TWO WEIGHT FUNCTION NORM INEQUALITIES FOR THE POISSON INTEGRAL

BY

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ABSTRACT. Let \( f(x) \) denote a complex valued function with period \( 2\pi \), let

\[
P_r(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2)f(y)dy}{1 - 2r \cos(x - y) + r^2}
\]

be the Poisson integral of \( f(x) \) and let \( |I| \) denote the length of an interval \( I \).

For \( 1 < p < \infty \) and nonnegative \( U(x) \) and \( V(x) \) with period \( 2\pi \) it is shown that there is a \( C_2 \), independent of \( f \), such that

\[
\sup_{0 < r < 1} \int_{-\pi}^{\pi} |P_r(f, x)|^p U(x) \, dx \leq C \int_{-\pi}^{\pi} |f(x)|^p V(x) \, dx
\]

if and only if there is a \( B \) such that for all intervals \( I \)

\[
\left[ \frac{1}{|I|} \int_I U(x) \, dx \right]^{1+p-1/(p-1)} \leq B.
\]

Similar results are obtained for the nonperiodic case and in the case where \( U(x)dx \) and \( V(x)dx \) are replaced by measures.

1. Introduction. In [4] Rosenblum showed that if \( 1 \leq p < \infty \), \( 0 \leq r < 1 \), \( f(x) \) has period \( 2\pi \) and \( \mu \) is a finite Borel measure with period \( 2\pi \), then there is a \( C \), independent of \( r \) and \( f \), such that

\[
\int_{-\pi}^{\pi} |P_r(f, x)|^p d\mu(x) \leq C \int_{-\pi}^{\pi} |f(x)|^p d\mu(x)
\]

if and only if \( d\mu(x) = V(x)dx \) is absolutely continuous and there is a constant \( K \), independent of \( x \) and \( h \), such that

\[
\frac{1}{h} \int_{x-h}^{x+h} \left[ \frac{1}{hV(t)} \right]^{1/(p-1)} \left\{ \frac{V(s)}{V(t)} \right\}^{1/(p-1)} dt \leq K
\]

for all \( h > 0 \) and all \( x \). It is easy to see that a necessary and sufficient condition for (1.2) to hold is that there is a \( B \) such that for every interval \( I \)

\[
\left[ \frac{1}{|I|} \int_I V(x) \, dx \right]^{1/(p-1)} \leq B;
\]
this follows from the fact that the left side of (1.2) is bounded below by

\[ \frac{1}{h} \int_{x-h/2}^{x+h/2} \left[ \frac{1}{hV(t)} \int_{x-h/2}^{x+h/2} V(s) \, ds \right]^{1/(p-1)} \, dt \]

and above by

\[ \frac{1}{h} \int_{x-2h}^{x+2h} \left[ \frac{1}{hV(t)} \int_{x-2h}^{x+2h} V(s) \, ds \right]^{1/(p-1)} \, dt. \]

Rosenblum also considered (1.1) when \( p = 1 \); he proved the same result with (1.2) replaced by

\[ \frac{1}{h} \int_{x-h}^{x+h} V(t) \, dt \leq KV(x). \]

The purpose of this paper is to give a simpler proof than Rosenblum’s of a more general result. Specifically, the following will be proved in §2 and 3.

**Theorem 1.** If \( 1 \leq p < \infty \), \( 0 \leq r < 1 \), \( \mu \) and \( \nu \) are Borel measures of period \( 2\pi \) and \( f(x) \) has period \( 2\pi \), then there is a \( C \), independent of \( f \) and \( r \), such that

\[ \int_{-\pi}^{\pi} |P_r(f, x)|^p \, d\mu(x) \leq C \int_{-\pi}^{\pi} |f(x)|^p \, d\nu(x) \]

if and only if for every interval \( I \)

\[ \left[ \frac{\mu(I)}{|I|} \right] \left[ \frac{1}{|I|} \int_I \left[ \frac{dv_a(x)}{dx} \right]^{-1/(p-1)} \, dx \right]^{p-1} \leq B, \]

where \( B \) is independent of \( I \) and \( v_a \) denotes the absolutely continuous part of \( \nu \).

In Theorem 1 and throughout this paper

\[ \left[ \frac{1}{|I|} \int_I \left[ \frac{dv_a(x)}{dx} \right]^{-1/(p-1)} \, dx \right]^{p-1} \]

is to be interpreted as ess sup_{x \in I} [dv_a(x)/dx]^{-1} if \( p = 1 \) and 0 · ∞ is to be interpreted as 0.

The nonperiodic version of Theorem 1 will follow from the same reasoning; this is stated as Theorem 2 in §4. The fact that for nonnegative \( U(x) \) and \( V(x) \)

\[ \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} |P_r(f, x)|^p \, U(x) \, dx \leq C \int_{-\pi}^{\pi} |f(x)|^p \, V(x) \, dx \]

holds if and only if for every interval \( I \)

\[ \left[ \frac{1}{|I|} \int_I U(x) \, dx \right] \left[ \frac{1}{|I|} \int_I \left[ V(x) \right]^{-1/(p-1)} \, dx \right]^{p-1} \leq B \]

is an immediate corollary of Theorem 1.

In the light of recent results concerning other operators, Theorem 1 and this
corollary are not as natural an extension of Rosenblum's result as they may seem. For example, if \( 1 < p < \infty \), \( U(x) \) and \( V(x) \) are nonnegative and \( U(x) = V(x) \), then there is a \( C \), independent of \( f \), such that

\[
(1.8) \quad \int_{-\pi}^{\pi} \left( \sup_{0 < r < 1} |P_r(f, x)| \right)^p U(x) \, dx \leq C \int_{-\pi}^{\pi} |f(x)|^p V(x) \, dx
\]

if and only if (1.3) is true; this strengthened version of Rosenblum's result follows from [2, Theorem 2, p. 215] since \( \sup_{0 < r < 1} |P_r(f, x)| \) is equivalent to the Hardy-Littlewood maximal function for nonnegative \( f \). If the assumption \( U(x) = V(x) \) is dropped, however, (1.7) does not imply (1.8); an example showing this is in §5 of [2]. In fact, the problem of characterizing the weight functions for which (1.8) is true is unsolved and evidently quite difficult. Similarly, it is shown in [1] that if \( 1 < p < \infty \), \( U(x) \) and \( V(x) \) are nonnegative and \( U(x) = V(x) \), then (1.3) is necessary and sufficient for

\[
(1.9) \quad \int_{-\infty}^{\infty} |\widetilde{f}(x)|^p U(x) \, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p V(x) \, dx
\]

where \( \widetilde{f}(x) = \lim_{\varepsilon \to 0^+} \int_{|t| > \varepsilon} (f(x - t)/t) \, dt \) is the Hilbert transform of \( f(x) \). Again, (1.7) does not imply (1.9) if the assumption \( U(x) = V(x) \) is dropped; for an example see [3]. It is rather surprising, therefore, that (1.7) is necessary and sufficient for (1.6) whether it is assumed that \( U(x) = V(x) \) or not.

It should be noted that Theorem 1 does imply all of Rosenblum's result with his assumptions that \( \mu = \nu \) and \( \mu([-\pi, \pi]) < \infty \). The only problem is to show that (1.5) implies that \( \mu \) is absolutely continuous since (1.3) is obviously equivalent to (1.5) once absolute continuity is proved. To prove that \( \mu \) is absolutely continuous, observe that since \( \mu([-\pi, \pi]) < \infty \), \( d\mu_a(x)/dx < \infty \) almost everywhere. If \( d\mu_a(x)/dx = 0 \) on a set of positive measure, then (1.5) implies that \( \mu([-\pi, \pi]) = 0 \) and \( \mu \) is absolutely continuous. Therefore, assume that \( 0 < d\mu_a(x)/dx < \infty \) almost everywhere. Then, for any interval \( I \),

\[
(1.10) \quad 1 \leq \left( \frac{1}{|I|} \int_I \left[ \frac{d\mu_a(x)}{dx} \right]^{1/p} \left[ \frac{d\mu_a(x)}{dx} \right]^{-1/p} \, dx \right)^p.
\]

Applying Hölder's inequality to the right side of (1.10) then shows that

\[
(1.11) \quad 1 \leq \left( \frac{\mu_a(I)}{|I|} \right) \left( \frac{1}{|I|} \int_I \left[ \frac{d\mu_a(x)}{dx} \right]^{-1/(p-1)} \, dx \right)^{p-1}.
\]

Multiplying (1.11) by \( \mu(I) \) and using (1.5) shows that \( \mu(I) \leq B\mu_a(I) \); since this is true for every interval \( I \), \( \mu \) is absolutely continuous.

2. Proof that (1.5) implies (1.4). The proof of this part of Theorem 1 will be done by proving two simple lemmas and then combining them. In this section and §4 the notation, \( f_h(x) = h^{-1} \int_{x-h}^{x+h} f(t) \, dt \), will be used.
Lemma 1. If $f(x)$ has period $2\pi$ and $\mu$ and $\nu$ are Borel measures with period $2\pi$ that satisfy (1.5), then for every $h > 0$,

$$
\int_{-\pi}^{\pi} [f_h(x)]^p \, d\mu(x) \leq 6^{p+1} B \int_{-\pi}^{\pi} |f(t)|^p \, d\nu(x).
$$

Assume first that $h \leq \pi$ and let $N$ be the least integer such that $Nh \geq \pi$. Then the left side of (2.1) is bounded by

$$
\sum_{k=-N}^{N-1} \int_{kh}^{(k+1)h} h^{-p} \left( \int_{x-h}^{x+h} |f(t)| \, dt \right)^p \, d\mu(x),
$$

and this is bounded by

$$
\sum_{k=-N}^{N-1} \left[ h^{-p} \int_{kh}^{(k+1)h} d\mu(x) \right] \left( \int_{(k-1)h}^{(k+2)h} |f(t)| \, dt \right)^p.
$$

Using Hölder's inequality on the second integral in (2.2) shows that (2.2) is bounded by

$$
\sum_{k=-N}^{N-1} \left( \mu([kh, (k+1)h]) \right) \left( \frac{1}{h} \int_{(k-1)h}^{(k+2)h} \left[ \frac{d\nu_a}{dx} \right]^{-1/(p-1)} \, dx \right)^{p-1} 
\cdot \left( \int_{(k-1)h}^{(k+2)h} |f(t)|^p \, d\nu_a(t) \right);
$$

note that if $d\nu_a/dx$ is 0 on a set of positive measure, $\mu([h, \pi]) = 0$ and (2.2) is still bounded by this. Now (1.5) with $I = [(k-1)h, (k+2)h]$ shows that this is bounded by

$$
3^p B \int_{-\pi}^{\pi} |f(t)|^p \, d\nu(t).
$$

Since all the intervals $[(k-1)h, (k+2)h]$ are subsets of $[-2\pi, 2\pi]$ and no point is in more than three of these intervals, (2.3) is bounded by

$$
3^{p+1} B \int_{-\pi}^{\pi} |f(t)|^p \, d\nu(t).
$$

This completes the proof if $h \leq \pi$. If $h > \pi$, $f_h(x) \leq 2f_\pi(x)$ and the result follows from the first part.

Lemma 2. If $f(x)$ has period $2\pi$ and $0 < r < 1$, then there is a constant $K$, independent of $f$ and $r$, such that

$$
|P_r(f, x)| \leq K \int_{1-r}^{2\pi} (1 - r)h^{-2} f_h(x) \, dh.
$$

By reversing the order of integration, the integral on the right side of (2.4) is greater than or equal to
\[ \int_{x-\pi}^{x+\pi} \left| f(t) \right| \left[ \int_{|x-t|\vee(1-r)}^{2\pi} (1-r)h^{-3} \, dh \right] \, dt \]

where \(|x-t|\vee(1-r)|\) denotes the larger of \(|x-t|\) and \(1-r\). The inner integral can be calculated and is clearly greater than a positive constant times

\[ \frac{1}{2\pi} \left( \frac{1-r^2}{(1-r)^2 + 2r[1 - \cos(x-t)]} \right) \]

since \(|x-t| \leq \pi\). This proves Lemma 2.

To show that (1.5) implies (1.4), use Lemma 2 to show that the left side of (1.4) is bounded above by

\[ (2.5) \quad K^p \int_{-\pi}^{\pi} \left[ \int_{1-r}^{2\pi} (1-r)h^{-2}f(x) \, dx \right]^p \, d\mu(x). \]

By Minkowski's integral inequality, (2.5) is bounded by

\[ (2.6) \quad K^p \left( \int_{1-r}^{2\pi} \left[ \int_{-\pi}^{\pi} \left| f(x) \right|^p \, d\mu(x) \right]^{1/p} \right)^p (1-r)h^{-2} \, dh \]

By Lemma 1, (2.6) is bounded by

\[ 6^{p+1}BK^p \left[ \int_{-\pi}^{\pi} \left| f(x) \right|^p \, d\nu(x) \right] \left[ \int_{1-r}^{2\pi} (1-r)h^{-2} \, dh \right]^p \]

Since the last integral is less than 1, (1.4) follows.

3. Proof that (1.4) implies (1.5). This proof need only be done for intervals \(I\) with \(|I| \leq 2\pi\), since if \(|I| > 2\pi\) the left side of (1.5) is bounded by \(2^p\) times its value for the interval \([-\pi, \pi]\). Given any \(f(x)\), let \(f_1(x) = 0\) on the support of the singular part of \(\nu\) and let \(f_1(x) = f(x)\) elsewhere. Then \(P_r(f_1, x) = P_r(f, x)\) and \(\int_{-\pi}^{\pi} \left| f_1(x) \right|^p \, d\nu(x) = \int_{-\pi}^{\pi} \left| f(x) \right|^p \, d\nu(x)\). Therefore, if \(d\nu(x)/dx\) is written as \(V(x)\), (1.4) with \(f\) replaced by \(f_1\) implies that

\[ (3.1) \quad \int_{-\pi}^{\pi} \left| P_r(f, x) \right|^p \, d\nu(x) \leq C \int_{-\pi}^{\pi} \left| f(x) \right|^p \, V(x) \, dx. \]

Consequently, the proof that (1.4) implies (1.5) can be completed by showing that (3.1) implies (1.5) for intervals \(I\) with \(|I| \leq 2\pi\).

Given \(I\) with \(|I| \leq 2\pi\), let \(Q = f_I[V(x)]^{-1/(p-1)} \, dx\) and let \(p' = p/(p-1)\). If \(Q = 0\), (1.5) follows because of the convention \(0 \cdot \infty = 0\). If \(Q = \infty\), \([V(x)]^{-1/p}\) is not in \(L^{p'}\) on \(I\) so there is a function \(g(x)\) in \(L^p\) on \(I\) such that \(g(x)[V(x)]^{-1/p}\) is not integrable on \(I\). Let \(f(x) = g(x)[V(x)]^{-1/p}\) on \(I\) and 0 elsewhere. Then \(P_r(f, x) = \infty\) for all \(x\) and the right side of (3.1) is finite since \(g(x)\) is in \(L^p\) on \(I\). Therefore, \(\mu(I) = 0\) and (1.5) is true.

If \(0 < Q < \infty\) and \(p > 1\), let \(f(x) = [V(x)]^{-1/(p-1)}\) if \(x + 2\pi n \in I\) for some integer \(n\) and 0 elsewhere. Let \(r\) be the larger of \(1 - |I|\) and 0. Then for \(x\) in \(I\),
$P_r(f, x) \geq \frac{A}{|I|} \int_I [V(y)]^{-1/(p-1)} \, dy$

where $A$ is a positive constant independent of $x$, $V$, $p$ and $I$. Then (3.1) implies that

$$(3.2) \quad \int_I \left[ \frac{A}{|I|} \int_I [V(y)]^{-1/(p-1)} \, dy \right]^p \, d\mu(x) \leq C \int_I [V(x)]^{-1/(p-1)} \, dx.$$  

Dividing by the integral on the right side of (3.2) then gives (1.5).

If $0 < Q < \infty$ and $p = 1$, choose $\epsilon > 0$ and let $E$ be the subset of $I$ where $V(x) < \epsilon + \text{ess inf}_{y \in I} V(y)$. Let $f(x)$ equal 1 if $x + 2n\pi \in E$ for some integer $n$ and 0 otherwise, and let $r$ be the larger of $1 - |I|$ and 0. Then for $x$ in $I$, $P_r(f, x) \geq A|E|/|I|$ where $A$ is a positive constant independent of $x$, $V$ and $I$, and $|E|$ denotes the Lebesgue measure of $E$. Then (3.1) implies that

$$\frac{A|E|}{|I|} \leq C|E| \left[ \epsilon + \text{ess inf}_{y \in I} V(y) \right].$$

Dividing by $A|E|$ and using the fact that $\epsilon$ was arbitrary gives

$$\frac{\mu(I)}{|I|} \leq \frac{C}{A} \text{ess inf}_{y \in I} V(y);$$

this is equivalent to (1.5) with the appropriate interpretation for $p = 1$.

4. The nonperiodic case. Given $f(x)$ defined on $(-\infty, \infty)$, let

$$f(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(y)dy}{t^2 + (x - y)^2}$$

be the usual Poisson integral. The nonperiodic theorem is the following.

**Theorem 2.** If $1 \leq p < \infty$, $t > 0$ and $\mu$ and $\nu$ are Borel measures, then there is a $C$, independent of $f$, such that

$$(4.1) \quad \int_{-\infty}^{\infty} |f(t, x)|^p \, d\mu(x) \leq C \int_{-\infty}^{\infty} |f(x)|^p \, d\nu(x)$$

if and only if for every interval $I$ (1.5) holds where $B$ is independent of $I$ and $\nu_a$ denotes the absolutely continuous part of $\nu$.

The proof that (1.5) implies (4.1) uses the following analogues of Lemmas 1 and 2.

**Lemma 3.** If $\mu$ and $\nu$ are Borel measures that satisfy (1.5), then for every $h > 0$

$$\int_{-\infty}^{\infty} [f_h(x)]^p \, d\mu(x) \leq 3^{p+1} B \int_{-\infty}^{\infty} |f(t)|^p \, d\nu(x).$$

**Lemma 4.** There is a constant $K$, independent of $f$ and $t$, such that

$$|f(t, x)| \leq Kf_t \frac{1}{t} f_h(x) \, dh.$$
The proof of Lemma 3 is the same as that of Lemma 1 except that the sum is taken from $-\infty$ to $\infty$ and the initial restriction on the length of $I$ is not needed. Lemma 4 is proved in the same way that Lemma 2 was. The rest of the proof that (1.5) implies (4.1) is the same as the proof that (1.5) implies (1.4) except that Lemmas 3 and 4 are used in place of Lemmas 1 and 2.

The proof that (4.1) implies (1.5) is essentially the same as the proof in §3; the reduction to intervals $I$ with $|I| \leq 2\pi$ is not needed and in the last two cases $t$ should be chosen equal to $|I|$ instead of $r$ being the larger of $1 - |I|$ and 0.

REFERENCES