \( M \) denote \( M(K; R) \); by 2.6 and 2.8 there exist a lattice embedding \( \psi: K \to M \) and an exact embedding functor \( G: A_M \to R\text{-Mod} \). By restricting the codomain of \( \psi \), we obtain a lattice embedding from \( K \) into the interval sublattice \( M[\psi(0), \psi(1)] \) of \( M \). If \( A \) denotes the object \( \psi(1)/\psi(0) \) of \( A_M \), then there exists a lattice isomorphism \( M[\psi(0), \psi(1)] \to \Gamma(A; A_M) \) by [7, 3.24, p. 178]. The exact embedding functor \( G \) induces an embedding \( \Gamma(A; A_M) \to \Gamma(G(A); R\text{-Mod}) \); see [7, p. 183] for relevant information. So, \( L \) is representable by \( R \)-modules via the following composite of lattice embeddings.
L \rightarrow K \rightarrow M[\psi(0), \psi(1)] \rightarrow \Gamma(A; A_M) \rightarrow \Gamma(G(A); R\text{-mod}).

(Of course, \(\Gamma(G(A); R\text{-Mod})\) is isomorphic to \(\Gamma(G(A); R)\).) This shows that 2.6 implies the reverse implication of the main theorem, and the forward implication was proved in 2.5.

Recall that every interval sublattice of an abelian lattice is representable by abelian groups [7, Theorem 4.3]. But then every \(\mathbb{Z}\)-constructible Horn formula must be satisfied in any abelian lattice, and so all abelian lattices are representable by abelian groups. In fact, we can simplify [7, Theorem 4.3] as follows: A lattice is representable by abelian groups if and only if it is embeddable in some abelian lattice.

3. The construction of \(M(K; R)\). Throughout this section, \(R\) will denote a fixed commutative ring with unit and \(K\) will denote a fixed \((0, 1)\) lattice. To avoid confusion with other zeros, we will denote the smallest element of \(K\) by \(\omega\).

In the following, we will label certain explanatory material as “interpretation”. Although this material is not needed for the formal definitions and calculations, it helps one to understand the ideas motivating the construction.

**Definition.** Let \(V\) be \(\{a_k, b_k: k = 1, 2, 3, \ldots\}\), a set of two denumerably infinite sequences of variables. We will sometimes write \(c_k\) to represent a general element \(a_k\) or \(b_k\) of \(V\). Let \(F_R(V)\) denote the free \(R\)-module with free generating set \(V\). We will represent elements of \(F_R(V)\) by coefficient functions. That is, \(h\) in \(F_R(V)\) will be any function \(h: V \rightarrow R\) such that \(h(v) = 0\) except for at most finitely many \(v\) in \(V\). A function \(\alpha: V \rightarrow K\) is a “lattice constraint function” if \(\alpha(v) = \omega\) except for at most finitely many \(v\) in \(V\). A “constraint system” is a pair \((G, \alpha)\) such that \(G\) is a finite (possibly empty) subset of \(F_R(V)\) and \(\alpha: V \rightarrow K\) is a lattice constraint function. Let \(D(K; R)\) denote the set of all constraint systems.

**Interpretation.** Suppose \(\iota: K \rightarrow \Gamma(M; R)\) is an embedding for some \(R\)-module \(M\), and \(\iota(\omega) = 0\). We can modify the definitions of §2 to interpret \(D(K; R)\). The variables \(a_k\) correspond to coordinate positions in \(M^N\), and the variables \(b_k\) \((k \geq 1)\) are similar to those in §2. The variable \(b_0\) of §2 should now be identified with \(a_1\). We obtain extended solutions in the \(R\)-module \(M^V\) similar to the extended solutions in \(M^B\) in §2, and project the extended solutions to obtain a solution set in \(\Gamma_f(M^N; R)\). More precisely, define the “\(R\)-linear set” \(g^*\) for \(g\) in \(F_R(V)\) by

\[g^* = \left\{h \in M^V: \sum_{k \geq 1} (g(a_k)h(a_k) + g(b_k)h(b_k)) = 0\right\}.

Define the “box” \(\iota_*(\alpha)\) for a lattice constraint function \(\alpha: V \rightarrow K\) by

\[\iota_*(\alpha) = \left\{h \in M^V: h(c_k) \in \iota(\alpha(c_k)), \text{ all } c_k \in V\right\}.

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Define the extended solution set \( \nu_0(G, \alpha, \iota) \) to be the submodule

\[ \iota_\ast(\alpha) \wedge g_1^* \wedge g_2^* \wedge \cdots \wedge g_n^* \]

of \( M^V \), if \( G = \{ g_1, g_2, \cdots, g_n \} \). (If \( G = \emptyset \), \( \nu_0(G, \alpha, \iota) = \iota_\ast(\alpha) \).) Define the solution set to be the projection of the extended solution set. That is, let \( a : \mathbb{N} \to V \) be given by \( a(k) = a_k \), and let \( \nu(G, \alpha, \iota) \) be the submodule \( \{ ha : h \in \nu_0(G, \alpha, \iota) \} \) of \( M^\mathbb{N} \). Because of the assumption \( \iota(\omega) = 0 \), \( \nu(G, \alpha, \iota) \) is in \( \Pi_f(M^\mathbb{N}; R) \). The subsequent definitions and calculations are all motivated by this interpretation of \( D(K; R) \).

**Definition.** The equivalence relation \( E(K; R) \) on \( D(K; R) \) is defined by means of seven “rules of equivalence" \( E_1 \) through \( E_7 \). That is, binary relations \( E_1 \) through \( E_7 \) are defined on \( D(K; R) \), and we say that \( s \) and \( t \) in \( D(K; R) \) are “directly equivalent" if \( s E_i t \) or \( t E_j s \) for some \( i, j \leq 7 \). For \( s \) and \( t \) in \( D(K; R) \), write \( s \sim t \) if there exists a sequence \( s_1, s_2, \cdots, s_n \) (\( n \geq 1 \)) in \( D(K; R) \) such that \( s_1 = s, s_n = t \) and \( s_{i+1} \) is directly equivalent to \( s_i \) for \( 1 \leq i < n \). Then \( E(K; R) = \{ (s, t) : s \sim t \} \) is an equivalence relation on \( D(K; R) \).

For \( G \subset F_R(V) \), say that \( v \) in \( V \) is “unused" in \( G \) if \( g(v) = 0 \) for all \( g \) in \( G \).

If \( g \) is in \( F_R(V) \), \( c_k \) is in \( V \) and \( \alpha : V \to K \), let \( \rho(g, c_k, \alpha) \) denote the join in \( K \) of the finitely many elements \( \alpha(v) \) such that \( v \neq c_k \) and \( g(v) \neq 0 \). If \( g(v) = 0 \) for all \( v \neq c_k \), let \( \rho(g, c_k, \alpha) \) denote \( \omega \).

If \( V_0 \subset V \) and \( g \) is in \( F_R(V) \), let \( \eta(g, V_0) \) denote the element \( h \) of \( F_R(V) \) such that \( h(v) = g(v) \) for \( v \) in \( V_0 \) and \( h(v) = 0 \) for \( v \) in \( V - V_0 \).

In the next paragraphs, assume that \( (G, \alpha) \) and \( (H, \beta) \) are in \( D(K; R) \).

Suppose \( c_k, b_p \) and \( b_q \) are distinct elements of \( V \) such that \( b_p \) and \( b_q \) are unused in \( G \), and suppose that \( x_1 \) and \( x_2 \) are in \( K \) such that \( \alpha(c_k) \subset x_1 \vee x_2 \). Write \( (G, \alpha)E_1(H, \beta) \) if \( H = G \cup \{ c_k - b_p - b_q \}, \beta(b_p) = x_1, \beta(b_q) = x_2 \) and \( \beta(v) = \alpha(v) \) if \( v \neq b_p, b_q \). Then \( (H, \beta) \) is called the “union augmentation of \( (G, \alpha) \) using \( c_k - b_p - b_q \) and \( x_1 \) and \( x_2 \) in \( K \)”, and \( (G, \alpha) \) is called a “union deletion of \( c_k - b_p - b_q \) from \( (H, \beta) \)”.

Suppose \( b_k \) is unused in \( G \) and \( g \) is in \( F_R(V) \) such that \( g(b_k) = 1 \). Write \( (G, \alpha)E_2(H, \beta) \) if \( H = G \cup \{ g \}, \beta(b_k) = \rho(g, b_k, \alpha) \) and \( \beta(v) = \alpha(v) \) if \( v \neq b_k \). Then \( (H, \beta) \) is called a “defined variable augmentation of \( (G, \alpha) \) using \( g \) at \( b_k \)”, and \( (G, \alpha) \) is called a “defined variable deletion of \( g \) from \( (H, \beta) \)”.

Suppose \( g \) in \( G \) and \( c_k \) in \( V \) such that \( g(c_k) = 1 \). Write \( (G, \alpha)E_4(H, \beta) \) if \( H = G \cup \{ c_k \}, \beta(c_k) = \alpha(c_k) \wedge \rho(g, c_k, \alpha) \) and \( \beta(v) = \alpha(v) \) for all \( v \neq c_k \). Then...
(H, β) is called the "constraint decrease of (G, α) at c_k using g", and (G, α) is called a "constraint increase of (H, β) at c_k using g".

Suppose α(c_k) = ω for some c_k in V. Write (G, α)E_5(H, β) if H = \{η(g, V - \{c_k\}): g ∈ G\} and β = α. (That is, g - g(c_k)c_k is in H if g is in G.) Then (H, β) is called the "null variable deletion of (G, α) at c_k", and (G, α) is called a "null variable augmentation of (H, β) at c_k".

Suppose V_0 is a finite nonempty subset of \{b_k: k ≥ 1\} such that \(η(g, V_0) = 0\) for all g in G and \(α(v) = ω\) for all v in V_0. Let G_0 be a finite (possibly empty) subset of FR(V) such that \(η(A, G_0) = h\) for all h in G_0. Write \((G, α)E_6(H, β)\) if \(H = G \cup G_0\) and \(β(v) = α(v)\) for all v not in V_0. Then (H, β) is called an "inessential variables augmentation of (G, α) by G_0 using V_0", and (G, α) is called the "inessential variables deletion of G_0 from (H, β) using V_0".

Let \(ξ\) be a "renumbering function", that is, a permutation of V such that \(ξ(a_k) = a_k\) for all k ≥ 1. Write \((G, α)E_7(ξ, β)\) if \(H = \{g_ξ: g ∈ G\}\) and β = α_ξ. Then (H, β) is called the "renumbering of (G, α) by ξ". Observe that then \((G, α)\) is the renumbering of \((H, β)\) by \(ξ^{-1}\). Let G_ξ denote \(\{g_ξ: g ∈ G\}\) if G ⊆ FR(V) and \(ξ\) is a renumbering function.

This completes the definition of E(K; R). The underlying space of M(K; R) is the quotient set D(K; R)/E(K; R). Let \([s]\) or \([G, α]\) denote the equivalence class modulo E(K; R) of s = (G, α) in D(K; R). For x in K, let \(ψ_x: V → K\) be the lattice constraint function given by \(ψ_x(a_1) = x\) and \(ψ_x(v) = ω\) for v in V - \{a_1\}. Define the function \(ψ: K → M(K; R)\) by \(ψ(x) = [∅, ψ_x]\) for all x in K.

**Interpretation.** Suppose \(ι: K → Γ(M; R)\) is a lattice embedding and \(ι(ω) = 0\). One can easily verify that \((G, α) ~ (H, β)\) in D(K; R) implies \(ν(G, α, ι) = ν(H, β, ι)\) in \(Γ_f(M^N; R)\). That is, equivalent constraint systems have the same solution set. So, it is reasonable to treat the quotient \(D(K; R)/E(K; R)\) as an abstraction of \(Γ_f(M^N; R)\). Furthermore, the earlier assertion that \(ψ\) abstracts \(ψ_M: Γ(M; R) → Γ_f(M^N; R)\) is motivated by the observation that \(ψ_M(ι(x)) = ν(∅, ψ_x, ι)\) for all x in K.

We can now begin the verification of 2.6 with a critical point. The R-constructible Horn formulas are designed to establish the next technical result. From this result it follows that \(ψ: K → M(K; R)\) is one-one if every R-constructible Horn formula is satisfied in K.

**Definition.** For s and t in D(K; R) such that \(sE_i t\) for some i, i ≤ 4, say that t is a "direct reduction" of s. (That is, t is a union augmentation, defined variable augmentation, linear combination augmentation or constraint decrease of s.) If \(s_1, s_2, \cdots, s_m\) for \(m ≥ 1\) is a sequence in D(K; R) such that \(s_{i+1}\) is a direct reduction of \(s_i\), for \(1 ≤ i ≤ m\), say that \(s_1, s_2, \cdots, s_m\) is a "reduction sequence" and that \(s_m\) is a "reduction" of \(s_1\). If s is in D(K; R) and x is in K,
say that \( s \) is “reducible below” \( x \) if there exists a reduction \( t = (H, \beta) \) of \( s \) such that \( \beta(a_1) \subset x \).

3.1. Suppose every \( R \)-constructible formula is satisfied in \( K \). If \( y, z \in K \) and \((\emptyset, \psi_y)\) in \( D(K; R) \) is reducible below \( z \), then \( y \subset z \).

**Proof.** Assume the hypotheses, and let \( t_1, t_2, \cdots, t_m \) be a reduction sequence in \( D(K; R) \) such that \( t_1 = (\emptyset, \psi_y) \) and \( \beta_m(a_1) \subset z \), where \( t_i = (H_i, \beta_i) \) for \( i \leq m \). Given \( h \in F_R(V) \) and \( B_0 \) with \( \{b_0\} \subset B_0 \subset B = \{b_j: j \geq 0\} \), let \( \eta_0(h, B_0) \) denote that \( g \in F_R(B) \) such that \( g(b_0) = h(a_1) \), \( g(b_j) = h(b_j) \) if \( j \geq 1 \) and \( b_j \in B_0 \), and \( g(b_j) = 0 \) otherwise. We intend to define an \( R \)-frame \( u_m = \langle \Psi_m, G_m, \alpha_m \rangle \) and a function \( f_m: X \to K \) satisfying \( \Delta(m) \) below, using induction on \( m \).

**\( \Delta(m) \):** The \( R \)-frame \( u_m = \langle \Psi_m, G_m, \alpha_m \rangle \) is the last term of some proper sequence of \( R \)-frames. If \( g \in F_R(B) \) and \( g \neq 0 \), then \( g \in G_m \) iff there exists \( h \in H_m \) such that \( g = \eta_0(h, \text{Dom}(\alpha_m)) \). For \( j \geq 1 \), if \( b_j \) is unused in \( G_m \), then \( b_j \) is not in \( \text{Dom}(\alpha_m) \) and \( x_j \) does not appear in the formula \( \Psi_m \) or in any lattice polynomial \( \alpha_m(b_k) \) for \( b_k \in \text{Dom}(\alpha_m) \). Also, \( f_m(x_0) = \omega, f_m(x_0) = y, f_m \) satisfies the formula \( \Psi_m, \bar{f}_m \alpha_m(b_0) \subset \beta_m(a_1) \) and \( \bar{f}_m \alpha_m(b_k) \subset \beta_m(b_k) \) if \( k \geq 1 \) and \( b_k \) is in \( \text{Dom}(\alpha_m) \).

For \( m = 1 \), let \( u_1 \) be the initial \( R \)-frame and define \( f_1: X \to K \) by \( f_1(x_0) = y, f_1(x_0) = \omega \) and \( f_1(x_j) = \omega \) for \( j \geq 1 \). Then \( \Delta(1) \) is satisfied.

For the induction step, assume that \( m > 1 \) and that \( u_{m-1} = \langle \Psi_{m-1}, G_{m-1}, \alpha_{m-1} \rangle \) and \( f_{m-1}: X \to K \) have been defined satisfying \( \Delta(m-1) \). Define \( u_m \) and \( f_m \) by cases, using the assumption that \( t_m \) is a direct reduction of \( t_{m-1} \).

Define \( f_m = f_{m-1} \) in all cases except one. The exception occurs when \( t_m \) is a union augmentation of \( t_{m-1} \) using \( c_n - b_p - b_q \) in \( F_R(V) \) and \( z_1 \) and \( z_2 \) in \( K \), and \( c_n = a_1 \) or \( c_n = b_n \) for \( n \geq 1 \) and \( b_n \) in \( \text{Dom}(\alpha_{m-1}) \). In that case, define \( f_m \) by \( f_m(x_p) = z_1, f_m(x_q) = z_2, \) and \( f_m(x_j) = f_{m-1}(x_j) \) for \( j \neq p, q \) or \( \omega \).

In the exceptional case above, let \( u_m \) be the \( R \)-frame union augmentation of \( u_{m-1} \) using \( b_0 - b_p - b_q \) if \( c_n = a_1 \), and let \( u_m \) be the union augmentation of \( u_{m-1} \) using \( b_n - b_p - b_q \) if \( c_n = b_n \) for \( n \geq 1 \) and \( b_n \) in \( \text{Dom}(\alpha_{m-1}) \). If \( t_m \) is the defined variable augmentation of \( t_{m-1} \) using \( h \) at \( b_n \), let \( u_m \) be the \( R \)-frame defined variable augmentation of \( u_{m-1} \) using \( b_n + \eta_0(h, \text{Dom}(\alpha_{m-1})) \) at \( b_n \). If \( t_m \) is the linear combination augmentation of \( t_{m-1} \) using \( h \) and \( g = \eta_0(h, \text{Dom}(\alpha_{m-1})) \neq 0 \), let \( u_m \) be the \( R \)-frame linear combination of \( u_{m-1} \) using \( g \). If \( t_m \) is the constraint decrease of \( t_{m-1} \) using \( h \) at \( c_n \) and \( c_n = a_1 \) or \( c_n = b_n \) for \( n \geq 1 \) and \( b_n \in \text{Dom}(\alpha_{m-1}) \), let \( u_m \) be the \( R \)-frame constraint decrease of \( u_{m-1} \) using \( \eta_0(h, \text{Dom}(\alpha_{m-1})) \) at \( b_0 \) if \( c_n = a_1 \), and let \( u_m \) be the constraint decrease of
\[ u_{m-1} \] at \( b_n \) using \( \eta_0(h, \text{Dom}(\alpha_{m-1})) \) if \( n \geq 1 \) and \( b_n \in \text{Dom}(\alpha_{m-1}) \). In all other cases, let \( u_m = u_{m-1} \).

We omit the verifications that the above definitions are proper and that \( \Delta(m) \) is satisfied in each case, completing the induction. The Horn formula \( \Psi_m \Rightarrow (x_0 = \alpha_m(b_0)) \) is \( R \)-constructible by \( \Delta(m) \), and so is satisfied in \( K \) by hypothesis. But \( f_m \) satisfies \( \Psi_m \) by \( \Delta(m) \), so \( f_m(x_0) = \bar{f}_m \alpha_m(b_0) \). But then

\[ y = f_m(x_0) = \bar{f}_m \alpha_m(b_0) \subseteq \beta_m(a_1) \subseteq z, \]

using \( \Delta(m) \) and our assumptions. This proves 3.1.

3.2. If every \( R \)-constructible Horn formula is satisfied in \( K \), then \( \psi: K \rightarrow M(K; R) \) is one-one.

Proof. Assume the following lemma: If \( s \) is directly equivalent to \( t \) in \( D(K; R) \) and \( s \) is reducible below some \( z \) in \( K \), then \( t \) is also reducible below \( z \).

Suppose \( \psi(y) = \psi(z) \) for \( y \) and \( z \) in \( K \), so \( (\emptyset, \psi_y) \sim (\emptyset, \psi_z) \) in \( D(K; R) \). Now \( (\emptyset, \psi_z) \) is trivially reducible below \( z \), and so \( (\emptyset, \psi_y) \) is reducible below \( z \). (Use the lemma on a sequence of directly equivalent elements of \( D(K; R) \) beginning with \( (\emptyset, \psi_z) \) and ending with \( (\emptyset, \psi_y) \).) But then \( y \subseteq z \) by 3.1 and the hypotheses, and a similar argument proves \( z \subseteq y \). Therefore, the lemma suffices to prove 3.2.

Let \( t = (H, \beta) \) in \( D(K; R) \) and \( z \in K \), and suppose \( s_1, s_2, \ldots, s_m \) is a reduction sequence in \( D(K; R) \), \( s_i = (G_i, \alpha_i) \) for \( i \leq m \), such that \( \alpha_m(a_1) \subseteq z \). To prove the lemma, we must show that \( t \) is reducible below \( z \) if \( s_1 \) and \( t \) are related by any of the thirteen types of direct equivalence.

If \( t \) is a union deletion, defined variable deletion, linear combination deletion or constraint increase of \( s_1 \), then \( t \) is reducible below \( z \) via the reduction sequence \( t, s_1, s_2, \ldots, s_m \).

If \( t \) is a renumbering of \( s_1 \) via the renumbering function \( \xi \), then \( t \) is reducible below \( z \) via the reduction sequence \( (G, \xi, \alpha, \xi), i \leq m \).

For the remaining cases, a sequence \( t_1, t_2, \ldots, t_m \) in \( D(K; R) \) and a sequence \( V_1, V_2, \ldots, V_m \) of subsets of \( V \) are constructed by induction on \( m \) so that the property \( \Delta_0(m) \) below is satisfied. Let \( t_i = (H_i, \beta_i) \) for \( i \leq m \). To avoid possible conflicts of variables in the inductive constructions, we choose a sufficiently large positive integer \( d \) so that \( n > d \) implies \( \alpha_i(a_n) = \alpha_i(b_n) = \omega \) and \( g(a_n) = g(b_n) = 0 \) if \( g \in G_i \), for all \( i \leq m \). Recall that \( b_k \) is "unused" in \( G \) if \( g \in G \) implies \( g(b_k) = 0 \), and \( b_k \) is "used" in \( G \) otherwise. Also, for \( g, h \) in \( F_R(V) \) and \( V_0 \subseteq V \), we write \( h = \eta(g, V_0) \) if \( h(v) = g(v) \) for \( v \in V_0 \) and \( h(v) = 0 \) for \( v \in V - V_0 \).

\[ \Delta_0(m): \ t_m \ \text{is a reduction of a renumbering of} \ t. \] For some \( c_k \) in \( V - (V_m \cup \{b_1, b_2, \ldots, b_d\}) \) such that \( \alpha_m(c_k) = f_m(c_k) = \omega \), each \( g \) in \( G_m \) satisfies

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either \( \eta(g, V_m) = 0 \) or \( \eta(g, V_m) + rc_k \) is in \( H_m \) for some \( r \) in \( R \). We have \( a_j \in V_m \), and \( b_j \in V_m \) implies \( b_j \) is used in \( G_m \). For \( j \leq d \) and \( c_j \in V_m \), \( \beta_m(c_j) \subseteq \alpha_m(c_j) \). If \( j \leq d \) and \( b_j \) is unused in \( G_m \), then \( b_j \) is unused in \( H_m \).

Without loss of generality, we may assume that \( s_1 \neq t \) and that \( t \) is not a null variable augmentation or deletion of \( s_1 \) at \( a_1 \). Let \( V' \) denote the set \( \{a_j: j \geq 1\} \cup \{b_j: b_j \text{ is used in } G_1\} \). If \( t \) is a null variable deletion of \( s_1 \) at \( c_n \), let \( t_1 = t \) and \( V_1 = V' - \{c_n\} \). If \( t \) is the inessential variables deletion of \( G_0 \) from \( s_1 \) using \( V_0 \), then let \( t_1 = t \) and \( V_1 = V' - V_0 \). In the six remaining cases, \( V_1 = V' \) and \( t_1 \) is a renumbering of \( t \) by some \( \xi \) such that \( \xi(b_k) = b_k \) if \( k \leq d \) and \( b_k \) is used in \( G_1 \), but \( \xi(b_k) = b_{u+k} \) for \( k > d \) and \( b_k \) unused in \( G_1 \), where \( u > d \) is sufficiently large so that \( n > u \) implies \( h(b_n) = 0 \) for all \( h \) in \( H \). We will omit the calculations showing that \( \Delta_0(1) \) is satisfied for the above choices of \( t_1 \) and \( V_1 \).

Assuming that \( m > 1 \) and that \( \Delta_0(m-1) \) is satisfied for \( t_{m-1} \) and \( V_{m-1} \) by the induction hypothesis, we use the condition that \( s_m \) is a direct reduction of \( s_{m-1} \). If \( s_m \) is a union augmentation of \( s_{m-1} \) using \( c_n - b_p - b_q \) and \( z_1 \) and \( z_2 \) in \( K \), and \( c_n \in V_{m-1} \), then let \( V_m = V_{m-1} \cup \{b_p, b_q\} \) and let \( t_m \) be the union augmentation of \( t_{m-1} \) using \( c_n - b_p - b_q \) and \( z_1 \) and \( z_2 \). If \( s_m \) is the defined variable augmentation of \( s_{m-1} \) using \( g \) at \( b_n \), then let \( V_m = V_{m-1} \cup \{b_n\} \) and let \( t_m \) be the defined variable augmentation of \( t_{m-1} \) using \( \eta(g, V_m) \) at \( b_n \). If \( s_m \) is the linear combination augmentation of \( s_{m-1} \) by \( g \), and \( \eta(g, V_{m-1}) \neq 0 \), then let \( V_m = V_{m-1} \) and let \( t_m \) be the linear combination augmentation of \( t_{m-1} \) by a suitable \( h \) of form \( \eta(g, V_{m-1}) + rc_k \). If \( s_m \) is the constraint decrease of \( s_{m-1} \) using \( g \) at \( c_n \), and \( c_n \in V_{m-1} \), then let \( V_m = V_{m-1} \) and let \( t_m \) be the constraint decrease of \( t_{m-1} \) by \( h \) at \( c_n \) for a suitable \( h \) of form \( \eta(g, V_{m-1}) + rc_k \). In all other cases, let \( t_m = t_{m-1} \) and \( V_m = V_{m-1} \). We will omit the verifications that the above definitions of \( t_m \) and \( V_m \) are proper and that \( \Delta_0(m) \) is satisfied in each case, completing the induction. The argument outlined above proves the lemma, since \( \Delta_0(m) \) and \( \alpha_m(a_1) \subseteq \tau \) imply that a renumbering of \( t \) is reducible below \( \tau \) in the final eight cases. This completes the proof of 3.2.

We make \( M(K; R) \) into a lattice by operations induced in the quotient \( D(K; R)/E(K; R) \). That is, binary meet and join operations are defined on \( D(K; R) \) so that \( E(K; R) \) is a congruence for these operations.

**Interpretation.** Suppose \( \iota: K \rightarrow \Gamma(M; R) \) is an embedding such that \( \iota(\omega) = 0 \). Define \( \nu_\iota: D(K; R) \rightarrow \Gamma_r(M^N; R) \) by \( \nu_\iota(s) = \nu(G, \alpha, \iota) \) if \( s = (G, \alpha) \). Then \( s \land t \) and \( s \lor t \) are defined in \( D(K; R) \) so that \( \nu_\iota(s \land t) = \nu_\iota(s) \cap \nu_\iota(t) \) and \( \nu_\iota(s \lor t) = \nu_\iota(s) + \nu_\iota(t) \) in \( \Gamma_r(M^N; R) \), independently of the choice of \( \iota \) and \( M \). That is, our purpose is to define formal meets and joins of constraint systems corresponding to
the usual meets and joins of their solution sets.

**Definition.** For \( s = (G, \alpha) \) in \( D(K; R) \), let the "length" of \( s \), denoted \(|s|\), equal the smallest integer \( n \), \( n \geq 1 \), such that \( k > n \) implies \( \alpha(a_k) = \alpha(b_k) = \omega \) and \( g(a_k) = g(b_k) = 0 \) for all \( g \) in \( G \). Clearly, \(|s|\) exists for every \( s \) in \( D(K; R) \).

For \( c_j, c_k \) in \( V \), let \( \text{tr}(c_j, c_k) : V \to V \) denote the bijection transposing \( c_j \) and \( c_k \) and leaving all other \( v \) in \( V \) fixed. If \( c_j = c_k \), \( \text{tr}(c_j, c_k) = 1_V \). Given \( m \geq 1 \) and \( i, j > 0 \), we will denote by \( \text{tr}_m(i, j) \) the product \( \prod_{k=1}^{m} \text{tr}(b_{lm+k}, b_{lm+k}) \), where the product operation is composition of functions. These functions \( \text{tr}_m(i, j) \) are frequently used as renumbering functions. We also define bijections \( \theta_{m,n} : V \to V \) given by the product \( \prod_{k=1}^{m} \text{tr}(b_{k}, b_{2kn+k}) \text{tr}(a_{k}, b_{(2n+1)m+k}) \), for any \( m \geq 1 \) and \( n > 0 \). For \( (G, \alpha) \) in \( D(K; R) \), we let \( G_0 \) denote the finite subset \( \{g0m>": g \in G\} \) of \( F_R(V) \), and observe that \( \alpha\theta_{m,n} \) is a lattice constraint function. (We will also let \( - G \) denote the set \( \{-g: g \in G\} \) for \( G \subseteq F_R(V) \).)

For \( m \geq 1 \) and \( i_1, i_2, \ldots, i_n > 0 \), let \( Q_{i_1,i_2,\ldots,i_n}(m) \) denote the following \( m \)-subset of \( F_R(V) \):

\[
\left\{ a_k - \sum_{j=1}^{n} b_{ijm+k} : k = 1, 2, \ldots, m \right\}
\]

For example, \( Q_{1,3}(m) = \{a_k - b_{m+k} - b_{3m+k} : k \leq m\} \).

If \( \alpha, \beta : V \to K \) are lattice constraint functions, let \( \alpha \lor \beta \) and \( \alpha \land \beta \) be the lattice constraint functions given by \( (\alpha \lor \beta)(v) = \alpha(v) \lor \beta(v) \) and \( (\alpha \land \beta)(v) = \alpha(v) \land \beta(v) \) for all \( v \) in \( V \). Let \( \alpha^a \) and \( \alpha^b \) be functions given as follows: \( \alpha^a(a_k) = \alpha(a_k), \alpha^a(b_k) = \omega, \alpha^b(a_k) = \omega \) and \( \alpha^b(b_k) = \alpha(b_k) \), for all \( k \geq 1 \).

Let \( s = (G, \alpha) \) and \( t = (H, \beta) \) in \( D(K; R) \), and suppose that \( n > |s|, |t| \). Define \( y_m(s, t) = (J_1, \gamma_1) \) and \( z_m(s, t) = (J_2, \gamma_2) \) in \( D(K; R) \) as follows:

\[
J_1 = Q_{1,3}(m) \cup G\theta_{m,0} \cup H\theta_{m,1}, \quad \gamma_1 = (\alpha \lor \beta)^a \lor a\theta_{m,0} \lor \beta\theta_{m,1},
\]

\[
J_2 = Q_1(m) \cup Q_3(m) \cup G\theta_{m,0} \cup H\theta_{m,1}, \quad \gamma_2 = (\alpha \land \beta)^a \lor a\theta_{m,0} \lor \beta\theta_{m,1}.
\]

Define \( s \lor t = y_m(s, t) \) and \( s \land t = z_m(s, t) \) for \( n = \max\{|s|, |t|\} \). Note that \( |y_m(s, t)| = |z_m(s, t)| = 4m \) for \( m \geq |s|, |t| \).

**Interpretation.** Suppose \( \iota : K \to M(K; R) \) is an embedding and \( \iota(\omega) = 0 \). Let \( s = (G, \alpha) \) and \( t = (H, \beta) \) in \( D(K; R) \), and let \( n = \max\{|s|, |t|\} \). The extended solutions \( h : V \to M \) in \( \nu_\iota(G, \alpha, \iota) \) or in \( \nu_\iota(H, \beta, \iota) \) can be represented in tabular form by pairs of sequences \( (h_a, h_b) \), where

\[
h_a = (h(a_1), h(a_2), \ldots, h(a_n), 0, 0, \ldots), \quad \text{and}
\]

\[
h_b = (h(b_1), h(b_2), \ldots, h(b_n), 0, 0, \ldots).
\]

(Note that \( h(a_k) = h(b_k) = 0 \) for \( k > n \) because \( \iota(\omega) = 0 \) and \( n \geq |s|, |t| \)).
solution corresponding to this extended solution \( h \) can be represented by the first sequence \( h_a \). Now \( s \lor t = (J_1, \gamma_1) \) is constructed so that \( f: V \to M \) is in \( \nu_0(J_1, \gamma_1, i) \) if and only if there exist \( g: V \to M \) in \( \nu_0(G, \alpha, i) \) and \( h: V \to M \) in \( \nu_0(H, \beta, i) \) such that

\[
\begin{align*}
  f_a &= (g(a_1) + h(a_1), g(a_2) + h(a_2), \ldots, g(a_n) + h(a_n), 0, 0, \ldots), \\
  f_b &= (g(b_1), g(b_2), \ldots, g(b_n), g(a_1), g(a_2), \ldots, g(a_n), h(b_1), h(b_2), \ldots, h(b_n), h(a_1), h(a_2), \ldots, h(a_n), 0, 0, \ldots).
\end{align*}
\]

So, the solutions \( f_a \) correspond precisely to sums of solutions \( g_a + h_a \), as was required. Similarly, for \( s \land t = (J_2, \gamma_2) \), \( f: V \to M \) is in \( \nu_0(J_2, \gamma_2, i) \) if and only if there exist \( g \) in \( \nu_0(G, \alpha, i) \) and \( h \) in \( \nu_0(H, \beta, i) \) such that \( f_a = g_a = h_a \) and \( f_b \) is again given by the formula above. Therefore, the solutions \( f_a \) corresponding to \( s \land t \) are just the solutions common to \( s \) and \( t \), as was required.

3.3. **For** \( s, t \) in \( D(K; R) \), \( s \lor t \sim t \lor s \) and \( s \land t \sim t \land s \).

**Proof.** Clearly \( t \lor s \) is the renumbering of \( s \lor t \) using \( tr_{2n}(0, 1) \) for \( n = \max\{|s|, |t|\} \), and similarly \( t \land s \) is the renumbering of \( s \land t \) using \( tr_{2n}(0, 1) \).

3.4. Let \((G \cup J, \alpha)\) and \((H \cup J, \alpha)\) be in \( D(K; R) \), and suppose that for every \( g \) in \( G \) there exists \( h \) in \( H \) such that \( g(c_k) = h(c_k) \) for all \( c_k \) in \( V \) such that \( \alpha(c_k) \neq \omega \), and that for every \( h \) in \( H \) there exists \( g \) in \( G \) such that \( g(c_k) = h(c_k) \) for all \( c_k \) in \( V \) such that \( \alpha(c_k) \neq \omega \). Then \((G \cup J, \alpha) \sim (H \cup J, \alpha)\) in \( D(K; R) \). In particular, if \( g \) in \( G \) and \( g(c_k) \neq 0 \) implies \( \alpha(c_k) = \omega \) for all \( c_k \) in \( V \), then \((G \cup J, \alpha) \sim (J, \alpha)\). Furthermore, if \( \alpha^a = \psi_\omega \), then \((J, \alpha) \sim (\varnothing, \psi_\omega)\).

**Proof.** Assume the hypotheses for \((G \cup J, \alpha)\) and \((H \cup J, \alpha)\), and choose \( n \geq |(G \cup J, \alpha)|, |(H \cup J, \alpha)| \). Perform null variable deletions at \( c_k \) for all values of \( k \leq n \) such that \( \alpha(c_k) = \omega \). So, we obtain \((G \cup J, \alpha) \sim (G_0 \cup J_0, \alpha)\) and \((H \cup J, \alpha) \sim (H_0 \cup J_0, \alpha)\), where \( G_0 \) is the set of equations obtained from equations \( g \) of \( G \) by dropping all nonzero terms \( g(c_k)c_k \) for which \( \alpha(c_k) = \omega \), and \( H_0 \) and \( J_0 \) are obtained from \( H \) and \( J \), respectively, in the same way. By the hypotheses on \( G \) and \( H \), however, we have \( G_0 = H_0 \), and so \((G \cup J, \alpha) \sim (H \cup J, \alpha)\).

Suppose that \( g(c_k) \neq 0 \) implies \( \alpha(c_k) = \omega \) for all \( g \) in \( G \) and \( c_k \) in \( V \). Then \((G \cup J, \alpha) \sim (\{0\} \cup J, \alpha)\) as above, for \( G \neq \varnothing \). If \( J \) is nonempty, then \((\{0\} \cup J, \alpha) \sim (J, \alpha)\) by a linear combination deletion. It is also easily checked that \((\{0\}, \alpha) \sim (\varnothing, \alpha)\), so \((G \cup J, \alpha) \sim (J, \alpha)\).

Finally, suppose \( \alpha^a = \psi_\omega \). Noting that \((J, \alpha) \sim (J_0, \alpha)\) by null variable deletions as above, we observe that \( h \) in \( J_0 \) implies \( h(a_k) = 0 \) for \( k \geq 1 \) because \( \alpha^a = \psi_\omega \). Also, \( h(b_k) = 0 \) if \( k > m = |(J_0, \alpha)| \). But then \((J_0, \alpha) \sim (\varnothing, \psi_\omega)\) by
an inessential variables deletion of $J_0$ using $\{b_1, b_2, \cdots, b_m\}$. This completes
the proof of 3.4.

3.5. If $s, t$ are in $D(K; R)$ and $m \geq |s|, |t|$, then $s \lor t \sim \gamma_m(s, t)$ and $s \land t \sim z_m(s, t)$.

**Proof.** Assume the hypotheses, and let $n = \max\{|s|, |t|\}$. If $\gamma_m(s, t) = (J, \gamma)$, then
$\gamma(a_k) = \gamma(b_{m+k}) = \omega$ for $n < k \leq m$. If $J_1 = \{a_k - b_{m+k} - b_{3m+k} : n < k \leq m\}$, then $(J, \gamma) \sim (J - J_1, \gamma)$ by 3.4. But $s \lor t = \gamma_n(s, t)$ is clearly a renumbering of $(J - J_1, \gamma)$, so $s \lor t \sim \gamma_m(s, t)$. The proof
that $s \land t \sim z_m(s, t)$ is similar.

3.6. $E(K; R)$ is a congruence for meet and join in $D(K; R)$.

**Proof.** Suppose $s, t, u$ are in $D(K; R)$, and $n = \max\{|s|, |t|, |u|\}$. It suf-
fices to prove the following lemma: If $s \in E_j$ for some $j, j \leq n$, then $\gamma_m(s, u) \sim \gamma_n(s, u)$ and $\gamma_m(s, u) \sim z_m(s, u)$ for some $m, m > n$. From this lemma and 3.5,
an induction argument proves that $s \sim t$ implies $s \land u \sim t \land u$ and $s \lor u \sim t \lor u$. Using this result and 3.3, it follows that $E(K; R)$ is a congruence.

If $t$ is a constraint decrease of $s$ at $a_k$ using $g$, then $\gamma_n(t, u)$ can be obtained
from $\gamma_n(s, u)$ by two constraint decreases (at $b_{n+k}$ using $g_{n, 0}$, then at $a_k$
using $a_k - b_{n+k} - b_{3n+k}$).

If $t$ is a null variable deletion of $s$ at $a_k$ and $m \geq k, n$, then $\gamma_m(t, u)$ can be
obtained from $\gamma_m(s, u)$ by a null variable deletion at $b_{m+k}$ followed by a null
variable augmentation at $b_{m+k}$ restoring the linking equations $a_k - b_{m+k} - b_{3m+k}$.

If $t$ is a renumbering of $s$ by a renumbering function $\xi$, then we can suppose
without loss of generality that $\xi(b_k) = b_k$ for all $k > n$. But then $\gamma_n(t, u)$ is a
renumbering of $\gamma_n(s, u)$ using $\xi$ also.

In the remaining cases, it is relatively easy to show that $s \in E_j \implies \gamma_m(s, u)
E_j \gamma_m(t, u)$ for some sufficiently large $m$. We will omit these cases, plus all the
similar arguments showing that $s \in E_j \implies z_m(s, u) \sim z_m(t, u)$. This outlines
the proof of the lemma, completing the proof of 3.6.

**Definition.** If $G_i \subset F_R(V)$ for $i = 1, 2, \cdots, n$, let $[[G_1, G_2, \cdots, G_n]]$
denote the $R$-submodule of $F_R(V)$ generated by $G_1 \cup G_2 \cup \cdots \cup G_n$.

3.7. If $(G \cup H, \alpha)$ and $(G \cup J, \alpha)$ are in $D(K; R)$ such that $[[G, H]] = [[G, J]]$, then $(G \cup H, \alpha) \sim (G \cup J, \alpha)$. In particular, $(G \cup H, \alpha) \sim (G \cup (-H), \alpha)$.

**Proof.** Assuming the hypotheses, both $(G \cup H, \alpha)$ and $(G \cup J, \alpha)$ are equiva-
et to $(G \cup H \cup J, \alpha)$ by some number of linear combination augmentations.

For the second part, observe that $[[G, H]] = [[G, -H]]$.

**Definition.** Suppose $J_0, J_1, \cdots, J_p$ and $G_0, G_1, \cdots, G_q$ are finite
subsets of $F_R(V)$, and $\alpha, \alpha_0, \alpha_1, \cdots, \alpha_q$ are lattice constraint functions. For
$m \geq 1$, $F_m(J_0, J_1, \ldots, J_p | G_0, G_1, \ldots, G_q)$ denotes

$$J_0 \cup J_1 \cup \cdots \cup J_p \cup G_0 \theta_{m,0} \cup G_1 \theta_{m,1} \cup \cdots \cup G_q \theta_{m,q},$$

and $f_m(\alpha | \alpha_0, \alpha_1, \ldots, \alpha_q)$ denotes

$$\alpha \lor \alpha_0 \lor \alpha_1 \lor \cdots \lor \alpha_q.$$

In general, we will use these notations only when $m \geq |(G_i, \alpha_i)|$ for $0 \leq i \leq q$, and $\alpha = \alpha^a$. In this case, we observe that $h \in G_{i\theta_{m,i}}$ for some $i$, $0 \leq i \leq q$, implies that $h(a_k) = 0$ for all $k \geq 1$, and $h(b_k) = 0$ unless $2im < k \leq (2i + 2)m$. Furthermore, $\gamma = f_m(\alpha | \alpha_0, \alpha_1, \ldots, \alpha_q)$ is then given by $\gamma(a_k) = \alpha(a_k)$ for all $k \geq 1$, $\gamma(b_{2im+k}) = \alpha_i(b_k)$ and $\gamma(b_{(2i+1)m+k}) = \alpha_i(a_k)$ for $k \leq m$ and $0 \leq i \leq q$, and $\gamma(b_k) = \omega$ for $k > (2q + 2)m$.

Suppose $i_1, i_2, \ldots, i_p$ and $j_1, j_2, \ldots, j_q$ are nonnegative integers, and $m \geq 1$. Define $P_{i_1,i_2,\ldots,i_p}(m)$ as the following subset of $F_R(V)$:

$$\left\{ \left( \sum_{k=1}^{p} b_{i_k m+d} \right) - \left( \sum_{k=1}^{q} b_{j_k m+d} \right) : d = 1, 2, \ldots, m \right\}.$$

For example, $P_{3,9}(m) = \{b_{3m+d} - b_{7m+d} - b_{9m+d} : d \leq m\}$. In the following, we will often let

$$Q_{i_1,i_2,\ldots,i_q}$$

and $P_{i_1,i_2,\ldots,i_p}$ abbreviate

$$Q_{i_1,i_2,\ldots,i_q}(m)$$

and $P_{i_1,i_2,\ldots,i_p}(m)$,

respectively, when $m$ can be understood from the context. In particular, $m$ is understood for an argument of the $F_m$ notation.

3.8. Let $H, G_0, G_1, \ldots, G_n$ be finite subsets of $F_R(V)$, let $m \geq |(G_i, \psi_{\omega})|$ for $0 \leq i \leq n$, and let $r_1, r_2, r_3 \in R$ such that $r_1 \in \{1, -1\}$. Suppose $i, j$ and $k$ are distinct integers, $0 \leq i, j, k \leq n$, such that the following equations are in $H$:

$$r_1 b_{2im+d} + r_2 b_{2jm+d} + r_3 b_{2km+d} \text{ whenever } d \leq m \text{ and } g(b_d) \neq 0$$

for some $g$ in $G_1$, and

$$r_1 b_{(2i+1)m+d} + r_2 b_{(2j+1)m+d} + r_3 b_{(2k+1)m+d} \text{ whenever } d \leq m \text{ and } g(a_d) \neq 0 \text{ for some } g \text{ in } G_j.$$

Furthermore, suppose either that $r_3 = 0$ or that $G_j \subseteq G_k$. Then $G_j \theta_{m,i} \subset \left[ \left[ F_m(H | G_0, G_1, \ldots, G_n) \right] \right]$. If in addition $r_2 \in \{1, -1\}$, and $H_i = G_i \cup G_j$ and $H_p = G_p$ for $p \neq i, j$, then

$$\left[ \left[ F_m(H | G_0, G_1, \ldots, G_n) \right] \right] \subset \left[ \left[ F_m(H | H_0, H_1, \ldots, H_n) \right] \right].$$
If $G, J, J_0, J_1, \ldots, J_n$ are finite subsets of $F_R(V)$ such that $m \geq |J, \psi_\omega|$ and $Q_j(m) \subseteq G$, then

$$[[F_m(G, J|J_0, J_1, \ldots, J_n)]] = [[F_m(G|J \cup J_0, J_1, \ldots, J_n)]].$$  

**Proof.** Assume the hypotheses of the first part, and observe that $g \in G_j$ implies

$$r_1g_{m,i} + r_2g_{m,j} + r_3g_{m,k}$$

$$- \sum_{d=1}^{m} g(b_d)(r_1b_2im+d + r_2b_2jm+d + r_3b_2km+d)$$

$$- \sum_{d=1}^{m} g(a_d)(r_1b(2i+1)m+d + r_2b(2j+1)m+d + r_3b(2k+1)m+d) = 0.$$  

But $r_3g_{m,k} \in [[G_k\theta_{m,k}]]$ because either $r_3 = 0$ or $G_j \subseteq G_k$, by hypothesis. So, $G_j\theta_{m,i} \subseteq [[H, G_j\theta_{m,i}, G_k\theta_{m,k}]]$ because $r_1 \in \{1, -1\}$, and

$$[[H, G_k\theta_{m,k}, G_j\theta_{m,i}]] = [[H, G_k\theta_{m,k}, G_j\theta_{m,i}]]$$

if we also assume $r_2 \in \{1, -1\}$. This proves the first part.

For the second part, observe that $g \in J$ implies that

$$g - g_{m,0} - \sum_{d=1}^{m} g(a_d)(a_d - b_{m+d}) = 0.$$  

Therefore, $[[G, J]] = [[G, J\theta_{m,0}]]$ if $Q_j(m) \subseteq G$, and the result follows. This completes the proof of 3.8.

**Definition.** We will use reverse functional notation for lattice polynomials $w(s_1, s_2, \ldots, s_q)$ with variables $s_i, i \leq q$, in $D(K; R)$. That is, a \textit{\textquotedblleft D(K; R) polynomial\textquotedblright} is recursively defined as (1) a one-term sequence $(x_0)$ with $x_0 \in D(K; R)$, or (2) a sequence formed from shorter $D(K; R)$ polynomials $e$ and $e_0$ by juxtaposing sequences $e \ast e_0 \ast (x_n)$, where $x_n \in \{\vee, \wedge\}$. (For example, $(s_1, s_2, \vee, s_3, s_1, \vee, \wedge)$ is the $D(K; R)$ polynomial usually written as $(s_1 \vee s_2) \wedge (s_3 \vee s_1)$.) Given a $D(K; R)$ polynomial $w(s_1, s_2, \ldots, s_q) = (x_0, x_1, \ldots, x_n)$, there is a unique $D(K; R)$ polynomial $w_i(s_1, s_2, \ldots, s_q)$ of form $(x_j, x_{j+1}, \ldots, x_i)$ for each $i, 0 \leq i \leq n$. If $x_i = s_p$ in $D(K; R)$, then $w_i(s_1, s_2, \ldots, s_q) = (x_i)$. If $x_i \in \{\vee, \wedge\}$, however, then there exists a unique integer $k, j < k < i$, such that

$$w_{k-1}(s_1, s_2, \ldots, s_q) = (x_j, x_{j+1}, \ldots, x_{k-1}),$$

$$w_{i-1}(s_1, s_2, \ldots, s_q) = (x_k, x_{k+1}, \ldots, x_{i-1})$$

and

$$w_i(s_1, s_2, \ldots, s_q) = w_{k-1}(s_1, s_2, \ldots, s_q)x_jw_{i-1}(s_1, s_2, \ldots, s_q).$$

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In this case, we say that \( x_i \) is "connected" to \( x_{k-1} \) and \( x_{i-1} \). Of course, \( w_n(s_1, s_2, \ldots, s_q) = w(s_1, s_2, \ldots, s_q) \). We observe that \( x_0 \) is in \( D(K; R) \) always, and that \( x_n \in \{\vee, \wedge\} \) if \( n > 0 \). Furthermore, there is a unique "path" from \( x_n \) to each \( x_i \), \( i < n \); that is, there is a sequence \( x_{i_1}, x_{i_2}, \ldots, x_{i_p} \) such that \( x_{i_1} = x_n, x_{i_p} = x_i \) and \( x_{i_j} \) is connected to \( x_{i_{j+1}} \) for \( 1 \leq j < p \).

Suppose \( w(s_1, s_2, \ldots, s_q) = (x_0, x_1, \ldots, x_n) \) is a \( D(K; R) \) polynomial, for \( s_i = (G_i, \alpha_i) \) in \( D(K; R) \), \( i \leq q \). We will recursively define \( F_m(x_0, x_1, \ldots, x_n) \) in \( D(K; R) \), for any \( m \geq |s_1|, |s_2|, \ldots, |s_q| \). If \( x_i = s_p \) for some \( p \leq q \), then define \( H_i = G_p \) and \( \beta_i = \alpha_p \). If \( x_i \in \{\vee, \wedge\} \), then let \( H_i = \emptyset \) and \( \beta_i = \psi_\omega \). We now define \( F_m(x_0, x_1, \ldots, x_n) = (J, \gamma) \) as follows:

\[
J = F_m(H_0, H_1, \ldots, H_n),
\gamma = f_m(w(\alpha_1, \alpha_2, \ldots, \alpha_q) \beta_0, \beta_1, \ldots, \beta_n).
\]

It remains to give the definition of \( H \). For any finite set \( I \) containing sequences \( \iota = (i_1, i_2, \ldots, i_p) \) of nonnegative integers, let \( Q(I) \) denote \( \bigcup_{\iota \in I} Q_{\iota} \). (For example, if \( I = \{(1), (5, 7)\} \), then \( Q(I) = Q_1 \cup Q_{5, 7} \).) If \( I \) and \( I' \) are sets of sequences of integers, let \( I \ast I' \) denote the juxtaposition set product \( \{\iota \ast \iota' : \iota \in I, \iota' \in I'\} \). (For example, \( (1, 5) \ast (1, 7, 9) = \{(1, 5), (1, 7, 9), (3, 5), (3, 7, 9)\} \).) We now define a sequence \( I_0, I_1, \ldots, I_n \) of such sets recursively, using \((x_0, x_1, \ldots, x_n)\). If \( x_i \) is in \( D(K; R) \), let \( I_i = \{(2i + 1)\} \). Otherwise, \( x_i \in \{\vee, \wedge\} \), so \( x_i \) is connected to \( x_{k-1} \) and \( x_{i-1} \) for a unique \( k < i \). In that case, define \( I_i = I_{k-1} \ast I_{i-1} \) if \( x_i \) equals \( \vee \), and define \( I_i = I_{k-1} \cup I_{i-1} \) if \( x_i \) equals \( \wedge \). This uniquely defines \( I_0, I_1, \ldots, I_n \), and the definition of \( F_m(x_0, x_1, \ldots, x_n) \) is completed by setting \( H = Q(I_n) \).

**Example.** Suppose \( w(s_1, s_2, s_3) = (s_1, s_2, \vee, s_3, s_1, \wedge) \), where \( s_i = (G_i, \alpha_i) \) for \( i = 1, 2, 3 \). By our definitions, \( I_0 = \{(1)\}, I_1 = \{(3)\}, I_2 = I_0 \ast I_1 = \{(1, 3)\} \), \( I_3 = \{\{7\}\}, I_4 = \{\{9\}\}, I_5 = I_3 \ast I_4 = \{(7, 9)\} \) and \( I_6 = I_2 \cup I_5 = \{(1, 3), (7, 9)\} \). Therefore, \( Q(I_6) = Q_{1, 3} \cup Q_{7, 9} \), so \( F_m(s_1, s_2, \vee, s_3, s_1, \wedge) = (J, \gamma) \) as given below:

\[
J = F_m(Q_{1, 3}, Q_{7, 9}, G_1, G_2, \emptyset, G_3, G_1),
\gamma = f_m(((\alpha_1 \vee \alpha_2) \wedge (\alpha_3 \vee \alpha_1))^{\alpha_1} | \alpha_1, \alpha_2, \psi_\omega, \alpha_3, \alpha_1).
\]

Note that we have replaced the union symbol in the expression for \( Q(I_6) \) by a comma, as our \( F_m \) notation permits. We will replace union symbols by commas similarly in subsequent evaluations of \( F_m \) applied to \( D(K; R) \) polynomials. We will also delete final null terms, as \( H_5 = H_6 = \emptyset \) and \( \beta_5 = \beta_6 = \psi_\omega \) were deleted above.

3.9. If \( w(s_1, s_2, \ldots, s_q) = (x_0, x_1, \ldots, x_n) \) is a \( D(K; R) \) polynomial and \( m \geq |s_1|, |s_2|, \ldots, |s_q| \), then
Proof. Assume the hypothesis, and let \( s_i = (G_i, \gamma_i) \) for \( i \leq q \). Let \( F_m(x_0, x_1, \cdots, x_n) = (J_0, \gamma_0) \) for \( J_0 = F_m(Q(I_n)|H_0, H_1, \cdots, H_n) \) and \( \gamma_0 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)\beta_0, \beta_1, \cdots, \beta_n) \), where the sequences \((H_0, H_1, \cdots, H_n), (\beta_0, \beta_1, \cdots, \beta_n)\) and \((I_0, I_1, \cdots, I_n)\) are formed as described above. Define a sequence \( U_0, U_1, \cdots, U_n \) of subsets of \( F_R(V) \) and a sequence \( \alpha_0, \alpha_1, \cdots, \alpha_n \) of lattice constraint functions as follows: If \( x_i = s_p \) in \( D(K; R) \), then \( U_i = H_i = G_p \), and \( \alpha_i = \beta_i = \alpha_p \). If \( x_i \in \{\lor, \land\} \), then \( x_i \) is connected to \( x_{k-1} \) and \( x_{i-1} \) for a unique \( k < i \). In this case, \( \alpha_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q) \), and \( U_i = Q_{2k-1,2k-1} \) if \( x_i \) equals \( \lor \), and \( U_i = Q_{2k-1} \cup Q_{2k-1} \) if \( x_i \) equals \( \land \).

Define a sequence \( (J_0, \gamma_0), (J_1, \gamma_1), \cdots, (J_n, \gamma_n) \) in \( D(K; R) \) as follows:

\[
J_i = F_m(Q(I_n)|U_0, U_1, \cdots, U_i, H_{i+1}, H_{i+2}, \cdots, H_n),
\]

\[
\gamma_i = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)\kappa_0, \kappa_1, \cdots, \kappa_i, \beta_{i+1}, \beta_{i+2}, \cdots, \beta_n).
\]

(Since \( x_0 \in D(K; R) \), the two definitions of \((J_0, \gamma_0)\) agree.) We intend to prove that \((J_{i-1}, \gamma_{i-1}) \sim (J_i, \gamma_i) \) in \( D(K; R) \) for \( 0 < i \leq n \).

Consider first the following lemma: For \( 0 < i < n \), if \( \tau = (i_1, i_2, \cdots, i_p) \in I_j \), and \( M_j = [F_m(\emptyset|U_0, U_1, \cdots, U_j)] \), then \( P_{i+j}^j \subset M_j \). Using induction on \( j \), we note that \( I_j = \{(2j+1)\} \) if \( x_j \) is in \( D(K; R) \), and so \( P_{i+j}^j = \{0\} \) for \( i \) in \( I_j \) in this case, and in the case \( i = 0 \) in particular. Assuming the induction hypothesis, suppose \( x_j \in \{\lor, \land\} \), where \( x_j \) is connected to \( x_{k-1} \) and \( x_{i-1} \), and \( i \in I_j \). If \( x_j \) equals \( \lor \), then \( \tau = \tau_1 \star \tau_2 \) for some \( \tau_1 \) in \( I_{k-1} \) and \( \tau_2 \) in \( I_{i-1} \). Furthermore, \( P_{i+j}^j = U_j \theta_{m,i} \subset M_j \), and \( P_{i+j}^j = M_j \subset P_{i+j}^j \subset M_j \) by the induction hypothesis. Therefore

\[
P_{i+j}^j \subset \left[ [P_{2k-1,2i-1}^j, P_{i+1,2i-1}^j, P_{2i-1}^j] \right] \subset M_j \quad \text{if } x_j \text{ equals } \lor.
\]

Finally, suppose \( x_j \) equals \( \land \), so \( i \in I_{k-1} \cup I_{i-1} \). Observing that \( P_{i+j}^j \subset \left( P_{2k-1}^j \cup P_{2i-1}^j \right) \) = \( U_j \theta_{m,i} \subset M_j \), we see that

\[
P_{i+j}^j \subset \left[ [P_{2k-1}^j, P_{i+1}^j, P_{2i-1}^j] \right] \subset M_j \quad \text{or} \quad P_{i+j}^j \subset \left[ [P_{2k-1}^j, P_{i+1}^j, P_{2i-1}^j] \right] \subset M_j,
\]

using the induction hypothesis. This completes the proof of the lemma.

If \( x_i \) is in \( D(K; R) \), then \((J_{i-1}, \gamma_{i-1}) = (J_i, \gamma_i) \). So, assume that \( x_i \in \{\lor, \land\} \), and let \( k \) be the unique integer, \( 0 < k < i \), such that \( x_i \) is connected to \( x_{k-1} \) and \( x_{i-1} \). If \( x_i \) equals \( \lor \), then make defined variable augmentations at \( b_{(2i+1)m+d} \) for \( d \leq m \) using the equations \( P_{2k-1,2i-1}^j(m) \). Since \( P_{2k-1,2i-1}^j = Q_{2k-1,2i-1}^j \theta_{m,i} = U_j \theta_{m,i} \) and \( \kappa_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q) = w_{k-1}(\alpha_1, \alpha_2, \cdots, \alpha_q) \lor w_{i-1}(\alpha_1, \alpha_2, \cdots, \alpha_q) = (\kappa_{k-1}) ^a \lor (\kappa_{i-1}) ^a \), the above argument proves \((J_{i-1}, \gamma_{i-1}) \sim (J_i, \gamma_i) \). Now suppose that \( x_i \) equals \( \land \). We make defined variable...
We now prove that $P_{2i}-1 \subseteq [[[J^*_i]]]$. Clearly $I_0, I_1, \cdots, I_n$ are nonempty sets, so we choose sequences $i_1$ in $I_{k-1}$ and $i_2$ in $I_{k-1}$ and note that $\{i_1, i_2\} \subseteq I_i$. Since $i = n$ or there is a “path” from $x_n$ to $x_i$, we can find (possibly empty) sequences $t_3$ and $t_4$ such that $i_5 = t_3 * t_4$ and $i_6 = t_3 * t_2 * t_4$ are in $I_n$.

Now, $P_{t_2} \subseteq [[[Q_{t_5}, Q_{t_6}]]) \subseteq [[[Q_{I_n}]]) \subseteq [[[J^*_i]]]$. (For example, $P_{5,7} \subseteq [[[Q_{139}, [0, s, 7, 9]]])$. But then $P_{2i+1} \subseteq [[[P_{t_1}^+, P_{t_2}^+, P_{t_2}^+, P_{2i}^+]]) \subseteq [[[J^*_i]]]$, using the lemma and the equation $Q_{2k-1} \theta_{m,i} = P_{2i+1}^+$. So $(J^*_i, \gamma^*_i) \sim (J_i, \gamma_i)$ by linear combination augmentations adding the equations $P_{2i+1}^+ = Q_{2i+1}^+ \theta_{m,i}$, followed by constraint decreases at $b_{(2i+1)m+d}$ for $d \leq m$ using these equations, since $\kappa_i = w_i(\alpha_1, \alpha_2, \cdots, \alpha_q)^a | \kappa_i, \beta_{i+1}, \beta_{i+2}, \cdots, \beta_n$. This completes the proof that $(J_{i+1}, \gamma_{i+1}) \sim (J_i, \gamma_i)$ for $0 < i < n$.

Let $J = F_m(Q_{2n+1}, U_0, U_1, \cdots, U_n)$, and recall that $J_n = F_m(Q_{I_n}, U_0, U_1, \cdots, U_n)$. For $t$ in $I_n$, note that $P_{t}^{n+1} \subseteq F_m(Q, U_0, U_1, \cdots, U_n)$ by the lemma, and also that $[[[Q_{2n+1}, P_{t}^{n+1}]]) \subseteq [[[Q_{t}, P_{t}^{n+1}]])$. But then $[[[J]]) = [[[J_n]])$ follows, and so $(J_n, \gamma_n) \sim (J, \gamma_n)$ by 3.7.

Finally, we prove that $(J, \gamma_n) \sim w(s_1, s_2, \cdots, s_q)$ by induction on $n$. If $n = 0$, then $(x_0) = (s_1)$ and $w(s_1, s_2, \cdots, s_q) = s_1$ for some $j$. So, $J = F_m(Q \cdot G_1)$ and $\gamma_n = (\alpha_j^a | \gamma_j^a)$, and $(J, \gamma_n) \sim (J', \gamma_n)$ for $J' = F_m(Q, G_1 \cdot G_1)$ by 3.8 and 3.7. Replacing $Q_1$ by $\neg Q_1$ using 3.7, and eliminating $Q_1$ by defined variable deletions at $b_{m+d}$ for $d \leq m$, we obtain $(J', \gamma_n) \sim s_j$. This proves $(J, \gamma_n) \sim w(s_1, s_2, \cdots, s_q)$ if $n = 0$.

Now suppose that $n > 0$, so $x_n \in \{V, \land\}$ and $x_n$ is connected to $x_{k-1}$ and $x_{k-1}$ for a unique $k$, $0 < k < n$. That is, $w = w_{k-1} x_n w_{n-1}$, where $w_{k-1} = (x_0, x_1, \cdots, x_{k-1})$ and $w_{n-1} = (x_k, x_{k+1}, \cdots, x_{n-1})$. Suppose that $w_{k-1} \sim w_0$ and $w_{n-1} \sim u_1$ are the equivalences obtained by using the induction hypothesis for $w_{k-1}$ and $w_{n-1}$. If $p \geq |w_0|, |u_1|$, then $w \sim y_p(u_0, u_1)$ if $x_n$ equals $V$ and $w \sim z_p(u_0, u_1)$ if $x_n$ equals $\land$.

Suppose $x_n$ equals $V$. Using 3.4 to eliminate null equations $a_d - b_{p+d} - b_{3p+d}$ for $m < d < p$ and then renumbering, we can show that $y_p(u_0, u_1) \sim$.
\[(W_1, \lambda_1)\] as given below:
\[
W_1 = F_m(Q_{2n+3,2n+5}, p^{2n+3}_{2k-1}, p^{2n+5}_{2n-1}|U_0, U_1, \cdots, U_{n-1}),
\]
\[
\lambda_1 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|k_0, k_1, \cdots, k_{n-1}, \psi, (k_{k-1})^a, (k_{n-1})^a).
\]
If we introduce equations \(p^{2n+1}_{2k-1,2n-1}\) by defined variable augmentations at \(b_{(2n+1)m+d}\) for \(d \leq m\), and then apply 3.7 by noting that:
\[
[[Q_{2n+3,2n+5}, p^{2n+1}_{2k-1,2n-1}, p^{2n+3}_{2k-1,2n-1}]] = [[Q_{2n+1,2n+3,2n+5}]],
\]
we obtain \((W_1, \lambda_1) \sim (W_2, \lambda_2)\), where
\[
W_2 = F_m(Q_{2n+3,2n+5}, p^{2n+3}_{2k-1,2n-1}, p^{2n+5}_{2k-1,2n-1}|U_0, U_1, \cdots, U_n),
\]
\[
\lambda_2 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|k_0, k_1, \cdots, k_{n-1}, (k_{k-1})^a, (k_{n-1})^a).
\]
(Recall that \(U_n \theta_{m,n} = p^{2n+1}_{2k-1,2n-1}\)). But \((W_2, \lambda_2) \sim (J, \gamma_n)\) by eliminating \(p^{2n+1}_{2k-1}\) and \(p^{2n+5}_{2n-1}\) by defined variable deletions. This proves that \((J, \gamma_n) \sim w(s_1, s_2, \cdots, s_q)\) if \(x_n\) equals \(\lor\).

Suppose \(x_n\) equals \(\land\). Using 3.4 to eliminate null equations \(a_d - b_{p+d}\) and \(a_d - b_{3p+d}\) for \(m < d \leq p\) and renumbering, we can show that \(z_p(u_0, u_1) \sim (W_3, \lambda_3)\) for
\[
W_3 = F_m(Q_{2n+1,2n+3,2n+5}, p^{2n+1}_{2k-1,2n-1}, p^{2n+3}_{2n-1}|U_0, U_1, \cdots, U_{n-1}),
\]
\[
\lambda_3 = f_m(w(\alpha_1, \alpha_2, \cdots, \alpha_q)^a|k_0, k_1, \cdots, k_{n-1}, (k_{k-1})^a, (k_{n-1})^a).
\]
Since \(U_n \theta_{m,n} = p^{2n+1}_{2k-1} \cup p^{2n+1}_{2n-1}\), we use 3.7 and the equation
\[
[[Q_{2n+1,2n+3,2n+5}]] = [[Q_{2n+1,2n+3,2n+5}]],
\]
to obtain \((W_3, \lambda_3) \sim (W_4, \lambda_3)\), where
\[
W_4 = F_m(Q_{2n+1,2n+3,2n+5}, p^{2n+1}_{2n-1,2n-1}, p^{2n+3}_{2n-1}|U_0, U_1, \cdots, U_n).
\]
Finally, \((W_4, \lambda_3) \sim (J, \gamma_n)\) as follows: Make constraint decreases at \(b_{(2n+1)m+d}\) and \(b_{(2n+3)m+d}\) for \(d \leq m\) using the equations \(-Q_{2n+1}\) and \(-Q_{2n+3}\), respectively. Then eliminate \(-Q_{2n+1}\) by linear combination deletions and eliminate \(-Q_{2n+3}\) by defined variable deletions. This completes the induction proving that \((J, \gamma_n) \sim w(s_1, s_2, \cdots, s_q)\), and so we have proved 3.9.

**Definition.** Let \(s = (G, \alpha)\) and \(t = (H, \beta)\) in \(D(K; R)\). Write \(s \ll t\) if \(G \supset H, \alpha^a \subset \beta^a\) and \(\alpha(b_k) \subset \beta(b_k)\) for all \(k \leq |t|\).

3.10. *If* \(s \ll t \in D(K; R)\), *then* \(t \lor s \sim t.\)

**Proof.** Assume \(s \ll t \in D(K; R)\) for \(s = (G, \alpha)\) and \(t = (H, \beta)\). Choosing
$m \geq |s|, |t|$ and using 3.9 and the renumbering function $\text{tr}_{2m}(0, 2)$, we obtain

$$t \vee s \sim F_m(t, s, \vee) \sim (J_1, \gamma_1),$$

where

$$J_1 = F_m(Q_{3,5}|\varnothing, G, H) \quad \text{and} \quad \gamma_1 = f_m((\alpha \vee \beta)^a|\psi_\omega, \alpha, \beta).$$

Introducing equations $J = \{b_k - b_{2m+k} - b_{4m+k} : k \leq |t| \} \subset P^0_{2,4}$ and equations $P^1_{3,5}$ by defined variable augmentations at $b_k$ for $k \leq |t|$ and at $b_{m+k}$ for $k \leq m$, we obtain $(J_1, \gamma_1) \sim (J_2, \gamma_2)$ for

$$J_2 = F_m(Q_{3,5}, J, P^1_{3,5}|\varnothing, G, H) \quad \text{and} \quad \gamma_2 = f_m((\alpha^a|\beta, \alpha, \beta).$$

(The hypotheses $\alpha^a \subset \beta^a$ and $\beta(b_k) \subset \alpha(b_k)$ for $k \leq |t|$ were used.) From 3.8 and the hypothesis $H \subset G$, we obtain $[[U_2]] = [[F_m(Q_{3,5}, J, P^1_{3,5}|H, G, \varnothing)]]$. Since $P^1_{3,5} = P^3_{1,5}$ and $[[Q_{3,5}, P^1_{3,5}]] = [[Q_1, P^3_{1,5}]]$, several applications of 3.7 show that $(J_2, \gamma_2) \sim (J_3, \gamma_2)$ for $J_3 = F_m(Q_1, - J, P^3_{1,5}|H, G, \varnothing)$. But $- J$ and $P^3_{1,5}$ can be eliminated by defined variable deletions at $b_{4m+k}$ for $k \leq |t|$ and at $b_{5m+k}$ for $k \leq m$, respectively, again using the hypotheses for $\alpha$ and $\beta$. So, $(J_3, \gamma_2) \sim (J_4, \gamma_2)$ for $J_4 = F_m(Q_1|H, G)$. Finally, observe that $(J_4, \gamma_2) \sim F_m(t)$ by the inessential variables deletion of $G\theta_{m,1}$ using $\{b_k : 2m < k \leq 6m\}$. Therefore, $t \vee s \sim t$ by 3.9 and transitivity, completing the proof of 3.10.

3.11. $M(K; R)$ is a modular lattice.

Proof. By 3.6 and 3.3, $M(K; R)$ has well-defined and commutative meet and join operations. Let $s = (G, \alpha)$, $t = (H, \beta)$ and $u = (J, \gamma)$ in $D(K; R)$, and $m = \max\{|s|, |t|, |u|\}$. To prove the associativity of join in $M(K; R)$, we use 3.9 and observe that $F_m(s, t, u, \vee, \vee) = (G_1, \alpha_1)$ for $G_1 = F_m(Q_{1,3,5}|G, H, J)$ and $\alpha_1 = f_m((\alpha \vee \beta \vee \gamma)^a|\alpha, \beta, \gamma)$, and that $F_m(s, t, u, \vee)$ is the renumbering of $(G_1, \alpha_1)$ by $\text{tr}_{2m}(Q, 3)$. Replacing $\vee$ by $\wedge$ and $Q_{1,3,5}$ by $Q_1, Q_3, Q_5$ above, a proof of the associativity of the meet operation in $M(K; R)$ is obtained.

We now prove that $s_0 \sim s_1$ if $s_0 = (s \vee t) \wedge (s \vee u)$ and $s_1 = s \vee (t \wedge (s \vee u))$. By 3.9 and the renumbering function $\text{tr}_{2m}(3, 4)$, we have $s_0 \sim F_m(s, t, \vee, s, u, \vee, \wedge) \sim (H_1, \beta_1)$, where

$$H_1 = F_m(Q_{1,3}, Q_{7,9}|G, H, \varnothing, J, G),$$

$$\beta_1 = f_m(((\alpha \vee \beta) \wedge (\alpha \vee \gamma))^a|\alpha, \beta, \psi_\omega, \gamma, \alpha).$$

If we introduce equations $P^0_{8,4}$ and $P^1_{9,5}$ by defined variable augmentations at $b_{4m+k}$ and $b_{5m+k}$ for $k \leq m$, and then apply 3.8 and 3.7 using these equations, we obtain $(H_1, \beta_1) \sim (H_2, \beta_2)$, where

$$H_2 = F_m(Q_{1,3}, Q_{7,9}, P^0_{8,4}, P^1_{9,5}|G, H, G, J, \varnothing), \quad \beta_2 = f_m((\beta_1)^a|\alpha, \beta, \gamma).$$
We can now use 3.7 to replace $P^8_{0,4}$ by $P^8_{0,4}$ and $P^9_{1,5}$ by $P^9_{1,5}$ (negation) and also replace $Q_{7,9}$ by $Q_{1,5,7}$, since $[[P^9_{1,5}, Q_{7,9}]] = [[P^9_{1,5}, Q_{1,5,7}]]$. But then it is possible to eliminate $P^8_{0,4}$ and $P^9_{1,5}$ by defined variable deletions at $b_{8m+k}$ and $b_{9m+k}$ for $k \leq m$, and obtain $(H_2, \beta_2) \sim (H_3, \beta_3)$ as follows:

$$
H_3 = F_m(Q_{1,3}, Q_{1,5,7} | G, H, G, J), \quad \beta_3 = f_m((\beta_1)^a | \alpha, \beta, \alpha, \gamma).
$$

We now add equations $P^3_{5,7} \subset [[Q_{1,3}, Q_{1,5,7}]]$ by linear combination augmentations, and then make constraint decreases at $b_{3m+k}$ for $k \leq m$, using these equations. Then $(H_3, \beta_3) \sim (H_4, \beta_4)$, where

$$
H_4 = F_m(Q_{1,3}, Q_{1,5,7}, P^3_{5,7} | G, H, G, J),
$$

$$
\beta_4 = f_m((\beta_1)^a | \alpha, \beta, \alpha, \gamma) \text{ for } (\beta_0)^a = (\beta \wedge (\alpha \vee \gamma))^a \text{ and } (\beta_0)^b = \beta^b.
$$

Finally, we make constraint decreases at $a_k$ for $k \leq m$ using the equations of $Q_{1,3}$, then make constraint increases at $b_{3m+k}$ for $k \leq m$ using $P^3_{5,7}$ equations and reversing the previous constraint decreases at $b_{3m+k}$, and then eliminate $P^3_{5,7}$ by linear combination deletions. This proves $(H_4, \beta_4) \sim (H_5, \beta_5)$ as given below:

$$
H_5 = F_m(Q_{1,3}, Q_{1,5,7} | G, H, G, J), \quad \beta_5 = f_m((\alpha \vee (\beta \wedge (\alpha \vee \gamma)))^a | \alpha, \beta, \alpha, \gamma).
$$

(We have used the fact that $(\beta_1)^a \wedge (\beta_5)^a = (\beta_5)^a$ because $x \vee (y \wedge (x \vee z)) \subset (x \vee y) \wedge (x \vee z)$ in the lattice $K$.) But $(H_5, \beta_5) = F_m(s, t, s, u, \vee, \wedge, \vee) \sim s_1$ by 3.9, so we have proved the required equivalence $s_0 \sim s_1$.

We now prove the absorption laws for $M(K; R)$. It is easily checked that $F_m(s, t, \wedge) \ll F_m(s)$. But then we obtain the absorption equivalence $s \vee (s \wedge t) \sim s$ by 3.10, 3.6 and 3.9. Furthermore, if we substitute $s \wedge t$ for $u$ in the equivalence $s_0 \sim s_1$ above, and then reduce both sides of the resulting equivalence using the above absorption equivalence, 3.3 and 3.6, we obtain a proof of the dual absorption equivalence $s \wedge (s \vee t) \sim s$. Therefore, $M(K; R)$ is a lattice. But then the equivalence $s_0 \sim s_1$ proved above implies that $M(K; R)$ is modular, completing the proof of 3.11.

3.12. For any $(0, 1)$ lattice $K$, $\psi: K \rightarrow M(K; R)$ is a lattice homomorphism such that $\psi(\omega)$ is a minimum element for $M(K; R)$.

PROOF. For $x, y$ in $K$, we can show that $(\emptyset, \psi_{x \lor y})$ is a union deletion of $(\emptyset, \psi_x) \lor (\emptyset, \psi_y)$, and so they are equivalent. Starting from $(\emptyset, \psi_x) \wedge (\emptyset, \psi_y)$, replace $a_j - b_j$ by $-a_j + b_j$ for $j = 2, 4$ using 3.7. Then constraint decreases and defined variable deletions at $b_2$ and $b_4$ lead to the result $(\emptyset, \psi_{x \lor y}) \sim (\emptyset, \psi_x) \wedge (\emptyset, \psi_y)$. Therefore, $\psi$ is a lattice homomorphism. If $s \wedge (\emptyset, \psi_\omega) = (J, \gamma)$ in $D(K; R)$, we observe that $\gamma^a = \psi_\omega$. Therefore, $s \wedge (\emptyset, \psi_\omega) \sim (\emptyset, \psi_\omega)$ by 3.4, and so $\psi(\omega)$ is a minimum element for $M(K; R)$. This proves 3.12.
**Definition.** Let \( m \geq 1 \). For \( g \in F_R(V) \), \( \lambda_m(g) \) denotes the \( h \) in \( F_R(V) \) such that \( h(a_k) = 0 \) for \( k \leq m \), \( h(a_k) = g(a_{k-m}) \) for \( k > m \), and \( h(b_k) = g(b_k) \) for \( k \geq 1 \). If \( G \subseteq F_R(V) \), let \( \lambda_m(G) \) denote \( \{ \lambda_m(g) : g \in G \} \). Similarly, if \( \alpha \) is a lattice constraint function, let \( \lambda_m(\alpha) \) denote the lattice constraint function \( \beta \) such that \( \beta(a_k) = 0 \) if \( k \leq m \), \( \beta(a_k) = \alpha(a_{k-m}) \) if \( k > m \), and \( \beta(b_k) = \alpha(b_k) \) for \( k \geq 1 \).

Let \( D_m(r_0, r_1, \ldots, r_n) \) for \( r_0, r_1, \ldots, r_n \in R \) denote
\[
\{ r_0a_k + r_1a_{m+k} + r_2a_{2m+k} + \cdots + r_na_{nm+k} : k = 1, 2, \ldots, m \}.
\]
For example, \( D_m(1, r) = \{ a_k + ra_{m+k} : k \leq m \} \).

If \( s = (G, \alpha) \) in \( (K; R) \), let \( \pi_m(s) \) denote \( (\lambda_m(G), \lambda_m(\alpha)) \) in \( (K; R) \), and let \( \delta_m(s, r) \) denote \( (\lambda_m(G) \cup D_m(1, r), \delta_m(\alpha)) \) in \( (K; R) \) for \( r \in R \), where \( \delta_m(\alpha)^b = \alpha^b \), \( \delta_m(\alpha)(a_k) = \alpha(a_{k-m}) \) for \( k \leq m \), and \( \delta_m(\alpha)(a_k) = \alpha(a_{k-m}) \) for \( k > m \). If \( m \geq |s| \), then \( \delta_m(\alpha) = \alpha^a \vee \lambda_m(\alpha) \).

If \( s \sim t \) in \( (K; R) \), then \( \pi_m(s) \sim \pi_m(t) \) and \( \delta_m(s, r) \sim \delta_m(t, r) \) for \( r \in R \) can be verified without much difficulty. If \( x = [s] \) in \( (K; R) \), therefore, we can define \( \pi_m(x) = [\pi_m(s)] \) and \( \delta_m(x, r) = [\delta_m(s, r)] \) in \( (K; R) \), independently of the choice of representative \( s \) of \( x \).

Let \( |x| \) denote \( \min(|s| : x = [s]) \) for \( x \) in \( (K; R) \). We will consider \( \pi_m(x) \) and \( \delta_m(x, r) \) only when \( m \geq |x| \). Observe that \( |\pi_m(x)| \leq m + |x| \). Furthermore, \( |\delta_m(x, r)| \leq m + |x| \), deleting null equations from \( D_m(1, r) \) by \( 3.4 \) if necessary.

**Interpretation.** Given an embedding \( \iota : K \to \Gamma(M; R) \) such that \( \iota(\omega) = 0 \), let the solution set \( \nu^*(x) \) of \( x \) in \( (K; R) \) be the common solution set \( \nu(G, \alpha, \iota) \) of all the representatives \( (G, \alpha) \) of \( x \), \( (G, \alpha) \) in \( (K; R) \). For \( k = |x| \), the solutions of \( x \) in \( \Gamma_f(M^N; R) \) are certain sequences of the form \( (v_1, v_2, \ldots, v_k, 0, 0, \cdots) \), where \( v_i \in M \) for \( i < k \). If \( m \geq |x| \), then \( (v_1, v_2, \ldots, v_k, 0, 0, \cdots) \) is in \( \nu^*(x) \) if and only if the two sequences below are in \( \nu^*(\pi_m(x)) \) and \( \nu^*(\delta_m(x, r)) \), respectively:

\[
(0, 0, \cdots, 0, v_1, v_2, \cdots, v_k, 0, 0, \cdots),
\]
\[
(-rv_1, -rv_2, \cdots, -rv_k, 0, \cdots, 0, v_1, v_2, \cdots, v_k, 0, 0, \cdots).
\]
(In both cases above, \( v_i \) appears in the \( (m + i) \)th position of the sequence.) That is, \( \nu^*(\pi_m(x)) \) is an \( R \)-module which is isomorphic to \( \nu^*(x) \) and is disjoint from \( \nu^*(x) \) in \( \Gamma_f(M^N; R) \), obtained by translating \( m \) coordinate positions to the right. Furthermore, \( \nu^*(\delta_m(x, r)) \) is the "negative graph" (see \([7, \S 2]\)) of the \( R \)-homomorphism \( rF \), where \( F : \nu^*(\pi_m(x)) \to \nu^*(x) \) is the translation isomorphism. (Since \( R \) is commutative, \( rF \) is \( R \)-linear.) In particular, \( \nu^*(\delta_m(x, 1)) \) is the negative graph of \( F \) and of its reciprocal \( F^{-1} : \nu^*(x) \to \nu^*(\pi_m(x)) \). In \( A_{(M; R)} \), maps can be specified by suitable pairs of lattice elements, each element acting like the negative graph of an \( R \)-homomorphism. To specify \( \lambda_A(r) : A \to A \) for \( A = x/\Psi(\omega) \).
in $A_{M(K;R)}$, we will use the elements $\delta_m(x, r)$ and $\delta_m(x, 1)$ for $m = |x|$. This corresponds to the composite

$$\nu^*(x) \xrightarrow{F^{-1}} \nu^*(\pi_m(x)) \xrightarrow{rF} \nu^*(x)$$

in the interpretation. The definition of $\xi_A(r)$ for arbitrary objects $A$ of $A_{M(K;R)}$ is a slight modification of the definition described for the case above.

3.13. If $x$ is in $M(K; R)$, $r$ is in $R$ and $m \geq |x|$, then $\delta_m(x, r) \subset x \lor \pi_m(x)$, $\pi_m(x) \subset x \lor \delta_m(x, r)$ and $r = 1$ implies that $x \subset \pi_m(x) \lor \delta_m(x, r)$. Furthermore, $x \land \delta_m(x, r) = \psi(\omega)$, and $r = 1$ implies that $\pi_m(x) \land \delta_m(x, r) = \psi(\omega)$. If $x_0, x_1, \ldots, x_d$ are in $M(K; R)$ and $n \geq |x_1|, |x_2|, \ldots, |x_d|$, then $\pi_n(x_0) \land (x_1 \lor x_2 \lor \cdots \lor x_d) = \psi(\omega)$. If $y \subset z$ in $M(K; R)$ and $p \geq 1$, then $\pi_p(y) \subset \pi_p(z)$ and $\delta_p(y, r) \subset \delta_p(z, r)$.

**Proof.** Assume the hypotheses for $x$, $r$ and $m$, and choose $s = (G, \alpha)$ in $D(K; R)$ such that $x = \lfloor s \rfloor$ and $m \geq |s|$. Using 3.9 and the renumbering $\text{tr}_{4m}(0, 2)$, we have $\delta_m(s, r) \sim F_{2m}(\delta_m(s, r)) \sim (J_1, \gamma_1)$, where

$$J_1 = F_{2m}(Q_5(2m)|\varnothing, \varnothing, D_m(1, r) \cup \lambda_m(G)),$$

$$\gamma_1 = f_{2m}((\alpha^a \lor \lambda_m(\alpha))^a|\psi_\omega, \psi_\omega, \alpha^a \lor \lambda_m(\alpha)).$$

By defined variable augmentations, the following sets of equations are introduced: $P_{10}^2(m), P_8^4(m), P_{11}^7(m)$ and $J' = \{b_k + rb_{8m+k}: k \leq m\}$. So, $(J_1, \gamma_1) \sim (J_2, \gamma_2)$ as given below.

$$J_2 = F_{2m}(Q_5(2m), P_{10}^2(m), P_8^4(m), P_{11}^7(m), J'|\varnothing, \varnothing, D_m(1, r) \cup \lambda_m(G)),$$

$$\gamma_2 = f_{2m}((\alpha^a \lor \lambda_m(\alpha))^a|\alpha, \lambda_m(\alpha), \alpha^a \lor \lambda_m(\alpha)).$$

Now $Q_2(m) \subset [[Q_5(2m), P_{10}^2(m)]]$ and $\gamma_2(b_{6m+k}) = \lambda_m(\alpha)\alpha_k = \omega$ for $k \leq m$, so $(J_2, \gamma_2) \sim (J_2 \cup Q_{2,5}(m), \gamma_2)$ by linear combination augmentations followed by null variable augmentations. Similarly, we can add the remaining $m$ equations of $Q_{1,3}(2m)$ not in $Q_{2,5}(m)$ by linear combination augmentations from $[[Q_5(2m), P_{11}^7(m)]]$ followed by null variable augmentations. Furthermore, we have

$$G_{\theta_{2m,0}} \subset [[\lambda_m(G)\theta_{2m,2}, J', P_{10}^2(m), D_m(1, r)\theta_{2m,2}]]$$

and

$$\lambda_m(G)\theta_{2m,1} \subset [[\lambda_m(G)\theta_{2m,2}, P_8^4(m), P_{11}^7(m)]].$$
and $b_{2m+k} + rb_{11m+k}$ is the sum of $b_{2m+k} - b_{10m+k}$ in $P_{10}^2(m)$ and $b_{10m+k} + rb_{11m+k}$ in $D_m(1, r)\theta_{2m,1}$. The second inclusion follows from 3.8, so $(J_2, \gamma_2) \sim (J_3, \gamma_2)$ for $J_3 = J_2 \cup F_{2m}(Q_{1,3}(2m)|G, \lambda_m(G))$. But

$$(J_3, \gamma_2) \ll F_{2m}(s, \pi_m(s), \nu),$$

and so $\delta_m(x, r) \subseteq x \lor \pi_m(x)$ by 3.9 and 3.10.

By arguments similar to the foregoing, we obtain $\pi_m(s) \sim (H_1, \beta_1) \sim (H_2, \beta_2)$, given as follows:

$$H_1 = F_{2m}(Q_5(2m)|\emptyset, \emptyset, \lambda_m(G)),$$

$$\beta_1 = f_{2m}(\lambda_m(\alpha^a|\psi, \psi_\omega, \lambda_m(\alpha)),$$

$$H_2 = F_{2m}(Q_5(2m), P_8^2(m), P_{11}^7(m), H', P_{2,6}^2(m)|\emptyset, D_m(1, r), \lambda_m(G)),$$

$$\beta_2 = f_{2m}(\lambda_m(\alpha^a|\alpha, \alpha^a \lor \lambda_m(\alpha), \lambda_m(\alpha)),$$

$$H' = \{b_k - rb_{8m+k}: k \leq m\}.$$ 

(Note that $D_m(1, r)\theta_{2m,1}$ is obtained by defined variable augmentations at $b_{6m+k}$ for $k \leq m$, and that these equations must be introduced before those in $P_{2,6}^2(m)$.)

We can obtain equations $Q_{2,6}^2(m)$ by negating $P_{2,6}^2(m)$ using 3.7 and then making null variable deletions and then augmentations at each $a_k, k \leq m$. The $m$ equations of $Q_{1,3}(2m) - Q_{2,6}^2(m)$ can be obtained by the method previously used. We see that

$$G_{\theta_{2m,0}} \subseteq [[\lambda_m(G)\theta_{2m,2}, H', D_m(1, r)\theta_{2m,1}, P_{2,6}^2(m), P_{11}^7(m)]]$$

observing that $b_{2m+k} - rb_{11m+k}$ is in $[[D_m(1, r)\theta_{2m,1}, P_{2,6}^2(m), P_{11}^7(m)]]$ for all $k \leq m$. Using the previous inclusion for $\lambda_m(G)\theta_{2m,1}$, we obtain $(H_2, \beta_2) \sim (H_3, \beta_2)$ for $H_3 = H_2 \cup F_{2m}(Q_{1,3}(2m)|G, \lambda_m(G))$. But then $(H_3, \gamma_2) \ll F_{2m}(s, \delta_m(s, r), \nu)$ implies $\pi_m(x) \subseteq x \lor \delta_m(x, r)$ by 3.9, 3.10. We will omit the similar proof that $r = 1$ implies $x \subseteq \pi_m(x) \lor \delta_m(x, r)$. (The condition $r = 1$ is needed so that the equations $b_{6m+k} + rb_{7m+k}$ of $D_m(1, r)\theta_{2m,1}$ can be introduced by defined variable augmentations at $b_{7m+k}$. This completes the proof of the first part.

By 3.9 and 3.8, $s \land \delta_m(s, r) \sim F_{2m}(\delta_m(s, r), s, \land) \sim (H, \beta)$, where $H = F_{2m}(Q_1, Q_3, D_m(1, r)|\lambda_m(G), G)$ and $\beta = f_{2m}(\alpha^a|\alpha^a \lor \lambda_m(\alpha), \alpha)$. But $(H, \beta) \sim (H, \beta_0)$ for $\beta_0 = \psi_\omega$ and $\beta_0 = \beta$ by constraint decreases at $a_k$ using $a_k + ra_{m+k}$ for each $k \leq m$. Since $(H, \beta_0) \sim (\emptyset, \psi_\omega)$ by 3.4, we have $x \land \delta_m(x, r) = \psi(\omega)$. The proof that $r = 1$ implies that $\pi_m(x) \land \delta_m(x, r) = \psi(\omega)$ is similar, completing the second part.

Assume the hypotheses for $x_1, x_2, \ldots, x_d$ and $n$. For each $i, 0 \leq i \leq d$, choose $s_i = (G_i, \alpha_i)$ such that $x_i = [s_i]$ and $n \geq |s_i|$ if $i \geq 1$. Using 3.9, $\pi_n(x_0) \land
(x₁ ∨ x₂ ∨ ⋯ ∨ xₙ) = [J, γ] for some (J, γ) in D(K; R) such that γᵣ = (λᵣ(α₀) ∨ (α₁ ∨ α₂ ∨ ⋯ ∨ αₙ))ᵣ. But γᵣ = ψ_ᵣ is easily verified, and so [J, γ] = [∅, ψ_ω] = ψ(ω) by 3.4, proving the third part.

Suppose y ∈ z and p ≥ 1. Choose s₀ and t₀ such that y = [s₀] and z = [t₀], and let q ≥ |s₀|, p + |t₀|. Then y = [s] and z = [t] for s = F_q(t₀, s₀, ∧) and t = F_q(t₀), using 3.9. By inspection, s ≪ t, π_p(s) ≪ π_p(t) and δ_p(s, r) ≪ δ_p(t, r). But then π_p(y) ⊂ π_p(z) and δ_p(y, r) ⊂ δ_p(z, r) by 3.10, completing the proof of 3.13.

3.14. M(K; R) is an abelian lattice.

**Proof.** By 3.11 and 3.12, M(K; R) is a modular lattice with smallest element ψ(ω). To prove that every x in M(K; R) can be “tripled”, we observe by 3.13 that m ≥ |x| implies

x ∨ n_m(x) = x ∨ δ_m(x, 1) = π_m(x) ∨ δ_m(x, 1),

and

x ∧ n_m(x) = x ∧ δ_m(x, 1) = π_m(x) ∧ δ_m(x, 1) = ψ(ω).

Therefore, M(K; R) is an abelian lattice by [7, 4.1, p. 181]. (Note that the above proof simply adapts the proof of [7, 4.2].)

We next prove two complex inclusion relations in M(K; R) that are needed to complete the verification of 2.6.

3.15. Suppose x, y, z are in M(K; R), m ≥ |x|, |y|, p ≥ 2m and z ⊂ x ∨ π_m(y). For r in R, let w(r) denote (δ_p(x, r) ∨ z) ∧ (π_p(x) ∨ π_m(y)) in M(K; R). Then for any r₁, r₂ in R, it follows that

(w(r₂) ∨ δ_m(y, r₁)) ∧ (π_p(x) ∨ y) ⊂ (w(r₁r₂) ∧ δ_m(y, 1) ∧ (π_p(x) ∨ y).

**Proof.** Assuming the hypotheses, choose s = (G, α), t = (H, β) and u₀ in D(K; R) such that x = [s], y = [t], z = [u₀] and m ≥ |s|, |t|. Because of the hypothesis z ⊂ x ∨ π_m(y), we have by 3.9 that, for a sufficiently large j, u₀ ∼ u = (J, γ) = F_j(u₀, s, π_m(t), ∨, ∧). Note that γᵣ ⊂ (α ∨ λ_m(β))ᵣ, and so γ(αᵣ) = ω if i > 2m.

Suppose k ≥ 2p, |u| and n = 8k, and let D₀(r) denote {r aᵢ − b₁₁k+i; i ≤ m}. We intend to define G₀, H₀ and α₀ independently of r such that w(r) = [w(r)] for w(r) = (D₀(r) ∪ G₀ ∪ H₀, α₀), g in G₀ implies g(αᵣ) = 0 unless p < i ≤ p + m and g(βᵣ) = 0 unless i ≤ n, h in H₀ implies h(αᵣ) = 0 unless m < i ≤ 2m and h(βᵣ) = 0 unless n < i ≤ 2n, α₀ ⊂ (λ(α) ∨ λ_m(β)) and α₀(βᵣ) = ω for i > 2n. Now w(r) = [t₁(r)] for t₁(r) = F_k(δ_p(s, r), u, ∨, π_p(s), π_m(t), ∨, ∧) by 3.9. Using the renumbering ṭₙ₁(1, S), we have t₁(r) ∼ (H₁, α₀), as given below.
\[ H_1 = F_k(Q_{1,11}, Q_{7,9} | D_p(1, r) \cup \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J), \]
\[ \alpha_0^a = ((\alpha^a \lor \lambda_p(\alpha) \lor \gamma) \land (\lambda_p(\alpha) \lor \lambda_m(\beta))^a, \]
\[ \alpha_0 = f_k(\alpha_0^a | \alpha^a \lor \lambda_p(\alpha), \psi_\omega, \psi_\omega, \lambda_p(\alpha), \lambda_m(\beta), \gamma). \]

Note that \( \alpha_0^a \subset (\lambda_p(\alpha) \lor \lambda_m(\beta))^a \) and \( \alpha_0(b_i) = \omega \) unless \( i \leq 12k < 2n \). Using 3.4, replace \( D_p(1, r) \theta_{k,0} \) by \( D_1(r) = \{ b_{k+i} + b_{k+p+i}: i \leq m \} \), by deleting \( p - m \) null equations. Then \( D_1(r) \) can be replaced by

\[ D_2(r) = \{ a_i - b_{11k+i} + ra_{p+i} - rb_{11k+p+i}: i \leq m \} \]

by 3.7, since \([D_1(r), Q_{1,11}(k)] = [D_2(r), Q_{1,11}(k)] \). But \( \alpha_0(a_i) = \alpha_0(b_{11k+p+i}) = \omega \) for \( i \leq m \), so \( D_2(r) \) can be replaced by \( D_0(r) \) by 3.4. So, we obtain \( (H_1, \alpha_0) \sim (H_2, \alpha_0) \), where

\[ H_2 = F_k(D_0(r), Q_{1,11}, Q_{7,9} | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J). \]

We now define \( H_3 = F_k(D_0(r), H_4, H_5, H_6, H_7, H_8 | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G), \lambda_m(H), J) \), where

\[ H_4 = \{ -b_{k+i} - b_{11k+i}: i \leq m \}, \quad H_5 = \{ a_{m+i} - b_{11k+m+i}: i \leq m \}, \]
\[ H_6 = \{ a_{p+i} - b_{k+p+i}: i \leq m \}, \quad H_7 = \{ a_{m+i} - b_{9k+m+i}: i \leq m \} \quad \text{and} \]
\[ H_8 = \{ a_{p+i} - b_{7k+p+i}: i \leq m \}. \]

By 3.4, we can replace \( Q_{1,11}(k) \) by \( H_4 \cup H_5 \cup H_6 \) and \( Q_{7,9}(k) \) by \( H_7 \cup H_8 \), proving \( (H_2, \alpha_0) \sim (H_3, \alpha_0) \). Furthermore, the equations of \( H_4 \) can be eliminated by first replacing \( H_4 \) by \( -H_4 \) by 3.7, and then using constraint decreases and defined variable deletions of \( b_{k+i} + b_{11k+i} \) at \( b_{k+i} \), since \( \gamma^a \subset (\alpha \lor \lambda_m(\beta))^a \). So, if we define

\[ G_0 = F_k(H_6, H_8 | \lambda_p(G), \emptyset, \emptyset, \lambda_p(G)) \quad \text{and} \]
\[ H_0 = F_k(H_5, H_7 | \emptyset, \emptyset, \emptyset, \lambda_m(H), J), \]

we obtain \( (H_3, \alpha_0) \sim (D_0(r) \cup G_0 \cup H_0, \alpha_0) \). This proves \( w(r) = [w(r)] \) for \( w(r) = (D_0(r) \cup G_0 \cup H_0, \alpha_0) \), and it is easily seen that \( G_0, H_0 \) and \( \alpha_0 \) have the required properties.

The inclusion relation we must prove follows from 3.9 and 3.10 if we can construct \( (J_0, \gamma_0) \) in \( D(K; R) \) such that

\[ u^*(r_1, r_2) = F_{2n}(w(r_2), \delta_m(t, r_1), \omega, \pi_p(s), t, \lor, \land) \sim (J_0, \gamma_0), \]
\[ (J_0, \gamma_0) \ll u^*(r_1 r_2, 1) = F_{2n}(w(r_1 r_2), \delta_m(t, 1), \lor, \pi_p(s), t, \lor, \land). \]
First, we use the renumbering $\text{tr}_4(n, 5)\text{tr}_4(n, 1, 6)$ to establish $u^*(r_1, r_2) \sim (J_1, \gamma_1)$, for $J_1$ and $\gamma_1$ as follows, respectively:

\begin{align*}
F_{2n}(Q_{7,9}, Q_{11,13} | \varnothing, \varnothing, \varnothing, \lambda_p(H), H, D_0(r_2) \cup G_0 \cup H_0, D_m(1, r_1) \cup \lambda_m(H)),
\end{align*}

where $\gamma_1^a = ((\alpha_0 \lor \beta^a \lor \lambda_m(\beta)) \land (\lambda_p(\alpha) \lor \beta))^a$.

We now introduce by defined variable augmentations the sets of equations $G_1, G_2, \ldots, G_7$, where

\begin{align*}
G_1 &= P_{2n}(n), \\
G_2 &= \{b_n + r_1b_{2n} \mid i < n\}, \\
G_3 &= \{b_{2n+m} + r_1b_{2n+m} \mid i < m\}, \\
G_4 &= \{b_{2n+p} + r_1b_{2n+p} \mid i < m\}, \\
G_5 &= \{b_{4n+i} - r_1b_{4n+i} \mid i < n\}, \\
G_6 &= \{b_{6n+i} - b_{26n+i} \mid i < m\}, \\
G_7 &= \{b_{6n+m+i} - r_1b_{6n+m+i} \mid i < m\}.
\end{align*}

Then we obtain $(J_1, \gamma_1) \sim (J_2, \gamma_2)$, where $J_2 = J_1 \cup G_1 \cup G_2 \cup \cdots \cup G_7$ and

\begin{align*}
\gamma_2 &= f_{2n}(\gamma_1^a | \alpha_0, \beta^a \lor \lambda_m(\beta), \psi_\omega, \lambda_p(\alpha), \beta, \alpha_0, \beta^a \lor \lambda_m(\beta)).
\end{align*}

By 3.8 and the properties of $G_0$, $H_0$ and $\lambda_m(H)$, we see that

\begin{align*}
G_0^{2n,0} \subset \left[ [G_0^{2n,5}, G_1, G_4] \right], \\
H_0^{2n,0} \subset \left[ [H_0^{2n,5}, G_2, G_3] \right] \\
\lambda_m(H)^{2n,1} \subset \left[ [\lambda_m(H)^{2n,6}, G_5, G_7] \right].
\end{align*}

Furthermore, $D_0(r_1, r_2)^{2n,0} \subset \left[ [D_0(r_2)^{2n,5}, G_2, G_4] \right]$ and $D_m(1, 1)^{2n,1} \subset \left[ [D_m(1, r_1)^{2n,6}, G_6, G_7] \right]$ are easily verified. Therefore, $(J_2, \gamma_2) \sim (J_3, \gamma_2)$, where

\begin{align*}
J_3 &= J_2 \cup F_{2n}(\varnothing | D_0(r_1, r_2) \cup G_0 \cup H_0, D_m(1, 1) \cup \lambda_m(H)),
\end{align*}

We define $J_0 = J_3 \cup Q_{1,3}(2n)$ and $\gamma_0 = \gamma_2$, and observe that $(J_0, \gamma_0) \ll u^*(r_1, r_2, 1)$. Now the equations $a_i - b_{6n+i} - b_{22n+i}$ are in $[[Q_{11,13}(2n), G_6]]$, and $\gamma_2(b_{2n+i}) = \gamma_2(b_{22n+i}) = \omega$, for each $i < m$. So, $(J_3, \gamma_2) \sim (J_4, \gamma_2)$ such that $J_4$ contains $J_3$ and the first $m$ equations of $Q_{1,3}(2n)$, by 3.4. We can add the equations $a_i - b_{2n+i} - b_{6n+i}$ to $(J_4, \gamma_2)$ for $m < i < 2m$ and for $p < i < p + m$ by similar arguments. The remaining equations of $Q_{1,3}(2n)$ are null equations for $\gamma_2$, and so can also be added to $(J_4, \gamma_2)$ by 3.4, giving the result $(J_4, \gamma_2) \sim (J_5, \gamma_5)$. So, $u^*(r_1, r_2) \sim (J_5, \gamma_5) \ll u^*(r_1, r_2, 1)$, completing the proof of 3.15.
3.16. Suppose \( x \in M(K; R) \) and \( m \geq |x| \). Let
\[
\begin{align*}
y_1 &= (\delta_m(x, 1) \lor \delta_{2m}(x, 1)) \land (\pi_m(x) \lor \pi_{2m}(x)), \\
y_2 &= (\delta_m(x, 1) \lor \delta_{3m}(x, 1)) \land (\pi_m(x) \lor \pi_{3m}(x)) \text{ and} \\
y_3 &= (y_1 \lor \pi_{3m}(x)) \land (y_2 \lor \pi_{2m}(x)).
\end{align*}
\]
Then for any \( r_1, r_2 \in R \), it follows that
\[
(y_3 \lor \delta_{2m}(x, r_1) \lor \delta_{3m}(x, r_2)) \land (x \lor \pi_m(x)) \subseteq \delta_m(x, r_1 + r_2).
\]

**Proof.** Assume the hypotheses. Choose \( s = (G, \alpha) \in D(K; R) \) such that \( x = [s] \) and \( m \geq |s| \). For \( p = 4m \), we have by 3.9 that \( y_1 = [j_1, \gamma_1] \), where \( J_1 \) and \( \gamma_1 \) equal, respectively,
\[
F_p(Q_{1,3}, Q_{7,9}\{D_m(1, 1) \lor \lambda_m(G), D_{2m}(1, 1) \lor \lambda_{2m}(G), \emptyset, \lambda_m(G), \lambda_{2m}(G)\), \] \[
f_p((\lambda_m(\alpha) \lor \lambda_{2m}(\alpha))^a | \alpha^a \lor \lambda_m(\alpha), \alpha^a \lor \lambda_{2m}(\alpha), \psi_\omega, \lambda_m(\alpha), \lambda_{2m}(\alpha)).
\]
For \( k \leq m \), observe that the equation \( a_k + a_{m+k} + a_{2m+k} - b_p + 2m + k - b_{3p+m+k} \) is in \([Q_{1,3}, D_m(1, 1) \theta_{p,0}, D_{2m}(1, 1) \theta_{p,1}]\). But \( \gamma_1(a_k) = \gamma_1(b_{p+2m+k}) = \gamma_1(b_{3p+m+k}) = \omega \) for \( k \leq m \), so by 3.7 and 3.4 we have \( y_1 = [t_1] \), where \( t_1 = (D_m(0, 1, 1) \lor J_1, \gamma_1) \). By a similar argument, one can show that \( y_2 = [t_2] \), where \( t_2 = (D_m(0, 1, 0, 1) \lor J_2, \gamma_2) \) for a suitable \( J_2 \) and \( \gamma_2 \) such that \( \gamma_2^a = (\lambda_m(\alpha) \lor \lambda_{3m}(\alpha))^a \).

Choosing \( n \geq 6m \), \( |t_1|, |t_2| \), we observe by 3.12 and 3.9 that
\[
(y_3 \lor \delta_{2m}(x, r_1) \lor \delta_{3m}(x, r_2)) \land (\pi_m(x) \lor x) = [F_n(x_0, x_1, \ldots, x_{14})],
\]
where \( (x_0, x_1, \ldots, x_{14}) \) is the \( D(K; R) \) polynomial:
\[
(\pi_m(s), s, \lor, t_1, \pi_{3m}(s), \lor, t_2, \pi_{2m}(s), \lor, \land, \delta_{2m}(s, r_1), \lor, \delta_{3m}(s, r_2), \lor, \land).
\]
Let \( F_n(x_0, x_1, \ldots, x_{14}) = (J_3, \gamma_3) \). Then
\[
J_3 = F_n(Q(I_{14})|G_0, G_1, \ldots, G_{14}),
\]
where \( G_0 = \lambda_m(G), D_m(0, 1, 1) \subseteq G_3, D_m(0, 1, 0, 1) \subseteq G_6, D_m(1, 0, r_1) \subseteq D_{2m}(1, r_1) \subseteq G_{10}, D_m(0, 1, 0, r_2) \subseteq D_{3m}(1, r_2) \subseteq G_{12}, \) and \( Q(I_{14}) = Q_{1,3} \cup Q_{7,9,21,25} \cup Q_{13,15,21,25} \). Furthermore,
\[
\gamma_3 = f_n((\lambda_m(\alpha) \lor \alpha)^a | \beta_0, \beta_1, \ldots, \beta_{14}),
\]
where \( \beta_0 = \alpha^a \lor \lambda_m(\alpha), \beta_1 = \alpha, \beta_3 = (\lambda_m(\alpha) \lor \lambda_{2m}(\alpha))^a, \beta_4 = \lambda_{3m}(\alpha), \beta_7 = \lambda_{2m}(\alpha), \beta_{10} = \alpha^a \lor \lambda_{2m}(\alpha) \) and \( \beta_{12} = \alpha^a \lor \lambda_{3m}(\alpha) \).

For \( g \) in \( G \), the equation \( \lambda_m(g) - \sum_{k=1}^{m} g(a_k)b_{3n+m+k} \) is in \([Q_{1,3}(n), \lambda_m(G)\theta_{n,0}]\), and \( \gamma_3(b_{3n+m+k}) = \beta_1(a_{m+k}) = \omega \) for all \( k \leq m \). Similarly, the
equation
\[ a_k + (r_1 + r_2)a_{m+k} + r_1a_{2m+k} + r_2a_{3m+k} - b_{7n+k} - b_{9n+k} - r_1b_{9n+m+k} \\
- r_2b_{15n+m+k} - (r_1 + r_2)b_{21n+m+k} - (r_1 + r_2)b_{25n+m+k} \\
- r_1b_{9n+2m+k} - r_1b_{25n+2m+k} - r_2b_{15n+3m+k} - r_2b_{21n+3m+k} \]
is in \[ [[Q_{7,9,21,25}, Q_{13,15,21,25}, G_3\theta_{n,3}, G_6\theta_{n,6}, G_{10}\theta_{n,10}, G_{12}\theta_{n,12}]] \] for each \( k < m \), using the properties given for \( G_3, G_6, G_{10} \) and \( G_{12} \) above. But
\[ \gamma_3(a_{2m+k}) = \gamma_3(a_{3m+k}) = \gamma_3(b_{7n+k}) = \gamma_3(b_{9n+k}) = \gamma_3(b_{9n+m+k}) \]
\[ = \gamma_3(b_{15n+m+k}) = \gamma_3(b_{21n+m+k}) = \gamma_3(b_{25n+m+k}) \]
\[ = \gamma_3(b_{9n+2m+k}) = \gamma_3(b_{25n+2m+k}) = \gamma_3(b_{15n+3m+k}) \]
\[ = \gamma_3(b_{21n+3m+k}) = \omega \quad \text{for } k \leq m. \]

Therefore, \((J_3, \gamma_3) \approx (J_4, \gamma_3)\) with \( J_4 = J_3 \cup \lambda_m(G) \cup D_m(1, r_1 + r_2)\), by 3.7 and 3.4. But \((J_4, \gamma_3) \approx (D_m(1, r_1 + r_2) \cup \lambda_m(G), \alpha^a \lor \lambda_m(\alpha)) = \delta_m(s, r_1 + r_2)\), since \( \gamma_3 = (\alpha^a \lor \lambda_m(\alpha))^a \) and \( \gamma_3(b_k) = \alpha(b_k) = (\alpha^a \lor \lambda_m(\alpha))(b_k) \) for \( k \leq 2m \).

The required inclusion then follows from 3.10, completing the proof of 3.16.

We now give a formula for addition in abelian categories constructed from abelian lattices. We can then define \( \xi_\mathcal{A} \) and complete the verification of 2.6. We will use the notations 2, \( T \), \( A^1/A^0 \), \( S(A, B) \), \( S(I(A, B), f^-), g \circ f, f^- \) and the terms “isorepresentative”, “left sequence” and “mixed sequence” taken from [7]. Also, the notations \( \sigma \) and \( \tau \) and the results established in [7, 3.19, p. 173] will be used without reference.

3.17. Suppose \( L \) is an abelian lattice, \( A, B, C, D \) is a left sequence of maps \( 2 \to L \), suppose \( E = E^1/E^0 \) for \( E^1 = C^1 \lor D^1 \), and let \( g \) be in \( SI(C, A) \) and \( h \) be in \( SI(D, A) \). For any \( f_1, f_2 \) in \( S(A, B) \), \( \tau(f_1) + \tau(f_2) = \tau(d \circ c) \) in \( A_L \), where \( c \) in \( S(A, E) \) satisfies \( c^- = (g^- \lor D^1) \land (h^- \lor C^1) \) and \( d \) in \( S(E, B) \) satisfies \( d^- = (f_1 \circ g)^- \lor (f_2 \circ h)^- \).

Proof. We show first that \( f \in S(X, Y) \) and \( f_0 \in S(Y, X) \) such that \( f^- \subset f_0^- \) implies \( \tau(f_0)\tau(f) = 1_X \) in \( A_L \). By [7, 3.11, p. 168], choose \( g_0 \) in \( SI(L, Z) \) for some \( Z \) such that \( X, Y, Z \) is a left sequence. Then \( (g_0 \circ f_0) \circ f \) is defined, and \( f^- \subset f_0^- \) and \( f^- \lor Y^1 = X^1 \lor Y^1 \) imply that \( g_0^- \subset (g_0^- \lor f_0^-) \land (X^1 \lor Y^1 \lor Z^1) = f^- \lor (g_0 \circ f_0)^- \), using modularity. But then \( g_0^- \subset ((g_0 \circ f_0) \circ f)^- \), and so \( g_0 = (g_0 \circ f_0) \circ f \) by [7, 3.4]. Then
\[ \tau(g_0) = \tau(g_0)\tau(f)\tau(f), \]
and the desired result \( \tau(f_0)\tau(f) = 1_X \) follows because \( \tau(g_0) \) is an isomorphism.
Now assume the hypotheses of 3.17. By [7, 3.23, p. 176], there exist $u_1, u_2$ in $S(A, E)$ and $p_1, p_2$ in $S(E, A)$ such that $u_1^- = g^- \lor D^1, u_2^- = h^- \lor C^1, p_1^- = g^- \lor D^1$ and $p_2^- = h^- \lor C^1$. Since $u_1^- \subset p_1^-$ and $u_2^- \subset p_2^-$, we have $\tau(p_1)\tau(u_1) = \tau(p_2)\tau(u_2) = 1_A$ in $A_L$. By modularity and [7, 3.2, 3.3, p. 163], we have $I(u_1) = C^1 \lor D^0 = K(p_2)$ and $I(u_2) = C^0 \lor D^1 = K(p_1)$, and so $(\tau(u_1), \tau(p_2))$ and $(\tau(u_2), \tau(p_1))$ are exact in $A_L$ by [7, 3.25, p. 178]. Therefore, $E = A \oplus A$ and $\tau(u_1), \tau(u_2), \tau(p_1)$ and $\tau(p_2)$ form a direct sum system [5, Theorem 2.42, p. 51]. Using [7, 3.3] and modularity, we can verify that $c$ in $S(A, E)$ and $d$ in $S(E, B)$ exist with $c^-$ and $d^-$ as given above. Now $c^- = p_1^- \land p_2^-$, so $\tau(p_1)\tau(c) = \tau(p_2)\tau(c) = 1_A$ by our previous result. Therefore, $\tau(c) = (1_A, 1_A): A \to A \oplus A$ in $A_L$. Furthermore, $f_1 = (f_1 \circ g) \circ g^-$, and so $f_1^- = (g^- \lor (f_1 \circ g)^-) \land (A^1 \lor B^1) \subset (u_1^- \lor d^-) \land (A^1 \lor B^1) = \tau(d \circ u_1)^-$. Therefore, $f_1 = d \circ u_1$ by [7, 3.4], and so $\tau(f_1) = \tau(d)^-\tau(u_1)$. Since $\tau(f_2) = \tau(d)^-\tau(u_2)$ similarly, we have the equation below, in $A_L$:

$$\tau(d) = \begin{pmatrix} \tau(f_1) \\ \tau(f_2) \end{pmatrix}, \quad A \oplus A \to B.$$ 

Now $A, B, E$ is a left sequence by [7, 3.1, 3.2, p. 163], so $d \circ c$ is defined. But then $\tau(f_1) + \tau(f_2) = \tau(d)^-\tau(c) = \tau(d \circ c)$, using the (left) definition of sum in an abelian category [5, p. 47]. This completes the proof of 3.17.

**Definition.** Suppose $r \in R$, $A$ is in $A_{M(K;R)}$ (that is, $A: 2 \to M(K; R)$) and $m \geq |A^1|$. Let $\pi_m(A)$ denote $\pi_m(A^1)/\pi_m(A^0)$ in $A_{M(K;R)}$, since $\pi_m(A^0) \subset \pi_m(A^1)$ by 3.13. Let $\xi^m_A(r)$ denote the map $f: T \to M(K; R)$ in $S(\pi_m(A), A)$ such that $f^-= A^0 \lor \delta_m(A^1, r)$. (By 3.13 and [7, 3.4, p. 164], such an $f$ exists uniquely.) Again by 3.13, $\xi^m_A(1)$ is isorepresentative, belonging to $S(\pi_m(A), A)$. Using [7, 3.19], define $\xi_A: R \to \text{Hom}(A, A)$ by setting $\xi_A(r)$ equal to $\tau\xi^m_A(r)\tau\xi^m_A(1)^{-1}: A \to A$ for $n = |A^1|$.

**3.18.** For every $A$ in $A_{M(K;R)}$, $\xi_A: R \to \text{Hom}(A, A)$ is a ring homomorphism preserving the unit. If $f: A \to B$ in $A_{M(K;R)}$ and $r \in R$, then $\xi_B(r)f = f\xi_A(r)$.

**Proof.** We first establish the following lemma: If $r_1, r_2$ are in $R$, $A, B$ are in $A_{M(K;R)}$ and $g: A \to \pi_m(B)$ for $m \geq |A^1|, |B^1|$ and $p \geq 2m$, then $\tau\xi^m_B(r_1)g\tau\xi^m_A(r_2) = \tau\xi^m_B(1)g\tau\xi^m_A(r_1 r_2)$. Assuming the lemma hypotheses, we note that $A, \pi_m(B), \pi_p(A)$ and $B, \pi_m(B), \pi_p(A)$ are left sequences in $M(K; R)$ by 3.13. Therefore, $f_1 = \xi^m_B(r_1) \circ (\sigma(g) \circ \xi_p^p(r_2))$ and $f_2 = \xi^m_B(1) \circ (\sigma(g) \circ \xi_p^p(r_1 r_2))$ both exist in $S(\pi_p(A), B)$. We now apply 3.15 with $x = A^1, y = B^1$ and $z = \sigma(g^-)$. Defining $w(r)$ as in 3.15 and observing that $A^0 \subset \sigma(g^-)$
= z, we find that \( w(r) = (g \circ \xi^p_A(r))^- \). But then

\[
f_1^- = (w(r_2) \lor B^0 \lor \delta_m(B^1, r_1)) \land (\pi_p(A^1) \lor B^1)
\]

\[
\subset (w(r_1, r_2) \lor B^0 \lor \delta_m(B^1, 1)) \land (\pi_p(A^1) \lor B^1) = f_2^-, \]

using modularity and 3.15. So, \( f_1 = f_2 \) by [7, 3.4], and \( \tau \xi^n_B(r_1)g \tau \xi^p_A(r_2) = \tau f_1 = \tau f_2 = \tau \xi^n_B(1)g \tau \xi^p_A(r_1, r_2) \) follows, completing the proof of the lemma.

Assume the hypotheses of 3.18, suppose \( r_1, r_2 \in R \), and let \( n = |A^1| \). We will compute \( \tau \xi^n_A(r_1) + \tau \xi^n_A(r_2) \) using 3.17. Now, \( \pi_n(A), A, \pi_{2n}(A), \pi_{3n}(A) \) is a left sequence in \( M(K; R) \) by 3.13, and we have \( g = f_1(l)^{-1} \circ f_1(l) \) in \( S(\pi_{2n}(A), \pi_{3n}(A)) \) and \( h = f_1(l)^{-1} \circ f_1(l) \) in \( SI(\pi_{3n}(A), \pi_{3n}(A)) \). Letting \( E_l = \pi_{2n}(A^l) \lor \pi_{3n}(A^l) \) for \( j = 0, 1 \), we have by 3.17 that \( \sigma(\tau \xi^n_A(r_1) + \tau \xi^n_A(r_2)) = d \circ c \), where \( c \) in \( S(\pi_n(A), E) \) satisfies \( c^- = (g^- \lor \pi_{3n}(A^1)) \land (h^- \lor \pi_{2n}(A^1)) \) and \( d \) in \( S(E, A) \) satisfies \( d^- = (\xi^n_A(r_1) \lor g^-) \lor (\xi^n_A(r_2) \lor h^-) \). Now, \( \tau \xi^n_A(r_1) \land \xi^n_A(r_2) = \tau \xi^n_A(r_1) \land \xi^n_A(r_2) \) by the lemma, and so \( \xi^n_A(r_1) \land \xi^n_A(r_2) = \xi^n_A(r_1) \land \xi^n_A(r_2) \). Similarly, \( \xi^n_A(r_2) \land \xi^n_A(r_1) = \xi^n_A(r_2) \land \xi^n_A(r_1) \), and so \( d^- = A^0 \lor \delta_{2n}(A^1, r_1) \lor \delta_{3n}(A^1, r_2) \). Define \( y_1, y_2 \) and \( y_3 \) as in 3.16, with \( A^1 \) replacing \( x \). Now, \( y_1 \lor \pi_{2n}(A^0) = g^- \), using \( \pi_{2n}(A^0) \lor \delta_{2n}(A^1, 1) = A^0 \lor \delta_{2n}(A^1, 1) \) from 3.13, and modularity. Similarly, \( y_2 \lor \pi_{3n}(A^0) = h^- \), and so \( c^- = y_3 \lor \pi_{3n}(A^0) \lor \pi_{3n}(A^0) \) by 3.13 and modularity again. But \( \pi_{2n}(A^0) \lor \pi_{3n}(A^0) = E^0 \subset d^- \), since \( d \) is in \( S(E, A) \). Therefore, using modularity and 3.16, we have

\[
\sigma(\tau \xi^n_A(r_1) + \tau \xi^n_A(r_2))^- = (d \circ c)^- = (y_3 \lor d^-) \land (\pi_n(A^1) \lor A^1)
\]

\[
= A^0 \lor [(y_3 \lor \delta_{2n}(A^1, r_1) \lor \delta_{3n}(A^1, r_2)) \land (\pi_n(A^1) \lor A^1)]
\]

\[
\subset A^0 \lor \delta_{n}(A^1, r_1 + r_2) = \xi^n_A(r_1 + r_2)^-. \]

But then \( \tau \xi^n_A(r_1 + r_2) = \tau \xi^n_A(r_1) + \tau \xi^n_A(r_2) \) by [7, 3.4], and so

\[
\xi_A(r_1 + r_2) = \tau \xi^n_A(r_1 + r_2) \tau \xi^n_A(1)^{-1} = (\tau \xi^n_A(r_1) + \tau \xi^n_A(r_2)) \tau \xi^n_A(1)^{-1}
\]

\[
= \tau \xi^n_A(r_1) \tau \xi^n_A(1)^{-1} + \tau \xi^n_A(r_2) \tau \xi^n_A(1)^{-1} = \xi_A(r_1) + \xi_A(r_2), \]

using distributivity of composition over sum in abelian categories [5, 2.37, p. 48].

If \( m \geq |A^1| \) and \( p \geq 2m \), we have

\[
\tau \xi^n_A(r) \tau \xi^n_A(1)^{-1} = \tau \xi^n_A(r) \tau \xi^n_A(1)^{-1} \tau \xi^p_A(1) \tau \xi^p_A(1)^{-1}
\]

\[
= \tau \xi^p_A(1) \tau \xi^n_A(1)^{-1} \tau \xi^p_A(r) \tau \xi^p_A(1)^{-1} = \tau \xi^p_A(r) \tau \xi^p_A(1)^{-1}, \]

using the lemma. So, \( \xi_A(r) = \tau \xi^n_A(r) \tau \xi^n_A(1)^{-1} \) for all \( m \geq |A^1| \).

If \( m \geq 2n \), then

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\[ \xi_A(r_1, r_2) = \tau \xi_A^n(r_1, r_2) \xi_A^n(1)^{-1} = \tau \xi_A^n(r_1) \xi_A^n(1)^{-1} \]

using the lemma and the result above. Since \( \xi_A(1) = \tau \xi_A^n(1)^{-1} = 1_A \), we have proved that \( \xi_A \) is a ring homomorphism preserving the unit.

Suppose \( f: A \to B \) in \( A_{M(K; R)} \) and \( r \in R \). Choosing \( m > |A| \), \( |P| \), and \( p > 2m \), we see that

\[ \xi_B(r)f = \tau \xi_B^m(r) \xi_B^m(1)^{-1} \xi A^p(r) \xi A^p(1)^{-1} = \xi B(r), \]

applying the lemma with \( g = \tau \xi_B^m(1)^{-1}f \). This completes the proof of 3.18.

Since 2.6 follows from 3.2, 3.12, 3.14, and 3.18, we have proved the main theorem. In conclusion, some additional properties of the \( M(K; R) \) construction are noted.

Let \( \text{Lat}_\omega \) denote the category of lattices with smallest element \( \omega \) and homomorphisms preserving \( \omega \), and let \( \text{CRng}_1 \) denote the category of commutative rings with unit and unit-preserving ring homomorphisms. Since the construction of \( M(K; R) \) did not use our previous assumption that \( K \) has a largest element, we can obtain a bifunctor \( \text{Lat}_\omega \times \text{CRng}_1 \to \text{Lat}_\omega \) by defining \( M(f; h): M(K; R) \to M(L; S) \) for \( f: K \to L \) in \( \text{Lat}_\omega \) and \( h: R \to S \) in \( \text{CRng}_1 \) as follows:

\[ M(f; h)(x) = \{ [hg: g \in G], f(x) \} \quad \text{for } x = [G, \alpha]. \]

(Recall that \( g \) in \( F_R(V) \) is a coefficient function \( V \to R \) such that \( g(v) = 0 \) for all but finitely many elements \( v \) in \( V \).) Then the following proposition can be verified directly.

3.19. Given \( f: K \to L \) in \( \text{Lat}_\omega \) and \( h: R \to S \) in \( \text{CRng}_1 \), the expression for \( M(f; h)(x) \) determines a well-defined homomorphism \( M(f; h): M(K; R) \to M(L; S) \), and \( M(K; R) \) and \( M(f; h) \) determine a covariant bifunctor \( M: \text{Lat}_\omega \times \text{CRng}_1 \to \text{Lat}_\omega \). If \( f \) and \( h \) are onto, then \( M(f; h) \) is onto. If \( P_1: \text{Lat}_\omega \times \text{CRng}_1 \to \text{Lat}_\omega \) is the projection functor \( (P_1(K, R) = K \) and \( P_1(f; h) = f \), then \( \psi(K, R): K \to M(K; R) \) determines a natural transformation \( \psi: P_1 \to M \).

Since \( L(R) \) is a quasivariety, it is a reflective subcategory of the category of all meet and join algebras and their homomorphisms. (See [16, pp. 88–90] and [19] for discussions of reflective subcategories and their relationship to quasivarieties of universal algebras; note that "reflection" here and in [16] is "coreflection" in [19].) In particular, every \( K \) in \( \text{Lat}_\omega \) has a smallest congruence \( q \) such that \( K/q \) is in \( L(R) \). The universal property for reflection implies the following: Every \( K \to L \) in \( \text{Lat}_\omega \) such that \( L \) is in \( L(R) \) can be uniquely factored \( K \to K/q \to L \).
in $\text{Lat}_\omega$, where the first factor is the canonical epimorphism. The reflection map $K \to K/q$ is closely related to $\psi$.

3.20. For $K$ in $\text{Lat}_\omega$ and $R$ in $\text{CRng}_1$, the reflection map $h: K \to K/q$ is the coimage of $\psi(K, R): K \to M(K; R)$. That is, $\psi(K, R)$ factors uniquely as $K \xrightarrow{h} K/q \xrightarrow{f} M(K; R)$ such that $f$ is one-one.

**Proof.** Since there is an exact embedding functor $G: A_{M(K; R)} \to R\text{-Mod}$ by 2.6 and 2.8, every interval sublattice of $M(K; R)$ is in $L(R)$ by [7, 3.24, p. 178]. But then $M(K; R)$ is in $L(R)$ because $L(R)$ is a quasivariety, since any universal Horn formula satisfied in every interval sublattice of $M(K; R)$ is satisfied in $M(K; R)$.

Assuming the hypotheses, we obtain the following diagram in $\text{Lat}_\omega$;

$\begin{array}{ccc}
K & \xrightarrow{\psi(K, R)} & M(K; R) \\
\downarrow{h} & & \downarrow{M(h; 1_R)} \\
K/q & \xrightarrow{\psi(K/q, R)} & M(K/q; R) \\
\end{array}$

The outer rectangle is commutative because $\psi$ is natural, and a unique $f$ exists making the upper triangle commutative by the universal property of $h$ for reflection. But then the lower triangle is commutative because $h$ is onto. Furthermore, $\psi(K/q, R)$ is one-one by 2.5 and 3.2, since $K/q$ is in $L(R)$. Therefore, $f$ is one-one, completing the proof.


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