TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 210, 1975

ON MINIMAL IMMERSIONS OF S^2 INTO S^{2m} (1)

BY

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ABSTRACT. The study of minimal immersions of the 2-sphere into the standard n-sphere of the euclidean space has been better accomplished by associating to each such immersion a certain holomorphic curve. This has been done in several ways in the literature. In the present paper we explore this technique applying some knowledge about the topological and analytical invariants of the particular set of holomorphic curves used to obtain further results. Some new examples are provided, a beginning of a general description of such immersions is given and a rigidity theorem is proved.

1. **Introduction.** In this paper we will consider generalized minimal immersions $x: S^2 \longrightarrow S^{2m}(1)$ where $S^{2m}(1)$ is the unit sphere of the euclidean space R^{2m+1} and S^2 is the 2-sphere which will always be considered as having the induced metric. With this metric S^2 acquires a conformal structure and becomes the Riemann sphere.

Following S. S. Chern [4] we associate to x a certain holomorphic curve Ξ from S^2 into CP^{2m} (the complex projective space of dimension $2m$) called the directrix curve of the minimal immersion. This curve is rational and the only condition it must satisfy is that of being totally isotropic, i.e., if ξ is any of its local representations in homogeneous coordinates, then ξ satisfies

$$(\xi, \xi) = (\xi', \xi') = \ldots = (\xi^{m-1}, \xi^{m-1}) = 0$$

where the upper indices stand for derivatives and (,) denotes the symmetrical product in C^{2m+1}.

In §4 we show how Ξ is defined for a given x and how we may construct a generalized minimal immersion from its directrix curve. In fact, given any totally isotropic curve $\Xi: S^2 \longrightarrow CP^{2m}$ we may consider its $(m-1)$th associated curve $\Xi_{m-1}: S^2 \longrightarrow CP^{N-1}$, $N = \binom{2m+1}{m}$, which assigns to each point z of S^2 the subspace $V(z)$ of C^{2m+1} spanned by the first $m-1$ derivatives of ξ at that point. This subspace is perpendicular to its own conjugate $\overline{V(z)}$. The complex

Received by the editors June 10, 1974.

AMS (MOS) subject classifications (1970). Primary 53B25, 32C25, 58D10; Secondary 14H45, 14H99, 14M15, 53C40.

(1) Research supported by Conselho Nacional de Pesquisas do Brasil, NSF Grant GP-34785X and Universidade Federal do Ceará–Brasil.

line perpendicular to both subspaces V and $\overline{V}$ is spanned by a unitary real vector $x(z)$. This vector x describes a generalized minimal immersion of S^2 into S^{2m}. It can be shown that Ξ is its directrix curve.

In §5 we consider the space H_m of all m-dimensional totally isotropic subspaces V of C^{2m+1}. Total isotropy means in this case that V and $\overline{V}$ are perpendicular. It turns out that H_m is a Kähler submanifold of CP^{N-1}, $N = \binom{2m+1}{m}$. We prove that H_m has second homology group (with integer coefficients) isomorphic to the integers. Then we show that the degree of the fundamental cycle of H_m, as a curve in CP^{N-1}, is 2. Therefore the degree of any holomorphic curve from S^2 into CP^{N-1} whose image lies in H_m will be a multiple of 2 and the same is true of degree(Ξ_{m-1}). We use this to show that Area(x) is a multiple of 4π. This is an improvement of a result obtained by Calabi in his basic paper on the subject [2].

In §6 a rigidity theorem for minimal immersions of S^2 into S^{2m} is obtained. We show that if x, y: $S^2 \longrightarrow S^{2m}$ are isometric generalized minimal immersions, then x and y differ by a rigid motion of the ambient space S^{2m}. This result was known for the case in which the induced metric in S^2 was the one with constant curvature. We begin the proof by showing that if Ξ and Z are the directrix curves of x and y respectively, then x and y are isometric if and only if Ξ and Z are also. But, in the latter case, a theorem from Calabi [1] tells us that Ξ and Z differ by a rigid motion of CP^{2m}. We then show that this rigid motion must be an element of $SO(2m + 1)$. This depends on the fact that both curves are totally isotropic. By definition of directrix curve, this implies that x and y differ by an element of $O(2m + 1)$.

§7 is devoted to the construction of examples. We begin by examining more closely the relations between Area(x) and the analytical invariants of the directrix curve of x. It is natural here to make use of the classical Plücker formulas. From this one can derive the basic result of Calabi [2] that Area(x) $\geqslant 2\pi m(m + 1)$. By considering a very particular local expression for the directrix curve we obtain a set of minimal immersions such that for any multiple of 4π bigger than or equal to $2\pi m(m + 1)$ there is one having that value as its area. This shows that the result obtained in §4 is the best possible. We point out that the group $SO(2m + 1, C)$ of all complex matrices A satisfying $\det(A) = 1$ and $A \cdot {}^tA = I$ acts on the space of totally isotropic curves. Consequently it acts on the space of minimal immersions x: $S^2 \longrightarrow S^{2m}$. We then show that if we identify minimal immersions that are isometric, we find that the ones with area $2\pi m(m + 1)$ form a space diffeomorphic to $SO(2m + 1, C)/SO(2m + 1, R)$.

I wish to express my deep gratitude to my advisor and teacher, Professor S. S. Chern, for the help, encouragement and suggestions he gave me during the

time I was working on this paper. I am indebted to Professors H. Blaine Lawson and S. Kobayashi for the many conversations we had which proved invaluable in the development of the ideas of this work.

2. **Definitions and preliminary remarks.** Let M be an oriented, compact differentiable surface and $x: M \longrightarrow S^n(1)$ a differentiable map into the unit n-sphere. The induced metric on M, together with its orientation, defines a covering of M by isothermal coordinates. Relative to a local isothermal parameter z, the metric in M takes the particular form:

$$ds^2 = 2F|dz|^2 \tag{2.1}$$

and the area form can then be represented by:

$$\mathcal{W} = iFdz \wedge d\bar{z}. \tag{2.2}$$

When x is an immersion, F is an everywhere positive valued (real analytic) function. Throughout this paper we will be working with maps that are (minimal) immersions at all but finitely many points of M. These will be called generalized (minimal) immersions. In local terms this means only that we consider F as having at most finitely many zeros.

All higher order derivatives of x with respect to z and $\bar{z}$ will be considered as functions with values in C^{n+1}. The complex osculating space of order m, at a point p of M, is the pull back of the subspace of C^{n+1} spanned by all the mixed derivatives $\partial^{j+k}x/\partial^j z\partial^k\bar{z}$ with $0 \leqslant j + k \leqslant m$.

In C^{n+1} the symmetrical product of two vectors $a = (a_0, \ldots, a_n)$ and $b = (b_0, \ldots, b_n)$ is defined by

$$(a, b) = a_0b_0 + \ldots + a_nb_n \tag{2.3}$$

and the Hermitian product of a and b is then defined by $(a, \bar{b})$.

If we set $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$ we have that:

(a) as a consequence of z being an isothermal parameter

$$(\partial x, \partial x) = 0; \tag{2.4}$$

(b) the function F (obtained in the expression of the induced metric in M) is given by

$$F = (\partial x, \overline{\partial x}); \tag{2.5}$$

(c) the Gauss curvature of M is

$$K = -\frac{1}{F}\partial\bar{\partial}\log F; \tag{2.6}$$

(d) the Laplacian operator is given by

$$\Delta = \frac{2}{F}\partial\bar{\partial}. \tag{2.7}$$

It is known that x is a minimal immersion into S^n if and only if x satisfies the equation

$$\Delta x = \lambda x. \tag{2.8}$$

(See for example [7, p. 31].)

If we notice that Δx is parallel to $\partial\bar{\partial}x$, we see that $\partial\bar{\partial}x = \delta x$ for some function δ. To express δ in terms of our notation we simply compute $\partial\bar{\partial}$ of both sides of the identity $(x, x) = 1$ and get by (2.5) that

$$\partial\bar{\partial}x = -Fx. \tag{2.9}$$

Let us notice that x is a generalized minimal immersion if and only if it satisfies this last identity with F having at most a finite number of zeros.

The equation (2.9) enables us to write any mixed derivative of x of order $\leqslant k$ in terms of the complex vectors $x, \partial x, \ldots, \partial^k x, \bar{\partial}x, \bar{\partial}^2 x, \ldots, \bar{\partial}^k x$ of C^{n+1}. Consequently the complex osculating space of order k at a point p of M is spanned by these $2k + 1$ vectors evaluated at p.

Let us now assume $M = S^2$, where S^2 stands for the standard sphere of R^3. Using (2.9) and the topology of S^2 Calabi obtained in [2] that:

$$(\partial^j x, \partial^k x) = 0, \qquad j + k > 0, \tag{2.10}$$

where the notation was extended by setting $\partial^0 x = x$. Geometrically this means that the subspace $V(x)$ of C^{n+1} spanned by the vectors $\partial x, \partial^2 x, \ldots, \partial^k x, \ldots$, at a point x of S^2 is totally isotropic and perpendicular to x. Totally isotropic means that $V(x)$ is perpendicular to its own conjugate $\bar{V}(x)$. Hence, if we set $m = \dim V(x)$, we must have $2m \leqslant n$.

Using this, one can easily obtain the following known result (see, for example, Calabi [3]):

(2.11) THEOREM. *If $x: S^2 \longrightarrow S^n$ is a generalized minimal immersion not lying in any lower dimensional subspace of R^{n+1}, then n is even.*

We should point out that this theorem is true for a general compact surface M whenever the immersion satisfies (2.10). Following Calabi [3] we call them

generalized pseudo holomorphic maps. In fact, all results in §§3, 4 and 5 are true if we replace "generalized minimal immersion x: $S^2 \longrightarrow S^{2m}$" with "generalized pseudo holomorphic map x: $M \longrightarrow S^{2m}$" where M is supposed to be a compact Riemann surface.

3. **The directrix curve.**

(3.1) Let x: $S^2 \longrightarrow S^{2m}$ be a generalized minimal immersion and consider S^2 covered by isothermal coordinates as before. In a coordinate neighborhood we can construct the following local vector-valued functions:

$$\begin{aligned} G_0 &= x, \\ G_1 &= \bar{\partial} x, \\ G_2 &= \bar{\partial}^2 x - a_2^1 G_1, \\ &\cdots\cdots\cdots\cdots \\ G_k &= \bar{\partial}^k x - \sum_{j=1}^{k-1} a_k^j G_j, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where the a_k^j are chosen in such a way that

$$(G_k, \bar{G}_j) = 0, \qquad j < k, \tag{3.2}$$

From (2.10) we see that

$$G_{m+k} = 0 \tag{3.3}$$

and the space spanned by $G_1, G_2, \ldots$ is totally isotropic and perpendicular to x, which give us:

$$(G_k, G_j) = 0, \qquad j + k \geqslant 1. \tag{3.4}$$

(3.5) The importance of defining these G_k's is that at each point where they are different from zero the direction of each one of them is invariant under change of coordinates. This is a consequence of the fact that, at each point of S^2, the subspace $O_k(x)$ of C^{2m+1} spanned by $\{x, \partial x, \partial^2 x, \ldots, \partial^k x\}$, which we could call the k-osculating holomorphic space of x, is well defined in the sense that it does not depend on the particular system of coordinates used to compute the derivatives. Also, the points where $G_k = 0$ are exactly those where the subspace $O_k(x)$ is degenerate, i.e., the ones where $O_k(x)$ has dimension less than $k + 1$, and these are also invariant.

(3.6) From the invariance of the direction of each G_k we have that, for

each fixed k, we may consider a well-defined map from S^2 minus the points where G_k is zero, taking values in the complex projective space CP^{2m}. The natural questions at this point are:

(I) Which of these functions may be extended to the whole Riemann sphere?

(II) Which of these functions are holomorphic?

It turns out that one such function is the answer to both of these questions simultaneously. But, not anticipating the final result, we will proceed in the most natural way, looking for an answer to question (II) by computing derivatives of the G_k's.

(3.7) LEMMA. (a) $\bar{\partial} G_k = G_{k+1} + (\bar{\partial} \log |G_k|^2) G_k$;
(b) $\partial G_k = -|G_k|^2 G_{k-1}/|G_{k-1}|^2$, $k \geqslant 1$.

PROOF. (a) First we prove by induction that G_k has components only in the subspace of C^{2m+1} generated by $\{G_1, G_2, \ldots, G_m\}$. In fact we prove that

$$\bar{\partial} G_k = G_{k+1} + (a_{k+1}^k - a_k^{k-1}) G_k$$
$$+ \text{ terms involving only } \{G_1, \ldots, G_{k-1}\}.$$

For the case $k = 1$ we proceed as follows. We have $G_1 = \bar{\partial} x$, and so $\bar{\partial} G_1 = \bar{\partial}^2 x$. Consider $\bar{\partial}^2 x$ on the expression of G_2. Then, $\bar{\partial}^2 x = G_2 + a_2^1 G_1$ is the desired result. This is almost a typical case, and for the general situation the only thing we do additionally is to use the induction hypothesis replacing $\bar{\partial} G_j$, $j < k$, by the corresponding expression.

To complete the proof of part (a) of this lemma we need to show that, for the expression we have obtained for $\bar{\partial} G_k$, all terms in G_j, $j < k$, are zero, and the coefficient of G_k is given by $\bar{\partial} \log |G_k|^2$. Or, equivalently, we need to prove that

$$(\bar{\partial} G_k, \overline{G_j}) = \begin{cases} 0, & j < k, \\ \bar{\partial} |G_k|^2, & j = k. \end{cases}$$

But, since $(\bar{\partial} G_k, \bar{G}_j) = \bar{\partial}(G_k, \bar{G}_j) - (G_k, \bar{\partial} \bar{G}_j)$, this follows from (3.2) and (in view of (2.9)), from the observation that $\bar{\partial} \bar{G}_k = \bar{\partial}(\partial^k x - \Sigma_{j=1}^{k-1} \bar{a}_k^j \bar{G}_k)$ belongs to the subspace of C^{2m+1} generated by the set $\{\partial x, \ldots, \partial^{k-1} x\}$ (or $\{\bar{G}_1, \ldots, \bar{G}_{k-1}\}$).

(b) As in the proof of (a), we first show by induction that ∂G_k lies in the subspace of C^{2m+1} generated by $\{x, G_1, \ldots, G_k\}$. Then, to compute each component of ∂G_k in this basis, we proceed as follows. We know that

$$(\partial G_k, \bar{G}_j) = \partial(G_k, \bar{G}_j) - (G_k, \partial \bar{G}_j) \tag{3.8}$$

and from (3.2) we have

$$\partial(G_k, \bar{G}_j) = \begin{cases} 0, & \text{for } j < k, \\ \partial|G_k|^2, & \text{if } j = k. \end{cases} \tag{3.9}$$

To compute $(G_k, \partial\bar{G}_j) = (G_k, \overline{\bar{\partial}G_j})$, just substitute for $\bar{\partial}G_j$ the expression obtained in part (a) of this lemma. Then we get

$$(G_k, \partial\bar{G}_j) = \begin{cases} 0 & \text{for } j < k-1, \\ |G_k|^2 & \text{for } j = k-1, \\ \partial|G_k|^2 & \text{for } j = k. \end{cases} \tag{3.10}$$

Putting (3.8), (3.9) and (3.10) together, we finally get

$$(\partial G_k, \bar{G}_j) = \begin{cases} 0 & \text{for } j < k-1, \\ -|G_k|^2 & \text{if } j = k-1, \\ 0 & \text{if } j = k, \end{cases}$$

completing the proof of (b).

Because $G_{m+k} = 0$ (see (3.3)), this lemma gives

$$\bar{\partial}G_m = (\bar{\partial}\log|G_m|^2)G_m. \tag{3.11}$$

From this it follows that $G_m/|G_m|^2$ is holomorphic wherever it is defined. Remembering the comments made in (3.6), this means that the global function, locally defined by G_m, is holomorphic, except for the points where $G_m = 0$ which are its singular points. The nature of these singularities is clarified by the following lemma.

(3.12) LEMMA. *$G_m/|G_m|^2$ has only isolated singularities. Moreover, they are poles.*

PROOF. Set $G_m = (y_0, y_1, \ldots, y_{2m})$. Then, for each k, $y_k/|G_m|^2$ is holomorphic wherever G_m is different from zero. If z_0 is a point where $G_m = 0$, then we have by the continuity of G_m

$$\lim_{z \to z_0} \frac{y_k}{|G_m|^2} = \infty, \quad \text{for some } k,\ 0 \leqslant k \leqslant 2m.$$

Choose a neighborhood U of z_0 where $y_k/|G_m|^2 > \epsilon > 0$. In such a neighborhood $|G_m|^2/y_k$ defines a continuous function that is zero where $G_m = 0$, and that is

holomorphic outside of these points. Hence, by Rado's theorem (see [8, p. 40]), this function is holomorphic in the interior of any closed disk centered in z_0 and contained in U. Consequently, z_0 is an isolated zero of $|G_m|^2/y_k$ and so is an isolated pole of $G_m/|G_m|^2$.

(3.13) The immediate consequence of this lemma is that the function defined by the projection of G_m in the complex projective space CP^{2m} extends through the points where G_m is zero, and so, by the observations made in (3.6), $G_m/|G_m|^2$ may be used as a local definition in homogeneous coordinates for a holomorphic map from S^2 with values in CP^{2m}. We call such a function the *directrix curve* of the minimal immersion x.

After having constructed the directrix curve our next step is to look for its properties. Since the main property we derived from the minimality of x was the total isotropy of the subspace generated by $\{G_1, \ldots, G_m\}$, we should obviously search for some reflex of this on the several derivatives of the directrix curve.

Using Lemma (3.7) we obtain the following local expression for the derivatives of the directrix curve. Let Ξ denote the directrix curve and let ξ be its local expression in terms of homogeneous coordinates given by $\xi = G_m/|G_m|^2$; then we have

$$\xi^k = \sum_{j=0}^{k-1} A^k_{m-j} G_{m-j} + (-1)^k \frac{1}{|G_{m-k}|^2} G_{m-k}, \tag{3.14}$$

$$\xi^{m+k} = \sum_{j=0}^{m-1} A^{m+k}_{m-j} G_{m-j} + \sum_{j=0}^{k-1} B^{m+k}_j \overline{G}_j + (-1)^m \overline{G}_k, \tag{3.15}$$

where $0 \leqslant k \leqslant m$ and the coefficients A^i_j, B^r_s are functions of z and $\bar{z}$. From this we have that the directrix curve Ξ is totally isotropic; that is, if ξ is any of its local representations in homogeneous coordinates, then

$$(\xi, \xi) = (\xi', \xi') = \ldots = (\xi^{m-1}, \xi^{m-1}) = 0. \tag{3.16}$$

Another condition that we have been assuming is that x does not lie in any lower dimensional subspace of R^{2m+1}. As a consequence of this, we have that Ξ does not lie in any complex hyperplane of CP^{2m}.

(3.17) The next natural question to ask is if these properties characterize the set of directrix curves among all the holomorphic ones from S^2 into CP^{2m}. This immediately leads us to consider the possibility of reversing the process described above and, starting from an arbitrary totally isotropic holomorphic curve $\Xi: S^2 \longrightarrow CP^{2m}$, that is not contained in any complex hyperplane of CP^{2m},

somehow to construct a minimal immersion $x: S^2 \longrightarrow S^{2m}$ that has Ξ as its directrix curve. This is indeed possible.

We begin by noticing that $\{\xi, \xi', \ldots, \xi^{m-1}\}$ spans the same subspace of C^{2m+1} as $\{G_1, \ldots, G_m\}$. Remember that x is perpendicular to this space and to its conjugate. Hence, if we consider the vector-valued function ψ defined by

$$\psi = \xi \wedge \xi' \wedge \ldots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \bar{\xi}' \wedge \ldots \wedge \bar{\xi}^{m-1},$$

we have that

(a) the function ψ is perpendicular to the space spanned by $\{\xi, \xi', \ldots, \xi^{m-1}, \bar{\xi}, \bar{\xi}', \ldots, \bar{\xi}^{m-1}\}$.

(b) $\psi = (-1)^m \bar{\psi}$.

From (a) it follows that ψ and x are parallel as complex vectors in C^{2m+1}. From (b) it follows that ψ is either real or pure imaginary. It will be real if m is even, and pure imaginary if m is odd. Set

$$\tilde{\psi} = \begin{cases} \psi, & \text{if } m \text{ is even,} \\ -i\psi, & \text{if } m \text{ is odd.} \end{cases}$$

Then $\tilde{\psi}$ is parallel to x in the real sense, and we may establish the following proposition.

(3.17) PROPOSITION. *The function $\tilde{\psi}/|\tilde{\psi}|$ is independent of the particular local coordinates used, and so it defines a global map x from S^2 into S^{2m}. Furthermore, we have, relative to a local coordinate z, that $(\partial x, \partial x) = 0$ and $\partial\bar{\partial}x$ is parallel to x.*

PROOF. If z, w are two local isothermal coordinates in S^2 then

$$\xi^k(w) = \xi^k(z)(dz/dw)^k + \text{terms in } \xi^j(z) \text{ with } j < k,$$

and so $\psi(w) = \psi(z)|dz/dw|^{(1+2+\ldots+(m-1))}$. Because ψ and $\tilde{\psi}$ differ only by a constant term, this implies $\tilde{\psi}(w)/|\tilde{\psi}(w)| = \tilde{\psi}(z)/|\tilde{\psi}(z)|$. We also have that $\tilde{\psi}/|\tilde{\psi}|$ is invariant under change of the local representation for Ξ. In fact, if ζ is another local representation then $\xi = \lambda\zeta$ and $\tilde{\psi}_\xi = |\lambda|^{2m}\tilde{\psi}_\zeta$. Consequently $\tilde{\psi}_\xi/|\tilde{\psi}_\xi| = \tilde{\psi}_\zeta/|\tilde{\psi}_\zeta|$. One should notice that $\tilde{\psi}(z)$ may have some isolated zeros. But, even at these points $\tilde{\psi}/|\tilde{\psi}|$ is well defined. Indeed, if for example $\tilde{\psi}(z_0) = 0$, this means that $\xi \wedge \xi' \wedge \ldots \wedge \xi^{m-1}$ has a zero of a certain order, say r, at z_0. We may then factor $\tilde{\psi}$ as $\tilde{\psi}(z) = |z - z_0|^r\varphi(z)$ with $\varphi(z_0) \neq 0$. Consequently, the functions $\tilde{\psi}(z)/|\tilde{\psi}(z)|$ are local expressions for a global function x from S^2 into S^{2m}.

To complete the proof of the proposition, it remains to show that:

(a) $(\partial x, \partial x) = 0$ (meaning that z is an isothermal parameter for S^2 relative to the induced metric),

(b) $\partial\bar{\partial}x$ is parallel to x (meaning that x is minimal).

To do this, let us first set up some machinery. Since

$$\begin{aligned} \psi &= \xi \wedge \ldots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \ldots \wedge \bar{\xi}^{m-1}, \\ \partial\psi &= \xi \wedge \ldots \wedge \xi^{m-2} \wedge \xi^m \wedge \bar{\xi} \wedge \ldots \wedge \bar{\xi}^{m-1}, \\ \bar{\partial}\psi &= \xi \wedge \ldots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \ldots \wedge \bar{\xi}^{m-2} \wedge \xi^m, \end{aligned} \tag{3.18}$$

then, if we set $T = \xi \wedge \xi' \wedge \ldots \xi^{m-1}$, we have

$$\begin{aligned} (T, T) &= 0, \\ (\psi, \psi) &= (-1)^m |T|^4, \\ (\psi, \bar{\psi}) &= |\psi|^2 = |T|^4, \\ (\psi, \partial\psi) &= (-1)^m |T|^2(\bar{T}, \partial T), \\ (\psi, \bar{\partial}\psi) &= (-1)^m |T|^2(T, \overline{\partial T}), \\ (\partial\psi, \bar{\partial}\psi) &= (-1)^m |T|^2 |\partial T|^2, \\ (\partial\psi, \partial\psi) &= (-1)^m (\partial T, \bar{T})^2, \\ \partial(1/|\psi|) &= -(\partial T, \bar{T})/|T|^4, \\ \bar{\partial}(1/|\psi|) &= -(T, \overline{\partial T})/|T|^4. \end{aligned} \tag{3.19}$$

Now, we may start proving (a) and (b).

(a) Set $x = a\psi/|\psi|$ where a is 1 if m is even and (-1) if m is odd. Then we have

$$\partial x = a\{\partial(1/|\psi|)\psi + (1/|\psi|)\partial\psi\} \tag{3.20}$$

and consequently

$$(\partial x, \partial x) = a^2\left\{\partial\left(\frac{1}{|\psi|}\right)\partial\left(\frac{1}{|\psi|}\right)(\psi, \psi) + \frac{2}{|\psi|}\partial\left(\frac{1}{|\psi|}\right)(\psi, \partial\psi) + \frac{1}{|\psi|^2}(\partial\psi, \partial\psi)\right\}.$$

Now, substitution of (3.19) in this expression yields $(\partial x, \partial x) = 0$.

(b) Computing $\bar{\partial}$ of (3.20) we obtain

$$\bar{\partial}\partial x = a\left\{\partial\bar{\partial}\left(\frac{1}{|\psi|}\right)\psi + \bar{\partial}\left(\frac{1}{|\psi|}\right)\partial\psi + \partial\left(\frac{1}{|\psi|}\right)\bar{\partial}\psi + \frac{1}{|\psi|}\partial\bar{\partial}\psi\right\}. \tag{3.21}$$

To prove that $\partial\bar{\partial}x$ is parallel to x we should prove that $(\partial\bar{\partial}x, \xi^k) = 0$ for $0 \leqslant$

$k \leqslant m-1$. From (3.18) we obtain:

$$
\begin{aligned}
&(\psi, \xi^k) = 0, && 0 \leqslant k \leqslant m-1,\\
&(\partial\psi, \xi^k) = 0, && 0 \leqslant k < m-1,\\
&(\partial\psi, \bar{\xi}^k) = 0, && 0 \leqslant k \leqslant m-1,\\
&(\partial\bar{\partial}\psi, \xi^k) = 0, && 0 \leqslant k < m-1.
\end{aligned}
\tag{3.22}
$$

Thus we have $(\partial\bar{\partial}x, \xi^k) = 0$ for $0 \leqslant k < m-1$ and it remains to prove only that $(\partial\bar{\partial}x, \xi^{m-1}) = 0$. From (3.21) and (3.22) follows that

$$
\begin{aligned}
(\partial\bar{\partial}x, \xi^{m-1}) &= a\left\{\bar{\partial}\left(\frac{1}{|\psi|}\right)(\partial\psi, \xi^{m-1}) + \frac{1}{|\psi|}(\partial\bar{\partial}\psi, \xi^{m-1})\right\}\\
&= a\bar{\partial}\left\{\frac{1}{|\psi|}(\partial\psi, \xi^{m-1})\right\}.
\end{aligned}
\tag{3.23}
$$

From (3.18) we also have

$$
\begin{aligned}
(\partial\psi, \xi^{m-1}) &= (\xi \wedge \ldots \wedge \xi^{m-2} \wedge \xi^m \wedge \bar{\xi} \wedge \ldots \wedge \bar{\xi}^{m-1}, \xi^{m-1})\\
&= (-1)^{m+1}\xi \wedge \ldots \wedge \xi^{m-2} \wedge \xi^{m-1} \wedge \xi^m \wedge \bar{\xi} \wedge \ldots \wedge \xi^{m-1}.
\end{aligned}
$$

Since $(\xi^m, \xi^k) = 0$ for $k = 0, 1, \ldots, m-1$ (as a consequence of (3.16)), it is possible to write

$$
\xi^m = \sum_{j=0}^{m-1} \alpha_j \xi^j + (\xi^m, x)x. \tag{3.24}
$$

Thus

$$
\begin{aligned}
(\partial\psi, \xi^{m-1}) &= -(\xi^m, x)\xi \wedge \ldots \wedge \xi^{m-1} \wedge \bar{\xi} \wedge \ldots \wedge \bar{\xi}^{m-1} \wedge x\\
&= -(\xi^m, x)(\psi, x).
\end{aligned}
$$

Using $x = a\psi/|\psi|$ together with (3.18) we get

$$
(\partial\psi, \xi^{m-1}) = a(-1)^{m+1}|T|^2(\xi^m, x).
$$

Substitution of this in (3.23) yields

$$
(\partial\bar{\partial}x, \xi^{m-1}) = a^2(-1)^{m+1}\bar{\partial}(\xi^m, x).
$$

From (3.24), by multiplying both sides of the equation by ξ^m we get: $(\xi^m, \xi^m) = (\xi^m, x)^2$. Since (ξ^m, ξ^m) is holomorphic, then $\bar{\partial}(\xi^m, x) = 0$. Q.E.D.

The next proposition gives a criterion for the regularity of the function $x\colon S^2 \longrightarrow S^{2m}$ obtained in Proposition (3.17).

(3.25) PROPOSITION. *Let Ξ and x be as in the previous proposition. Then x satisfies, in terms of the local coordinate z, the relation*

$$(\partial x, \bar{\partial} x) = |\xi_{m-1} \wedge \xi^1_{m-1}|^2 / |\xi_{m-1}|^4.$$

PROOF. From the expression (3.20) for ∂x, it follows that

$$(\partial x, \bar{\partial} x) = a^2 \left\{ \partial\left(\frac{1}{|\psi|}\right) \bar{\partial}\left(\frac{1}{|\psi|}\right)(\psi, \psi) + \frac{1}{|\psi|}\, \partial\left(\frac{1}{|\psi|}\right)(\psi, \bar{\partial}\psi) \right.$$
$$\left. + \frac{1}{|\psi|}\, \bar{\partial}\left(\frac{1}{|\psi|}\right)(\psi, \partial\psi) + \frac{1}{|\psi|^2}(\partial\psi, \bar{\partial}\psi) \right\}.$$

By applying to this, the formulas obtained on (3.19)

$$(\partial x, \bar{\partial} x) = a^2(-1)^m (1/|T|^4)\{|T|^2 |\partial T|^2 - |(\partial T, \bar{T})|^2\}.$$

Using the definition of a, we have $a^2(-1)^m = 1$ and we may rewrite the previous formula as

$$(\partial x, \bar{\partial} x) = \frac{|T \wedge \partial T|^2}{|T|^4} = \frac{|\xi_{m-1} \wedge \xi_{m-1}|^2}{|\xi_{m-1}|^4}.$$

(3.26) The consequence of this proposition is that x and Ξ_{m-1} are isometric and therefore x will be regular in all points where Ξ_{m-1} is. Hence x will be regular in all but finitely many points and so x is a generalized minimal immersion.

It would be nice if this construction of a minimal immersion from a holomorphic curve were canonical. Unfortunately there is a choice of sign on the definition of $\tilde{\psi}$ that is arbitrary and, depending upon it, we may end up with x or $-x$. (Notice also that $\pm x$ have the same directrix curve.) To make our construction canonical we should identify the elements of the set of generalized minimal immersions under the action of the multiplicative group $\{+1, -1\}$. In fact this is the way in which this set of generalized minimal immersions will always be considered in this work. Under this identification the construction we have made is a canonical left inverse of the process of obtaining the directrix curve for a minimal immersion. Consequently the latter is 1-1.

(3.27) PROPOSITION. *Suppose* Ξ, Z: $S^2 \longrightarrow CP^{2m}$ *are totally isotropic holomorphic curves which do not lie in any complex hyperplane of* CP^{2m} *and which give rise to the same minimal immersion* x: $S^2 \longrightarrow S^{2m}$ *by the process described above. Then* $\Xi = Z$.

PROOF. Let ξ and ζ be local representations for Ξ and Z respectively. Since Ξ and Z give rise to the same minimal immersion, then, locally, the hyperplanes spanned by $\{\xi, \ldots, \xi^{m-1}, \bar{\xi}, \ldots, \bar{\xi}^{m-1}\}$ and $\{\zeta, \ldots, \zeta^{m-1}, \bar{\zeta}, \ldots, \bar{\zeta}^{m-1}\}$ coincide for each z. Consequently we may write

$$\zeta = \sum_{j=0}^{m-1} \alpha_j \xi^j + \sum_{j=0}^{m-1} \beta_j \bar{\xi}^j$$

where α_j's and β_j's are complex valued functions of z and $\bar{z}$. We have to prove that ξ and ζ are parallel; that is, we have to prove that $\beta_j = 0$ for all j and $\alpha_j = 0$ for j bigger than 0.

Since ζ is holomorphic, then $\bar{\partial}\zeta = 0$. This implies

$$\sum_{j=0}^{m-1} \bar{\partial}\alpha_j \xi^j + \bar{\partial}\beta_0 \bar{\xi} + \sum_{j=1}^{m-1} (\bar{\partial}\beta_j + \beta_{j-1})\bar{\xi}^j + \beta_{m-1}\bar{\xi}^m = 0.$$

But, from (3.24) we may write $\bar{\xi}^m = \Sigma_{j=0}^{m-1}\theta_j\bar{\xi}^j + (\bar{\xi}^m, x)x$. Then, the previous equation yields

$$\begin{aligned} &\bar{\partial}\alpha_j = 0, \qquad 0 \leqslant j \leqslant m-1, \\ &\bar{\partial}\beta_0 + \beta_{m-1}\theta_0 = 0, \\ &\bar{\partial}\beta_j + \beta_{j-1} + \beta_{m-1}\theta_j = 0, \\ &\beta_{m-1}(\bar{\xi}^m, x)x = 0. \end{aligned} \tag{3.28}$$

The last equation implies that $\beta_{m-1} = 0$ or $(\xi^m, x) = 0$ at each point. But we know that $(\xi^m, x)^2 = (\xi^m, \xi^m)$ (see end of the proof of Proposition (3.17)) and from the total isotropy of ξ we also have that

$$(\xi \wedge \ldots \wedge \xi^{2m})^2 = (\xi^m, \xi^m)^{2m+1}. \tag{3.29}$$

Since ξ does not lie in any complex hyperplane of CP^{2m} it follows that (ξ^m, ξ^m) has at most isolated zeros. Hence $\beta_{m-1} = 0$ and now, the second and third equations of (3.28) give us $\beta_j = 0$ for $0 \leqslant j \leqslant m-1$.

To prove that $\alpha_j = 0$ for $1 \leqslant j \leqslant m-1$, we proceed by induction. Using the total isotropy of ξ we find $0 = (\zeta', \zeta') = \alpha_{m-1}^2(\xi^m, \xi^m)$. Since (ξ^m, ξ^m) has only isolated zeros, this implies $\alpha_{m-1} = 0$. Assume $\alpha_{m-1} = \ldots = \alpha_{m-k+1} = 0$; then using (3.16) we get $0 = (\zeta^k, \zeta^k) = \alpha_{m-k}^2(\xi^m, \xi^m)$. By the same reason as before, this implies $\alpha_{m-k} = 0$. This argument can be carried on until we reach $k = m-1$. Thus $\zeta = \alpha_0\xi$ and consequently $\Xi = Z$. Q.E.D.

Now, suppose $Z: S^2 \longrightarrow CP^{2m}$ is an arbitrary totally isotropic holomorphic curve not lying in any complex hyperplane of CP^{2m}. Let $x: S^2 \longrightarrow S^{2m}$ be the minimal immersion associated with Z by the process described before, and let $\Xi: S^2 \longrightarrow CP^{2m}$ be the directrix curve of x. Then, by Proposition (3.27) we have $Z = \Xi$.

Propositions (3.17), (3.25) and (3.27), plus the comments we have made along the way, prove the following theorem:

(3.30) THEOREM. *There exists a canonical* 1-1 *correspondence between the set of generalized minimal immersions* $x\colon S^2 \longrightarrow S^{2m}$ *which are not contained in any lower dimensional subspace of* R^{2m+1} *and the set of totally isotropic holomorphic curves* $\Xi\colon S^2 \longrightarrow CP^{2m}$ *which are not contained in any complex hyperplane of* CP^{2m}. *The correspondence is the one that associates to each minimal immersion* x *its directrix curve.*

The principal consequence of this theorem is that it allows us to identify these two spaces; and the space of totally isotropic holomorphic curves is easier to study than the one of minimal immersions.

4. The area of a minimal immersion $x\colon S^2 \longrightarrow S^{2m}$.

(4.1) It is our intention to apply some classical results of the theory of holomorphic curves to the directrix curve to see what results can be drawn from this.

Let Ξ be an arbitrary holomorphic curve from S^2 into CP^{2m}. Then, if ξ is its local representation in homogeneous coordinates, we have that

$$\Omega_k = \frac{i}{2\pi} \frac{|\xi_k \wedge \xi_k'|^2}{|\xi_k|^4} dz \wedge d\bar{z}$$

for each k, $0 \leqslant k \leqslant 2m-1$, is a globally defined form in S^2 of bidegree (1, 1), called the curvature form of the curve Ξ_k (the kth associated curve of Ξ). Furthermore, if we denote by ν_k the degree of the holomorphic curve Ξ_k, we have

$$\nu_k = \int_{S^2} \Omega_k.$$

For reference see, for example, [10].

(4.2) Now, we return to the situation where we have that $x\colon S^2 \longrightarrow S^{2m}$ is a generalized minimal immersion not lying in any proper subspace of R^{2m+1} and Ξ its directrix curve. By Proposition (3.25) it follows that x and Ξ_{m-1} are isometric; that is, if ξ is a local representation for Ξ in homogeneous coordinates, then

$$F dz \wedge d\bar{z} = (\partial x, \bar{\partial} x) dz \wedge d\bar{z} = -2i\pi\Omega_{m-1}.$$

Consequently $\mathrm{Area}(x) = \int_{S^2} iF dz \wedge d\bar{z} = 2\pi \int_{S^2} \Omega_{m-1}$. Hence

$$\text{Area}(x) = 2\pi\nu_{m-1} \tag{4.3}$$

and so, Area(x) must be a multiple of 2π. This was already known to Calabi [2]. Since Ξ is not a general curve in CP^{2m}, but a totally isotropic one, we may ask ourselves what values ν_{m-1} can assume. The objective of the remainder of this chapter is to show that ν_{m-1} can assume only even values.

(4.4) The total isotropy of Ξ means that, at each point of S^2, the subspace of C^{2m+1} spanned by $\xi, \xi', \ldots, \xi^{m-1}$ is totally isotropic. So it makes sense to consider the set H_m of all m-dimensional subspaces of C^{2m+1} that are totally isotropic. (It is a linear algebra exercise to show that a totally isotropic subspace V of C^{2m+1} is maximal, if and only if V has dimension m.) This set H_m is a subset of the Grassmannian $G_{m,2m+1}$ and so it is a subset of the complex projective space CP^{N-1}, $N = \binom{2m+1}{m}$.

(4.5) PROPOSITION. *H_m is a Kähler submanifold of the complex projective space CP^{N-1}.*

(This result is known; for example, see [9, p. 235].)

(4.6) PROPOSITION. *The manifold H_m is diffeomorphic to the homogeneous space $SO(2m+1)/U(m)$.*

PROOF. Given $(e_0, e_1, \ldots, e_{2m}) \in SO(2m+1)$, we define vectors $E_1, E_2, \ldots, E_m$ by $E_j = (e_j + ie_{jm})/\sqrt{2}$, $1 \leqslant j \leqslant m$, where $i = \sqrt{-1}$. Define a map φ_1 from $SO(2m+1)$ into the Stiefel manifold $U(2m+1)/U(m+1)$ by

$$\varphi_1(e_0, e_1, \ldots, e_{2m}) = (E_1, \ldots, E_m).$$

Notice that $E_1, \ldots, E_m$ span a maximal totally isotropic subspace of C^{2m+1}. We know that an element of H_m is completely determined by an orthonormal basis of it up to an action of $U(m)$. On the other hand, $U(m)$ can be realized as a subgroup of $SO(2m+1)$ by the inclusion

$$A + iB \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & -B & A \end{bmatrix}.$$

Now we may verify that the action of $U(m)$ on an element of $SO(2m+1)$ on the right is compatible with the change of coordinates on the corresponding totally isotropic subspace. So φ_1 induces a map φ_2 from $SO(2m+1)/U(m)$ into $G_{m,2m+1}$ and we have the following commutative diagram:

$$\begin{array}{ccc} SO(2m+1) & \xrightarrow{\varphi_1} & U(2m+1)/U(m+1) \\ \pi_1 \Big\downarrow & & \Big\downarrow \pi_2 \\ SO(2m+1)/U(m) & \xrightarrow{\varphi_2} & U(2m+1)/U(m) \times U(m+1) \end{array}$$

It is easy to verify that φ_1 (and consequently φ_2) is 1-1. We have φ_1 differentiable, and since π_1 on the above diagram defines a fiber bundle, it follows that φ_2 is also differentiable. Since φ_2 is 1-1 and $SO(2m+1)/U(m)$ is compact, φ_2 is a homeomorphism onto its image. Thus, to complete the argument we need only show that $(\varphi_2)_*$, the differential of φ_2, is injective at each point. We will do this by defining natural metrics in both domain and range of φ_2, and then we will compare the pull back of both to $SO(2m+1)$ using the above diagram.

We first define a metric on $SO(2m+1)/U(m)$. If X is a general element in $SO(2m+1)$, then $\mathcal{W} = dX \cdot X^{-1}$ is the matrix of Maurer-Cartan forms of $SO(2m+1)$. The matrix $\mathcal{W}$ is antisymmetric and so, if we set $\mathcal{W} = (\mathcal{W}_{AB})$, $0 \leqslant A, B \leqslant 2m$, we then have $\mathcal{W}_{AB} + \mathcal{W}_{BA} = 0$. Define on $SO(2m+1)$ a Riemannian structure by

$$ds^2 = -\frac{1}{4} \operatorname{trace} \mathcal{W} \otimes {}^t\mathcal{W} = \frac{1}{4} \sum_{A,B} \mathcal{W}_{AB} \otimes \mathcal{W}_{BA}. \tag{4.7}$$

Let $\mathcal{G}$ be the Lie algebra of $SO(2m+1)$ endowed with this inner product. Let $\mathcal{M}$ be its subalgebra tangent to $U(m)$ and set $\mathcal{M}^\perp$ for its orthogonal complement. Consider also the dual spaces $\mathcal{G}^*$, $\mathcal{M}^*$ and $(\mathcal{M}^\perp)^*$.

Using the Maurer-Cartan forms we may construct an orthonormal basis for $(\mathcal{M}^\perp)^*$. In fact if we consider the set of indices j, k, $1 \leqslant j, k \leqslant m$, then the matrix β given by

$$\beta = \begin{bmatrix} 0 & \beta_1 & \beta_2 \\ -{}^t\beta_1 & \beta_3 & \beta_4 \\ -{}^t\beta_2 & \beta_4 & -\beta_3 \end{bmatrix},$$

where β_1, β_2 are line matrices and β_3, β_4 are square matrices given by:

$$\beta_1 = (\mathcal{W}_{0j}) \quad \text{and} \quad \beta_2 = (\mathcal{W}_{0j+m}), \qquad 1 \leqslant j \leqslant m,$$
$$\beta_3 = (1/\sqrt{2})(\mathcal{W}_{jk} - \mathcal{W}_{j+m\ k+m}), \quad \text{and}$$
$$\beta_4 = (1/\sqrt{2})(\mathcal{W}_{j\ k+m} - \mathcal{W}_{k\ j+m}), \qquad 1 \leqslant j, k \leqslant m,$$

will be the antisymmetrical matrix whose entries form such an orthonormal basis for $(\mathcal{M}^\perp)^*$. Set

$$ds_1^2 = \tfrac{1}{4} \text{ trace } \beta \otimes {}^t\beta. \tag{4.8}$$

Then ds_1^2 will define an inner product in $(M^\perp)^*$, in fact the restriction of ds^2 to $(M^\perp)^*$. The induced metric in the homogeneous space $SO(2m+1)/U(m)$ is such that its pull back to $SO(2m+1)$ is exactly ds_1^2. In terms of W_{AB}'s, we have

$$ds_1^2 = \frac{1}{4}\sum_{j,k} [(W_{jk} - W_{j+m\ k+m})^2 + (W_{j\ k+m} - W_{k\ j+m})^2]$$
$$+ \frac{1}{2}\sum_j [W_{j\ 0}^2 + W_{j+m\ 0}^2].$$

Now we look at the metric on $G_{m,2m+1}$ and consider its pull back to $SO(2m+1)$. If Y is a general element on $U(2m+1)$, then $\theta = dY \cdot Y^{-1} = (\theta_{AB})$ is the matrix of Maurer-Cartan forms on $U(2m+1)$.

A Hermitian structure can then be defined in $G_{m,2m+1}$ by

$$ds_2^2 = \sum_{j,k} \theta_{j\ k+m} \otimes \bar{\theta}_{j\ k+m} + \sum_j \theta_{j0} \otimes \bar{\theta}_{j0}$$

(see for example [6]) where $1 \leqslant j,\ k \leqslant m$. Using now the definition of φ_2 we compute that

$$\theta_{j0} = \tfrac{1}{2}(W_{j0} + iW_{j+m\ 0}),$$
$$\theta_{jk} = \tfrac{1}{2}(W_{jk} + W_{j+m\ k+m}) + i(W_{j+m\ k} - W_{j\ k+m}),$$
$$\theta_{j\ k+m} = \tfrac{1}{2}(W_{jk} - W_{j+m\ k+m}) + i(W_{j+m\ k} + W_{j\ k+m}).$$

A direct substitution of this in the expression of ds_2^2 shows that the real part of ds_2^2 is equal to ds_1^2. This means that if we consider $G_{m,2m+1}$ as a real Riemannian manifold, then φ_2 becomes an isometry and so

$$H_m = SO(2m+1)/U(m). \quad \text{Q.E.D.}$$

The next theorem and its corollary are among the main results of this work.

(4.9) THEOREM. *Let Γ: $S^2 \longrightarrow H_m \subset CP^{N-1}$ be a holomorphic curve. Then its degree in CP^{N-1} is even.*

PROOF. First we will prove that the second homology group of H_m with coefficients on $\mathbf{Z}$ is equal to $\mathbf{Z}$. Then it makes sense to talk about the degree of Γ in H_m. The degree of Γ in CP^{N-1} will be equal to the degree of Γ in H_m times the degree of the fundamental cycle of H_m in CP^{N-1}. The conclusion of the theorem will follow when we show that degree of the fundamental cycle of H_m in CP^{N-1} is 2. Each one of these steps will be covered by a lemma.

(4.10) LEMMA. *$H_2(H_m, \mathbf{Z})$, the second homology group of H_m with integer coefficients, is isomorphic to* $\mathbf{Z}$.

PROOF. We first remark that $SO(2n)/U(n) = SO(2n-1)/U(n-1)$ for all $n > 1$. In fact, from the following inclusion diagram:

$$\begin{tikzcd} & SO(2n) \arrow[dl, dash] \arrow[dr, dash] & \\ SO(2n-1) \arrow[dr, dash] & & U(n) \arrow[dl, dash] \\ & U(n-1) & \end{tikzcd}$$

we can conclude that $SO(2n-1)/U(n-1) \subset SO(2n)/U(n)$. Since the groups involved are compact, each one of the homogeneous spaces also is and, consequently, the inclusion is an imbedding. If we consider $SO(2n)$ endowed with an invariant metric and $SO(2n-1)$, and both homogeneous spaces, with the induced metric, the inclusion then becomes an isometry. Now computing dimensions one sees that they are the same for both homogeneous spaces. They are therefore equal.

Using the fact we have just proved, we get that H_m is a bundle over S^{2m} with fiber H_{m-1}. Hence, by standard results of algebraic topology we have $H_2(H_m, \mathbf{Z}) = H_2(H_{m-1}, \mathbf{Z})$. Since m is general, we have

$$H_2(H_m, \mathbf{Z}) = H_2(H_1, \mathbf{Z}). \tag{4.11}$$

But, $H_1 = SO(3)/U(1) = SO(3)/SO(2) \cong S^2$. Thus $H_2(H_m, \mathbf{Z}) = H_2(S^2, \mathbf{Z}) = \mathbf{Z}$.

(4.12) LEMMA. *There are canonical inclusions such that $H_1 \subset H_2 \subset \ldots \subset H_m$. The space H_1 is a generator for $H_2(H_m, \mathbf{Z})$ and the degree of H_1 as a curve in CP^{N-1} is* 2.

PROOF. We certainly have for each $n \geqslant 2$ that

$$\begin{aligned} H_{n-1} &= SO(2n-1)/U(n-1) = SO(2n)/U(n) \\ &\subset SO(2n+1)/U(n) = H_n. \end{aligned} \tag{4.13}$$

If we consider H_n and H_{n-1} endowed with the natural metric ds_1^2 defined in (4.8), then this inclusion becomes an isometry. This follows from the fact that the metrics we have just considered on H_{m-1} and H_m can be obtained by simply considering the metric defined in (4.7) on $SO(2n+1)$ and the induced metric on all homogeneous spaces which show up in (4.13).

In fact we may prove further that the inclusion of H_{m-1} into H_m given in (4.13) is a holomorphic imbedding. This will imply that H_{m-1} sits naturally in H_m as a complex Kähler submanifold.

To do this we begin by considering an orthonormal basis $E_1, \ldots, E_{2n}$, E_{2n+1} of C^{2n+1} where the two last vectors E_{2n} and E_{2n+1} are chosen in such a way that we have $E_{2n+1} = \overline{E}_{2n}$ and $(E_{2n}, E_{2n+1}) = 0$. Consider now $C^{2n-1} \subset C^{2n+1}$, where C^{2n-1} is spanned by the first $2n-1$ elements of the basis of C^{2n+1}. Then define a linear map $A\colon C^M \longrightarrow C^N$, $M = \binom{2n-1}{n-1}$, $N = \binom{2n+1}{n}$, by

$$A(E_{i_1} \wedge \ldots \wedge E_{i_{n-1}}) = E_{i_1} \wedge \ldots \wedge E_{i_{n-1}} \wedge E_{2n},$$

where $0 \leqslant i_1 < i_2 < \ldots < i_{n-1} < 2n$. This map is in fact an isometry and so defines a natural inclusion of C^M into C^N. Since it is linear, it is projective, and consequently also defines a natural inclusion of CP^{M-1} into CP^{N-1}. Its restriction to $G_{n-1,2n-1}$ is also a natural inclusion into $G_{n,2n+1}$. Using the particular choice we have made for E_{2m} and E_{2m+1} it follows that if $\{V_1, \ldots, V_{n-1}\}$ spans a maximal totally isotropic subspace of C^{2n-1} then $\{V_1, \ldots, V_{n-1}, E_{2n}\}$ will also span a maximal totally isotropic subspace of C^{2n+1}. Consequently, the function A restricted to H_{n-1} defines a holomorphic imbedding of H_{n-1} into H_n. One may verify that this map is the inclusion defined on (4.13). Hence, for each $m > n > 1$, H_n can be realized as a Kähler submanifold of H_m and the inclusion is an isometry.

From (4.11) we have that H_1 is a generator for $H_2(H_m, \mathbf{Z})$. Computing its degree in CP^{N-1}, $N = \binom{2m+1}{m}$, is the same as computing its degree in CP^2 (as we have just seen). Now, by definition H_1 is the set of all 1-dimensional subspaces of C^3 that are totally isotropic. But such space is the hyperquadric in CP^2 whose degree in CP^2 is 2. This completes the proof of Theorem (4.9).

(4.14) COROLLARY. *The area of a generalized minimal immersion $x\colon S^2 \longrightarrow S^{2m}$ is a multiple of 4π.*

PROOF. This follows from the above theorem in view of equation (4.3).

5. A rigidity theorem.

(5.1) Let $x, y\colon S^2 \longrightarrow S^{2m}$ be generalized minimal immersions not lying in any subspace of R^{2m+1} whose induced metrics are the same, i.e. $ds_x^2 = ds_y^2$. Let Ξ and Z be the corresponding directrix curves. From Proposition (3.25) it follows that Ξ_{m-1} and Z_{m-1} are also isometric. We would like to know, in this situation, what relation there is between the metrics induced in S^2 by Ξ and Z. The answer is given by the following proposition.

(5.2) PROPOSITION. *Let $x\colon S^2 \longrightarrow S^{2m}$ be generalized minimal immersions not lying in any proper subspace of R^{2m+1}. Then x and y are isometric if and only if the corresponding directrix curves are also isometric.*

PROOF. As above let Ξ and Z denote the directrix curves of x and y respectively. If Ξ and Z are isometric then, by general facts about holomorphic curves, we have that Ξ_k and Z_k are also isometric for each k, $0 \leqslant k \leqslant 2m-1$, in particular for $k = m-1$. Therefore, it follows that x and y are also isometric.

The other implication follows from the next lemma.

(5.3) LEMMA. *Let Ξ: $S^2 \longrightarrow CP^{2m}$ be a totally isotropic curve not lying in any complex hyperplane of CP^{2m}. Then the metric of Ξ_k for any $0 \leqslant k < m-1$, is completely determined by the metric of Ξ_{m-1}.*

PROOF. From Theorem (3.30) we know that Ξ is the directrix curve of a certain generalized minimal immersion x: $S^2 \longrightarrow S^{2m}$. Using the notation of §3 we set $\xi = G_m/|G_m|^2$ as a local expression for Ξ in homogeneous coordinates. Using (3.14) we have that

$$\xi_k = \frac{1}{|G_m|^2 \ldots |G_{m-k}|^2} G_m \wedge \ldots \wedge G_{m-k}, \tag{5.4}$$

$$\xi'_k = \frac{1}{|G_m|^2 \ldots |G_{m-k+1}|^2 |G_{m-k-1}|^2} G_m \wedge \ldots \wedge G_{m-k+1} \wedge G_{m-k-1}, \tag{5.5}$$

where $0 \leqslant k \leqslant m$. Since by definition, the G_k's form an orthogonal set of vectors, ξ_k and ξ'_k are perpendicular to each other and consequently

$$|\xi_k \wedge \xi'_k|^2 = |\xi_k|^2 |\xi'_k|^2 = (|G_m|^4 \ldots |G_{m-1+1}|^2 |G_{m-k-1}|^2)^{-1}$$

and

$$|\xi_k|^4 = (|G_m|^4 \ldots |G_{m-k}|^4)^{-1}.$$

We have then

$$\Omega_k = \frac{i}{2\pi} \frac{|\xi_k \wedge \xi'_k|^2}{|\xi_k|^4} dz \wedge d\bar{z} = \frac{i}{2\pi} \frac{|G_{m-k}|^2}{|G_{m-k-1}|^2} dz \wedge d\bar{z}. \tag{5.6}$$

Set

$$A_k = |G_{m-1}|^2/|G_{m-k-1}|^2, \qquad 0 \leqslant k \leqslant m-1.$$

To prove this lemma it remains to prove that A_k can be obtained from the A_j's with $k < j \leqslant m-1$ by a prescribed rule. To do this we begin by considering the curvature form Ω_k given by

$$\Omega_k = (i/2\pi)\partial\bar{\partial} \log |\xi_k|^2 \, dz \wedge d\bar{z}.$$

(See for example [6, p. 80].) Then we have

$$\Omega_k = (i/2\pi)\partial\bar{\partial}\log(|G_m|^2 \ldots |G_{m-k}|^2)^{-1}\, dz \wedge d\bar{z}. \tag{5.7}$$

Comparing (5.7) and (5.6) in the particular case of $k = m - 1$ we get

$$A_{m-1} = \partial\bar{\partial}\log(|G_m|^2 \ldots |G_1|^2)^{-1}.$$

Multiplying and dividing the product between parentheses in (5.7) by $(|G_{m-k-1}|^2|G_{m-k-2}|^2 \ldots |G_1|^2)$ and using the properties of log we can rewrite (5.7) as

$$\Omega_k = (i/2\pi)\{A_{m-1} + \partial\bar{\partial}\log(|G_1|^2 \ldots |G_{m-k-1}|^2)\}dz \wedge d\bar{z} \tag{5.8}$$

for the case $k < m - 1$. One can prove by induction that

$$|G_k|^2 = A_{m-k}A_{m-k-1} \ldots A_{m-1}.$$

This, together with (5.6) and (5.8), implies that for $k < m - 1$ we have $A_k = A_{m-1} - \partial\bar{\partial}\log(A_{k+1}A_{k+2}^2 \ldots A_{m-1}^{m-k-1})$. This completes the proof of the lemma.

Now, returning to the proof of Proposition (6.2) it is clear that if Ξ_{m-1} and Z_{m-1} are isometric, then Ξ_k and Z_k will also be isometric for any k less than $m - 1$. Q.E.D.

(5.9) Let x, y: $S^2 \longrightarrow S^{2m}$ be two generalized minimal immersions not lying in any subspace of R^{2m+1}, and suppose that they are isometric. By Proposition (5.2) their corresponding directrix curves Ξ and Z are also isometric. Calabi proved in his thesis [1] that given two isometric linearly full holomorphic curves in CP^{2m}, say Ξ and Z, there is a unitary matrix U with the determinant of U equal to 1 such that $Z = U \cdot \Xi$. Since Ξ and Z are totally isotropic curves, the matrix U cannot be a general matrix; it is in fact very special.

(5.10) PROPOSITION. *Let Ξ, Z: $S^2 \longrightarrow CP^{2m}$ be totally isotropic curves not lying in any complex hyperplane of CP^{2m}. Suppose there exists $U \in U(2m + 1)$ such that $Z = U \cdot \Xi$ and* $\det(U) = 1$. *Then $U \in SO(2m + 1)$.*

PROOF. Since $U \in U(2m + 1)$ and $\det(U) = 1$ we need only show that U is a real matrix. In fact we will show that $U \cdot {}^tU = I$ which implies that U is real.

Let ξ and ζ be local representations for Ξ and Z respectively in homogeneous coordinates.

Since $Z = U \cdot \Xi$ we may assume that $\zeta = U \cdot \xi$. Since U is a constant matrix, we have that $\zeta^k = U \cdot \xi^k$, $k \geqslant 0$. Proving that $U \cdot {}^tU = I$ is now equivalent to

proving that U preserves the symmetrical product of C^{2m+1}. Choosing as a basis of C^{2m+1} the vectors $\xi, \xi', \ldots, \xi^{2m}$ we have to show that $(U\cdot\xi^j, U\cdot\xi^k) = (\xi^j, \xi^k)$, $0 \leqslant j, k \leqslant 2m$, or equivalently, that $(\zeta^j, \zeta^k) = (\xi^j, \xi^k)$ for $0 \leqslant j, k \leqslant 2m$. We will prove this by induction on $j + k$.

Since ξ and ζ are totally isotropic we have that

$$(\zeta^j, \zeta^k) = (\xi^j, \xi^k) = 0 \quad \text{for } 0 \leqslant j + k < 2m, \tag{5.11}$$

as a consequence of (3.15). Since $\det(U) = 1$ it follows that $\zeta \wedge \zeta' \wedge \ldots \wedge \zeta^{2m} = \xi \wedge \xi' \wedge \ldots \wedge \xi^{2m}$. From (3.29) this implies that

$$(\zeta^m, \zeta^m) = (\xi^m, \xi^m). \tag{5.12}$$

Differentiating the equations defined by (5.11) when $j + k = 2m - 1$ and using (5.12) we have

$$(\zeta^j, \zeta^k) = (\xi^j, \xi^k) \quad \text{for } j + k = 2m. \tag{5.13}$$

Suppose we have that $(\zeta^j, \zeta^k) = (\xi^j, \xi^k)$ for $j + k = 2s$. By differentiating we get

$$(\zeta^s, \zeta^{s+1}) = (\xi^s, \xi^{s+1}) \quad \text{for } j = k = s,$$

$$(\zeta^{j+1}, \zeta^k) + (\zeta^j, \zeta^{k+1}) = (\xi^{j+1}, \xi^k) + (\xi^j, \xi^{k+1}) \quad \text{if } j \neq k.$$

Solving this linear system of equations we obtain

$$(\zeta^j, \zeta^k) = (\xi^j, \xi^k), \qquad j + k = 2s + 1. \tag{5.14}$$

For the next case, $j + k = 2s + 2$, we start by computing derivatives of (5.13). The resulting linear system of equations has $s + 1$ equations and $s + 2$ unknowns. We reduce the number of unknowns by the following:

Write $\xi^{2s+2} = \Sigma_{j=0}^{2m} \alpha_j \xi^j$. We then have

$$\zeta^{2s+2} = U\cdot\xi^{2s+2} = \sum_{j=0}^{2m} \alpha_j U\cdot\xi^j = \sum_{j=0}^{2m} \alpha_j \zeta^j,$$

and using the total isotropy of ζ we can compute

$$(\zeta^{2s+2}, \zeta) = \sum_{j=0}^{2m} \alpha_j (\zeta^j, \zeta) = \alpha_{2m}(\zeta^{2m}, \zeta).$$

Similarly for ξ we get $(\xi^{2s+2}, \xi) = \alpha_{2m}(\xi^{2m}, \xi)$. Using (5.13) we can conclude

that $(\zeta^{2s+2}, \zeta) = (\xi^{2s+2}, \xi)$. This equation provides the reduction we needed to solve the linear system of equations and we obtain from it $(\zeta^j, \zeta^k) = (\xi^j, \xi^k)$ for $j + k = 2s + 2$. This completes the induction and the proof of the proposition.

(5.15) THEOREM. *Let x, y: $S^2 \longrightarrow S^{2m}$ be generalized minimal immersions not lying in any subspace of R^{2m+1}. Then, x and y are isometric if and only if they differ by a rigid motion of S^{2m}.*

PROOF. Let Ξ and Z be respectively the directrix curves of x and y. We have seen in Proposition (5.2) that if x and y are isometric then there exists a unitary matrix U with $\det(U) = 1$ such that $Z = U \cdot \Xi$. From Proposition (5.10) it follows that $U \in SO(2m + 1)$.

Consider now the minimal immersion $\tilde{y} = U \cdot x$. The complex osculating spaces of $\tilde{y}$ will be obtained from those of x by the action of U. Thus, the directrix curve of $\tilde{y}$ will be given by $U \cdot \Xi$. That is, it will be equal to the directrix curve of y. Then, from the comments in (3.26) we have $\tilde{y} = \pm y$ and so y and x will differ by an element of $O(2m + 1)$. Q.E.D.

6. Examples.

(6.1) In §4 we proved that the area of a generalized minimal immersion from S^2 to S^{2m} is a multiple of 4π. It is also known from Calabi [2] that the area is either 4π or a value bigger than or equal to $2\pi m(m + 1)$. In this section we wish to prove that this is in fact the best possible result, that is, we wish to show examples of minimal immersions such that for each prescribed multiple of 4π allowed by Calabi's result there is one having that value as its area. This will be done by constructing examples of linearly full totally isotropic holomorphic curves from S^2 into CP^{2m} for which the corresponding minimal immersions will have the prescribed value as its area.

We will begin by analyzing more closely the relations between the area of a minimal immersion x: $S^2 \longrightarrow S^{2m}$ which is not contained in any subspace of R^{2m+1}, and the invariants of its directrix curve. As usual let Ξ be the directrix curve of x and ξ be its local representation in homogeneous coordinates.

(6.2) By a stationary point of Ξ_k we mean a point where the induced metric is degenerate, that is, a point z_0 where $(\xi_k \wedge \xi_k')(z_0) = 0$. Locally we may write $\xi_k \wedge \xi_k' = (z - z_0)^\delta \varphi(z)$, where $\varphi(z_0) \neq 0$, and we call δ the multiplicity of the stationary point. If $z_0, z_1, \ldots, z_n$ are the stationary points of Ξ_k and $\delta^0, \delta^1, \ldots, \delta^n$ are the respective multiplicities, then we call $\sigma_k = \delta^0 + \ldots + \delta^n$ the stationary index of Ξ_k or the k stationary index of Ξ.

Let ν_k be the degree (order) of Ξ_k. Then, if we set $\nu_{-1} = \nu_{2m} = 0$ we have the following relation between the ν_k's and σ_k's called Plücker formulas:

$$\nu_{k-1} - 2\nu_k + \nu_{k+1} = -(2 + \sigma_k). \tag{6.3}$$

For reference see [10, p. 123].

In §5 we have computed expressions for the curvature forms Ω_k, $0 \leqslant k \leqslant m - 1$. Using (3.15) one can compute in the same way expressions for Ω_{k+m} to see that $\Omega_k = \Omega_{2m-k-1}$, $0 \leqslant k \leqslant m - 1$. From this follows $\nu_k = \nu_{2m-k-1}$. Then we may reduce (6.3) to

$$\begin{aligned} -2\nu_0 + \nu_1 &= -(2 + \sigma_0), \\ \nu_0 - 2\nu_1 + \nu_2 &= -(2 + \sigma_1), \\ &\cdots\cdots\cdots\cdots \\ \nu_{m-2} - 2\nu_{m-1} + \nu_m &= -(2 + \sigma_{m-1}), \\ \nu_m &= \nu_{m-1}. \end{aligned} \tag{6.4}$$

This gives us

$$\nu_{m-k} = \nu_{m-k-1} + 2k + (\sigma_{m-1} + \sigma_{m-2} + \ldots + \sigma_{m-k})$$

for $0 \leqslant k \leqslant m - 1$. Since we have by (5.3) that $\mathrm{Area}(x) = 2\pi\nu_{m-1}$ we then may write

$$\mathrm{Area}(x) = 2\pi(\nu_{m-k} + k(k-1) + N_k) \tag{6.5}$$

where $N_k = \Sigma_{s=1}^{k-1} \Sigma_{j=1}^{s} \sigma_{m-j}$ and $0 \leqslant k \leqslant m - 1$. In particular we have that

$$\mathrm{Area}(x) = 2\pi\left(\nu_0 + m(m-1) + \sum_{s=1}^{m-1} \sum_{j=1}^{s} \sigma_{m-j}\right). \tag{6.6}$$

Since we are dealing with linearly full holomorphic curves we must have $\nu_0 \geqslant 2m$. So, (6.6) gives immediately $\mathrm{Area}(x) \geqslant 2\pi m(m+1)$, which is Calabi's result which we referred to in the begining of this section. But the main use of (6.6) will be to compute $\mathrm{Area}(x)$ when we have the directrix curve and do not want to compute ν_{m-1}.

(6.7) Returning to the problem of constructing examples, one should first observe that any holomorphic curve from S^2 into CP^{2m} is a rational curve. By this we mean that given any point of S^2, we may always represent this curve around this point in homogeneous coordinates by a rational function, and therefore by a polynomial. Let Ξ: $S^2 \longrightarrow CP^{2m}$ be a linearly full totally isotropic holomorphic curve. Consider S^2 covered by isothermal coordinates given by stereographic projections. Then, we can represent Ξ with respect to any of these coordinates by a polynomial ξ: $C \longrightarrow C^{2m+1}$, given by $\xi(z) = \Sigma_{i=0}^{n} a_i z^i$, $a_i \in$

C^{2m+1}, where $a_0, a_1, \ldots, a_n$ must span C^{2m+1}. Furthermore we may assume $\xi(z) \neq 0$ for each $z \in C$ since we are allowed to factor such zeros. With such a representation we have $\nu_0 = \text{degree } \xi = n$.

As we have seen before, the total isotropy of Ξ implies that

$$(\xi^i, \xi^j) = 0, \qquad 0 \leqslant i + j \leqslant 2m - 1. \tag{6.8}$$

By simply computing each of these products for $z = 0$ we have that

$$(a_i, a_j) = 0, \qquad 0 \leqslant i + j \leqslant 2m - 1. \tag{6.9}$$

Observing that $\zeta(z) = z^n \xi(1/z) = \Sigma_{i=0}^n a_{n-i} z^i$ is also a totally isotropic polynomial, we obtain by the same argument

$$(a_{n-i}, a_{n-j}) = 0, \qquad 0 \leqslant i + j \leqslant 2m - 1. \tag{6.10}$$

Furthermore the products (a_i, a_j) with $2m < i + j < 2(n - m)$ must satisfy all the linear equations obtained by successive derivation of both sides of the identities

$$\begin{aligned} (\xi^{k-1}, \xi^{2m-k}) &= 0, \\ (\zeta^{k-1}, \zeta^{2m-k}) &= 0, \end{aligned} \qquad 1 \leqslant k \leqslant 2m.$$

From now on we will be trying to construct examples of linearly full totally isotropic polynomials from C into C^{2m+1} of the following particular form:

$$\begin{aligned} \xi_{km}(z) = a_0 &+ a_{k-m+1} z^{k-m+1} + a_{k-m+2} z^{k-m+2} + \ldots \\ &+ a_{k+m-1} z^{k+m-1} + a_{2k} z^{2k}, \end{aligned} \tag{6.11}$$

that is, linearly full totally isotropic polynomials of degree $2k$ having only $2m + 1$ terms different from zero, the zero coefficients being a_j with $1 \leqslant j \leqslant k - m$ or $m + k \leqslant j \leqslant 2k - 1$.

(6.12) PROPOSITION. *ξ_{km} is totally isotropic if and only if the following conditions are satisfied*:

(a) $(a_i, a_j) = 0$ *except for* $i + j = 2k$,

(b) $(a_{k-r}, a_{k+r}) = \lambda_{k-r}(a_k, a_k)$, $1 \leqslant r \leqslant m - 1$, *where*

$$\lambda_{k-r} = (-1)^r \frac{k^2}{k^2 - r^2} \frac{(m-1)(m-2)\ldots(m-r)}{(m+r-1)(m+r-2)\ldots(m)}.$$

(c) $(a_0, a_{2k}) = \lambda_0(a_k, a_k)$ *where*

$$\lambda_0 = \frac{1}{2} + \sum_{r=1}^{m-1} (-1)^r \frac{k^2}{k^2 - r^2} \frac{(m-1)(m-2)\ldots(m-r)}{(m+r-1)(m+r-2)\ldots(m)}.$$

PROOF. Suppose ξ_{km} is totally isotropic. Using (6.9) and the definition of ξ_{km} we have that

$$(a_i, a_j) = 0, \quad 0 \leqslant i + j \leqslant 2(k-m) + 1. \tag{6.13}$$

Also by the definition of ξ_{km} we have $\xi'_{km} = z^{k-m} P_{km}(z)$, where

$$P_{km} = \sum_{j=1}^{2m-1} (k - m + j) a_{k-m+j} z^{j-1} + 2k a_{2k} z^{k+m-1}.$$

The total isotropy of ξ_{km} then implies $(P_{km}^{(i)}, P_{km}^{(j)}) = 0$ for $0 \leqslant i + j \leqslant 2m - 2$. By considering each one of these products at $z = 0$ we obtain $(a_i, a_j) = 0$, $2(k-m) \leqslant i + j \leqslant 2k - 1$. This together with (6.13) yields $(a_i, a_j) = 0$, $0 \leqslant i + j \leqslant 2k - 1$. Now, carrying out the same argument with respect to ζ_{km} we obtain $(a_{2k-i}, a_{2k-j}) = 0$, $0 \leqslant i + j \leqslant 2k - 1$, completing the proof of (a).

We prove (b) by induction on r. Begin by noticing that in view of (a) the equation $(P_{km}^{(m-2)}, P_{km}^{(m-2)}) = 0$ implies that

$$2(k-1)(m-2)!(k+1)\frac{m!}{2}(a_{k-1}, a_{k+1}) + k(m-1)!(m-1)!(a_k, a_k) = 0,$$

and this gives

$$(a_{k-1}, a_{k+1}) = -\frac{k^2}{k^2 - 1} \frac{m-1}{m} (a_k, a_k),$$

and so

$$\lambda_{k-1} = -\frac{k^2}{k^2 - 1} \frac{m-1}{m}.$$

Assume λ_{k-j} is given by the desired formula for $j < r$. Then, in view of (a) the equation $(P_{km}^{(m-r-1)}, P_{km}^{(m-r-1)}) = 0$ implies

$$\sum_{j=1}^{r} 2(k^2 - j^2) \frac{(m-j-1)!}{(r-j)!} \frac{(m+j-1)!}{(r+j)!} (a_{k-j}, a_{k+j})$$

$$+ k^2 \frac{(m-1)!}{r!} \frac{(m-1)!}{r!} (a_k, a_k) = 0.$$

Taking the value of (a_{k-r}, a_{k+r}) from this and using the induction hypothesis we obtain

$$(a_{k-r}, a_{k+r}) = -\frac{k^2}{k^2 - r^2} \frac{(m-1)!\,(m-1)!}{(m-r-1)!\,(m+r-1)!} A_r (a_k, a_k)$$

where

$$A_r = \sum_{j=1}^{r-1} \frac{(2r)!}{(r-j)!\,(r+j)!} + \frac{1}{2}\frac{(2r)!}{r!r!}.$$

Using that $\Sigma_{j=0}^{2r}(-1)^j\binom{2r}{j} = 0$ we see that $A_r = (-1)^{r-1}$. Hence (b) is proved. To prove (c) take the equation $(\xi_{km}, \xi_{km}) = 0$ and using (a) obtain that

$$2(a_0, a_{2k}) + \sum_{j=1}^{m-1} 2(a_{k-m+j}, a_{k+m-j}) + (a_k, a_k) = 0.$$

Now, using (b) we have

$$(a_0, a_{2k}) = \left[\frac{1}{2} + \sum_{j=1}^{m-1} \lambda_{m-j}\right](a_k, a_k).$$

Hence, (c) is proved.

To prove the converse one has only to verify that if ξ_{km} is given by (6.11) and its coefficients satisfy (a), (b) and (c), then ξ_{km} is totally isotropic. But this is just a straightforward computation.

(6.14) Once this proposition has been proved, we consider the possibility of effectively assigning values to $a_0, \ldots, a_{2k}$ to get examples of the polynomials ξ_{km}. We can do this as follows. First take $(e_0, e_1, \ldots, e_{2m}) \in SO(2m+1)$ and construct with them the vectors $E_1, E_2, \ldots, E_m$ by setting $E_j = [e_j + ie_{j+m}]/\sqrt{2}$ where $i = \sqrt{-1}$. Then consider the basis of C^{2m+1} given by $E_m, E_{m-1}, \ldots, E_1, e_0, \overline{E}_1, \overline{E}_2, \ldots, \overline{E}_m$ and define the vectors a_j by

$$\begin{aligned}
a_0 &= \lambda_0 E_m, & a_{2k} &= \overline{E}_m,\\
a_{k-m+1} &= \lambda_{k-m+1}E_{m-1}, & a_{k+m-1} &= \overline{E}_{m-1},\\
a_{k-m+2} &= \lambda_{k-m+2}E_{m-2}, & a_{k+m-2} &= \overline{E}_{m-2},\\
&\cdots\cdots & &\cdots\cdots\\
a_{k-1} &= \lambda_{k-1}E_1, & a_{k+1} &= \overline{E}_1,
\end{aligned}$$

$$a_k = e_0.$$

These vectors satisfy conditions (a), (b) and (c) of Proposition (6.12) and so the polynomial of type ξ_{km}, defined as in (6.11) using these a_j's, is a concrete example of a linearly full totally isotropic polynomial.

(6.15) PROPOSITION. *ξ_{km} induces a regular minimal immersion x_{km}: $S^2 \longrightarrow S^{2m}$ whose area is $2\pi(2k + m(m-1))$.*

PROOF. Choose two arbitrary antipodal points over S^2, say p_1 and p_2, and take S^2 covered by isothermal coordinates z and w defined by the stereographic projection at these points. Then consider the holomorphic curve Ξ_{km}: $S^2 \longrightarrow CP^{2m}$ defined by $\xi_{km}(z)$ and $\zeta_{km}(w)$ where $\zeta_{km}(w) = w^n \xi_{km}(1/w)$ and where each one of the local functions is supposed to represent Ξ_{km} in the corresponding coordinate neighborhood. One can verify that this definition makes sense and that Ξ_{km} is then a linearly full totally isotropic holomorphic curve. From Theorem (3.30) we have that Ξ_{km} is the directrix curve for a certain minimal immersion $x_{km}: S^2 \longrightarrow S^{2m}$. The latter is then uniquely associated to ξ_{km} up to the choice of points p_1 and p_2.

We are now interested in computing the area of x_{km}. This will be done by using (6.6), but we first have to compute the stationary indices of ξ_{km} and ζ_{km}. Using the definition of ξ_{km} we have $\xi_{km} \wedge \xi'_{km} = z^{k-m} \xi_{km} \wedge P_{km}$. Thus ξ_{km} has a stationary point of multiplicity $(k - m)$ at the origin. Similarly we find that ζ_{km} has also a stationary point at the origin of multiplicity $(k - m)$. Thus we can conclude that

$$\sigma_0^{km} \geqslant 2(k - m). \tag{6.16}$$

In fact we will show that $\sigma_0^{km} = 2(k - m)$ and that $\sigma_j^{km} = 0$ for $1 \leqslant j \leqslant m - 1$. We do this as follows. From (3.9) we have that

$$(\xi_{km}^m, \xi_{km}^m)^{2m+1} = (\xi_{km} \wedge \xi'_{km} \wedge \ldots \wedge \xi_{km}^{2m})^2. \tag{6.17}$$

Thus, (ξ_{km}^m, ξ_{km}^m) is a polynomial that has a zero at each stationary point of each one of the associated curves of Ξ_{km} restricted to its domain. The same kind of argument applies to $(\zeta_{km}^m, \zeta_{km}^m)$. A computation gives

$$\begin{aligned} (\xi_{km}^m, \xi_{km}^m) &= k^2(m-1)!(m-1)!(a_k, a_k) z^{2(k-m)}, \\ (\zeta_{km}^m, \zeta_{km}^m) &= k^2(m-1)!(m-1)!(a_k, a_k) w^{2(k-m)}. \end{aligned} \tag{6.18}$$

From this we conclude that only $z = 0$ and $w = 0$ are possible stationary points for any one of the associated curves of Ξ_{km}. Furthermore we must have

$$2(k - m) \geqslant \sigma_0^{km} + \sigma_1^{km} + \ldots + \sigma_{m-1}^{km}.$$

But this together with (6.16) yields

$$\sigma_0^{km} = 2(k - m) \quad \text{and} \quad \sigma_j^{km} = 0 \quad \text{if} \quad 1 \leqslant j \leqslant m - 1.$$

Thus, using formula (6.6) we have $\text{Area}(x_{km}) = 2\pi(2k + m(m - 1))$. The regularity of x_{km} is a consequence of (3.26) and the fact that $\sigma_{m-1} = 0$. Q.E.D.

As a consequence of (6.14) and Proposition (6.15) we have the following theorem.

(6.19) THEOREM. *For each multiple of* 4π *bigger than or equal to* $2\pi m(m+1)$ *there is at least one example of a regular minimal immersion* x: $S^2 \longrightarrow S^{2m}$ *having that value as its area.*

(6.20) The statement of this theorem certainly suggests some questions such as: Are those examples the only ones? Does the space of minimal immersions consist of isolated points? The answer for both questions is no! The following is a proof of this.

Let $SO(n, C)$ denote the set of $n \times n$ complex matrices satisfying:

(a) $A \cdot {}^tA = I$ (identity),

(b) determinant$(A) = 1$.

Because of (a), $SO(n, C)$ preserves the symmetrical product of C^n and consequently it acts on the space of totally isotropic holomorphic curves into CP^{n-1}. Because of (b) this action has orbits diffeomorphic to $SO(n, C)$. In terms of Theorem (3.30), we may then say that $SO(2m+1, C)$ acts on the space of generalized minimal immersions x: $S^2 \longrightarrow S^{2m}$ not lying in any lower dimensional subspace of R^{2m+1}, and the action has orbits diffeomorphic to $SO(2m+1, C)$. So, a number of generalized minimal immersions exist and they are not isolated.

Suppose we decide to restrict our attention to minimal immersions x: $S^2 \longrightarrow S^{2m}$ that are isometrically different. In so doing, we would consider the space of minimal immersions modulo the equivalence relation defined by: $x \approx y$ if and only if x and y are isometric. Then we would find that each orbit is diffeomorphic to $SO(2m+1, C)/SO(2m+1, R)$ where $SO(2m+1, R)$ is the special orthogonal group over R. This follows from (5.9) and (5.10) or from (5.9) and the observation that

$$U(2m+1) \cap SO(2m+1, C) = SO(2m+1, R).$$

For the particular case of minimal immersions having area $2\pi m(m+1)$ we obtain the following proposition.

(6.21) PROPOSITION. *The space of generalized minimal immersions* x: $S^2 \longrightarrow S^{2m}$, *not lying in any lower dimensional subspace of* R^{2m+1}, *that are isometrically different and have area* $2\pi m(m+1)$ *is diffeomorphic to* $SO(2m+1, C)/SO(2m+1, R)$.

PROOF. It is enough to prove that the action of the group $SO(2m+1, C)$, in this particular case, has only one orbit. Let x, y: $S^2 \longrightarrow S^{2m}$ be generalized

minimal immersions not lying in any lower dimensional subspace of R^{2m+1} and let Ξ, Z: $S^2 \longrightarrow CP^{2m}$ be respectively their corresponding directrix curves. Assume that Area(x) = Area(y) = $2\pi m(m+1)$. This implies, using formula (6.6), that degree(Ξ) = degree(Z) = $2m$. Let ξ and ζ be local representations for Ξ and Z respectively in homogeneous coordinates. As we have seen we may assume that ξ and ζ are polynomials without zeros, linearly full, and with algebraic degree equal to the analytical degree of the corresponding curve. Thus: degree(ξ) = degree(ζ) = $2m$. Notice that ξ and ζ are polynomials of type ξ_{km} discussed before with $k = m$. Then, the proof of this proposition follows from the following lemma.

(6.22) LEMMA. *Given ξ and ζ polynomials of type ξ_{km} then there exist a linear map $A \in SO(2m+1, C)$ and a complex number λ such that $\zeta = \lambda \cdot A \cdot \xi$.*

PROOF. Set

$$\xi = \sum_{i=0}^{2k} a_i z^i, \quad \zeta = \sum_{i=0}^{2k} b_i z^i,$$

where $a_i = b_i = 0$ for $1 \leqslant i \leqslant k - m$ or $k + m \leqslant i \leqslant 2k - 1$. Let us assume for the moment that ξ and ζ also satisfy

$$\xi \wedge \ldots \wedge \xi^{2m} = \zeta \wedge \ldots \wedge \zeta^{2m}. \tag{6.23}$$

Then, using (6.17) and (6.18) we have $(a_k, a_k) = (b_k, b_k)$. By Proposition (6.12), this implies

$$(a_i, a_j) = (b_i, b_j), \qquad 0 \leqslant i, j \leqslant 2k. \tag{6.24}$$

Since there are only $2m + 1$ a_i's different from zero, and since they are linearly independent, we may define a linear map A: $C^{2m+1} \longrightarrow C^{2m+1}$ by $A(a_i) = b_i$, $0 \leqslant i \leqslant 2k$. But then, (6.24) implies that

$$(A(a_i), A(a_j)) = (a_i, a_j), \qquad 0 \leqslant i, j \leqslant 2k.$$

Since the nonzero a_i's form a basis for C^{2m+1}, this implies that A preserves the symmetrical product of C^{2m+1} and so, $A \cdot {}^tA = I$. We also have $A\xi = \zeta$ and consequently $A\xi^k = \zeta^k$ for any $k \geqslant 0$. Thus

$$\zeta \wedge \ldots \wedge \zeta^{2m} = \det(A)\xi \wedge \ldots \wedge \xi^{2m}.$$

Comparing this with (6.23) we conclude that $\det(A) = 1$. Therefore $A \in SO(2m+1, C)$.

Now suppose (6.23) is not true for ξ and ζ. Then we may define a new polynomial $\tilde{\xi} = \lambda\xi$ where λ is chosen such that $(b_k, b_k) = \lambda^2(a_k, a_k)$ and (6.23) holds for $\tilde{\xi}$ and ζ. Thus everything follows and we find $A \in SO(2m+1, C)$ such that $\zeta = A \cdot \tilde{\xi}$. Therefore $\zeta = \lambda A\xi$.

(6.25) The polynomials of type ξ_{km} are examples of linearly full totally isotropic curves having $\sigma_0 \neq 0$ and $\sigma_1 = \sigma_2 = \ldots = \sigma_{m-1} = 0$. Furthermore the value of σ_0 is distributed over only two points. As a matter of fact there does not exist a linearly full holomorphic curve Ξ: $S^2 \longrightarrow CP^{2m}$ having exactly one stationary point; it either has none or has more than one. A proof of this is as follows:

Let ξ: $C \longrightarrow C^{2m+1}$ be a local representation for Ξ in homogeneous coordinates as usual, that is, ξ is a polynomial without zeros and $\deg(\xi) = \nu_0$. Suppose Ξ has only one stationary point. We may assume that this point corresponds to $z = 0$. For a suitable choice of coordinates in C^{2m+1} we may put the equations that define ξ into the normal form, that is, we may write $\xi = (w_0, w_1, \ldots, w_{2m})$ and

$$\begin{aligned}
w_0 &= 1 + \ldots \\
w_1 &= \quad z^{\delta_0+1} + \ldots \\
w_2 &= \qquad\quad z^{\delta_0+\delta_1+2} + \ldots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
w_{2m} &= \qquad\qquad\quad z^{\delta_0+\ldots+\delta_{2m-1}+2m} + \ldots
\end{aligned}$$

where δ_k's have been defined in (6.2). From this we see that degree(ξ) satisfies:

$$\nu_0 \geqslant \delta_0 + \delta_1 + \ldots + \delta_{2m-1} + 2m. \tag{6.26}$$

Now, if we remember that total isotropy of Ξ implies that $\Omega_k = \Omega_{2m-k-1}$, then we have not only that σ_k and σ_{2m-k-1} are equal, but also that each stationary point of Ξ_k and Ξ_{2m-k-1} are the same and have the same multiplicity. Thus we have $\delta_k = \delta_{2m-k-1}$, $0 \leqslant k \leqslant m-1$. Consequently we may rewrite (6.26) as

$$\nu_0 \geqslant 2(\delta_0 + \delta_1 + \ldots + \delta_{m-1} + m).$$

On the other hand, we obtain from the use of the Plücker formulas in (6.4) that

$$\nu_0 = 2m + \sigma_0 + \sigma_1 + \ldots + \sigma_{m-1}. \tag{6.27}$$

Comparison of these two equations gives

$$\delta_0 + \ldots + \delta_{m-1} \leqslant \tfrac{1}{2}(\sigma_0 + \ldots + \sigma_{m-1}) \tag{6.28}$$

and equality occurs if and only if $\delta_k = \frac{1}{2}\sigma_k$ for each k. Thus, it is impossible that Ξ has exactly one stationary point because this would imply that $\delta_k = \sigma_k$. Q.E.D.

(6.29) This result in some way justifies the construction of the polynomials ξ_{km}. Indeed, if we decide to construct polynomials that represent totally isotropic holomorphic curves having two stationary points then, from (6.27) and (6.28), at each one of the points and for each k we have that $\delta_k = \frac{1}{2}\sigma_k$. Furthermore we must have $\nu_0 = 2(\delta_0 + \ldots + \delta_{m-1} + m)$. Now, the normal form of ξ suggests the simplest form this polynomial shall have. In the case of the construction of the ξ_{km} we took the simplest case in which only σ_0 was different from zero. But it is clear that one can do the same for arbitrary values of σ_k's. Similar results to Propositions (6.12) and (6.15) and Lemma (6.22) are true for this kind of polynomial. In fact, it would be interesting to know if some form of Lemma (6.22) is true in general.

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