A CHARACTERIZATION OF MANIFOLDS

BY

LOUIS F. McAULEY

ABSTRACT. The purpose of this paper is (1) to give a proof of one general theorem characterizing certain manifolds and (2) to illustrate a technique which should be useful in proving various theorems analogous to the one proved here.

THEOREM. Suppose that \( f: X \to [0, 1] \), where \( X \) is a compactum, and that \( f \) has the properties:

1. for \( 0 < x < \frac{1}{2} \), \( f^{-1}(x) = S^n \cong M_0 \),
2. \( f^{-1}(\frac{1}{2}) = S^n \) with a tame (or flat) \( k \)-sphere \( S^k \) shrunk to a point,
3. for \( \frac{1}{2} < x < 1 \), \( f^{-1}(x) \cong a \) compact connected \( n \)-manifold \( M_1 \cong S^{n-(k+1)} \times S^{k+1} \) (a spherical modification of \( M_0 \) of type \( k \)), and
4. there is a continuum \( C \) in \( X \) such that (letting \( C_x = f^{-1}(x) \cap C \))
   a. \( 0 < x < \frac{1}{2} \), \( C_x \cong S^k \),
   b. \( C_{\frac{1}{2}} = \{p\} \) a point,
   c. for \( \frac{1}{2} < x < 1 \),
   \( C_x \cong S^{n-(k+1)} \),
   d. each of \( f(I - C), f \mid f^{-1}(0, \frac{1}{2}), \) and \( f \mid f^{-1}(\frac{1}{2}, 1) \)
   is completely regular.

Then \( X \) is homeomorphic to a differentiable \( (n + 1) \)-manifold \( M \) whose boundary is the disjoint union of \( \overline{M}_0 \) and \( \overline{M}_1 \) where \( \overline{M}_i = \overline{M}_i, i = 0, 1 \).

1. Introduction. There are a number of interesting theorems in differential topology that characterize spheres, but their proofs use "smoothness" of both the manifold and the mapping. For example, the theorem of Reeb [15] and Milnor [10], later generalized by Milnor [11] and Rosen [16], is such a characterization.

THEOREM 1 (REEB-MILNOR-ROSEN). Suppose that \( M \) is smooth \((C^\infty)\) compact manifold and that \( f \) is a smooth real-valued function on \( M \) with exactly two critical points (degenerate or not). Then \( M \) is homeomorphic to a sphere.

This theorem has a topological version which we gave in [8]. It is as follows.

THEOREM 2 (MCAULEY). Suppose that \( M \) is a continuum (compact connected metric space) and that \( f: M \to I = [0, 1] \) is a (continuous) mapping.
Furthermore, $f^{-1}(0) = a$ (point), $f^{-1}(1) = b$ (point), $f(M - \{a, b\})$ is completely regular, and $f^{-1}(x)$ is homeomorphic to an $n$-sphere $S^n$ for each $x \in (0, 1)$. Then $M$ is homeomorphic to $S^{n+1}$.

The condition that $f^{-1}(x)$ be an $n$-sphere $S^n$ is quite natural in view of the following theorem from differential topology.

**Theorem 3.** If $f : M \to N$ is a smooth mapping between smooth manifolds of dimensions $m$ and $n$, respectively, where $m \geq n$ and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subseteq M$ is a smooth manifold of dimension $m - n$.

One wonders just what are the topological properties of differential mappings? Also, under what reasonable (topological) conditions is a mapping differential? In the case of nonconstant analytic mappings from the complex plane to the complex plane, the properties of openness and lightness actually characterize them. Whyburn [21] and Stoilow [18] have shown that if $f : M^2 \to N^2$ is a light open mapping between 2-manifolds, then $f$ is topologically equivalent to an analytic mapping. Several researchers, Church in particular, have made considerable progress in obtaining topological properties of differentiable mappings. For references and results, see [2].

At the topology conference held at the University of Oklahoma, March, 1972, I gave a talk containing outlines of proofs of theorems for which Theorems 2 and 4 (below) are special cases. The manuscript for that talk has appeared in the PROCEEDINGS, TOPOLOGY CONFERENCE, University of Oklahoma, 1972.

**Theorem 4 (McAuley).** Suppose $M$ is a continuum and that $f : M \to [0, 1]$ is a mapping such that (1) $f^{-1}(0) = a$ (point), (2) $f^{-1}(1) = b$ (point), (3) $f^{-1}(\frac{3}{4}) = f^{-1}(\frac{1}{4}) = a$ figure eight (two circles with exactly one common point), (4) for $0 < x < \frac{3}{4}$ or $\frac{1}{4} < x < 1$, $f^{-1}(x) \cong a$ circle, (5) for $\frac{1}{4} < x < \frac{3}{4}$, $f^{-1}(x) \cong a$ pair of disjoint circles, and (6) for $0 < x < 1$, there is a "triangulation" of $f^{-1}(x)$ which contains exactly four 1-simplexes (simple arcs) and $f$ is completely regular with respect to the collection of all 1-simplexes. Then $M \cong$ torus or Klein bottle.

The purpose of this paper is to give a proof of one general theorem characterizing certain manifolds. Perhaps a more important objective is our illustration of the methods of proof which should be useful in proving various theorems of this kind.

**Theorem 5.** Suppose that $X$ is a compact metric space and that $f : X \to [0, 1]$ has the following properties:
(1) for $0 < x < \frac{1}{2}$, $f^{-1}(x) = S^n \cong M_0$,
(2) $f^{-1}(\frac{1}{2})$ is homeomorphic to $S^n$ with a tame or flat $k$-sphere $S^k$ shrunk to a point,
(3) for $\frac{1}{2} < x \leq 1$, $f^{-1}(x) \cong S^n - (k+1) \times S^{k+1}$ which is a spherical modification of $M_0$ of type $k$ (a regular neighborhood of $S^k \subset M_0$, i.e., $S^k \times I^{n-k}$, is replaced by $S^n - (k+1) \times I^{k+1}$), and
(4) there is a continuum $C$ in $X$ such that (letting $C_x = f^{-1}(x) \cap C$)
(a) for $0 < x < \frac{1}{2}$, $C_x \cong S^k$, (b) $C_{\frac{1}{2}} = p$, a point—the topological critical point of $f$, (c) for $\frac{1}{2} < x \leq 1$, $C_x \cong S^n - (k+1)$, and (d) each of $f|_{(X - C)}$, $f|_{[0, \frac{1}{2}]}$, and $f|_{[\frac{1}{2}, 1]}$ is completely regular.

Then $X$ is homeomorphic to a differentiable $(n + 1)$-manifold $M$ whose boundary is the disjoint union of $M_0$ and $M_1$ where $M_i \cong M_i$, $i = 0, 1$.

Proof. Let $\overline{M}_0$ and $\overline{M}_1$ be differentiable manifolds homeomorphic to $M_0$ and $M_1$, respectively. There is a differentiable manifold $M$ whose boundary is the disjoint union of $\overline{M}_0$ and $\overline{M}_1$ and a differentiable function $g$ on $M$ equal to 0 on $\overline{M}_0$, equal to 1 on $\overline{M}_1$, and otherwise having values between 0 and 1 and having exactly one nondegenerate critical point $q$ (with critical value $\frac{1}{2}$, say) with type number $k + 1$ [24]. Now, for $0 < x < \frac{1}{2}$, $g^{-1}(x) \cong S^n \cong M_0 \cong \overline{M}_0$, $g^{-1}(\frac{1}{2}) \cong S^n$ with a $k$-sphere shrunk to a point, and for $\frac{1}{2} < x \leq 1$, $g^{-1}(x) \cong M_1 \cong \overline{M}_1$. Furthermore, there is a “smooth” closed and connected set $Z$ such that (1) for $0 < x < \frac{1}{2}$, $Z_x = Z \cap g^{-1}(x) \cong S^k$, (2) $g^{-1}(\frac{1}{2}) \cap Z = q$, the critical point of $g$, (3) for $\frac{1}{2} < x \leq 1$, $Z_x = Z \cap g^{-1}(x) \cong S^n - (k+1)$, and (4) $Z$ is “canonical” in the sense of Wallace [24, p. 88]. Consider the trajectories to the level sets of $g$. The trajectories starting at points of $Z_0 \cong S^k$ all end at $q$. As we move through the levels of $g$ from $\overline{M}_0$ to $\overline{M}_1$, the $Z_x \cong S^k$ shrink to $q$ along the orthogonal trajectories. As we continue above the critical level, $g^{-1}(\frac{1}{2})$, $Z_x \cong S^n - (k+1)$ grows along the orthogonal trajectories from $q$ to $Z_1 \subset \overline{M}_1$. Thus, in this sense, $Z$ is “canonical”.

Clearly, $Z$ is homeomorphic to $C$ (in the hypothesis) and $M$ is homeomorphic to $P = (S^n \times [0, \frac{1}{2}]) \cup ((S^n - (k+1) \times S^{k+1}) \times [\frac{1}{2}, 1])$ where $(S^n, \frac{1}{2})$ and $(S^n - (k+1) \times S^{k+1}, \frac{1}{2})$ are sewed together in the obvious manner (indicated below). Thus, there is (I) a tame $k$-sphere $S^k$ in $S^n$, (II) a tame $n - (k + 1)$ sphere $S^n - (k+1)$ in $(S^n - (k+1) \times S^{k+1})$, and (III) a continuous mapping $m: P \Rightarrow M$ such that (1) $m|(S^n, x)$, $0 < x < \frac{1}{2}$, is a homeomorphism taking $(S^n, x)$ onto $g^{-1}(x)$, (2) $m|(S^n - (k+1), x)$, $\frac{1}{2} < x \leq 1$, is a homeomorphism taking $(S^n - (k+1), x)$ onto $g^{-1}(x)$, and (3) each of $m|(S^n, \frac{1}{2})$ and $m|(S^n - (k+1), \frac{1}{2})$ is a homeomorphism off $(S^k, \frac{1}{2})$ and $(S^n - (k+1), \frac{1}{2})$, respectively,
which takes \((S^n - S^k)\) and \((S^{n-(k+1)} \times S^{k+1} - S^n-(k+1))\) onto \(g^{-1}(\frac{1}{2}) - \{q\}\) and takes \((S^k, \frac{1}{2})\) and \((S^n-(k+1), \frac{1}{2})\) onto \(q\). In the following, it is more convenient to work with \(P\) than with \(M\).

Let \(h_1\) denote a mapping of \(S^n \times [0, \frac{1}{2}]\) into \(f^{-1}[0, \frac{1}{2}]\) (actually, onto \(f^{-1}[0, \frac{1}{2}] \cap C\)) such that \(h_1\) takes \((S^k, t)\) homeomorphically onto \(C_t = f^{-1}(t) \cap C\) for \(0 \leq t < \frac{1}{2}\) and takes \((S^k, \frac{1}{2})\) onto \(f^{-1}(\frac{1}{2}) \cap C = p\). Similarly, let \(h_2: S^n-(k+1) \times [\frac{1}{2}, 1] \to f^{-1}[\frac{1}{2}, 1]\) take \((S^n-(k+1), t)\) homeomorphically onto \(C_t = f^{-1}(t) \cap C\) for \(\frac{1}{2} < t \leq 1\) and takes \((S^n-(k+1), \frac{1}{2})\) onto \(p = C_{\frac{1}{2}} = f^{-1}(\frac{1}{2}) \cap C\).

For \(0 \leq t < \frac{1}{2}\), let \(K_t\) be the space of all homeomorphisms of \(S^n\) onto \(f^{-1}(t)\) taking \(x \in S^k\) onto \(h_1(x, t)\). Similarly, let \(K_t\) be the space of all homeomorphisms of \(S^n-(k+1) \times S^{k+1}\) onto \(f^{-1}(t)\) taking \(x \in S^n-(k+1)\) onto \(h_2(x, t)\) for \(\frac{1}{2} < t \leq 1\).

Let \(K^0_{\frac{1}{2}}\) be the space of all mappings \(w\) of \(S^n\) onto \(f^{-1}(\frac{1}{2})\) taking \(x \in S^k\) to \(h_1(x, \frac{1}{2}) = p\) such that \(w|\(S^n - S^k)\) is a homeomorphism. Similarly, let \(K^1_{\frac{1}{2}}\) be the space of mappings \(w\) of \(S^n-(k+1) \times S^{k+1}\) onto \(f^{-1}(\frac{1}{2})\) taking \(x \in S^n-(k+1)\) to \(h_2(x, \frac{1}{2}) = p\) such that \(w|\(S^n-(k+1) \times S^{k+1} - S^n-(k+1)\) is a homeomorphism.

We shall consider the collection \(L_0\) of all \(K_t\), \(0 \leq t < \frac{1}{2}\) plus \(K^0_{\frac{1}{2}}\) and the collection \(L_1\) of all \(K_t\), \(\frac{1}{2} < t \leq 1\) plus \(K^1_{\frac{1}{2}}\). Now, \(L_i^*\) will denote the union of the elements of \(L_i\). Next, we define a metric for \(L_i^*\). If \(m \in L_i^*\), let \(\hat{m}\) denote the graph of \(m\) in \(P \times X\). Thus, for each pair \(m, n \in L_i^*\) where \(m \in K_a\) and \(n \in K_b\), let \(D(m, n) = H(\hat{m}, \hat{n})\) where \(H\) denotes the Hausdorff metric on the space of all closed subsets of \(P \cap X\). Now, \((L_i^*, D)\) is a topologically complete metric space. For a proof, see an argument in [7, Theorem 1] for an analogous result. We let \(\rho\) denote a complete metric for \(L_i^*\).

**Lemma 1.** Each \(K_t\) and \(K^i_{\frac{1}{2}}, i = 0, 1\), is LC0 (in the homotopy sense). Indeed, each is locally contractible.

**Proof.** For \(0 \leq t < \frac{1}{2}\), it should be clear that \(K_t\) is homeomorphic to the space of all homeomorphisms of \(S^n\) onto itself with a tame (or flat) \(k\)-sphere \(S^k\) fixed. Thus, by [4], it follows that \(K_t\) is locally contractible. Similarly, for \(\frac{1}{2} < t \leq 1\), \(K_t\) is locally contractible. Now, \(K^i_{\frac{1}{2}}\) is the space of all homeomorphisms of a compact polyhedron \(T\) onto itself keeping a point \(s\) fixed where \(T\) is the result of shrinking \(S^k\) (or \(S^n-(k+1)\)) in \(S^n\) (or \(S^n-(k+1) \times S^{k+1}\)) to a point. It follows from [22] that \(K^i_{\frac{1}{2}}\) is locally contractible.

**Lemma 2.** The collections \(L_i, i = 0, 1\), are equi-LCn.

**Proof.** Each \(L_i^*\) is a complete metric space with metric \(\rho\). Note that
$f|^{-1}[0, \frac{1}{2})$ and $f|^{-1}(\frac{1}{2}, 1]$ are completely regular in the sense of Dyer and Hamstrom [3]. It follows by an argument analogous to that given in [3] that the collection of all $K_t$ is equi-$LC^n$ for each $n$. To show that $L_0$ is equi-$LC^n$, we need only consider $\epsilon > 0$ and $g \in K_0^\epsilon$.

Since $K_0^\epsilon$ is $LC^n$, there is a $\delta_1 > 0$ such that each mapping $r: S^k \rightarrow K_0^\epsilon \cap N_{\delta_1}(g)$, for $0 \leq k \leq n$, can be extended to a mapping $R: I^{k+1} \rightarrow K_0^\epsilon \cap N_{\epsilon/2}(g)$. Since $f(X - C)$ is completely regular, there is $\alpha > 0$ such that if $\frac{1}{2} - b < \alpha$, $b \in [0, \frac{1}{2}]$, there is a mapping $m: f^{-1}(b) \rightarrow f^{-1}(\frac{1}{2})$ such that $m(C_b) = C_{\frac{1}{2}}$, $m(f^{-1}(b) - C_b)$ is a homeomorphism, and $m$ moves no point as much as $\delta_1/2$.

Choose $\delta_1 > \delta < \min(\delta_1/2, \frac{1}{2})$, such that if $K_b \cap N_\delta(g) \neq \emptyset$, then $\frac{1}{2} - b < \alpha$. Now, let $\phi: S^k \rightarrow K_b \cap N_\delta(g)$. We wish to show that $\phi$ can be extended to $\phi: I^{k+1} \rightarrow K_b \cap N_\delta(g)$. Let $c = \frac{1}{2}$. We can define a 1-1 mapping $H_{bc}: K_b \rightarrow K_c$ as follows: For $e \in K_b$, let $H_{bc}(e) = me \in K_c$. Clearly, $H_{bc}(K_b \cap N_\delta(g))$ maps $K_b \cap N_\delta(g)$ into $K_c \cap N_{\delta_1}(g)$. In fact, $H_{bc}$ maps $K_b$ onto $K_c$. Furthermore, $r = [H_{bc}, \phi(S^k)] \phi$ maps $S^k$ into $K_c \cap N_{\delta_1}(g)$ and can be extended to a mapping $R: I^{k+1} \rightarrow K_c \cap N_{\epsilon/2}(g)$ such that for each $p \in I^{k+1}$, $R(p) \in H_{bc}(K_b) \subset K_c$ since $H_{bc}(K_b)$ is $LC^n$. Now, define $H_{cb}: H_{bc}(K_b) \rightarrow K_b$ as $H_{cb}(me) = e$. Clearly, $H_{cb}$ is the inverse of $H_{bc}$ and $H_{bc}$ is a homeomorphism. Now, $\Phi = [H_{cb}, H_{bc}(K_b) \cap N_{\epsilon/2}(g)]$ maps $I^{k+1}$ into $K_b \cap N(g)$ and agrees with $\phi$ on $S^k$ the boundary of $I^{k+1}$. Thus, $L_0$ is equi-$LC^n$. Similarly, it follows that $L_1$ is equi-$LC^n$.

**Lemma 3.** The collections $L_i$ are lower semicontinuous (lsc) in the sense that if $\{x_i\} \rightarrow x$ in $[0, \frac{1}{2})$ or $[\frac{1}{2}, 1]$, then $K_x$ is in the closure of $\bigcup K_{x_i}$.

A proof follows easily from the fact that each of $f|^{-1}[0, \frac{1}{2}), f|^{-1}(\frac{1}{2}, 1]$, and $f(X - C)$ is completely regular.

Next, let $F: L_0^* \rightarrow [0, \frac{1}{2})$ be the function defined by $F(k) = x$ iff $k \in K_x$. Thus, the collection of point inverses under $F$ is the collection $L_0$ which is lsc and equi-$LC^n$. Also, $L_0^*$ is a complete metric space. Given $x \in [0, \frac{1}{2}]$, let $\phi(x) \in K_x$. By Michael's section theorem [9], there is an open set $U$ of $[0, \frac{1}{2}]$ with $x \in U$ and a continuous extension of $\phi$ to $U$ (denote it by $\Phi$) with the property that $\Phi(u) \in K_u$ for each $u \in U$. Clearly, $[0, \frac{1}{2}]$ is covered by a finite number of closed intervals $[a_i, b_i]$ where $a_0 = 0 < b_0 = a_1 < b_1 = a_2 < b_2 \ldots < b_t = \frac{1}{2}$ with mappings $m_i: S^n \times [a_i, b_i] \rightarrow f^{-1}[a_i, b_i]$ where $m_i$ is a homeomorphism for $i = 1, 2, \ldots, t - 1$ and $m_t$ is a homeomorphism off $(S^k, \frac{1}{2})$ and takes $(S^k, \frac{1}{2})$ to $p$. Next, we sew the pieces together in the obvious way. Identify $h_i(x, a_i)$ with $h_{i+1}(x, a_i)$ for $i = 0, 1, \ldots, t - 1$. We obtain a mapping
$H_0: S^n \times [0, \frac{1}{2}] \Rightarrow f^{-1}[0, \frac{1}{2}]$ which is a homeomorphism except on $(S^k, \frac{1}{2})$.

In a similar way, we obtain a mapping $H_1: (S^n-(k+1) \times S^{k+1}) \times [\frac{1}{2}, 1] \Rightarrow f^{-1}[\frac{1}{2}, 1]$ which is a homeomorphism except on $(S^n-(k+1), \frac{1}{2})$ which maps to $p$. We sew these together to obtain a mapping $H: P \Rightarrow f^{-1}[0, 1] = X$ (recalling that $P = (S^n \times [0, \frac{1}{2}]) \cup ((S^n-(k+1) \times S^{k-1}) \times [\frac{1}{2}, 1])$) such that $h = Hm^{-1}: M \Rightarrow X$ is a homeomorphism (again, recalling that $m: P \Rightarrow M$ has certain properties). Consequently, $X$ is homeomorphic to the differentiable $(n + 1)$-manifold $M$. Theorem 5 is proved.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901