GLOBAL DIMENSION OF DIFFERENTIAL OPERATOR RINGS. II

BY

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ABSTRACT. The aim of this paper is to find the global homological dimension of the ring of linear differential operators $R[\theta_1, \ldots, \theta_u]$ over a differential ring $R$ with $u$ commuting derivations. When $R$ is a commutative noetherian ring with finite global dimension, the main theorem of this paper (Theorem 21) shows that the global dimension of $R[\theta_1, \ldots, \theta_u]$ is the maximum of $k$ and $q + u$, where $q$ is the supremum of the ranks of all maximal ideals $M$ of $R$ for which $R/M$ has positive characteristic, and $k$ is the supremum of the sums rank$(P) + \text{diff dim}(P)$ for all prime ideals $P$ of $R$ such that $R/P$ has characteristic zero. [The value $\text{diff dim}(P)$ is an invariant measuring the differentiability of $P$ in a manner defined in §3.] In case we are considering only a single derivation on $R$, this theorem leads to the result that the global dimension of $R[\theta]$ is the supremum of $\text{gl dim}(R)$ together with one plus the projective dimensions of the modules $R/J$, where $J$ is any primary differential ideal of $R$. One application of these results derives the global dimension of the Weyl algebra in any degree over any commutative noetherian ring with finite global dimension.

1. Introduction. As in [5], we reserve the term differential ring for a nonzero associative ring $R$ with unit together with a single specified derivation $\delta$ on $R$. In case we have specified a finite collection $\delta_1, \ldots, \delta_u$ of commuting derivations on $R$, we shall refer to $R$ as a $u$-differential ring. The ring of differential operators over a $u$-differential ring $R$ is additively the group of all polynomials over $R$ in indeterminates $\theta_1, \ldots, \theta_u$, with multiplication subject to the requirements $\theta_i\theta_j = \theta_j\theta_i$ for all $i, j$, and $\theta_ia = a\theta_i + \delta_ia$ for all $i$, all $a \in R$. We denote this ring by $R[\theta_1, \ldots, \theta_u]$, or by $R[\theta]$ in the case of a single derivation. The elements of $R[\theta_1, \ldots, \theta_u]$ are normally written as sums of monomials of the form $rp$, where $r \in R$ and $p$ is a product of powers of the $\theta_i$, although for some arguments it is more convenient to use right-hand coefficients. (Note that when an element of $R[\theta_1, \ldots, \theta_u]$ is written with left-hand coefficients, these coefficients will in general be different from those used to express...
the element with right-hand coefficients.) In particular, any element \( x \in R[\theta] \) is written as \( x = r_0 + r_1\theta + \ldots + r_n\theta^n \) for suitable \( r_i \in R \), and when \( r_n \neq 0 \) we say that \( n \) is the degree of \( x \) and that \( r_n \) is the leading coefficient of \( x \). Finally, for induction purposes we note that

\[
R[\theta_1, \ldots, \theta_u] = R[\theta_1, \ldots, \theta_{u-1}] [\theta_u],
\]

where \( \delta_u \) has been implicitly extended to \( R[\theta_1, \ldots, \theta_{u-1}] \) by setting \( \delta_u \theta_i = 0 \) for all \( i \).

The objective of this paper is to derive formulas for the global dimension of \( R[\theta_1, \ldots, \theta_u] \), where \( R \) is a commutative noetherian \( u \)-differential ring with finite global dimension. Basically, the task breaks down into the problems of finding suitable lower bounds and upper bounds for the global dimension of \( R[\theta_1, \ldots, \theta_u] \). Since these two problems require relatively different techniques, we allot separate sections of the paper to them. In both cases we also require the techniques of localization: namely ordinary localization of the commutative ring \( R \) at a prime ideal, which induces a natural noncommutative localization on the ring \( R[\theta_1, \ldots, \theta_u] \).

Our notation for the various homological dimensions involved with a ring \( S \) is as follows: \( r \text{ gl dim } S \) denotes the right global dimension of \( S \), and \( \text{GWD}(S) \) denotes the global weak dimension of \( S \). For any \( S \)-module \( A \), we use \( \text{pd}_S(A) \) and \( \text{wd}_S(A) \) to stand for the respective projective and weak dimensions of \( A \).

The reason that weak dimensions are useful is that we shall be dealing mostly with noetherian rings. For if \( R \) is a right and left noetherian differential ring, then \( R[\theta] \) is right and left noetherian, as observed in \([2, \text{p. 68}]\). By induction, \( R[\theta_1, \ldots, \theta_u] \) is right and left noetherian also. Our basic estimates on homological dimensions are given in the following two propositions, which follow automatically by induction from \([5, \text{Propositions 2, 3}]\).

**Proposition 1.** Let \( R \) be any \( u \)-differential ring, and set \( T = R[\theta_1, \ldots, \theta_u] \). If \( A \) is any right \( T \)-module, then

\[
\text{pd}_R(A) \leq \text{pd}_T(A) \leq u + \text{pd}_R(A).
\]

**Proposition 2.** If \( R \) is any \( u \)-differential ring with \( r \text{ gl dim } R < \infty \), then

\[
r \text{ gl dim } R \leq r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \leq u + r \text{ gl dim } R.
\]

The left-hand inequality in Proposition 2 may fail if \( r \text{ gl dim } R = \infty \), as shown in \([5, \text{§2}]\).

We close this section with two propositions which give the basic results on the localization procedures needed later. The first of these is proved in exactly the same manner as \([5, \text{Lemma 7}]\).
Proposition 3. Let $R$ be any commutative $u$-differential ring, and set $T = R[\theta_1, \ldots, \theta_u]$. If $S$ is any multiplicatively closed subset of $R$, then the following are true:

(a) Each $\delta_i$ induces a derivation on $R_S$ according to the rule $\delta_i(r/s) = [(\delta_i r) s - r (\delta_i s)]/s^2$.

(b) The natural map $T \to R_S[\theta_1, \ldots, \theta_u]$ makes $R_S[\theta_1, \ldots, \theta_u]$ into a flat right and left $T$-module such that the multiplication map $R_S[\theta_1, \ldots, \theta_u] \otimes_T R_S[\theta_1, \ldots, \theta_u] \to R_S[\theta_1, \ldots, \theta_u]$ is an isomorphism.

(c) $r \text{ gl dim } R_S[\theta_1, \ldots, \theta_u] < r \text{ gl dim } T$.

Proposition 4. Let $R$ be a commutative noetherian $u$-differential ring with $\text{gl dim } R < \infty$. Then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] = \sup \{r \text{ gl dim } R_M[\theta_1, \ldots, \theta_u] \mid M \text{ is a maximal ideal of } R\}.$$  

Proof. Inasmuch as all rings involved in this proposition are right noetherian, it suffices to prove the corresponding statement for global weak dimension. Just as in the proof of [5, Lemma 7], we see that each of the rings $R_M[\theta_1, \ldots, \theta_u]$ is a classical localization of $R[\theta_1, \ldots, \theta_u]$ with respect to the multiplicative set $R \setminus M$. It is easily checked that these localizations satisfy the hypotheses of [13, Proposition 1], from which we obtain the desired result.

2. Lower bounds. In this section we set up our basic tool for finding lower bounds for the global dimension of $R[\theta_1, \ldots, \theta_u]$. This is Theorem 7, which allows us to compute the projective dimensions of those $R[\theta_1, \ldots, \theta_u]$-modules which happen to be finitely generated as $R$-modules. As one consequence, we find that $r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \geq u + \text{rank}(M)$ for any maximal ideal $M$ of $R$ such that $R/M$ has positive characteristic. We begin with two lemmas, the first of which is essentially a special case of [6, Lemma, p. 68].

Lemma 5. Let $R$ be any differential ring, and let $A$ be a right $R[\theta]$-module. If $E: 0 \to K \to F \to A \to 0$ is an exact sequence of right $R$-modules with $F_R$ free, then $K$ and $F$ can be made into right $R[\theta]$-modules such that $E$ becomes an exact sequence of $R[\theta]$-modules.

Proof. Let $f: K \to F$ and $g: F \to A$ denote the maps in $E$. Choosing a decomposition of $F$ as a direct sum of copies of $R$, and applying $\delta$ to each copy of $R$, we obtain an additive map $d: F \to F$ such that $d(xr) = (dx)r + x(\delta r)$ for all $x \in F, r \in R$. Define a map $h: F \to A$ by the rule $hx = g(dx) + (gx)\theta$, and check that $h$ is an $R$-homomorphism. Then $h$ lifts to an $R$-homomorphism $k: F \to F$ such that $gk = h$.
Now \( d' = k - d \) is an additive endomorphism of \( F \) such that \( d'(xr) = (d'x)r - x(\delta r) \) for all \( x \in F, \ r \in R \), from which we infer that \( F \) can be made into a right \( R[\theta] \)-module by defining \( x\theta = d'x \) for all \( x \in F \). Computing that now \( g(x\theta) = (gx)\theta \) for all \( x \in F \), we see that \( g \) is an \( R[\theta] \)-homomorphism. As a consequence, \( \ker g \) is an \( R[\theta] \)-submodule of \( F \), hence \( K \) can be made into a right \( R[\theta] \)-module so that \( f \) is an \( R[\theta] \)-homomorphism.

**Lemma 6.** Let \( R \) be a semiprime left Goldie differential ring. If \( J \) is any essential left ideal of \( R[\theta] \), then \( J \) contains an element of \( R[\theta] \) whose leading coefficient is a regular element of \( R \).

**Proof.** Since \( R \) is left Goldie, it must contain a finite direct sum \( A_1 \oplus \ldots \oplus A_k \) of nonzero uniform left ideals which is essential in \( _RR \). The essentiality of \( J \) implies that each of the left ideals \( R[\theta]A_i \) must contain a nonzero element \( x_i \) from \( J \). After multiplying the \( x_i \) on the left by suitable powers of \( \theta \), we may assume that the \( x_i \) all have the same degree, say \( n \). Inasmuch as \( R[\theta] = R + \theta R + \theta^2 R + \ldots \), we see that \( R[\theta]A_i = A_i + \theta A_i + \theta^2 A_i + \ldots \). Noting that the degree of \( x_i \) remains the same when \( x_i \) is written with coefficients on the right, we see that \( x_i = x_{i0} + \theta x_{i1} + \ldots + \theta^{n-1}x_{in-1} + \theta^n a_i \) for some \( x_{ij}, \ a_i \in A_i, \ a_i \neq 0 \). Changing back to left-hand coefficients, the leading coefficient of \( x_i \) is still \( a_i \), although the other coefficients need not even belong to \( A_i \).

Now \( Ra_i \) is a nonzero submodule of the uniform left ideal \( A_i \) and hence is essential in \( A_i \), from which we deduce that \( Ra_1 \oplus \ldots \oplus Ra_k \) is an essential left ideal of \( R \). Inasmuch as \( R \) is a semiprime left Goldie ring, [8, Lemma 7.2.5] says that \( Ra_1 \oplus \ldots \oplus Ra_k \) must contain a regular element \( a \) of \( R \), say \( a = r_1 a_1 + \ldots + r_k a_k \). Since each \( x_i \) has leading term \( a\theta^n \), we now conclude that \( r_1 x_1 + \ldots + r_k x_k \) is an element of \( J \) whose leading coefficient is \( a \).

**Theorem 7.** Let \( R \) be a semiprime right and left noetherian \( u \)-differential ring, and set \( T = R[\theta_1, \ldots, \theta_u] \). If \( A \) is any nonzero right \( T \)-module such that \( A_R \) is finitely generated, then \( \text{pd}_T(A) = u + \text{pd}_R(A) \).

**Proof.** Each of the rings \( T_j = R[\theta_1, \ldots, \theta_j] \) is right and left noetherian, and it is easily checked that each \( T_j \) is semiprime as well. Now \( A \) is a finitely generated right \( T_j \)-module for each \( j \), and we are done if we show that the projective dimension of \( A \) over each \( T_{j+1} \) is exactly one greater than the projective dimension of \( A \) over \( T_j \). Thus it suffices to consider only the 1-differential case: here \( R \) is a semiprime right and left noetherian differential ring, \( A \) is a nonzero right \( R[\theta] \)-module such that \( A_R \) is finitely generated, and we must prove that \( \text{pd}_{R[\theta]}(A) = 1 + \text{pd}_R(A) \).
The case \( \text{pd}_R(A) = \infty \) is taken care of by Proposition 1, hence we may assume that \( \text{pd}_R(A) = n < \infty \), and we induct on \( n \). As noted above, \( R[\theta] \) is a semiprime right and left noetherian ring, hence the maximal right quotient ring \( Q \) of \( R[\theta] \) coincides with the maximal left quotient ring of \( R[\theta] \) (and is a classical right and left quotient ring). Also, \( R[\theta] \) is a semiprime right Goldie ring, hence [4, Theorem 1.7] shows that \( R[\theta] \) is a right nonsingular ring.

If \( n = 0 \), then \( \text{pd}_{R[\theta]}(A) \leq 1 \) by Proposition 1; hence it remains to show that \( A_{R[\theta]} \) is not projective. Inasmuch as \( A \neq 0 \) and all projective right \( R[\theta] \)-modules are nonsingular, it suffices to show that \( A_{R[\theta]} \) is singular. Given any \( a \in A \), set \( J = \{ x \in R[\theta] \mid ax = 0 \} \) and note that \( R[\theta]/J \) is noetherian as an \( R \)-module. Now any nonzero right ideal \( K \) of \( R[\theta] \) contains elements of arbitrarily high degree, whence \( K_R \) cannot be finitely generated. Thus the natural map \( K \rightarrow R[\theta] \rightarrow R[\theta]/J \) cannot be a monomorphism, i.e., \( K \cap J \neq 0 \).

Therefore \( J \) is an essential right ideal of \( R[\theta] \) and so \( A \) is indeed a singular \( R[\theta] \)-module.

Next assume that \( n = 1 \), and choose a positive integer \( k \) such that \( A \) can be generated by \( k \) elements. If \( S \) denotes the ring of all \( k \times k \) matrices over \( R \), then we obtain a Morita equivalence between the category of all right \( R \)-modules and the category of all right \( S \)-modules, where any right \( R \)-module \( B \) gets taken to \( B \otimes_R R^k \), i.e., to \( B^k \). We intend to use this equivalence to transfer our problem to \( S \)-modules, since \( A^k \) is a cyclic right \( S \)-module. Now \( \delta \) can be extended to a derivation of \( S \) by letting \( \delta \) act on each entry of any matrix in \( S \), and then \( S[\theta] \) may be identified with the ring of all \( k \times k \) matrices over \( R[\theta] \). With this identification, we get another Morita equivalence between the category of all right \( R[\theta] \)-modules and the category of all right \( S[\theta] \)-modules, where any right \( R[\theta] \)-module \( B \) gets taken to \( B^k \). Because of these equivalences, \( \text{pd}_S(A^k) = 1 \) and \( \text{pd}_{R[\theta]}(A) = \text{pd}_{S[\theta]}(A^k) \), hence we may assume without loss of generality that \( A_R \) is cyclic.

Therefore we may assume that \( A = R/I \) for some right ideal \( I \) of \( R \). Inasmuch as \( A \) is also a right \( R[\theta] \)-module, we have \( \overline{\theta} = \overline{\alpha} \) for some \( \alpha \in R \). Then \( \overline{\theta} = (\alpha - \delta)r \) for all \( r \in R \) and consequently \( (\alpha - \delta)(I) \subseteq I \). Noting that \( R[\theta] = R + (\theta - \alpha)R[\theta] \), we see that \( A \cong R[\theta]/J \), where \( J = I + (\theta - \alpha)R[\theta] \).

We claim that for any \( R[\theta] \)-homomorphism \( f: J \rightarrow R[\theta], f \big|_I \) must be left multiplication by some element of \( R[\theta] \). Since \( R[\theta] \) is a right nonsingular ring, its maximal right quotient ring \( Q \) is the injective hull of \( R[\theta]_{R[\theta]} \), hence \( f \) must be left multiplication by some \( t \in Q \). Noting that \( t(\theta - \alpha) \in R[\theta] \), we see that \( t = x(\theta - \alpha)^{-1} \) for some \( x \in R[\theta] \). This element \( x \) can be put in the form \( x = x_0 + x_1(\theta - \alpha) \) for suitable \( x_0 \in R \) and \( x_1 \in R[\theta] \), whence \( t = x_0(\theta - \alpha)^{-1} + x_1 \). If \( x_0 = 0 \), then \( f \) itself is left multiplication by the
element $x_1 \in R[\theta]$ and the claim holds, hence we may assume that $x_0 \neq 0$.
We have $tJ = fJ \subseteq R[\theta]$, and clearly $x_1 J \subseteq R[\theta]$ as well, whence $x_0(\theta - \alpha)^{-1}J \subseteq R[\theta]$.

Inasmuch as $Q$ is also the maximal left quotient ring of $R[\theta]$, we must have $Kx_0(\theta - \alpha)^{-1} \subseteq R[\theta]$ for some essential left ideal $K$ of $R[\theta]$, and by Lemma 6, $K$ must contain an element $y$ whose leading coefficient is a regular element of $R$. Now $y$ is clearly a regular element of $R[\theta]$ and so is invertible in $Q$, hence we obtain $x_0(\theta - \alpha)^{-1} = y^{-1}z$ for some $z \in R[\theta]$, or $yx_0 = z(\theta - \alpha)$. Since $x_0 \neq 0$ we have $z \neq 0$, too, which makes it possible to talk about the degrees of the elements in this last equation. Obviously $\deg[z(\theta - \alpha)] = 1 + \deg(z)$, and since the leading coefficient of $y$ is a regular element we obtain $\deg(yx_0) = \deg(y)$; thus $\deg(y) = 1 + \deg(z)$. Given any $r \in I$, we have $y^{-1}zr = x_0(\theta - \alpha)^{-1}r \in R[\theta]$ (because $r \in J$), whence $zr \in yR[\theta]$. Since $\deg(y) > \deg(z)$, and since $\deg(yw) > \deg(y)$ for all nonzero $w \in R[\theta]$, this is possible only when $zr = 0$. Thus we obtain $zI = 0$, from which we infer that $x_0(\theta - \alpha)^{-1}I = 0$. It follows that $f|_J$ is just left multiplication by the element $x_1 \in R[\theta]$, as claimed.

As right $R$-modules, $J = I \oplus (\theta - \alpha)R[\theta]$, from which we see that $I$ can be made into a right $R[\theta]$-module so that the projection $p: J \rightarrow I$ is an $R[\theta]$-homomorphism. Choose an $R[\theta]$-epimorphism $g: F \rightarrow I$, where $F$ is a finitely generated free right $R[\theta]$-module. If we assume that $J_{R[\theta]}$ is projective, then $p$ must lift to an $R[\theta]$-homomorphism $h: J \rightarrow F$ such that $gh = p$. In view of the claim just proved above, we see that $h|_J$ must be left multiplication by some $w \in F$, from which we compute that $(gw)r = r$ for all $r \in I$. Consequently $gw$ is an idempotent and $(gw)R = I$, hence $(R/I)_R$ must be projective. However, this contradicts the assumption that $pd_{R}(A) = 1$, and thus $J_{R[\theta]}$ cannot be projective. This gives us $pd_{R[\theta]}(A) > 1$, so by Proposition 1 we conclude that $pd_{R[\theta]}(A) = 2$.

Finally, let $n > 1$ and assume the theorem holds for $n - 1$. Choose an exact sequence $E: 0 \rightarrow K \rightarrow F \rightarrow A \rightarrow 0$ of right $R$-modules with $F_R$ finitely generated free, and use Lemma 5 to make $E$ into an exact sequence of right $R[\theta]$-modules. Now $K$ is a right $R[\theta]$-module which is finitely generated as an $R$-module, and $pd_{R}(K) = n - 1 > 0$ (so that in particular $K \neq 0$), hence we obtain $pd_{R[\theta]}(K) = n$ from the induction hypothesis. Inasmuch as $n > 1$ and $pd_{R[\theta]}(F) \leq 1$ by Proposition 1, it now follows from the long exact sequence for Ext that $pd_{R[\theta]}(A) = n + 1$. \square

Using more homological methods, a stronger version of Theorem 7 has been proved in [12, Corollary 1.7(b)].
I am grateful to the referee for pointing out the necessity of condition (b) in the following corollary.

**Corollary 8.** Let $R$ be a semiprime right and left noetherian $u$-differential ring, let $J$ be a proper right ideal of $R$, and let $\alpha_1, \ldots, \alpha_u \in R$ such that

(a) $(\delta_i - \alpha_i)(J) \subseteq J$ for $i = 1, \ldots, u$,

(b) $(\delta_j - \alpha_j)(\alpha_i) - (\delta_j - \alpha_j)(\alpha_i) \in J$ for $i, j = 1, \ldots, u$.

Then $\text{gl dim } R[\theta_1, \ldots, \theta_u] > u + \text{pd}_R(R/J)$.

**Proof.** Let $A = R/J$, which is a nonzero finitely generated right $R$-module. Using (a) and (b), we infer that $A$ can be made into a right $R[\theta_1, \ldots, \theta_u]$-module by setting $\theta_i r = (\alpha_i - \delta_i)r$ for all $i$ and all $r \in R$. (Condition (a) ensures that $x\theta_i$ is well defined, and condition (b) ensures that $x\theta_i \theta_j = x\theta_j \theta_i$. The details are very straightforward.) Consequently, Theorem 7 says that $A$ is a right $R[\theta_1, \ldots, \theta_u]$-module with projective dimension $u + \text{pd}_R(A)$. $\square$

Corollary 8 applies in particular to the case when $J$ is a differential right ideal of $R$, i.e., $\delta_i(J) \subseteq J$ for all $i$. In this case, condition (b) is trivially satisfied.

In order to apply Theorem 7 or Corollary 8 in the case when $R$ is a commutative noetherian ring of finite global dimension, we must know that $R$ is semiprime. This is probably well known, as are the other facts in the following proposition, which we include for completeness.

**Proposition 9.** Let $R$ be any commutative noetherian ring with $\text{gl dim } R = n < \infty$.

(a) $R$ is a finite direct product of integral domains, and thus is a semiprime ring.

(b) If $M$ is any maximal ideal of $R$, then $\text{gl dim } R_M = \text{rank}(M) = \text{pd}_R(R/M) \leq n$.

(c) The (classical) Krull dimension of $R$ is $n$.

**Proof.** (a) For each maximal ideal $M$ of $R$ [9, Part III, Theorem 11] says that $\text{gl dim } R_M \leq n < \infty$, hence it follows from [9, Part III, Theorem 13] that $R_M$ is a regular local ring. Thus $R_M$ is an integral domain for every maximal ideal $M$ [10, Theorem 164], whence [10, Theorem 168] says that $R$ is a finite direct product of integral domains.

(b) As seen in (a), $R_M$ is a regular local ring. According to [9, Part III, Theorem 12], $\text{gl dim } R_M$ is the same as the Krull dimension of $R_M$, i.e., $\text{gl dim } R_M = \text{rank}(M)$. In view of [10, Theorem 176], we also see that the projective dimension of $R_M/\text{MR}_M$ over $R_M$ is equal to $\text{rank}(M)$. Inasmuch as
R is noetherian, the projective dimension of any finitely generated $R$-module $A$ is the supremum of the projective dimensions of the $R_K$-modules $A_K$, where $K$ ranges over all maximal ideals of $R$. For the case $A = R/M$, we have $A_M = R_M/MR_M$ and $A_K = 0$ for all other $K$, from which we conclude that $\text{pd}_R(R/M) = \text{rank}(M)$.  

(c) Since $R$ is noetherian, $n$ is the supremum of the numbers $\text{gl dim } R_M$ over all maximal ideals $M$, hence (c) follows immediately from (b).

We conclude this section by deriving the lower bound $u + \text{rank}(M) \leq r \text{ gl dim } R[\theta_1, \ldots, \theta_u]$, where $M$ is any maximal ideal of $R$ such that $R/M$ has positive characteristic. We must also derive lower bounds for $r \text{ gl dim } R[\theta_1, \ldots, \theta_u]$ related to maximal ideals $M$ such that $R/M$ has characteristic zero, but this depends on the differential dimension of $M$, which we develop in the next section.

**PROPOSITION 10.** Let $R$ be a commutative noetherian $u$-differential ring with $\text{gl dim } R < \infty$, and let $M$ be a maximal ideal of $R$. If $R/M$ has characteristic $p > 0$, then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \geq u + \text{rank}(M).$$

**PROOF.** According to Proposition 9, the simple module $R/M$ satisfies the property $\text{pd}_R(R/M) = \text{rank}(M) < \infty$. If $A$ is any nonzero $R$-module with a composition series such that all the composition factors are isomorphic to $R/M$, then it follows from the long exact sequence for $\text{Ext}$ (by induction on length) that $\text{pd}_R(A) = \text{rank}(M)$.

Now let $J$ be the ideal of $R$ generated by $pR$ and $\{x^p | x \in M\}$, and note that $\delta_i(J) \subseteq J$ for all $i = 1, \ldots, u$. Since $\text{char}(R/M) = p$, we see that $J \subseteq M$, whence $R/J \neq 0$. Inasmuch as $M/J$ is a nil ideal in the noetherian ring $R/J$, Levitzki's Theorem says that $M/J$ must be nilpotent, from which we infer that $R/J$ has a composition series with all composition factors isomorphic to $R/M$. Now $\text{pd}_R(R/J) = \text{rank}(M)$, hence the desired inequality follows from Corollary 8.

**COROLLARY 11.** Let $R$ be a commutative noetherian $u$-differential ring with $\text{gl dim } R = n < \infty$. If $R$ has positive characteristic, then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] = n + u.$$  

**PROOF.** In view of Proposition 9, we must have $\text{rank}(M) = n$ for some maximal ideal $M$, whence Proposition 10 yields $r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \geq n + u$. According to Proposition 1, we also have $r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \leq n + u$.  

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3. Differential dimension. The purpose of this section is to introduce a concept of differential dimension for prime ideals \( P \) of \( R \), and to obtain the lower bounds

\[
\text{rank}(P) + \text{diff dim}(P) \leq \text{r gl dim } R[\theta_1, \ldots, \theta_u].
\]

This differential dimension of \( P \) is meant to measure the "differentiability" of \( P \) in the sense that it indicates how large a collection of \( R \)-linear combinations of the derivations \( \delta_1, \ldots, \delta_u \) can map \( P \) into itself. In particular, the differential dimension of \( P \) will be \( u \) if and only if \( P \) is closed under all the \( \delta_i \). The details follow.

Given any commutative \( u \)-differential ring \( R \), make \( \text{Hom}_Z(R, R) \) into a left \( R \)-module by defining \((rf)(x) = r(fx)\) for all \( r, x \in R \), \( f \in \text{Hom}_Z(R, R) \), and let \( \Delta \) denote the left \( R \)-submodule of \( \text{Hom}_Z(R, R) \) generated by \( \delta_1, \ldots, \delta_u \). For any prime ideal \( P \) of \( R \), the set \( D(P) = \{f \in \Delta \mid f(P) \subseteq P\} \) is a left \( R \)-submodule of \( \Delta \), and it is clear that \( \Delta/D(P) \) is a torsion-free left \((R/P)\)-module. We define the differential codimension of \( P \), abbreviated \( \text{diff codim}(P) \), to be the rank of this torsion-free \((R/P)\)-module \( \Delta/D(P) \), i.e., the vector space dimension \([Q[\Delta/D(P)]: Q]\), where \( Q \) stands for the quotient field of \( R/P \). [Alternately, \( \text{diff codim}(P) \) may be defined as the Goldie dimension of the left \( R \)-module \( \Delta/D(P) \).] Finally, we define the differential dimension of \( P \), denoted \( \text{diff dim}(P) \), to be \( u - \text{diff codim}(P) \).

**Proposition 12.** Let \( R \) be a commutative \( u \)-differential ring. Let \( P \) be any prime ideal of \( R \), and set \( S = R_P \), \( M = PR_P \). Then each \( \delta_i \) induces a linear transformation \( \delta_i^* \) in the dual space \( V = \text{Hom}_{S/M}(M/M^2, S/M) \), and the subspace \( W \) of \( V \) spanned by \( \delta_1^*, \ldots, \delta_u^* \) has dimension exactly \( \text{diff codim}(P) \).

**Proof.** Each \( \delta_i \) induces a derivation on \( S \) as in Proposition 3, and this gives us additive maps \( \delta_i: M \to S \). Observing that \( \delta_i(M^2) \subseteq M \), we see that \( \delta_i \) induces an additive map \( \delta_i^*: M/M^2 \to S/M \), and an easy check confirms that \( \delta_i^* \) is an \((S/M)\)-homomorphism.

There is a left \( R \)-homomorphism \( \phi: \Delta \to W \) such that \( \phi(\delta_i) = \delta_i^* \) for each \( i \), and an easy computation shows that \( \ker \phi = D(P) \). Now \( \phi \Delta \) is a left module over the domain \( T = (R + M)/M \cong R/P \), from which we infer that \( \tau(\phi \Delta) \) and \( R/P[\Delta/D(P)] \) have the same rank, i.e., \( \tau(\phi \Delta) \) has rank \( \text{diff codim}(P) \). Inasmuch as \( \tau(\phi \Delta) \) is torsion-free and \( S/M \) is the quotient field of \( T \), the rank of \( \tau(\phi \Delta) \) is just \( [S(\phi \Delta): S/M] \). Observing that \( S(\phi \Delta) = W \), we conclude that \([W: S/M] = \text{diff codim}(P)\).

**Corollary 13.** Let \( R \) be a commutative \( u \)-differential ring. If \( P \subseteq Q \) are prime ideals of \( R \), then \( \text{diff codim}(PR_Q) = \text{diff codim}(P) \).
Proof. Inasmuch as the localization of $R_\mathcal{Q}$ at the prime ideal $PR_\mathcal{Q}$ is just $R_\mathcal{P}$, this follows immediately from Proposition 12.

In particular, Corollary 13 shows that $\text{diff codim}(P) = \text{diff codim}(PR_\mathcal{P})$ for any prime ideal $P$, which makes it possible to carry out some computations using the maximal ideal $PR_\mathcal{P}$ in the local ring $R_\mathcal{P}$. Before proving the inequality $\text{rank}(P) + \text{diff dim}(P) \leq r \text{ gl dim } R[\theta_1, \ldots, \theta_u]$, we introduce the following easy lemma, which will also be useful later.

Lemma 14. (a) Let $R$ be any ring such that $r \text{ gl dim } R = n < \infty$. If $A \subseteq B$ are right $R$-modules with $\text{pd}_R(A) = n$, then $\text{pd}_R(B) = n$. 

(b) Let $R$ be any ring such that $\text{GWD}(R) = n < \infty$. If $A \subseteq B$ are $R$-modules with $\text{wd}_R(A) = n$, then $\text{wd}_R(B) = n$.

Proof. (a) If $\text{pd}_R(B) < n$, then it follows from the long exact sequence for Ext that $\text{pd}_R(B/A) = n + 1$, which is impossible. (b) is proved similarly.

Proposition 15. Let $R$ be a commutative noetherian $u$-differential ring with $\text{gl dim } R < \infty$. If $P$ is any prime ideal of $R$, then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \geq \text{rank}(P) + \text{diff dim}(P).$$

Proof. The local ring $R_\mathcal{P}$ is a commutative noetherian $u$-differential ring with $\text{gl dim } R_\mathcal{P} < \infty$ and certainly $\text{rank}(PR_\mathcal{P}) = \text{rank}(P)$. Inasmuch as $\text{diff dim}(PR_\mathcal{P}) = \text{diff dim}(P)$ by Corollary 13 and $r \text{ gl dim } R_\mathcal{P}[\theta_1, \ldots, \theta_u] \leq r \text{ gl dim } R[\theta_1, \ldots, \theta_u]$ by Proposition 3, it suffices to consider the case when $R$ is local and $P$ is its maximal ideal. According to Proposition 9, we have $\text{gl dim } R = \text{rank}(P) = \text{pd}_R(R/P)$; let $n$ denote this common value.

If $s = \text{diff codim}(P)$, then Proposition 12 shows that the subspace $W$ of $\text{Hom}_{R_\mathcal{P}}(R/P^2, R/P)$ spanned by the induced linear transformations $\delta_1^*, \ldots, \delta_u^*$ has dimension $s$. Thus $W$ must have a basis consisting of $s$ of the $\delta_i^*$, hence we may arrange the indices $1, \ldots, u$ so that $\delta_1^*, \ldots, \delta_s^*$ is a basis for $W$.

Since $R$ is semiprime by Proposition 9, the ring $Q = R[\theta_1, \ldots, \theta_s]$ must be a semiprime ring, as well as right and left noetherian, and of course $R[\theta_1, \ldots, \theta_u] = Q[\theta_{s+1}, \ldots, \theta_u]$. Now $PQ$ is a right ideal of $Q$ and $Q/PQ \cong (R/P) \otimes_R Q$, whence $\text{pd}_Q(Q/PQ) \leq \text{pd}_R(R/P) = n$. On the other hand, since $Q/PQ$ contains an $R$-submodule isomorphic to $R/P$, we obtain $\text{pd}_R(Q/PQ) = n$ from Lemma 14, and then Proposition 1 says that $\text{pd}_Q(Q/PQ) \geq n$. Therefore $\text{pd}_Q(Q/PQ) = n$.

Given any $j \in \{s + 1, \ldots, u\}$, we must have $\delta_j^* = r_{j1}\delta_1^* + \ldots + r_{js}\delta_s^*$ for suitable $r_{ji} \in R$, whence $(\delta_j - r_{j1}\delta_1 - \ldots - r_{js}\delta_s)(P) \subseteq P$. Setting...
$q_j = r_{j1} \theta_1 + \ldots + r_{js} \theta_s \in Q$, we compute that $(\delta_j - q_j)(PQ) \subseteq PQ$. Given any $i, j \in \{s + 1, \ldots, u\}$, we have

$$\left(\delta_i - \sum_{k=1}^{s} r_{ik} \delta_k\right)(P) \subseteq P \quad \text{and} \quad \left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)(P) \subseteq P,$$

from which it follows that

$$\left[\left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\left(\delta_i - \sum_{k=1}^{s} r_{ik} \delta_k\right) - \left(\delta_i - \sum_{k=1}^{s} r_{ik} \delta_k\right)\left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\right](P) \subseteq P.$$

We compute that

$$\left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\left(\delta_i - \sum_{k=1}^{s} r_{ik} \delta_k\right) - \left(\delta_i - \sum_{k=1}^{s} r_{ik} \delta_k\right)\left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)$$

$$= \sum_{k=1}^{s} \left[\left(\delta_i - \sum_{t=1}^{s} r_{it} \delta_t\right)\left(r_{jk}\right) - \left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\left(r_{ik}\right)\right] \delta_k;$$

hence we obtain

$$\sum_{k=1}^{s} \left[\left(\delta_i - \sum_{t=1}^{s} r_{it} \delta_t\right)\left(r_{jk}\right) - \left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\left(r_{ik}\right)\right] \delta_k^* = 0.$$

Inasmuch as $\delta_1^*, \ldots, \delta_s^*$ are linearly independent over $R/P$, we see that

$$\left(\delta_i - \sum_{t=1}^{s} r_{it} \delta_t\right)\left(r_{jk}\right) - \left(\delta_j - \sum_{t=1}^{s} r_{jt} \delta_t\right)\left(r_{ik}\right) \in P \quad \text{for } k = 1, \ldots, s,$$

from which we compute that $(\delta_i - q_i)(q_j) - (\delta_j - q_j)(q_i) \in PQ$. According to Corollary 8, we obtain $r \text{ gl dim } Q[\theta_{s+1}, \ldots, \theta_u] \geq u - s + n$. Inasmuch as $u - s = \text{ diff dim}(P)$ and $n = \text{ rank}(P)$, we are done.

4. Upper bounds. The purpose of this section is to introduce two kinds of upper bounds which are needed in the computation of the global dimension of $R[\theta_1, \ldots, \theta_u]$. First, we prove a theorem which shows that the global dimension of $R[\theta_1, \ldots, \theta_u]$ is the supremum of the projective dimensions of its simple modules. The second upper bound, which is needed only in the case
that \( R \) is an algebra over the rationals, shows that, for any maximal ideal \( M \) of \( R \), all factor modules of \( R[\theta_1, \ldots, \theta_u]/MR[\theta_1, \ldots, \theta_u] \) have projective dimension at most \( \text{rank}(M) + \text{diff dim}(M) \).

For the first theorem, we need the concepts of Krull dimension (for non-commutative rings) and critical modules, as defined in [7].

**Theorem 16.** Let \( R \) be any nonzero right noetherian, left coherent ring. If \( \text{r gl dim } R = n < \infty \), then \( n = \sup \{ \text{pd}_R(A) | A_R \text{ is simple} \} \).

**Proof.** Since this is clear for \( n = 0 \), we may assume that \( n > 0 \). Inasmuch as \( R \) is right noetherian, we have \( \text{GWD}(R) = n \) and \( \text{pd}_R(A) = \text{wd}_R(A) \) for all simple modules \( A_R \), hence it suffices to show that \( R \) has a simple right module with weak dimension \( n \). According to [3, Theorem 2.1], all direct products of flat right \( R \)-modules are flat, from which we infer that the weak dimension of any direct product of right \( R \)-modules equals the supremum of the weak dimensions of the factors.

In view of [7, Proposition 1.3], all finitely generated right \( R \)-modules have Krull dimension, and there certainly exist finitely generated right \( R \)-modules with weak dimension \( n \). Now let \( \alpha \) be minimal among the Krull dimensions of those finitely generated right \( R \)-modules which have weak dimension \( n \), and choose some finitely generated right \( R \)-module \( B \) such that \( \text{K dim}(B) = \alpha \) and \( \text{wd}_R(B) = n \). Since \( n > 0 \), we have \( B \neq 0 \). All factor modules of \( B \) are finitely generated and hence have Krull dimension, whence [7, Theorem 2.1] says that every nonzero factor module of \( B \) contains a critical submodule. Thus \( B \) must have a chain of submodules \( B_0 = 0 < B_1 < \ldots < B_k = B \) such that each \( B_i/B_{i-1} \) is critical. Inasmuch as \( \text{wd}_R(B) = \sup \{ \text{wd}_R(B_i/B_{i-1}) \} \), we must have \( \text{wd}_R(B_i/B_{i-1}) = n \) for some \( i \). Setting \( A = B_i/B_{i-1} \), we see by [7, Lemma 1.1] that \( \text{K dim}(A) \leq \alpha \), hence it follows from the minimality of \( \alpha \) that \( \text{K dim}(A) = \alpha \).

We now have a finitely generated \( \alpha \)-critical right \( R \)-module \( A \) such that \( \text{wd}_R(A) = n \). We claim that \( \alpha = 0 \), i.e., that \( A \) is simple.

Assume on the contrary that \( \alpha > 0 \). Then every nonzero submodule of \( A \) is \( \alpha \)-critical too [7, Proposition 2.3], and thus is not simple; so \( A \) has no simple submodules. Thus the intersection of all nonzero submodules of \( A \) is zero, hence we obtain an embedding \( A \hookrightarrow P \), where \( P \) is the direct product of all proper factors of \( A \). Since \( A \) is \( \alpha \)-critical, each proper factor of \( A \) is a finitely generated module with Krull dimension strictly less than \( \alpha \), so by the minimality of \( \alpha \) we see that each proper factor of \( A \) has weak dimension at most \( n - 1 \). However, this implies that \( \text{wd}_R(P) \leq n - 1 \), which contradicts Lemma 14. Therefore \( \alpha = 0 \) and \( A \) is simple.

We now turn to considering factors of \( R[\theta_1, \ldots, \theta_u]/MR[\theta_1, \ldots, \theta_u] \).
where $M$ is a maximal ideal of $R$, and $R$ is an algebra over the rationals. For conciseness, we here use the term \textit{u-differential Ritt algebra} to stand for a commutative \textit{u}-differential ring which is an algebra over the rationals. In such a case, the rings $R[\theta_1, \ldots, \theta_j]$ will also be algebras over the rationals, but we do not refer to them as Ritt algebras since they are usually not commutative.

\textbf{Lemma 17.} Let $R$ be any differential ring which is an algebra over the rationals, and let $M$ be any maximal right ideal of $R$. If $(\delta + a)(M) \subset M$ for all $a \in R$, then $MR[\theta]$ is a maximal right ideal of $R[\theta]$.

\textbf{Proof.} Suppose on the contrary that $R[\theta]$ has a right ideal $J$ such that $MR[\theta] < J < R[\theta]$, and pick an element $x \in J - MR[\theta]$ of minimal degree. Observing that $J \cap R = M$, we see that $x$ must have degree $n > 0$, and we write $x = x_0 + \ldots + x_n\theta^n$ with $x_0, \ldots, x_n \in R$ and $x_n \neq 0$. In view of the minimality of $n$, we infer that $x_n \notin M$, whence $x_n r + y = 1$ for some $r \in R$, $y \in M$. Then $xr + y\theta^n$ has leading term $\theta^n$, hence $xr + y\theta^n$ is an element of $J - MR[\theta]$ with degree $n$. Thus, replacing $x$ by $xr + y\theta^n$, we may assume that $x_n = 1$.

Given any $m \in M$, it is clear that $xm - m\theta^n \in J$. Observing that $xm - m\theta^n$ has degree at most $n - 1$, we obtain $xm - m\theta^n \in MR[\theta]$, by the minimality of $n$. Since the coefficient of $\theta^{n-1}$ in $xm - m\theta^n$ is $x_{n-1} m + n(\delta m)$, we thus get $x_{n-1} m + n(\delta m) \in M$. But now $(\delta + x_{n-1}/n)(M) \subseteq M$, which is impossible.

\textbf{Lemma 18.} Let $R$ be a \textit{u}-differential Ritt algebra, and let $M$ be a maximal ideal of $R$. Assume that $s$ is a nonnegative integer such that the induced maps $\delta_1^*, \ldots, \delta_s^* \in \text{Hom}_R(M/M^2, R/M)$ are linearly independent over $R/M$. Then $MR[\theta_1, \ldots, \theta_s]$ is a maximal right ideal of $R[\theta_1, \ldots, \theta_s]$.

\textbf{Proof.} We first prove the following series of statements $P_0, \ldots, P_{s-1}$.

\textbf{P}_j: If $a \in R[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in R$ such that

$$(a + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j],$$

then $a \in R + MR[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in M$.

To prove $P_0$, assume that we have $a, r_1, \ldots, r_s \in R$ such that

$$(a + r_1\delta_1 + \ldots + r_s\delta_s)(M) \subseteq M.$$ 

Since $aM \subseteq M$ as well, we obtain

$$(r_1\delta_1 + \ldots + r_s\delta_s)(M) \subseteq M,$$

for which it follows that $r_1\delta_1^* + \ldots + r_s\delta_s^* = 0$. In view of the linear independence of $\delta_1^*, \ldots, \delta_s^*$ over $R/M$, this implies that $r_1, \ldots, r_s \in M$. Therefore $P_0$ holds.

Now let $0 < j < s - 1$ and assume that $P_{j-1}$ holds. If $P_j$ fails, then there exist elements $a \in R[\theta_1, \ldots, \theta_j]$ and $r_{j+1}, \ldots, r_s \in R$ such that
but either \( a \in R + MR[\theta_1, \ldots, \theta_j] \) or else some \( r_i \notin M \). In case \( a \in R + \)
\( MR[\theta_1, \ldots, \theta_j] \), then \( aM \subseteq MR[\theta_1, \ldots, \theta_j] \) and hence
\( (r_{j+1}^\delta_{j+1} + \ldots + r_s^\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j] \), from which we obtain
\( (r_{j+1}^\delta_{j+1} + \ldots + r_s^\delta_s)(M) \subseteq M \). In this situation, however, \( P_0 \) says that
\( r_{j+1}, \ldots, r_s \in M \), which is impossible. Thus we must have \( a \notin R + \)
\( MR[\theta_1, \ldots, \theta_j] \), and in particular \( a \neq 0 \). We may also assume that \( a \) has the
lowest degree in \( \theta_j \) of those elements of \( R[\theta_1, \ldots, \theta_j] \) for which there exist
\( r_{j+1}, \ldots, r_s \in R \) with
\[ (a + r_{j+1}^\delta_{j+1} + \ldots + r_s^\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_j]. \]

Now write \( a = a_0 + a_1^\theta_j + \ldots + a_k^\theta_j \), where \( a_0, \ldots, a_k \in R[\theta_1, \ldots, \theta_j-1] \)
and \( a_k \neq 0 \). In view of \( P_{j-1} \), we must have \( k > 0 \), and then it follows from
the minimality of \( k \) that \( a_k \notin MR[\theta_1, \ldots, \theta_j] \).

If \( k \geq 2 \), then for any \( m \in M \) we compute that
\[ (a + r_{j+1}^\delta_{j+1} + \ldots + r_s^\delta_s)(m) \]
leads off with the terms \( a_km^{\theta_j^k} + [a_{k-1}m + ka_k(\delta_j m)]^{\theta_j^{k-1}} \), from which we obtain
\[ a_km, a_{k-1}m + ka_k(\delta_j m) \in MR[\theta_1, \ldots, \theta_j-1]. \]

First, we have \( a_kM \subseteq MR[\theta_1, \ldots, \theta_j] \), hence \( P_{j-1} \) says that \( a_k = r + b \)
for some \( r \in R, b \in MR[\theta_1, \ldots, \theta_j-1] \). Inasmuch as \( a_k \notin MR[\theta_1, \ldots, \theta_j] \),
we see that \( r \notin M \). Second, we have \( (a_{k-1} + ka_k)(M) \subseteq MR[\theta_1, \ldots, \theta_j] \),
and clearly \( (ka_k)(M) \subseteq MR[\theta_1, \ldots, \theta_j] \) as well, whence
\[ (a_{k-1} + kr\delta_j)(M) \subseteq MR[\theta_1, \ldots, \theta_j-1]. \]

According to \( P_{j-1} \), we obtain \( kr \in M \), and then \( r \in M \) (because \( R \) is a Ritt algebra).
This is a contradiction.

Therefore \( k < 2 \), so the only possibility left is \( k = 1 \). Now \( a = a_0 + a_1^\theta_j \),
hence for any \( m \in M \) we have
\[ (a + r_{j+1}^\delta_{j+1} + \ldots + r_s^\delta_s)(m) = a_1m^{\theta_j} + [a_0m + a_1(\delta_j m) + r_{j+1}(\delta_j m) + \ldots + r_s(\delta_s m)]. \]

Thus \( a_1M \subseteq MR[\theta_1, \ldots, \theta_j-1] \) and also
As above, it follows from the first inclusion that \( a_1 = r + b \) for some \( r \in R - M \), \( b \in MR[\theta_1, \ldots, \theta_{j-1}] \), and then we infer from the second inclusion that

\[
(a_0 + r\delta_j + r_{j+1}\delta_{j+1} + \ldots + r_s\delta_s)(M) \subseteq MR[\theta_1, \ldots, \theta_{j-1}]
\]

But now \( P_{j-1} \) gives us \( r \in M \), which is impossible.

Therefore \( P_j \) must hold, and the induction works. We now return to the proof of the lemma and show that for \( j = 0, \ldots, s \), \( MR[\theta_1, \ldots, \theta_j] \) is a maximal right ideal of \( R[\theta_1, \ldots, \theta_j] \). For \( j = 0 \), this is part of our hypotheses. Now let \( 0 < j < s \) and assume that \( MR[\theta_1, \ldots, \theta_{j-1}] \) is a maximal right ideal of \( R[\theta_1, \ldots, \theta_{j-1}] \). In view of \( P_{j-1} \), we must have

\[
(\delta_j + a)(MR[\theta_1, \ldots, \theta_{j-1}]) \not\subseteq MR[\theta_1, \ldots, \theta_{j-1}]
\]

for all \( a \in R[\theta_1, \ldots, \theta_{j-1}] \), whence Lemma 17 shows that \( MR[\theta_1, \ldots, \theta_j] \) is a maximal right ideal of \( R[\theta_1, \ldots, \theta_j] \).

**Proposition 19.** Let \( R \) be a noetherian \( u \)-differential Ritt algebra, and set \( T = R[\theta_1, \ldots, \theta_u] \). Let \( M \) be any maximal ideal of \( R \). If \( J \) is any right ideal of \( T \) which contains \( M \), then \( \text{pd}_T(T/J) \leq \text{rank}(M) + \text{diff dim}(M) \).

**Proof.** If \( s = \text{diff codim}(M) \), then according to Proposition 12 the subspace \( W \) of \( \text{Hom}_{R/M}(M/M^2, R/M) \) spanned by \( \delta_1^*, \ldots, \delta_u^* \) has dimension \( s \); hence we may arrange the indices \( 1, \ldots, u \) so that \( \delta_1^*, \ldots, \delta_s^* \) is a basis for \( W \). Setting \( Q = R[\theta_1, \ldots, \theta_s] \), we now see from Lemma 18 that \( MQ \) is a maximal right ideal of \( Q \).

Given any \( j \in \{s + 1, \ldots, u\} \), we must have \( \delta_j^* = r_{j1}\delta_1^* + \ldots + r_{js}\delta_s^* \) for suitable \( r_{ji} \in R \), whence \( (\delta_j - r_{j1}\delta_1 - \ldots - r_{js}\delta_s)(M) \subseteq M \). Setting \( q_j = r_{j1}\theta_1 + \ldots + r_{js}\theta_s \in Q \), we compute that \( (\theta_j - q_j)M \subseteq MT \). If now \( X \) denotes the set of all products of nonnegative powers of \( \theta_{s+1} - q_{s+1}, \ldots, \theta_u - q_u \), then we obtain \( XMT \subseteq MT \).

In particular, \( XMQ \subseteq MT \subseteq J \). Observing that \( T \) is generated as a right \( Q \)-module by \( X \), we infer that \( (T/J)_Q \) is a sum of homomorphic images of \( Q/MQ \). Inasmuch as \( Q/MQ \) is a simple right \( Q \)-module, it follows that \( (T/J)_Q \) is isomorphic to a direct sum of copies of \( Q/MQ \), whence \( \text{pd}_Q(T/J) \leq \text{pd}_R(Q/MQ) \). Since \( Q/MQ \cong (R/M) \otimes_R Q \), we also have \( \text{pd}_R(Q/MQ) \leq \text{pd}_R(R/M) \). In addition, \( \text{pd}_R(R/M) = \text{rank}(M) \) by Proposition 9, and thus \( \text{pd}_R(T/J) \leq \text{rank}(M) \). According to Proposition 1, \( \text{pd}_T(T/J) \leq u - s + \text{rank}(M) \).

Inasmuch as \( u - s = \text{diff dim}(M) \), this gives us the required inequality.
5. Global dimension formulas.

**Theorem 20.** Let $R$ be a noetherian $u$-differential Ritt algebra with $\text{gl dim } R < \infty$. Then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] = \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R\}.$$  

**Proof.** If $S = R[\theta_1, \ldots, \theta_u]$, $n = r \text{ gl dim } S$, and

$$k = \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R\},$$

then $n \geq k$ by Proposition 15. According to Proposition 2, $n \leq u + \text{gl dim } R < \infty$. Inasmuch as $S$ is right and left noetherian, Theorem 16 says that there exists a simple right $S$-module $A$ with $\text{pd}_S(A) = n$, and we note that $\text{wd}_S(A) = n$ also.

Choose a nonzero element $x \in A$ whose $R$-annihilator $P = \{r \in R \mid xr = 0\}$ is maximal among the $R$-annihilators of all nonzero elements of $A$. According to [10, Theorem 6], $P$ is a prime ideal of $R$. If $T = R_p[\theta_1, \ldots, \theta_u]$, then the right $R_p$-module $A_p$ can be made into a right $T$-module by defining $(a/s)\theta_i = [a\theta_i s + a(\delta_i,s)]/s^2$ for all $i$ and all $a/s \in A_p$. Since the $R$-annihilator of $x$ is $P$, the natural map $A \to A_p$ is not zero. However, this map is an $S$-homomorphism and $A$ is a simple $S$-module, hence $A \to A_p$ must be a monomorphism. In view of Lemma 14, we thus obtain $\text{wd}_S(A_p) = n$.

Now $A = xS$ and thus $A_p = (x/1)T$, from which we infer that $A_p \cong T/J$ for some right ideal $J$ of $T$ which contains $PR_p$. According to Proposition 19, $\text{pd}_T(A_p) \leq \text{rank}(PR_p) + \text{diff dim}(PR_p)$. In view of Corollary 13, we now obtain $\text{wd}_T(A_p) \leq \text{rank}(P) + \text{diff dim}(P) \leq k$. Inasmuch as $T_S$ is flat by Proposition 3, $\text{wd}_S(A_p) \leq \text{wd}_T(A_p)$, and therefore $n \leq k$.

**Theorem 21.** Let $R$ be any commutative noetherian $u$-differential ring such that $\text{gl dim } R < \infty$. Set

$$k = \sup\{\text{rank}(P) + \text{diff dim}(P) \mid P \text{ is a prime ideal of } R \text{ and } \text{char}(R/P) = 0\},$$

$$q = \sup\{\text{rank}(M) \mid M \text{ is a maximal ideal of } R \text{ and } \text{char}(R/M) > 0\}.$$  

[In either case, if there are no ideals of the type required, the supremum is considered to be $-\infty$.] Then

$$r \text{ gl dim } R[\theta_1, \ldots, \theta_u] = \max\{k, q + u\}.$$  

**Proof.** In view of Propositions 10 and 15, we have $r \text{ gl dim } R[\theta_1, \ldots, \theta_u] \geq \max\{k, q + u\}$. According to Proposition 4, the reverse inequality will hold.
provided \( r \text{ gl dim } R_M[\theta_1, \ldots, \theta_u] \leq \max\{n, q + u\} \) for each maximal ideal \( M \) of \( R \).

First consider the case when \( \text{char}(R/M) > 0 \). According to Proposition 9, \( \text{gl dim } R_M = \text{rank}(M) \leq q \), hence Proposition 2 shows that 
\[
r \text{ gl dim } R_M[\theta_1, \ldots, \theta_u] \leq q + u.
\]

Now assume that \( \text{char}(R/M) = 0 \). Here \( nR_M \not\subset MR_M \) for all nonzero integers \( n \), hence all nonzero integers are invertible in \( R_M \). Thus \( R_M \) is a Ritt algebra, and so Theorem 20 is applicable. According to [10, Theorem 34], any prime ideal of \( R_M \) must have the form \( PR_M \) for some prime ideal \( P \) of \( R \) which is contained in \( M \), and since \( \text{char}(R/M) = 0 \) we see that \( \text{char}(R/P) = 0 \), too. In view of Corollary 13, we obtain

\[
\text{rank}(PR_M) + \text{diff dim}(PR_M) = \text{rank}(P) + \text{diff dim}(P) \leq k,
\]
and therefore Theorem 20 shows that \( r \text{ gl dim } R_M[\theta_1, \ldots, \theta_u] \leq k \). \( \square \)

In particular, Theorem 21 gives a formula for the global dimension of \( R[\theta] \) when \( R \) is only a 1-differential ring. For this case, the formula can be improved somewhat as follows, since the differential dimension of any prime ideal \( P \) depends only on whether or not \( P \) is a differential ideal. Also, for this case it is possible to restrict attention to just the maximal ideals of \( R \).

**Theorem 22.** Let \( R \) be any commutative noetherian differential ring with \( \text{gl dim } R = n < \infty \). Let \( M \) denote the collection of all differential maximal ideals of \( R \), together with all maximal ideals \( M \) such that \( \text{char}(R/M) > 0 \), and set \( k = \sup\{\text{rank}(M) | M \in M\} \). [If \( M \) is empty, then \( k \) is considered to be \( -\infty \).] Then

\[
r \text{ gl dim } R[\theta] = \max\{n, k + 1\}.
\]

**Proof.** According to Proposition 2, \( r \text{ gl dim } R[\theta] \geq n \). Inasmuch as \( \text{diff dim}(M) = 1 \) for any differential maximal ideal \( M \) of \( R \), Theorem 21 shows that \( r \text{ gl dim } R[\theta] \geq k + 1 \).

Suppose that \( P \) is any prime ideal of \( R \) with \( \text{char}(R/P) = 0 \). If \( P \) is not maximal, then it is clear from Proposition 9 that \( \text{rank}(P) < n \). Since \( \text{diff dim}(P) \leq 1 \), we get \( \text{rank}(P) + \text{diff dim}(P) \leq n \) in this case. Now assume that \( P \) is a maximal ideal. If \( P \) is not a differential ideal, then \( \text{diff dim}(P) = 0 \) and \( \text{rank}(P) + \text{diff dim}(P) \leq n \), using Proposition 9 again. On the other hand, if \( P \) is a differential ideal, then \( \text{rank}(P) + \text{diff dim}(P) = 1 + \text{rank}(P) \leq k + 1 \), by definition of \( k \).

Thus we have \( \text{rank}(P) + \text{diff dim}(P) \leq \max\{n, k + 1\} \) for all prime ideals \( P \) of \( R \) such that \( \text{char}(R/P) = 0 \). In view of Theorem 21, we conclude that

\[
r \text{ gl dim } R[\theta] \leq \max\{n, k + 1\}.
\]

We conclude this section by using Theorem 22 to derive a formula for the global dimension of \( R[\theta] \) which involves only differential ideals of \( R \). We recall
that a proper ideal $J$ in a commutative ring $R$ is said to be primary provided all zero-divisors in the ring $R/J$ are nilpotent.

**Theorem 23.** Let $R$ be any commutative noetherian differential ring with $\text{gl dim } R = n < \infty$, and set $k = \sup \{\text{pd}_R(R/J) \mid J$ is a primary differential ideal of $R\}$. [If $R$ has no primary differential ideals, then $k$ is considered to be $-\infty$.] Then $r \text{ gl dim } R[\theta] = \max\{n, k + 1\}$.

**Proof.** According to Proposition 2, $r \text{ gl dim } R[\theta] \geq n$. Inasmuch as $R$ is semiprime by Proposition 9, Corollary 8 shows that $r \text{ gl dim } R[\theta] \geq k + 1$.

Now consider any maximal ideal $M$ of $R$ such that $\text{char}(R/M) = p > 0$. If $J$ is the ideal of $R$ generated by $pR$ and $\{x^p \mid x \in M\}$, then as in Proposition 10 we see that $M/J$ is nilpotent and that $\text{pd}_R(R/J) = \text{pd}_R(R/M)$. Inasmuch as $M/J$ is nilpotent, $R/J$ must be local, from which we infer that $J$ is a primary ideal of $R$. Also, $J$ is clearly a differential ideal, whence $\text{pd}_R(R/J) \leq k$. Since $\text{pd}_R(R/M) = \text{rank}(M)$ by Proposition 9, we thus obtain $\text{rank}(M) \leq k$.

Thus we have $\text{rank}(M) \leq k$ for all maximal ideals $M$ of $R$ such that $\text{char}(R/M) > 0$. Since any differential maximal ideal $M$ of $R$ is a primary differential ideal, we also have $\text{rank}(M) \leq k$ for all differential maximal ideals $M$. According to Theorem 22, we thus obtain $r \text{ gl dim } R[\theta] \leq \max\{n, k + 1\}$.

6. Applications. For any ring $S$ and any positive integer $u$, the Weyl algebra of degree $u$ over $S$ is the ring $A_u(S) = S[x_1, \ldots, x_u][\theta_1, \ldots, \theta_u]$, where the $x_i$ are ordinary polynomial indeterminates, and we use the derivations $\delta_i = \partial/\partial x_i$ on $S[x_1, \ldots, x_u]$. J.-E. Roos has shown that for a field $F$ of characteristic 0, $r \text{ gl dim } A_u(F) = u$ [13, Théorème 1], while G. S. Rinehart has shown that, for a field $F$ of positive characteristic, $r \text{ gl dim } A_u(F) = 2u$ [11, Theorem, p. 345]. We generalize these results in the following theorem, which has also been proved (using entirely different methods) in [12, Theorem 2.6].

**Theorem 24.** Let $S$ be any commutative noetherian ring with $\text{gl dim } S = n < \infty$, and set $k = \sup \{\text{rank}(M) \mid M$ is a maximal ideal of $S$ and $\text{char}(S/M) > 0\}$. [If $S$ has no such maximal ideals, then $k$ is considered to be $-\infty$.] Then for any positive integer $u$, $r \text{ gl dim } A_u(S) = \max\{n + u, k + 2u\}$.

**Proof.** Set $R = S[x_1, \ldots, x_u]$ and $\delta_i = \partial/\partial x_i$ for $i = 1, \ldots, u$. Since $\text{gl dim } R = n + u$, Proposition 2 shows that $r \text{ gl dim } A_u(S) \geq n + u$.

If $S$ has any maximal ideals $M$ such that $\text{char}(S/M) > 0$, then we may choose such an $M$ with $\text{rank}(M) = k$. Inasmuch as $S/M$ is a field, the ring $R/MR \cong (S/M)[x_1, \ldots, x_u]$ has Krull dimension $u$, whence $R/MR$ must have a maximal ideal $K/MR$ of rank $u$. Then $K$ is a maximal ideal of $R$ such that $\text{char}(R/K) > 0$, and clearly $\text{rank}(K) > k + u$, hence Theorem 21 says that $r \text{ gl dim } A_u(S) \geq k + 2u$. 

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Therefore \( r \text{ gl dim } A_u(S) \geq \max\{n + u, k + 2u\} \). According to Theorem 21, to prove the reverse inequality it is enough to show that \( \text{rank}(M) < k + u \) for any maximal ideal \( M \) of \( R \) with \( \text{char}(R/M) > 0 \), and that \( \text{rank}(P) + \text{diff dim}(P) \leq n + u \) for any prime ideal \( P \) of \( R \) such that \( \text{char}(R/P) = 0 \).

First consider any maximal ideal \( M \) of \( R \) for which \( \text{char}(R/M) > 0 \). Choosing a maximal ideal \( K \) of \( S \) which contains \( S \cap M \), we have \( \text{char}(S/K) > 0 \) and so \( \text{rank}(S \cap M) \leq \text{rank}(K) \leq k \). By induction on \([10, \text{Theorem 149}]\), we find that \( \text{rank}(M) \leq k + u \).

Now consider any prime ideal \( P \) of \( R \) with \( \text{char}(R/P) = 0 \), and set \( s = \text{diff dim}(P) \). If \( T = R_p \), \( M = PR_p \), and \( W \) is the subspace of \( \text{Hom}_{T/M}(M/M^2, T/M) \) spanned by \( \delta_1, \ldots, \delta_u \), then by Proposition 12 \( W \) has dimension \( u - s \). Thus we may arrange the indices \( 1, \ldots, u \) so that \( \delta_u^* \) is a basis for \( W \).

Set \( Q = P \cap \langle S[x_1, \ldots, x_s] \rangle \) and note that \( S[x_1, \ldots, x_s]/Q \) has characteristic 0. We claim that \( \delta_i(Q) \subseteq Q \) for \( i = 1, \ldots, s \). Given \( 1 \leq i \leq s \), we must have \( \delta_i = t_{s+1} \delta_{s+1} + \cdots + t_u \delta_u^* \) for suitable \( t_j \in T \). Multiplying out the denominators in this equation, we obtain

\[
a \delta_i^* = r_{s+1} \delta_{s+1}^* + \cdots + r_u \delta_u^* \quad \text{for some } a \in R - P, r_{s+1}, \ldots, r_u \in R.
\]

Thus \( (a \delta_i - r_{s+1} \delta_{s+1} - \cdots - r_u \delta_u)(M) \subseteq M \), from which we infer that \( (a \delta_i - r_{s+1} \delta_{s+1} - \cdots - r_u \delta_u)(P) \subseteq P \).

Since \( Q \subseteq P \) and \( \delta_{s+1}, \ldots, \delta_u \) all vanish on \( Q \), we thus obtain \( a \delta_i(Q) \subseteq P \). Now \( P \) is a prime ideal of \( R \) and \( a \in R - P \), hence it follows that \( \delta_i(Q) \subseteq P \), from which we conclude that \( \delta_i(Q) \subseteq Q \), as claimed.

All of the rings \( S[x_1, \ldots, x_i]/(Q \cap S[x_1, \ldots, x_i]) \) \( (i = 1, \ldots, s) \) have characteristic 0, hence with the help of the relations \( \delta_i(Q) \subseteq Q \) an easy induction shows that \( Q \cap (S[x_1, \ldots, x_i]) = (Q \cap S)[x_1, \ldots, x_i] \) for each \( i = 1, \ldots, s \). Consequently \( Q = (Q \cap S)[x_1, \ldots, x_s] \), whence \([10, \text{Theorem 149}]\) shows that \( \text{rank}(Q) = \text{rank}(Q \cap S) \). That same theorem also shows that \( \text{rank}(P) \leq u - s + \text{rank}(Q) \), and it is clear from Proposition 12 that \( \text{rank}(Q \cap S) \leq n \), hence we obtain \( \text{rank}(P) \leq n + u - s \). Therefore \( \text{rank}(P) + \text{diff dim}(P) \leq n + u \).

**Corollary 25.** Let \( S \) be any commutative noetherian ring with \( \text{gl dim } S = n < \infty \), and let \( u \) be any positive integer. If \( S \) is an algebra over the rationals, then \( r \text{ gl dim } A_u(S) = n + u \).

Corollary 25 has also been obtained in \([1, \text{Corollary 2.6}]\).

Given any ring \( S \) and any positive integer \( u \), then following \([2]\) we can define a ring \( F_u(S) = S[[x_1, \ldots, x_u]][\theta_1, \ldots, \theta_u] \) analogous to the Weyl algebra \( A_u(S) \). If \( S \) is a commutative noetherian ring with \( \text{gl dim } S = n < \infty \), and if \( S \) is an algebra over the rationals, then J.-E. Björk has shown in \([2, \text{The-}

orem 4.2] that \( \text{r gl dim } F_u(S) = n + u \). We shall generalize this result, but first some facts about power series rings must be developed. [We note that our proofs do not depend on Björk's result, and our methods are completely different from his.]

**Lemma 26.** Let \( S \) be a commutative noetherian ring with \( \text{gl dim } S = n < \infty \). If \( u \) is any positive integer, then \( \text{gl dim } S[[x_1, \ldots, x_u]] = n + u \).

**Proof.** It obviously suffices to prove the case \( n = 1 \). The indeterminate \( x \) lies in the Jacobson radical of \( S[[x]] \), and \( x \) is not a zero-divisor in \( S[[x]] \). Since \( S[[x]]/xS[[x]] \cong S \), [9, Part III, Theorem 10] shows that \( \text{r gl dim } S[[x]] = n + 1 \).

**Lemma 27.** If \( F \) is a field and \( u \) any positive integer, then \( F[[x_1, \ldots, x_u]] \) has Krull dimension \( u \).

**Proof.** This is immediate from Lemma 26 and Proposition 9.

For use in the next lemma, we recall that if \( S \) is a commutative local ring with maximal ideal \( M \), then \( S[[x]] \) is a local ring with maximal ideal generated by \( M \) and \( x \). Clearly, the ideal \( J \) of \( S[[x]] \) generated by \( M \) and \( x \) is a maximal ideal. Also, if \( p \) is any element of \( S[[x]] \) which does not belong to \( J \), then the constant term of \( p \) is not in \( M \) and so is invertible in \( S \), whence \( p \) is invertible in \( S[[x]] \).

**Lemma 28.** Let \( S \) be a commutative noetherian ring, and let \( Q \) be any prime ideal of \( S \), \( u \) any positive integer. Then \( Q[[x_1, \ldots, x_u]] \) is a prime ideal of \( S[[x_1, \ldots, x_u]] \) with rank equal to \( \text{rank}(Q) \). Also, if \( P \) is any prime ideal of \( S[[x_1, \ldots, x_u]] \) such that \( P \cap S = Q \), then \( \text{rank}(P) \leq u + \text{rank}(Q) \).

**Proof.** We may obviously assume that \( \text{rank}(Q) < \infty \). Also, we clearly need only prove the case \( u = 1 \). Finally, since \( P \) is disjoint from \( S - Q \), all the ranks we are interested in remain the same after localizing at \( Q \), hence we may assume, without loss of generality, that \( S \) is local with maximal ideal \( Q \). As remarked above, it follows that \( S[[x]] \) is local with maximal ideal \( M \) generated by \( Q \) and \( x \).

Now \( M \) is a prime ideal in the noetherian ring \( S[[x]] \), and \( x \) is an element of \( M \) which is not a zero-divisor in \( S[[x]] \), hence [10, Theorem 155] says that the rank of \( M/xS[[x]] \) in \( S[[x]]/xS[[x]] \) equals \( \text{rank}(M) - 1 \). Inasmuch as \( S[[x]]/xS[[x]] \cong S \), we infer that \( \text{rank}(M/xS[[x]]) = \text{rank}(Q) \), and thus \( \text{rank}(M) = 1 + \text{rank}(Q) \). Observing that \( P \subseteq M \), we obtain \( \text{rank}(P) \leq 1 + \text{rank}(Q) \). Finally, since \( \text{rank}(M) = 1 + \text{rank}(Q) < \infty \) and \( Q[[x]] \) is properly contained in \( M \), we must have \( \text{rank}(Q[[x]]) \leq \text{rank}(Q) \), from which we conclude that \( \text{rank}(Q[[x]]) = \text{rank}(Q) \).
With the help of these three lemmas, we may use the proof of Theorem 24, mutatis mutandis, to prove the following generalization of Björk's theorem:

**Theorem 29.** Let $S$ be any commutative noetherian ring with $\text{gl dim } S = n < \infty$ and set $k = \sup\{\text{rank}(M) \mid M \text{ is a maximal ideal of } S \text{ and } \text{char}(S/M) > 0\}$. [If $S$ has no such maximal ideals, then $k$ is considered to be $-\infty$.] Then, for any positive integer $u$,

$$r \text{ gl dim } F_u(S) = \max\{n + u, k + 2u\}.$$  

J. Cozzens and J. Johnson have shown that for any $u$-differential field $F$,

$$r \text{ gl dim } F[\theta_1, \ldots , \theta_u] = u \quad [4, \text{Theorem 1(b)}].$$  

In view of Corollary 8 and Proposition 2, this result generalizes to semisimple artinian rings:

**Theorem 30.** If $R$ is any semisimple artinian $u$-differential ring, then

$$r \text{ gl dim } R[\theta_1, \ldots , \theta_u] = u.$$  

**REFERENCES**


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