RESULTS ON SUMS OF CONTINUED FRACTIONS

BY

JAMES L. HLAVKA

ABSTRACT. Let $F(m)$ be the (Cantor) set of infinite continued fractions with partial quotients no greater than $m$ and let $F(m) + F(n) = \{a + \beta: a \in F(m), \beta \in F(n)\}$. We show that $F(3) + F(4)$ is an interval of length $1.14 \ldots$ so every real number is the sum of an integer, an element of $F(3)$ and an element of $F(4)$. Similar results are given for $F(2) + F(7)$, $F(2) + F(2) + F(4)$, $F(2) + F(3) + F(3)$ and $F(2) + F(2) + F(2) + F(2)$. The techniques used are applicable to any Cantor sets in $\mathbb{R}$ for which certain parameters can be evaluated.

Marshall Hall, Jr. [3] proved that $F(4) + F(4) \equiv \mathbb{R}$ (mod 1) (all notation is defined in the next paragraph) and posed the question: is $F(3) + F(4) \equiv \mathbb{R}$ (mod 1)? In this paper we prove $F(3) + F(4) \equiv \mathbb{R}$ (mod 1) and several other results, summarized in Table 1. Only two questions concerning when a sum of $F(m)$, $m \in \mathbb{R}$ remain open: $F(2) + F(5) \equiv \mathbb{R}$? and $F(2) + F(6) \equiv \mathbb{R}$? We conjecture that they are both false.

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Table 1. All congruences are modulo 1

We let $\mathbb{N}$ be the natural numbers and $\mathbb{R}$ the real numbers. Lower case Roman letters except $g$ and $h$ will be elements of $\mathbb{N}$; Greek letters elements of $\mathbb{R}$. Let

$$\langle a_1, a_2, \ldots \rangle = \frac{1}{a_1} + \frac{1}{a_2} + \ldots ,$$

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and

\[ F(m) = \{ (a_1, a_2, \ldots) : 1 \leq a_i \leq m \text{ for all } i \in \mathbb{N} \}. \]

When working with continued fractions it is convenient to write intervals without ordering their endpoints, so we define

\[ (a, \beta) = \{ \xi \in \mathbb{R} : \min(a, \beta) < \xi < \max(a, \beta) \} \]

and

\[ [a, \beta] = \{ \xi \in \mathbb{R} : \min(a, \beta) \leq \xi \leq \max(a, \beta) \}. \]

If \( A \) and \( B \) are subsets of \( \mathbb{R} \), let \( \mathbb{A} \) = the span of \( A = \sup(a - \beta) \) over all \( a, \beta \in A \) and \( A + B = \{ a + \beta : a \in A, \beta \in B \} \). Write "\( A + B = \mathbb{R} \text{ (mod 1)} \)" to mean "\( \xi \in \mathbb{R} \) implies \( \xi \equiv a + \beta \pmod{1} \) for some \( a \in A, \beta \in B. \)" Let \( P(m) \) be the special closed interval \( I(\frac{m}{1}, \frac{1}{1}, \frac{1}{m}) \).

Note that \( I(\frac{m}{1}, \frac{1}{m}) \) and \( I(\frac{1}{1}, \frac{1}{m}) \) are the least and greatest elements of \( F(m) \), respectively, so \( F(m) \subset P(m) \). Moreover \( F(m) \subset F(m + 1) \). The latter inclusion immediately shows that every question of the form \( \sum F(m_i) = \mathbb{R} \) is covered in Table 1.

\( F(m) \) is a Cantor set so the natural approach to computing \( \sum F(m_i) \) is by deleting intervals from the \( P(m_i) \). Our objective is to devise an algorithm, called a construction of \( F(m) \), which controls the order of these deletions sufficiently to establish that \( \sum F(m_i) = \sum P(m_i) \) whenever this is true. (This is the same approach Hall used to investigate \( F(4) + E(4) \).)

**Lemma 1.** \( F(m) = P(m) \setminus O(m) \) = the set-theoretic difference of \( P(m) \) and \( O(m) \), where \( O(m) \) is defined by

\[ O(m) = \bigcup \{ (a_1, \ldots, a_s, 1, m), (a_1, \ldots, a_s + 1, m) : s \in \mathbb{N}, \]

\[ 1 \leq a_i \leq m \text{ for } i \leq s \text{ and } a_s \neq m \}. \]

**Proof.** We show that \( O(m) \) is composed of precisely those intervals which are deleted from \( P(m) \) to form \( F(m) \).

First assume \( a \in P(m) \setminus F(m) \). Then \( a \) has a first partial quotient \( a_{r+1} \) which is greater than \( m \), with \( r > 0 \) and \( a_r \neq 1 \) when \( r = 1 \). Now if \( a_r = 1 \) then

\[ a = (a_1, \ldots, a_{r-1}, a_r, \ldots) \]

\[ \in \{ (a_1, \ldots, a_{r-1} + 1, m, 1), (a_1, \ldots, a_{r-1}, 1, m) \}. \]
and if $a_r > 1$ then

$$\alpha \in \langle (a_1, \ldots, a_r - 1, 1, m), (a_1, \ldots, a_r, m, 1) \rangle,$$

so $\alpha \in O(m)$. Conversely if $\alpha \in \langle (a_1, \ldots, a_s, 1, m), (a_1, \ldots, a_s + 1, m, 1) \rangle \subset O(m)$, then

1. $\alpha \in \langle (a_1, \ldots, a_s, 1, m), (a_1, \ldots, a_s + 1) \rangle$,

2. $\alpha = (a_1, \ldots, a_s + 1)$, or

3. $\alpha \in \langle (a_1, \ldots, a_s + 1), (a_1, \ldots, a_s + 1, m, 1) \rangle$.

If (2) then $\alpha = (a_1, \ldots, a_s, \alpha')$ where $(1, m) < \alpha' < 1$ so $\alpha' \notin F(m)$. Hence $\alpha \notin F(m)$. If (4) then $\alpha = (a_1, \ldots, a_s + 1, \alpha')$ where $\alpha' < (m, 1)$ and again $\alpha \notin F(m)$. Lastly, (3) implies $\alpha \notin F(m)$ since $F(m)$ contains only infinite continued fractions. □

We now define a construction of $F(m)$ as follows. Let $I_m^2 = P(m)$. Choose an interval $O_m^2$ of $O(m)$ and delete it from $I_m^2$, leaving two new intervals $I_m^3$ and $I_m^4$. From each delete an interval of $O(m)$, say $O_m^3$ and $O_m^4$ respectively. $I_m^3$ will be split into two intervals $I_m^5$ and $I_m^6$; $I_m^4$ will be split into $I_m^7$ and $I_m^8$. From each of $I_m^5, \ldots, I_m^8$ delete the interval $O_m^5, \ldots, O_m^8$ respectively. Continue in this way. This procedure is demonstrated in Figures 1a and 1b. If $O(m) = \bigcup_{i=2}^{\infty} O_i^i$ we call this procedure a construction $C$ of $F(m)$. We call $F_m^k = \bigcup_{j=2}^{k} I_j^i$ the $k$th step in the construction of $F(m)$.

**Figure 1a**

**Figure 1b**
Lemma 2. If C is any construction of F(m), m > 1, and F_m^k is the kth step in this construction, then F(m) = \bigcap_{k=1}^{\infty} F_m^k.

Proof. Obvious from Figure 1b. □

Definition. If C is a construction of F(m), then

\[ g_m = g_m(C) = \sup_i \left( \frac{O^i_{m}}{m^i} \right), \]

\[ h_m = h_m(C) = \inf \left( \inf_i \left( \frac{1^{2i-1}}{m^i} \right), \inf_i \left( \frac{1^{2i}}{m^i} \right) \right), \]

and

\[ h'_m = h'_m(C) = \sup_i \left( \sup_i \left( \frac{1^{2i-1}}{m^i} \right), \sup_i \left( \frac{1^{2i}}{m^i} \right) \right). \]

Theorem 3. If there exist constructions C and C' of F(m) and F(n) respectively such that

(5) \( g_m(C) \cdot g_n(C') \leq h_m(C) \cdot h_n(C') \)

(6) \( g_m(C) \cdot P(m) \leq P(n) \) and \( g_n(C') \cdot P(n) \leq P(m) \)

then \( F(m) + F(n) = P(m) + P(n) \).

Proof. Let \( \{i^i_m\}_{i=2}^{\infty} \) and \( \{j^j_n\}_{j=2}^{\infty} \) be the intervals appearing in the constructions C and C', respectively. Call the intervals \( i^i_m \) and \( j^j_n \) compatible (with respect to C and C'), written \( i^i_m \sim j^j_n \), iff

(7) \( g_m \cdot \overline{i^i_m} \leq \overline{j^j_n} \) and \( g_n \cdot \overline{j^j_n} \leq \overline{i^i_m} \).

Call the intervals \( i^i_m \) and \( j^j_n \) M-divisible, written \( i^i_m \sim^M j^j_n \), iff (8) or (9) is true, where (8) and (9) are the following (symmetric) conditions.

(8.1) \( i^{2i-1}_m \sim j^i_n \) and \( i^{2i}_m \sim j^i_n \),

(8.2) \( (i^{2i-1}_m + j^i_n) \cup (i^{2i}_n + j^i_n) = i^i_m + j^i_n \), and

(8.3) \( M \cdot \overline{i^i_m} \leq \overline{j^i_n} \).

(9.1) \( i^i_m \sim i^{2j-1}_n \) and \( i^i_m \sim i^{2j}_n \),

(9.2) \( (i^i_m + i^{2j-1}_n) \cup (i^i_m + i^{2j}_n) = i^i_m + j^i_n \), and

(9.3) \( M \cdot \overline{i^i_m} \leq \overline{i^i_n} \).
The four pairs of intervals appearing in (8.1) and (9.1) are said to be derived from the pair \((\ell^i_m, \ell^j_n)\).

It suffices to show that for some \(M \in \mathbb{R}^+\), \(i^i_m \sim j^i_n\) implies \(i^i_m \sim j^i_n\) for all \(i, j \geq 2\). To prove this, set \(S_0 = \{(i^2_m, j^2_n)\}\) and

\[ S_{r+1} = \{(l, j): l \sim j \text{ and } (l, j) \text{ is derived from a pair } (l_0, j_0) \in S_r\}. \]

Clearly

\[ \bigcup \{l + J: (l, j) \in S_{r+1}\} = \bigcup \{l + J: (l, j) \in S_r\} = \cdots = i^2_m + j^2_n = P(m) + P(n). \]

If \((l, j) \in S_r\), then \(\overline{I} \cdot J \leq \lambda^r \cdot \overline{i^2_m} \cdot \overline{j^2_n} \to 0\) as \(r \to \infty\), where \(\lambda = \max(1 - h_m, 1 - h_n) < 1\) (if \(b_m\) or \(b_n\) = 0 then \(g_m\) or \(g_n\) = 0 by (5) so \(F(m)\) or \(F(n)\) is not a Cantor set—contradiction). Since \(l \sim j\), the ratio \(\overline{I}/J\) is bounded so \(\overline{I} \to 0\) and \(J \to 0\). Therefore for each \(i\) there is an \(r_0\) such that \(O_i^m\) has been deleted from every \(I\) appearing in a pair \((l, j) \in S_r, r > r_0\).

Since \(O(m) = \bigcup_{i=2}^{\infty} O_i^m\),

\[ (10) \quad O(m) \cap \left( \bigcap_{r=0}^{\infty} \left( \bigcup l: (l, j) \in S_r \right) \right) = \emptyset. \]

But for all \(r\), \(F(m) \subseteq \bigcup \{l: (l, j) \in S_r\} \subseteq P(m)\) so \((10)\) and Lemma 1 yield

\[ F(m) = \bigcap_{r=0}^{\infty} \left( \bigcup l: (l, j) \in S_r \right). \]

Similarly for \(F(n)\). Since the sequence \(\bigcup \{l: (l, j) \in S_r\}_{r=0}^{\infty}\) is a nested sequence of compact sets, we obtain directly the result

\[ F(m) + F(n) = \bigcap_{r=0}^{\infty} \left( \bigcup l + J: (l, j) \in S_r \right) = \bigcap_{r=0}^{\infty} (P(m) + P(n)) = P(m) + P(n). \]

Now fix \(M \geq \max(h_m/g_n, b_n/g_m)\) and assume \(i^i_m \sim j^i_n\). Since \(g_m g_n \leq b_m b_n\), we must have

\[ (11) \quad \frac{i^i_m}{j^i_n} > \frac{g_n}{b_m} \]

or

\[ (12) \quad \frac{i^i_m}{j^i_n} \leq \frac{h_n}{g_m}. \]

Assuming \((11)\) we will verify \((8)\). Similarly \((9)\) will follow from \((12)\), so this will show \(i^i_m \sim j^i_n\). So assume \((11)\) and set \(k = 2i - 1\) or \(2i\). Then recalling the definition of \(h_m\),
so \( \overline{t}_m \sim \overline{t}_n \). To check (8.2), let \( \overline{t}_{m-1}^i = [\alpha, \beta], \overline{t}_m^i = [y, \delta] \) and \( \overline{t}_n^i = [\alpha_0, \delta_0] \), with \( \alpha < \beta < y < \delta \) and \( \alpha_0 < \delta_0 \). Since

(13)

\[
\overline{t}_m^i > g_m \cdot \overline{t}_n^i > \overline{g}_m \cdot \overline{t}_m^i
\]

we have \( \delta_0 - \alpha_0 \geq y - \beta \) or \( \beta + \delta_0 \geq y + \alpha_0 \). Then

\[
(\overline{t}_{m-1}^i + \overline{t}_n^i) \cup (\overline{t}_m^i + \overline{t}_n^i) = [\alpha + \alpha_0, \beta + \delta_0] \cup [y + \alpha_0, \delta + \delta_0]
\]

\[
= [\alpha + \alpha_0, \delta + \delta_0] = \overline{t}_m^i + \overline{t}_n^i.
\]

Lastly, (8.3) is satisfied by our choice of \( M \). □

**Lemma 4.** If \( a = (a_1, \ldots, a_s, a') \), \( \beta = (a_1, \ldots, a_s, \beta') \), \( a' > 0 \), \( \beta' > 0 \),

\[
(\alpha' - \beta')/(\alpha - \beta) = (Q + \alpha')(Q + \beta')(1)^{s+1}q_s^2.
\]

**Proof.** We have

\[
\alpha - \beta = \frac{p_s a' + p_{s-1} \alpha' + p_s \beta' + p_{s-1} \beta}{q_s a' + q_{s-1} \beta' + q_s \beta}
\]

\[
= \frac{p_s}{q_s} \left\{ \frac{\alpha' + Q + p_{s-1} / q_s - Q}{\alpha' + q_{s-1} / q_s} - \frac{\beta' + Q + p_{s-1} / q_s - Q}{\beta' + q_{s-1} / q_s} \right\}
\]

\[
= \frac{p_s}{q_s} \left\{ \frac{p_{s-1} / q_s - Q}{q_s / q} \right\} \left\{ \frac{1}{\alpha' + Q} - \frac{1}{\beta' + Q} \right\}
\]

\[
= \frac{(-1)^{s+1}(\alpha' - \beta')}{q_s^2(\alpha' + Q)(\beta' + Q)}
\]

since \( p_{s-1} q_s - p_s q_{s-1} = (-1)^s \). The result follows immediately. □

**Lemma 5.** If \( y = (a_1, \ldots, a_s, y') \), \( \delta = (a_1, \ldots, a_s, \delta') \), \( y' > 0 \), \( \delta' > 0 \), and \( a, \beta, \alpha', \beta', Q \) are as in Lemma 4, then

(14)

\[
\frac{a - \beta}{y - \delta} = \frac{\alpha' - \beta'}{y' - \delta'} \frac{(Q + \gamma')(Q + \delta')}{(Q + \alpha')(Q + \beta')}
\]

and \( Q \in [1/(a_s + 1), 1] \).

**Proof.** Statement (14) is an immediate corollary of Lemma 4. The restriction on \( Q \) follows from the well-known result that \( Q = (a_s, \ldots, a_1) \). □
Theorem 6. \( F(3) + F(4) = P(3) + P(4) = [(\frac{3}{1}, 1) + (\frac{4}{1}, 1), (\frac{1}{3}, 1) + (\frac{1}{4}, 1)] = [.4709\ldots, 1.6197\ldots] \).

Proof. We produce constructions of \( F(3) \) and \( F(4) \) satisfying the hypotheses of Theorem 3. Let us begin by defining a canonical construction, \( C_m \), of \( F(m) \) for any \( m \), as follows. \( I^2_m \) must be \([ \frac{m}{1}, 1 \), \( \frac{1}{m} \)). If
\[
I^m = [(a_1, \ldots, a_s, j, m, 1), (a_1, \ldots, a_s, m, 1)]
\]
with \( s \geq 0 \) and \( j \neq m \) then
\[
O^j_m = (a_1, \ldots, a_s, j, 1, m), (a_1, \ldots, a_s, j + 1, m, 1))
\]
so that
\[
I^{2i-1}_m = [(a_1, \ldots, a_s, j, m, 1), (a_1, \ldots, a_s, j + 1, m, 1)]
\]
and
\[
I^{2i}_m = [(a_1, \ldots, a_s, j + 1, m, 1), (a_1, \ldots, a_s, m, 1)].
\]
It is relatively easy to show that \( \bigcup_{i=2}^{\infty} O^i_m = O(m) \) so this does define a construction of \( F(m) \). The value of the constructions \( C_m \) is that we can readily calculate \( g_m(C_m) \) and \( h_m(C_m) \) (and \( h' = h_m(C_m) \) which will be needed later). We have
\[
g_m(C_m) = \max_{i \geq 2} \left( \frac{O^i_m}{I^2_m} \right)
= \max \frac{(a_1, \ldots, a_s, j, 1, m) - (a_1, \ldots, a_s, j + 1, m, 1)}{(a_1, \ldots, a_s, j, m, 1) - (a_1, \ldots, a_s, m, 1)}
\]
over \( 1 \leq j \leq m, s \geq 0 \) and \( 1 \leq a_i \leq m \) for \( i \leq s \). Using Lemma 5, we obtain
\[
g_m(C_m) \leq \max \left\{ \frac{(j, 1, m) - (j + 1, m)}{(j, m) - (m, 1)} \cdot \frac{(Q + j, m)}{(Q + j + 1, 1)}, \frac{(Q + j, 1, m)}{(Q + j + 1, m)} \right\}
\]
over \( 1 \leq j \leq m \) and \( Q \in [1/(m + 1), 1] \). For each allowable value of \( j \) this expression is a rational function in \( Q \) whose maximum on the interval \([1/(m + 1), 1]\) can be readily calculated. A Univac 1108 was used to perform these calculations and then maximize over \( j \) (and for similar calculations arising later). We thus obtain the bounds
\[
g_3(C_3) \leq .2992\ldots \quad \text{and} \quad g_4(C_4) \leq .2278\ldots.
\]
Similarly
\[ h_m(C_m) = \min_{i \geq 2} \left( \min \left( \frac{l^{2i-1}_m}{i^2_m}, \frac{l^{2i}_m}{i^2_m} \right) \right) \]

\[ = \min_{i, a_i} \left( \frac{\langle a_1, \ldots, a_s, j, 1, m \rangle - \langle a_1, \ldots, a_s, j, m, 1 \rangle}{\langle a_1, \ldots, a_s, m, 1, m \rangle - \langle a_1, \ldots, a_s, j, m, 1 \rangle} \right) \]

\[ \geq \min \left( \frac{\langle j, 1, m \rangle - \langle j, m, 1 \rangle}{\langle m, 1, m \rangle - \langle j, m, 1 \rangle} \cdot \frac{Q + \langle m, 1, m \rangle}{Q + \langle j, 1, m \rangle} \right) \]

\[ \cdot \frac{\langle m, 1, m \rangle - \langle j + 1, m, 1 \rangle}{\langle m, 1, m \rangle - \langle j, m, 1 \rangle} \cdot \frac{Q + \langle j + 1, m, 1 \rangle}{Q + \langle j, m, 1 \rangle} \]

from which we obtain

\[ h_3(C_3) \geq .2471 \ldots \quad \text{and} \quad h_4(C_4) \geq .2963 \ldots \]

A simple multiplication shows that

\[ g_3(C_3) \cdot g_4(C_4) \leq .0667 < .0731 < h_3(C_3) \cdot h_4(C_4) \]

Also

\[ g_3 \cdot P(3) \leq .2992 \times .5276 < P(4) = .6212 \ldots < \frac{.5276}{.2278} \leq \frac{P(3)}{g_4} \]

so by Theorem 3 we have the result \( F(3) + F(4) = P(3) + P(4) \).

Corollary 7. \( F(3) + F(4) \equiv R \pmod{1} \).

Proof. This is obvious since \( F(3) + F(4) \) contains an interval of length greater than one.

The values of \( g_3, g_4, h_3 \) and \( h_4 \) are in fact equal to the bounds given because these bounds arise from \( Q = 1/(m+1) \) or \( Q = 1 \), which are possible values of \( Q \). For \( g_2 \) and \( h_2 \), below, this does not happen.

Applying Theorem 3 to the canonical constructions of \( F(2) \) and \( F(12) \) as above we can establish \( F(2) + F(12) = P(2) + P(12) \), but now we find that the canonical construction of \( F(12) \) is not an optimal construction in terms of minimizing the ratio \( g_{12}/b_{12} \). This is because the maximal value of \( \frac{O_i^2}{l_m^{i^2}} \) always occurs at \( j = m - 1 \) but the minimal value of \( \min(l_m^{2i-1}, l_m^{2i})/l_m^{i^2} \)
only occurs at \( j = m - 1 \) if \( m \leq 4 \). A noncanonical construction can allow us to lower the number 12, but the best result is obtained by extending Theorem 3.

**Theorem 8.** If there exist constructions \( C \) and \( C' \) of \( F(m) \) and \( F(n) \) respectively, such that

\[
\begin{align*}
(15) & \quad \frac{O_l}{m} \cdot \frac{O_l}{n} < \min \left( \frac{I^2_{i-1}}{m}, \frac{I^2_j}{n} \right), \quad \min \left( \frac{I^2_{i-1}}{n}, \frac{I^2_j}{m} \right) \quad \text{for all } i, j \geq 2, \\
(16) & \quad g_m \cdot \frac{P(m)}{m} \leq \frac{P(n)}{n} \quad \text{and} \quad g_n \cdot \frac{P(n)}{n} \leq \frac{P(m)}{m}, \\
(17) & \quad \max \left( \frac{h_m}{m}, \frac{h_n}{n} \right) \cdot g_m g_n \leq h_m h_n,
\end{align*}
\]

then \( F(m) + F(n) = P(m) + P(n) \).

**Proof.** Call the intervals \( I^i_m \) and \( I^i_n \) M-divisible* iff (8), (9) or

\[
(18.1) \quad (I^2_{i-1} + I^2_{j-1}) \cup (I^2_{i-1} + I^2_j) \cup (I^2_i + I^2_{j-1}) \cup (I^2_i + I^2_j) = I^i_m + I^i_n
\]

holds. The proof now parallels the proof of Theorem 3; the only significant difference being to show that a compatible pair is M-divisible* when neither (11) nor (12) holds. So assume

\[
\begin{align*}
(19) & \quad \frac{h_n}{g_n} \leq \frac{\bar{I}_m}{\bar{I}_n} \leq \frac{g_m}{h_m}.
\end{align*}
\]

Let \( I^2_{i-1} = [\alpha, \beta], I^2_i = [\gamma, \delta], I^2_{j-1} = [\alpha_0, \beta_0], \) and \( I^2_j = [\gamma_0, \delta_0] \). From (15) we obtain

\[
\frac{O_l}{m} \leq \min \left( \frac{I^2_{i-1}}{m}, \frac{I^2_j}{n} \right) \quad \text{or} \quad \frac{O_l}{n} \leq \min \left( \frac{I^2_{i-1}}{n}, \frac{I^2_j}{m} \right);
\]

assume for simplicity the latter. Then \( \gamma_0 - \beta_0 \leq \min(\beta - \alpha, \delta - \gamma) \) so \( \alpha + \gamma_0 \leq \beta + \beta_0 \) and \( \gamma + \gamma_0 \leq \delta + \delta_0 \). Then the LHS of (18.1) is

\[
\begin{align*}
[\alpha + \alpha_0, \beta + \beta_0] \cup [\alpha + \gamma_0, \beta + \delta_0] \cup [\gamma + \alpha_0, \delta + \beta_0] \cup [\gamma + \gamma_0, \delta + \delta_0] \\
= [\alpha + \alpha_0, \beta + \delta_0] \cup [\gamma + \alpha_0, \delta + \delta_0] = I^i_m + I^i_n,
\end{align*}
\]

For \( k = 2i - 1 \) or \( 2i, \ l = 2j - 1 \) or \( 2j \), we have

\[
\begin{align*}
\frac{h_m}{m} \cdot \frac{h_n}{n} \geq g_m g_n \cdot I^i_m \geq \frac{h_m}{m} \cdot \frac{h_n}{n} \cdot g_m g_n \cdot I^i_n \geq g_m g_n \cdot I^i_n.
\end{align*}
\]
Theorem 9. \( F(2) + F(7) = P(2) + P(7) = [.4928..., 1.6195...] \equiv R \pmod{1} \).

Proof. The canonical constructions can be used. We again apply Lemma 5 to obtain the bounds

\[ g_2 < .4456..., h_2 > .1686..., h_2' < .4589..., g_7 < .1343..., \]

\[ h_7 > .2906..., h_7' < .6359..., \quad \text{and} \quad \frac{O_7^j}{\min(\frac{i_7^{2j-1}}{1}, \frac{i_7^{2j}}{1})} \leq .3594... \]

Since \( \min(\frac{i_2^{2j-1}}{1}, \frac{i_2^{2j}}{1})/O_2^j \geq h_2/g_2 \), it is now easy to verify the hypotheses of Theorem 8. □

For any constructions of \( F(2) \) and \( F(6) \), equation (15) fails whenever an interval of the type ((\( a_1, a_s, 5, 6 \), (\( a_1, a_s, 6, 6, 1 \)) is deleted from \( F(6) \). Thus, as (15) is intuitively a "best possible" condition in the sense that no weakening approximations were made, it is probable that \( F(2) + F(6) \neq P(2) + P(6) \). Moreover, since (15) fails infinitely often, \( F(2) + F(6) \) may possibly be a Cantor set.

The negative results \( F(3) + F(3) \not\in R \pmod{1} \) and \( F(2) + F(4) \not\in R \pmod{1} \) can be verified directly. \( P(2) + P(4) \) has length less than one and \( F(3) \subset [\.5274..., .62178... \cup [.62200..., 1.5826...] \)

We now look at sums of \( F(m_1) + ... + F(m_s) \) with \( s > 2 \).

Theorem 10. Let \( C_1, ..., C_s \) be constructions of \( F(m_1), ..., F(m_s) \) respectively (not necessarily canonical constructions). If

\[ \sum_{i=1}^{s} h_{m_i} + h_{m_i} \leq \sum_{i=1}^{s} \frac{1}{m_{i}} \]

then \( F(m_1) + ... + F(m_s) = P(m_1) + ... + P(m_s) \).

Proof. Again we mimic the proof of Theorem 3. Let \( (i_{m_1}^1, ..., i_{m_s}^s) \) be compatible iff

\[ h_{m_i} \cdot i_{m_k}^k \leq i_{m_i}^i \quad \text{for} \quad 1 \leq i, k \leq s. \]
Call \((i_{m_1}^1, \ldots, i_{m_s}^s)\) dividable iff
\[
(i_{n}^{2j-1} + (i_{m_1}^1 + \cdots + \hat{i}_{n}^j + \cdots + i_{m_s}^s)) \cup (i_{n}^{2j} + (i_{m_1}^1 + \cdots + \hat{i}_{n}^j + \cdots + i_{m_s}^s))
\]
\[
= i_{m_1}^1 + \cdots + i_{m_s}^s,
\]
and \((i_{m_1}^1, \ldots, i_{n}^{2j-1}, \ldots, i_{m_s}^s)\) and \((i_{m_1}^1, \ldots, i_{n}^{2j}, \ldots, i_{m_s}^s)\) are compatible, where \(\sim\) means omission and \(n\) is such that
\[
\frac{i_{n}^j}{m_i} ≥ \frac{g_n}{m_i} \quad \text{for} \quad 1 ≤ i ≤ s.
\]

Hypothesis (20) says the beginning \(s\)-tuple \((i_{m_1}^1, \ldots, i_{m_s}^s)\) is compatible, so the proof reduces to showing that every compatible \(s\)-tuple is dividable. Let \(k = 2j - 1\) or \(2j\). Then \(\frac{h_n}{m_i} ≥ \frac{h_n}{m_i} \cdot \frac{i_{n}^j}{m_i}\) for \(1 ≤ i ≤ s\), and the other combinations of subscripts occurring in (22) trivially produce correct inequalities, so \((i_{m_1}^1, \ldots, i_{m_1}^k, \ldots, i_{m_s}^s)\) is compatible. From (21), we have
\[
g_n < h_{m_1} + \cdots + \hat{h}_{n} + \cdots + h_{m_s}
\]
so that
\[
\frac{i_{m_1}^1}{m_1} + \cdots + \frac{i_{n}^j}{m_n} + \cdots + \frac{i_{m_s}^s}{m_s} ≥ h_{m_1} \cdot \frac{i_{m_1}^1}{m_1} + \cdots + h_{m_s} \cdot \frac{i_{m_s}^s}{m_s}
\]
\[=(h_{m_1} + \cdots + \hat{h}_{n} + \cdots + h_{m_s}) \cdot \frac{i_{n}^j}{m_n} > g_n \cdot \frac{i_{n}^j}{m_n} > 0.
\]

This is the analog of equation (13) and is precisely the inequality needed to establish (23). \(\Box\)

**Theorem 11.** The following are all true.

\[
F(2) + F(2) + F(4) = P(2) + P(2) + P(4) \equiv R \pmod{1},
\]
\[
F(2) + F(3) + F(3) = P(2) + P(3) + P(3) \equiv R \pmod{1},
\]
and
\[
F(2) + F(2) + F(2) + F(2) = P(2) + P(2) + P(2) + P(2) \equiv R \pmod{1}.
\]

**Proof.** Apply Theorem 10 to the canonical constructions of \(F(2), F(3)\) and \(F(4)\). \(\Box\)

The final result listed in Table 1, \(F(2) + F(2) + F(3) \not \equiv R\), results from inspecting the first few subdivisions of \(F(2)\) and \(F(3)\).
T. W. Cusick and R. A. Lee [1], [2] have investigated $\sum_i S(m_i)$, where

$$S(m) = \{(a_1, a_2, \ldots): a_i \geq m \text{ for all } i\}$$

$$\cup\{(a_1, \ldots, a_s): a_i \geq m \text{ for } 1 \leq i \leq s, s \geq 1\} \cup \{0\}.$$

They have shown [2] that

$$\sum_{i=1}^{m} S(m) = [0, 1].$$

Our Theorem 10 can be applied to $\sum_i S(m_i)$ in place of $\sum_i F(m_i)$, whereupon

(25)

follows as a relatively easy special case.

More generally, Theorems 3, 8 and 10 are applicable to any Cantor sets for which $g_m$, $h_m$ and $h_m'$ can be evaluated.

BIBLIOGRAPHY


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53705