A GENERALIZATION OF JARŇÍK'S THEOREM
ON DIOPHANTINE APPROXIMATIONS
TO RIDOUT TYPE NUMBERS

BY

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ABSTRACT. Let \( s \) be a positive integer, \( c > 1 \), \( \mu_0, \ldots, \mu_s \) reals in \([0, 1]\), \( \sigma = \sum_{i=0}^{s} \mu_i \), and \( t \) the number of nonzero \( \mu_i \). Let \( \Pi_i \) \((i = 0, \ldots, s)\) be \( s + 1 \) disjoint sets of primes and \( S \) the set of all \((s + 1)\)-tuples of integers \((p_0, \ldots, p_s)\) satisfying \( p_0 > 0 \), \( p_i = p_i^k p_i' \), where the \( p_i' \) are integers satisfying \( |p_i'| \leq c |p_i|^\mu_i \), and all prime factors of \( p_i' \) are in \( \Pi_i \) \((i = 0, \ldots, s)\). Let \( X > 0 \) if \( t = 0 \), \( X \geq \sigma / \min(s, t) \) otherwise, \( E_\lambda \) the set of all real \( s \) -tuples \((a_1, \ldots, a_s)\) satisfying \( |a_i - p_i / p_0| < p_0^{-\lambda} \) \((i = 1, \ldots, s)\) for an infinite number of \((p_0, \ldots, p_s) \in S\). The main result is that the Hausdorff dimension of \( E_\lambda \) is \( \sigma / \lambda \). Related results are obtained when also lower bounds are placed on the \( p_i' \). The case \( s = 1 \) was settled previously (Proc. London Math. Soc. 15 (1965), 458-470). The case \( \mu_i = 1 \) \((i = 0, \ldots, s)\) gives a well-known theorem of Jarňík (Math. Z. 33 (1931), 505-543).

1. Introduction. Jarňík [3] proved that the Hausdorff dimension of the set \( E \) of all real \( s \) -tuples \((a_1, \ldots, a_s)\) satisfying \( |a_i - p_i / q| < q^{-\lambda} \), \( i = 1, \ldots, s \), for an infinite number of \((s + 1)\)-tuples \((q, p_1, \ldots, p_s)\) of integers with \( q > 0 \), is \((s + 1)\lambda^{-1}\) provided that \( \lambda > 1 + s^{-1} \).

In this paper we investigate the case where \( q, p_1, \ldots, p_s \) are restricted to certain sets of integers which were considered by Ridout in his extension of Roth's theorem [6]. In [1] it was proved that the set \( E \) in this case has Lebesgue measure 0. The Hausdorff dimension for the one-dimensional case of the problem was determined by the authors in [2].

2. Definitions and notation. Let \( s \) be a positive integer, \( \mu_0, \mu_1, \ldots, \mu_s \) reals in \([0, 1]\) and \( \sigma = \sum_{i=0}^{s} \mu_i \). Let \( \Pi_i = \{P_{i,1}, \ldots, P_{i,n_i}\} \((i = 0, \ldots, s)\) be \( s + 1 \) sets of distinct primes, \( C_i \) the set of integers all of whose prime factors belong to \( \Pi_i \).

We say that condition 1 is satisfied, if there exists \( P_i \in \Pi_i \) for \( i = 0, \ldots, s \), such that

\[
(Ia) \quad P_i \neq P_0 \quad (i = 1, \ldots, s).
\]

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(lb) Those among the numbers \((1 - \mu_i)/\log P_0, \ldots, (1 - \mu_s)/\log P_s\)
which are not zero are linearly independent over the field of rational numbers.

In particular, condition (lb) is satisfied if \(\mu_i = 1\), \(i = 0, \ldots, s\).

Let \(c > 1\). We define \(S = S(c; \mu_0, \ldots, \mu_s; C_0, \ldots, C_s)\) to be the set of all \((s + 1)\)-tuples of integers \((p_0, \ldots, p_s), p_0 > 0\), satisfying

(i) \((p_i, p_0) = 1, \ i = 1, \ldots, s\).

(ii) \(p_i = p_i^* \cdot p_i'\) with \(p_i' \in C_i\) and \(p_i^*\) any integer satisfying \(|p_i^*| < c|p_i|^{\mu_i}\), \(i = 0, \ldots, s\).

Similarly we define \(S'^T = S^T(c; \mu_0, \ldots, \mu_s; C_0, \ldots, C_s)\) by replacing (ii) by the requirement

(ii)' \(p_i = p_i^* \cdot p_i'\) where \(p_i' \in C_i\) and \(p_i^*\) is any integer satisfying \(|p_i^*| < c|p_i|^{\mu_i}\), \(i = 0, \ldots, s\).

Let \(\mu_0', \mu_1', \ldots, \mu_s'\) be reals satisfying (a) \(0 \leq \mu'_i \leq \mu_i\); (b) if \(\sigma > 0\),
then \(0 \leq \mu'_i < \mu_i\) for some \(i\). We define a set \(S'\) in a similar way to \(S\) and \(S^T\), but replacing this time condition (ii) by the requirement

(ii)' \(p_i = p_i^* \cdot p_i'\) where \(p_i' \in C_i\) and \(p_i^*\) is any integer satisfying \(|p_i^*| < c|p_i|^{\mu_i}\), \(i = 0, \ldots, s\).

Let \(\lambda, D\) be positive reals, \(W\) an \(s\)-dimensional interval with edges parallel to the axes. We define the set \(E = E(\lambda, W, S, D)\) to be the set of all \(s\)-tuples \((\alpha_0, \ldots, \alpha_s) \in W\) satisfying \(\alpha_i - p_i \cdot p_0^{-1} < D p_0^{-\lambda}\), \(i = 1, \ldots, s,\)
for an infinite number of \((s + 1)\)-tuples \((p_0, \ldots, p_s)\) from \(S\). Similarly we define \(E^T = E^T(\lambda, W, S^T, D)\) and \(E' = E'(\lambda, W, S', D)\).

By \(R^s\) we denote the Euclidean space of \(s\) dimensions, and by \(d(x, y)\)
the distance between two points \(x, y\) of \(R^s\). By \(\delta(E), \alpha - m^*E, \dim E\) we denote, respectively, the diameter, the Hausdorff measure with respect to the function \(r^\alpha\) and the Hausdorff dimension of the set \(E\). By a cube we mean an \(s\)-dimensional interval with edges parallel to the axes.

3. Main results. The main results of this paper are

Theorem I. \(\dim E^T \leq \dim E' \leq \dim E \leq \sigma/\lambda\).

Theorem II. Let \(t\) be the number of \(\mu_i\) which are not zero \((i = 0, \ldots, s)\).
Let \(\lambda\) satisfy

\[
\lambda > \begin{cases} 0 & \text{if } t = 0, \\ \sigma/\min(s, t) & \text{if } t > 0. \end{cases}
\]
If condition I holds, then
\[ \dim E \geq \dim E' \geq \dim E^T \geq \sigma/\lambda. \]

Theorem III. If (1) and (Ia) hold then \( \dim E \geq \dim E' \geq \sigma/\lambda. \)

These results imply \( \dim E = \dim E' = \sigma/\lambda \) if (1) and (Ia) hold and \( \dim E = \dim E' = \dim E^T = \sigma/\lambda \) if (1) holds and condition I is satisfied. The case \( \mu_i = 1, \ i = 0, \ldots, s \), gives Jarník's result.

4. Proof of Theorem I. Let \( b_i > 0, \ i = 1, \ldots, s \). By symmetry, it is enough to prove the theorem when \( W \) is defined by
\[ W = \{(x_1, \ldots, x_s) | 0 \leq x_i \leq b_i, \ i = 1, \ldots, s\}. \]

We shall prove that, for every \( \sigma > 0 \), if \( \rho = (\sigma + \delta)\lambda^{-1} \) then \( \rho - m^*E = 0 \). We may also assume that \( \delta < 1 - \mu_0 \) if \( \mu_0 < 1 \).

Let \( \epsilon > 0 \). The set of all cubes whose center is \( (p_1/p_0, \ldots, p_s/p_0) \in W \) with \( (p_0, \ldots, p_s) \in S, \ p_0 > q_0, \) and length of edge \( 2Dp_0^\lambda \), is obviously a covering for \( E \). If \( q_0 \) is large enough, the diameter of each cube is smaller than \( \epsilon \). It remains to prove that the series \( M = \sum (p_i/p_0)^\rho = \sum p_i^{-\sigma - \delta} \) converges, where the summation is over all sets \( (p_0, \ldots, p_s) \in S \) such that \( (p_1/p_0, \ldots, p_s/p_0) \in W \). Since \( p_i = p_i^* p_i' \) for \( i = 0, \ldots, s \), the summation can be broken up into a summation over \( p_1^*, \ldots, p_s^* \), and over \( p_1', \ldots, p_s' \).

Therefore,
\[ M = \sum_{p_0} M_1, \quad M_1 \leq \sum_{p_0} \{2\} p_0^{-\sigma - \delta} \sum_{1} 1, \]
where \( \{1\} \) and \( \{2\} \) indicate summations over \( p_1^*, \ldots, p_s^* \) and \( p_1', \ldots, p_s' \), respectively. Positive constants depending only on \( c, \delta, \mu_i, b_i, \Pi_i \) \( (0 \leq i \leq s) \) are denoted by \( A \) below. Since \( p_i^* < c p_i^\mu_i < c b_i^\mu_i p_0^\mu_i \) \( (1 \leq i \leq s) \), we have \( \Sigma \Pi_1 < A p_0^{-\sigma - \mu_0} \). Putting \( \eta = \delta/2 \), we thus obtain
\[ M_1 \leq A p_0^{-\mu_0 - \eta} \sum_{p_0} \{2\} p_0^{-\eta} = A p_0^{-\mu_0 - \eta} \prod_{i=1}^s \sum_{3} p_i^{-\eta/s}, \]
where \( \{3\} \) denotes summation over \( p_i' \in C_i \). Since \( p_i' \leq p_i \leq b_i p_0 \) \( (1 \leq i \leq s) \), we obtain
\[ \sum_{3} p_i^{-\eta/s} \leq A \sum_{3} p_i'^{-\eta/s} \leq A \prod_{i=1}^s (1 - p_i^{-\eta/s})^{-1} \leq A. \]

Therefore
\[ M_1 \leq A p_0^{-\mu_0 - \eta} \quad \text{and} \quad M \leq A \sum_3 p_i'^{-\mu_0 - \eta} \sum_4 p_0^{-\mu_0 - \eta}. \]
where \( \{4\} \) and \( \{5\} \) denote summations over all \( p^*_0 \leq R = C^{1/(1-\mu_0)} p^*_0 \mu_0/(1-\mu_0) \) \((\mu_0 < 1) \) and \( p^*_0 \in C_0, \) respectively. (If \( \mu = 1, M < A \sum_{i=1}^{\infty} p^*_0 - 1 \leq A.) \)

\[
\sum_{i=1}^{\infty} p^*_0^{-\mu_0-\eta} < 1 + \int_1^R x^{-\mu_0-\eta} dx \leq A p^*_0^{-\mu_0-\eta} \mu_0/(1-\mu_0). 
\]

Therefore \( M \leq A \sum_{i=1}^{\infty} p^*_0-\eta A \leq \infty, \) completing the proof.

5. Proof that Theorem II implies Theorem III. We may assume that \( \sigma > 0, \) because otherwise Theorem III is trivially true. Let \( P_i \in \Pi_i, \) \( i = 0, \ldots, s \) and \( P_i \neq P_0, i = 1, \ldots, s. \) If condition I is not satisfied, then

\[
(1 - \mu_0)/\log P_0, \ldots, (1 - \mu_s)/\log P_s
\]

are linearly dependent over the rationals.

Let \( \epsilon > 0. \) There exists \( j \) such that \( 0 \leq \mu^*_j < \mu_j. \) Choose \( \mu^*_i \) such that \( \mu^*_j < \mu^*_j < \mu_j, \mu_j - \mu^*_j < \epsilon, \) and such that the nonzero members among

\[
(1 - \mu_0)/\log P_0, \ldots, (1 - \mu_s)/\log P_s
\]

are linearly independent over the rationals. Let \( \mu^*_i = \mu_i \) for \( i \neq j, \) and let \( S^T \) and \( S^m \) be the same as \( S^T \) and \( S^i \) respectively, except that in (ii) \( T \) and (ii)' \( \mu_i \) is replaced by \( \mu^*_i \) \((0 \leq i \leq s). \) Then

\[
S^T \subset S^m \subset S^i < S, \quad E^T \subset E^m \subset E^i < E.
\]

By Theorem II,

\[
\dim E \geq \dim E^i \geq \dim E^m \geq \dim E^T \geq (A - \epsilon)/\lambda.
\]

Since this holds for every \( \epsilon > 0, \) we have \( \dim E \geq \dim E^i \geq \sigma/\lambda, \) which is Theorem III.

Remark. Condition I is, however, essential in proving \( \dim E^T \geq \sigma/\lambda, \) as is shown by the following example. Let \( P_0 \) and \( P_1 \) be two distinct primes, \( C_0 = \{ p_0^m \}, C_1 = \{ p_1^m \}, m_0, m_1 \) nonnegative integers. There exist \( \mu_0 \) and \( \mu_1 \) in \([0,1]\) such that \( P_1^{1/(1-\mu_1)} = P_0^{1/(1-\mu_0)} = A > 1. \) Let \( 0 < \epsilon < (A - 1)/(A + 1), \) and

\[
1 < c < \min \{ (1 + \epsilon)^{1-\mu_1}, (1 - \epsilon)^{1-\mu_0} \}.
\]

If \( (p_0, p_1) \in S^T(c; \mu_0, \mu_1; C_0, C_1) \) and \( p_0, p_1 > 0, \) then

\[
p_i = p^*_i \mu_i, \quad p_i^* \leq p^*_i < c^{-\mu_i}, \quad p_i^* = p_i^m_i, \quad i = 0, 1.
\]

This gives

\[
p_i^{m_i/(1-\mu_i)} \leq p_i < c^{1/(1-\mu_i)} p_i^{m_i/(1-\mu_i)}, \quad i = 0, 1.
\]
and

\[(1 - \varepsilon)A_k < c \frac{1}{(1 - \varepsilon_0)} A_k < \frac{1}{(1 - \varepsilon_0)} A_k < (1 + \varepsilon)A_k, \]

where \( k = m_1 - m_0. \)

The requirement for \( \varepsilon \) implies that \( A(1 - \varepsilon) > 1 + \varepsilon. \) By (2), the interval 
\((1 + \varepsilon, A(1 - \varepsilon))\) does not contain any \( p_1/p_0 \) with \((p_0, p_1) \in S^T\) because, if \( k \leq 0, \) then 
\((1 + \varepsilon)A_k \leq 1 + \varepsilon, \) and if \( k > 0, \) then 
\( A(1 - \varepsilon) \leq (1 - \varepsilon)A_k. \)

6. Lemmas for Theorem II. It suffices to prove Theorem II for an interval \( W \) of the form

\[ W = \{(x_1, \ldots, x_s)|a_i \leq x_i \leq b_i \quad i = 1, \ldots, s\}, \]

where the \( a_i \) are arbitrary positive reals, \( b_i = a_i + L_0, \) and \( L_0 \) is any sufficiently small real number, to be chosen later in the proof (Lemma 4).

Lemma 1. It is enough to prove Theorem II for the case \( \mu_i \geq \mu_0, \ i = 1, \ldots, s. \)

Proof. If \( \mu_i < \mu_0 \) for some \( i > 0, \) we may assume that \( \mu_s = \min(\mu_0, \ldots, \mu_s). \)

Let \( \nu_i = \mu_i \) if \( i \neq 0, s, \nu_0 = \mu_s \) and \( \nu_s = \mu_0. \) Let \( \psi: W \rightarrow \mathbb{R}^s \) be defined by

\[ \psi(x_1, \ldots, x_{s-1}, x_s) = (x_1/x_s, \ldots, x_{s-1}/x_s, 1/x_s), \]

and let \( \psi(W_1) \subseteq W. \) It is easily seen that \( \psi \) has Jacobian \( a^{-s-1}_s, \) which is bounded away from 0 and \( \infty \) on \( W; \) and therefore preserves Hausdorff dimension.

Let \( S^T, E^T \) be as defined in \( \S 2, \)

\[ S^T = S^T(c; \nu_0, \ldots, \nu_s; C_s, C_1, \ldots, C_{s-1}, C_0), \quad E^T = E^T(\lambda, W_1, S^T, D_1), \]

where \( D_1 > 0 \) is sufficiently small. The conditions of Theorem II hold for \( E^T, \) and we have, moreover, \( \nu_i \geq \nu_0 (1 \leq i \leq s). \) Therefore, assuming the validity of the theorem for this case, \( \dim E^T \geq s/\lambda. \) We now prove that for a suitable choice of \( D_1 \) we have \( \psi(E^T_1) \subseteq E^T. \) Let \( (\beta_1, \ldots, \beta_s) \in \psi(E^T_1). \) There exists \( (a_1, \ldots, a_s) \in E^T_1 \) such that \( (a_i/a_s, \ldots, a_{s-1}/a_s, 1/a_s) = (\beta_2, \ldots, \beta_{s-1}, \beta_s), \) and an infinity of \( (p_s, \ldots, p_{s-1}, p_0) \in S^T_1 \) \( (p_0 \in C_i, \ i = 0, \ldots, s), \) satisfying \( |a_i - p_i/p_s| < D_1p_s^{-\lambda}, \ i \leq s - 1, \)

\[ |a_s - p_0/p_s| < D_1p_s^{-\lambda}. \]

Let \( a_i = p_i/p_s + \eta_i, \ i \leq s - 1, \) \( a_s = p_0/p_s + \eta_s, \)

\[ |\eta_i| < D_1p_s^{-\lambda} (0 \leq i \leq s). \]

For \( 1 \leq i \leq s - 1 \) we then have

\[ \frac{a_i}{a_s} = \frac{p_i}{p_0} \cdot \frac{1 + \eta_i p_s/p_i}{1 + \eta_s p_s/p_0}, \]
\[ \left| \frac{\alpha_i}{\alpha_s} - \frac{p_i}{p_0} \right| < \frac{p_i}{p_0} (1 - D_1 p_s^{-\lambda} / p_0)^{-1} \left( |\eta_1| \frac{p_s}{p_i} + |\eta_s| \frac{p_s}{p_0} \right) \]

\[ \leq 2 \left( \frac{b_i}{a_i} \right) (1 - D_1 p_s^{-\lambda} / p_0)^{-1} D_1 p_s^{-\lambda} < D p_0^{-\lambda}, \]

if \( D_1 \) is sufficiently small. A similar computation shows that \(|\alpha_{s-1} p_s p_0^{-1}| < D p_0^{-\lambda}\) for \( D \) small enough. Thus

\[ |\beta_i - p_i/p_0| < D p_0^{-\lambda}, \quad i = 1, \ldots, s, \]

which shows that \( \psi(E^T_1) \subset E^T \). Therefore,

\[ \dim E^T \geq \dim \psi(E^T_1) = \dim E^T_1 \geq \sigma/\lambda. \]

From now on we shall assume \( \mu \geq \mu_0 (1 \leq i \leq s) \). We may also assume that every \( \Pi_i \) contains only one prime \( P_i \) such that condition I is satisfied, that not all \( \mu_i \) are 1 because this is Jarník's theorem, and that not all \( \mu_i \) are zero because then Theorem II is trivial. These assumptions are not essential but permit a simpler exposition.

Let \( \delta > 0, \rho = (\sigma - \delta)/\lambda \). In order to prove that \( \rho - m^*(E^T) > 0 \), we use the following special case of a theorem due to P. A. P. Moran [5].

**Lemma 2.** Let \( s \) be a positive integer, \( E \) a bounded set in \( \mathbb{R}^s \) and \( 0 \leq \rho \leq s \). A sufficient condition for \( \rho - m^*(E) \) to be positive is the existence of a closed subset \( F \) of \( E \) and an additive function \( \phi \) defined on the ring \( \mathcal{R} \) generated by the semiopen cubes of \( \mathbb{R}^s \), satisfying the following properties:

(a) \( \phi \) is nonnegative.

(b) For every \( R \in \mathcal{R} \) and \( R \supset F \) we have \( \phi(R) > b > 0 \) for some fixed \( b \).

(c) There exists a positive constant \( k \) such that for every semiopen cube \( R \) we have \( \phi(R) < k d(R)^{\rho} \).

**Lemma 3.** Let \( \theta_1, \ldots, \theta_s \) be reals such that \( 1, \theta_1, \ldots, \theta_s \) are linearly independent over the rationals, \( \delta, \eta, n_0 > 0 \). There exist real numbers \( b, B \) such that for every set of real numbers \( \alpha_1, \ldots, \alpha_s \) there is an \( (s + 1) \)-tuple of integers \( (m_0, \ldots, m_s) \) satisfying \( |m_0 \theta_i - m_i - \alpha_i| < \delta, 1 \leq i \leq s, n_0 < b < m_0 < B < (1 + \eta)b \).

Except for the explicit bound on \( m_0 \), this is Kronecker's theorem. The bound can be obtained by introducing a slight change in one of the proofs of Kronecker's theorem, for example, Lettenmeyer's proof [4].
Let \( t' \) be the number of nonzero \( \mu_i \) \((1 \leq i \leq s)\), \( 0 < \mu < \min \mu_i \neq 0 \mu_i \). We shall now formulate the main lemma.

**Lemma 4.** Let \( L < L_0, \theta, \eta \) be positive real, \( q_0 = q_0(a, b, \pi, \mu, L, \eta) \) a sufficiently large real number. There exist reals \( A, a \) such that for every cube \( I \subset W \) with edge \( L \), there is a subset \( S_i \subset S^T \) with the following properties:

(i) If \((p_{01}, \ldots, p_{0s}) \in S_i\), then \((p_{1}/p_{01}, \ldots, p_{s}/p_{0s}) \in I\), \( q_0 < a < p_i < A < a^{1+\eta}, (p_i, p_0) = 1, \) all the \((p_0, \ldots, p_s) \in S_i \) share the same fixed \((s + 1)\)-tuple \((p_{01}, \ldots, p_{0s})\).

(ii) If \( p_{0i}^{(1)} < p_{0j}^{(2)} \) and \((p_{0i}^{(1)}, \ldots, p_{0s}^{(1)}) \in S_i \) \((i = 1, 2)\), then there exists at least one \( j \) such that

\[
|p_{0j}^{(1)} / p_0^{(1)} - p_{0j}^{(2)} / p_0^{(2)}| \geq (p_0^{(1)}) - (\sigma/s) - \theta.
\]

(iii) Let \( a^{-\mu} < l \leq L, I, \) any cube with edge length \( l \) contained in \( I \), \( V_l \) the number of elements \((p_{01}, \ldots, p_{0s})\) of \( S_i \) such that \((p_{1}/p_{01}, \ldots, p_{s}/p_{0s}) \in I_l\). Then

\[
V_l < K l^{t'} p_0^{\sigma/(1-\mu)} / Y,
\]

where

\[
Y = \begin{cases} 
\log p_0' & \text{if } \mu_0 > 0, \\
1 & \text{if } \mu_0 = 0,
\end{cases}
\]

\( K \) a suitable positive constant depending on \( S^T, W, \lambda, D, \eta, \theta. \)

(iv) The total number \( V_L \) of elements of \( S_i \) satisfies

\[
V_L > KL^{t'} p_0^{\sigma/(1-\mu)} / Y \geq KL^{t'} a^{\sigma / X},
\]

where

\[
X = \begin{cases} 
\log a & \text{if } \mu_0 > 0, \\
1 & \text{if } \mu_0 = 0.
\end{cases}
\]

Remark. The convention on \( K \) will be used for the rest of the paper, for the sake of simplicity of notation.

**Proof.** Let \( \epsilon > 0 \) be sufficiently small,

\[
I = \{(x_1, \ldots, x_s) \mid a_i + \epsilon < y_i \leq x_i \leq y_i + L < b_p, 1 \leq i \leq s\},
\]

\[
1 < c_0 < c_1 < c, \quad c_1 < 1 + \min_i (\epsilon/a_i), \quad c_1/c_0 < 2, \quad c_0 < 2.
\]
Since $\mu_i \geq \mu_0$ and not all $\mu_i$ are 1, we have $\mu_0 < 1$. Suppose that $\mu_0, \ldots, \mu_h$ ($h \leq s$) are all the $\mu_i$ which are not 1. We assume first $b > 0$. Let

$$
\delta = \min_{1 \leq i \leq h} \frac{1 - \mu_i}{2 \log P_i} \log \left(1 + \frac{L}{b_i}\right),
$$

$$
\theta_i = \frac{(1 - \mu_i)}{(1 - \mu_0)} \log P_0, \quad \xi_i = -\frac{1 - \mu_i}{2 \log P_i} \log \left(\frac{\gamma_i (\gamma_i + L)}{c_i^2}\right), \quad 1 \leq i \leq h.
$$

Condition I implies that $\theta_1, \ldots, \theta_h$ are linearly independent over the rationals. By Lemma 3, there exist numbers $b, B$ and an $(h + 1)$-tuple of integers $(m_0, \ldots, m_h)$ satisfying

$$
(1 - \mu_0) \log P_0 (q_0 / c_0) < b < m_0 < B < (1 + \eta) b,
$$

(6)

$$
|m_0 \theta_i - m_i - \xi_i| < \delta, \quad 1 \leq i \leq h.
$$

This with the definition of $\delta$ implies

$$
\gamma_i < c_1^{m_i/(1-\mu_i)} P_0^{\mu_0/(1-\mu_0)} < \gamma_i + L, \quad 1 \leq i \leq h.
$$

(7)

Define a set $T_i$ of $(s + 1)$-tuples $(p_0^*, \ldots, p_s^*)$ of integers with $p_i = p_i^* p_i^\prime$ ($0 \leq i \leq s$) satisfying:

1. $p_i^\prime = p_i^{m_i}$ ($0 \leq i \leq h$), where $(m_0, \ldots, m_h)$ is a fixed $(h + 1)$-tuple of integers satisfying (7), and $p_i^\prime = 1$ for $i > h$.
2. If $\mu_0 > 0$, $p_0^*$ ranges over all primes $> \max_i P_i$ satisfying

$$
c_0^{\mu_0/(1-\mu_0)} \leq p_i^* \leq c_1^{\mu_0/(1-\mu_0)}.
$$

The existence of such $p_0^*$ is guaranteed if $q_0$ is sufficiently large. If $\mu_0 = 0$, put $p_0^* = 1$.
3. If $\mu_i > 0$, $p_i^*$ ranges over all integers satisfying

$$
\gamma_i \frac{P_0}{p^\prime} < p_i^* < (\gamma_i + L) \frac{P_0}{p}, \quad (p_i^*, p_0^* p_i^\prime) = 1, \quad 1 \leq i \leq s.
$$

(9)

Since every interval of length $> 5$ contains an integer relatively prime to the product of three given primes, integers $p_i^*$ satisfying (9) will exist if $L P_0 / p_i^\prime > 6$. By (7) this condition is easily seen to hold if $q_0$ is sufficiently large. If $\mu_i = 0$, put $p_i^* = 1$.

Now assume $b = 0$. Choose $b = m_0 - 1 > (1 - \mu_0) \log P_0 (q_0 / c_0)$, $B = m_0 + 1$, $p_i^\prime = P_0^{m_0}$, $p_i^\prime = 1$ ($1 \leq i \leq s$), and $p_0^*, p_i^*$ as above. It is clear

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that such $p_i^* = p_i$ satisfying (9) do in fact exist. Moreover, for $q_0$ sufficiently
large, (6) holds.

The definition of $T_i$ implies that if $(p_0, \ldots, p_s) \in T_i$, then $a_i < p_i/p_0 < b_i$, and $(p_i, p_0) = 1$ ($1 \leq i \leq s$). This follows from (9) if $h \mu_i > 0$ or $h = 0$. If $b > 0$, $\mu_i = 0$, it follows from (7) and (5). For $h > 0$, $\mu_i = 0$, $\mu_0 > 0$, we have by (4), (5), (7) and (8),

$$a_i < \frac{a_i + \epsilon}{c_1} < \frac{\gamma_i}{c_1} < \frac{p_i + \gamma_i}{p_0} < \frac{\gamma_i + L}{c_0} < \gamma_i + L.$$  

Let $a = c_0 p_0^{1-\mu_0}$, $A = c_0 p_0^{1-\mu_0}$. If $q_0$ is sufficiently large, we obtain, by (6), (8) and (5) ($\mu_0 \geq 0$, $h \geq 0$),

$$q_0 < a < p_0 < A < a^{1+\eta}, \quad a^{-\mu} < L.$$  

For $\mu_0 > 0$, (8) implies $p_i > c_0^{1-\mu_0} p_0^{\mu_0} < p_i^* < c_1 p_0^{\mu_0} < c_0 p_0^{\mu_0}$, and for $\mu_0 = 0$, $p_i^* = p_0^{\mu_0}$. To prove that $T_i \subseteq S_T$, it remains to show that

$$p_i^* < p_i < c p_i^*, \quad 1 \leq i \leq s.$$  

We may assume $0 < \mu_i < 1$ ($1 \leq i \leq s$), because otherwise (10) is trivial. If $\mu_0 > 0$, we obtain, from (7), (8), (9),

$$(c_0 c_1)^{1-\mu_i} \frac{\gamma_i}{\gamma_i + L} p_i^{\mu_i} < p_i^* < c_1 \frac{(1-\mu_i)^2 \gamma_i + L}{\gamma_i} p_i^{\mu_i},$$

and for $\mu_0 = 0$, we obtain, from (7) and (9),

$$\frac{\gamma_i}{\gamma_i + L} c^{1-\mu_i} p_i^{\mu_i} < p_i^* < \frac{\gamma_i + L}{c_1} p_i^{\mu_i}.$$  

Therefore (10) will hold by choosing $L$ to satisfy

$$0 < L < L_0 < \min_{1 \leq i \leq s} (a_i (c/c_1 - 1), a_i (c_1 - 1)).$$

We thus proved that $T_i \subseteq S_T$. Let

$$I_i = \{ (x_1, \ldots, x_s) | \gamma_i \beta_i \leq x_i \leq \beta_i + l \leq \gamma_i + L, 1 \leq i \leq s \}, \quad a^{-\mu} < l \leq L.$$  

Let $p_0$ be fixed. For $\mu_i > 0$ ($i > 0$), denote by $W^*_i(p_0)$ the number of integers $p_i^*$ relatively prime to $p_0^* P_0 P_i$, which satisfy $\beta_i p_0 / p_i^* < (\beta_i + h) p_0^* / p_i^*$. Lemma 4 of [2] implies
\[ \left( \frac{lp_0}{p_i^*} - 1 \right) \left( 1 - \frac{1}{p_i^*} \right) \left( 1 - \frac{1}{p_0^*} \right) - 2^3 < w_i(p_0^*) \]
\[ < \left( \frac{lp_0}{p_i^*} + 1 \right) \left( 1 - \frac{1}{p_i^*} \right) \left( 1 - \frac{1}{p_0^*} \right) \left( 1 - \frac{1}{p_0^*} \right) + 2^3, \]

except that the factor \( 1 - \frac{1}{p_0^*} \) is dropped if \( p_0^* = 0 \). Since \( l > a^{-\mu} > p_0^{-\mu} \), (9) and (10) imply \( lp_0/p_i^* > Kp_0^{-\mu_i} \). Since \( \mu_i - \mu > 0 \), 1 is absorbed by \( lp_0/p_i^* \). Thus

\[ (11) \quad Kl^i < w_i(p_0) < Kl^i. \]

For fixed \( p_0 \), denote by \( W_i(p_0) \) the number of elements \((p_0, \ldots, p_s) \in T_I \) such that \((p_1/p_0, \ldots, p_s/p_0) \in I_I \). Multiplying together the \( t \) inequalities (11) and defining \( w_i(0) = 1 \) for \( \mu_i = 0 \), we obtain

\[ (12) \quad Kl^{i\sigma - \mu_0} < W_i(p_0) < Kl^{i\sigma - \mu_0}. \]

It is easily seen that if \( s = 1 \), the set \( T_I \) satisfies all the conditions of the lemma for \( S_I \). For \( s > 1 \), however, condition (ii) is not necessarily satisfied. Let \((p_0, p_1^{(1)}, \ldots, p_s^{(1)}) \) and \((p_0, p_1^{(2)}, \ldots, p_s^{(2)}) \) be two distinct elements of \( T_I \) with the same \( p_0 \). By (9) and (10),

\[ \left| \frac{p_i^{(1)}}{p_0} - \frac{p_i^{(2)}}{p_0} \right| = \frac{p_i^*}{p_0} \left| p_i^{(1)} - p_i^{(2)} \right| \geq \frac{p_i^*}{p_0} > Kp_0^{-\mu_i}. \]

There exists \( j \) such that

\[ \mu_j \leq \frac{1}{s} \sum_{i=1}^{s} \mu_i < \frac{\sigma}{s} + \theta; \]

hence

\[ \left| \frac{p_j^{(1)}}{p_0} - \frac{p_j^{(2)}}{p_0} \right| \geq Kp_0^{-\mu_j} > Kp_0^{-\sigma/s - \theta}. \]

Condition (ii) of the lemma is therefore satisfied for two elements of \( T_I \) with the same \( p_0 \). If \( \mu_0 = 0 \), then all the elements of \( T_I \) have the same \( p_0 \) and we define \( S_I = T_I \) in this case. If \( \mu_0 > 0 \), we define \( S_I \subset T_I \) by excluding all those elements \((p_0, \ldots, p_s) \) of \( T_I \) for which there exists \( p_0^{(1)} < p_0 \) and \((p_0^{(1)}, \ldots, p_s^{(1)}) \in T_I \) such that for \( i = 1, \ldots, s \) we have
Clearly, $S_i$ satisfies condition (ii) of the lemma. We shall now count the number of elements of $T_i$ which are not in $S_i$. Let $N(p_0, p_0^{(1)})$ be the number of elements of $T_i$ for a fixed $p_0$ and fixed $p_0^{(1)} < p_0$, for which (13) holds for some $i$. For fixed $p_0$, let $N(p_0)$ denote the number of those elements $(p_0, \ldots, p_s)$ of $T_i$ for which there exists an element $(p_0^{(1)}, \ldots, p_s^{(1)})$ of $T_i$ such that (13) holds for every $i$. Clearly,

$$N(p_0) \leq \sum_{p_0^{(1)} \leq p_0} \prod_{i=1}^s N_i(p_0, p_0^{(1)}).$$

From (13),

$$|p_i^{*} - p_0^{*}| < \frac{1}{p_0^{(1)}}/p_i^{(1)}(\sigma/s) + \theta.$$

The expression $p_i^{*} - p_0^{*}$ can therefore assume at most

$$2p_0^{(1)}/p_i^{(1)}(\sigma/s) + \theta$$

different values. Let $u$ be a fixed integer. The equation $p_i^{*} - p_0^{*} = u$ implies

$$(14)\quad p_i^{*} = u \pmod{p_0^{*}}.$$

Since $p_0^{*}$ is a prime, this congruence has exactly one solution $p_i^{*}$ in each interval of length $p_0^{*}$. The integer $p_i^{*}$ is to be chosen in the interval $[\gamma_i p_0/p_i^{(1)}(\gamma_i + L)/p_0^{(1)}]$ of length $Lp_0^{(1)}p_i^{(1)} = KLp_0^{\mu_i}$. Since $p_0^{*} > c_0^{1-\mu_0}p_0^{\mu_0}$ and $\mu_i \geq \mu_0$, the number of solutions of (14) is $Lp_0^{(1)}p_i^{(1)} < KLp_0^{\mu_i-\mu_0}$. Therefore

$$N_i(p_0, p_0^{(1)}) \leq KL \frac{p_0^{(1)}p_0^{(1)}(\sigma/s) + \theta}{\mu_i(p_0^{(1)}(\sigma/s) + \theta)},$$

and hence

$$N(p_0) \leq KL \sum_{p_0^{(1)} < p_0} \frac{p_0^{(1)}p_0^{(1)}(\sigma/s) + \theta}{\mu_i(p_0^{(1)}(\sigma/s) + \theta)}.$$
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The last sum converges as was shown in the proof of Theorem I. Therefore,

\[ N(p_0) \leq KL^s p_0^{\sigma - \mu_0 - \theta s/2}. \]

Let \( V_I(p_0) \) denote the number of elements \((p_0, \ldots, p_s)\) of \( S_I \) such that 
\( (p_1/p_0^\gamma, \ldots, p_s/p_0^\gamma) \in I_I \) for fixed \( p_0 \), and let \( V_I \) be the total number of 
those elements in \( S_I \). By (12),

\[ V_I(p_0) \leq W_I(p_0) \leq KL^t p_0^{\sigma - \mu_0}, \]

\[ V_L(p_0) = W_L(p_0) - N(p_0) \geq KL^t p_0^{\sigma - \mu_0}. \]

Therefore,

\[ V_I < KL^t \sum p_0^{\sigma - \mu_0}, \quad V_L > KL^t \sum p_0^{\sigma - \mu_0}, \]

where \( \sum \) denotes summation over all \( p_0 \) so that \((p_0, \ldots, p_s) \in S_I \). By (8),

\[ K p_0^{(\sigma - \mu_0)/(1 - \mu_0)} \sum 1 < \sum p_0^{\sigma - \mu_0} < K p_0^{(\sigma - \mu_0)/(1 - \mu_0)} \sum 1, \]

where \( \sum 1 = 1 \) if \( \mu_0 = 0 \). If \( \mu_0 > 0 \), we obtain from (8) and the Prime Number Theorem,

\[ K p_0^{(\sigma - \mu_0)/(1 - \mu_0)/\log p_0} < \sum p_0^{\sigma - \mu_0} < K p_0^{(\sigma - \mu_0)/(1 - \mu_0)/\log p_0}. \]

Therefore we obtain \( (\mu_0 \geq 0) \)

\[ V_I < KL^t p_0^{\sigma/(1 - \mu_0)/Y}, \]

\[ V_L > KL^t p_0^{\sigma/(1 - \mu_0)/Y} > KL^t a^{\sigma/X}, \]

completing the proof of Lemma 4.

7. Proof of Theorem II. By (1), \( \lambda = \sigma/\min(s, t) + r \), for some \( r > 0 \). We 
shall construct by induction a sequence of closed sets \( F_0 \supset F_1 \supset \cdots \) and 
a sequence of additive functions \( \phi_n \) on \( \mathbb{R} \) such that the set 
\( F = \bigcap_{n=1}^{\infty} F_n \subset E \), 
and the function \( \phi = \lim_{n \to \infty} \phi_n \) satisfy the hypothesis of Lemma 2 with \( \rho = (\sigma - \delta)/\lambda \). Let \( F_0 = \emptyset, \) the set whose unique element is \( F_0 \). Let \( A_0 > \)
\((L_0/D)^{-1/\lambda} \) be sufficiently large. For every \( l \in \mathbb{R} \) and \( I \subset W \) we define \( \phi_0(l) = V(l)/L_0^s \), where \( V(l) \) denotes the \( s \)-dimensional volume of \( l \).

Suppose that for \( k = 0, \ldots, n - 1 \), a suitable increasing sequence of 
positive numbers \( A_k \) and sets \( G_k \) of disjoint closed cubes all with edge \( L_k = 2D(2A_k)^{-\lambda} \) have already been defined such that every element of \( G_k \) is con-
tained in some element of \( G_{k-1} \). Let \( F_k \) be the union of all elements of \( G_k \).
Suppose also that a sequence $\phi_k$ of additive functions on $\mathbb{R}$ has already been defined for all $k < n$.

Let $I \in G_{n-1}$, $I'$ the cube concentric with $I$ with edge $L_{n-1}/2$. We apply Lemma 4 with $\delta$, $\eta$ satisfying $0 < \theta < \min(\delta, \beta)$, $0 < \eta < \delta/(\sigma - \delta)$, where $0 < \delta < \sigma$; $L = L_{n-1}/2$, $A_{n-1}$ as $q_0$ and $I'$ as $I$. There exist reals $a_n$, $A_n$ and a subset $S_{I'} \subset S^T$ of $(s + 1)$-tuples of integers $(p_0, \ldots, p_s)$ satisfying
\[
(p_1/p_0, \ldots, p_s/p_0) \in I', \quad A_n - 1 < a_n < A_n < a_n^{1+\eta},
\]
and (3). Let $G_n$ be the set of all closed cubes with centers $(p_1/p_0, \ldots, p_s/p_0) \in I'$ and length of edge $2D(2A_n)^{-\lambda}$ where $I$ ranges over all cubes of $G_{n-1}$.

Note that each $I'$ has its own unique $p_0$, which induces a number of $p_0$ as specified by (8) (if $\mu_0 > 0$), but by Lemma 3 all of these $p_0$ satisfy the inequalities of (i) of Lemma 4 for the same $a = a, A = A$.

By (3), all cubes in $G_n$ are disjoint if $A_n$ is sufficiently large, as we shall assume. Let $F_n$ be the union of all cubes in $G_n$. Then $F_n$ is closed and $F_n \subset F_{n-1}$. If $I \in G_n$, then $I \subset J \in G_{n-1}$. Letting $N_J$ be the number of elements of $G_n$ contained in $J$, we define $\phi_n(I) = \phi_{n-1}(J)/N_J$. If $I \in \mathbb{R}$ and $I \subset J \in G_n$, let $\phi_n(I) = \phi_n(J) - V(J)/V(J)$. If $I \subset W$ is an arbitrary element of $\mathcal{R}$, then $I = \bigcup_{i \in H} I \cup Q$, where $I_i = I \cap J_i, J_i \in G_n, Q \cap F_n = \emptyset$. In this case we define $\phi_n(I) = \sum_i \phi_n(I_i)$. The following properties of the functions $\phi_n$ are obvious: They are nonnegative finite additive functions on $\mathbb{R}$, and for $I \in G_{n-1}$, $\phi_n(I) = \phi_{n-1}(I)$. If $I \in \mathbb{R}$, $I \subset F_n$, then $\phi_n(I) = 1$. Let $\delta_i, i = 0, 1, 2, \ldots, \beta_i$ be positive reals such that the product $\Pi_{i=0}^\infty (1 + \delta_i)$ converges and $\delta_0, \delta_1$ sufficiently large. Let $k_n = \Pi_{i=0}^\infty (1 + \delta_i)$. We shall prove by induction on $n$ that the sequence $A_i$ can be chosen such that for every cube $I \subset W$,
\[
(15) \quad \phi_n(I)/\delta(I) < k_n.
\]

For $n = 0$,
\[
\frac{\phi_0(\delta)^n}{\delta(I)^n} = \frac{V(l)}{L_0^s \delta(I)^n} = s^{-s/2}L_0^{-s}a^{-\sigma} \leq KL_0^{-\rho} < 1 + \delta_0.
\]

Let $\Delta_n = \max_{I \in G_n} \phi_n(I)$. By (iv) of Lemma 4,
\[
\Delta_n < KL_{n-1}^{-L_{n-1}} \Delta_{n-1} X_n a^{-\sigma}, \quad X_n = \begin{cases} 
\log a_n & \text{if } \mu_0 > 0, \\
1 & \text{if } \mu_0 = 0.
\end{cases}
\]

For proving (15) we distinguish several cases.

(a) $I \in G_n$. Then
\[
\frac{\phi_n(l)}{\delta(l)^\rho} < \frac{\Delta_n}{L_n^\rho} < KL_{n-1}^{-\alpha} \Delta_{n-1}^{-1} X_n^{-\sigma} A_n^{\lambda \rho} < KL_{n-1}^{-\alpha} \Delta_{n-1}^{-1} X_n^{-\sigma(1+\eta)(\sigma-\delta)},
\]

The exponent of \( a_n \) is negative. For \( a_n \) large enough, \( \phi_n(l)/\delta(l)^\rho \) can thus be made as small as desired.

(b) \( I \subseteq J \subseteq G_n \). Then

\[
\frac{\phi_n(l)}{\delta(l)^\rho} = \frac{\phi_n(j)}{\delta(i)^\rho} \frac{V(l)}{V(j)} = \frac{\phi_n(j)}{\delta(j)^\rho} \left( \frac{\delta(i)}{\delta(j)} \right)^{s-\rho} \leq \frac{\phi_n(j)}{\delta(j)^\rho},
\]

which is reduced to the previous case.

(c) \( I \subseteq J \subseteq G_{n-1} \) and the length \( l \) of the edge of \( I \) is greater than \( a_n^{-\mu} \). Let \( N_I \) and \( N_J \) denote the number of elements of \( G_n \) with nonempty intersection with \( I \) and \( J \) respectively. By (iii) and (iv) of Lemma 4,

\[
\frac{\phi_n(l)}{\delta(l)^\rho} \leq \frac{\phi_n(I)}{N_J} \cdot \frac{V(I)}{V(J)} \leq K \frac{\phi_n(I)}{\delta(I)^\rho} \frac{L_I^\rho}{L_J^\rho} \leq K \frac{\phi_n(I)}{\delta(I)^\rho},
\]

since inequality (1) on \( \lambda \) implies \( t'-\rho > 0 \). For \( n > 1 \), the last expression can be made as small as desired if \( a_{n-1}^{-\mu} \) is large enough, as was shown in case (a). For \( n = 1 \),

\[
\frac{\phi_1(l)}{\delta(l)^\rho} < K \frac{\phi_0(I)}{\delta(I)^\rho} < K \frac{1}{L_0^\rho} \leq 1 + \delta_1,
\]

if \( \delta_1 \) is sufficiently large.

(d) \( I \subseteq J \subseteq G_{n-1} \) but the edge \( l \) of \( I \) is not greater than \( a_n^{-\mu} \). The cubes concentric to the cubes of \( G_n \) and with edge of length \( A_n^{-\sigma/s-\theta} \) are disjoint by (3), so the number \( N_I \) of cubes of \( G_n \) with nonempty intersection with \( I \) is at most \( N_I \leq K \delta(I)^s A_n^{\sigma-\theta}s \). Therefore,

\[
\frac{\phi_n(l)}{\delta(l)^\rho} \leq \frac{N_I \Delta_n}{\delta(l)^\rho} \leq K \Delta_{n-1}^{-\alpha} L_{n-1}^{-\alpha} a_n^{-\mu(s-\rho)+(1+\eta)(\sigma+\theta)s} A_n^{\sigma(1+\eta)(\sigma-\delta)}.
\]

For \( \theta, \eta \) small enough and \( a_n \) large enough, this can be made as small as desired.
(e) \( I \) is an arbitrary cube of edge length \( l \). We may assume \( n > 1 \), as the case \( n = 1 \) is settled by the previous cases. We may also assume \( l > \frac{1}{2} A_{n-1}^{-\left(\frac{\sigma}{s}\right)-\theta} \), since otherwise, for \( A_{n-1} \) large enough, \( I \) intersects at most one element of \( G_{n-1} \), which is also subsumed by the previous cases. Let \( J \) be a cube with the same center as \( I \) and edge length \( l + 4A_{n-1}^- \). For \( A_{n-1} \) large enough we have

\[
(\delta(J)/\delta(I))^{\rho} < 1 + \delta_n,
\]

\[
\frac{\phi_n(I)}{\delta(I)^{\rho}} \leq \frac{\phi_{n-1}(I)}{\delta(I)^{\rho}} = \frac{\phi_{n-1}(J)}{(\delta(J)/\delta(I))^{\rho}} < (1 + \delta_n)k_{n-1} = k_n,
\]

which proves (15).

Now let \( \epsilon_i, i \geq 2 \), be any sequence of positive integers such that \( \sum_{i=2}^{\infty} \epsilon_i \) converges. For every cube \( I \in \mathbb{R} \), we have

\[
\phi_n(I) = \phi_0(I) + (\phi_1(I) - \phi_0(I)) + \cdots + (\phi_n(I) - \phi_{n-1}(I)).
\]

The difference \( \phi_k(I) - \phi_{k-1}(I) \) is contributed by those elements of \( G_{k-1} \) which intersect the boundary of \( I \). Let \( N_k \) be the number of those elements of \( G_{k-1} \). The cubes concentric to the elements of \( G_{k-1} \) and whose length of edge is \( \frac{1}{2} A_{k-1}^{-\left(\frac{\sigma}{s}\right)-\theta} \) are disjoint. Therefore,

\[
|\phi_k(I) - \phi_{k-1}(I)| \leq N_k^{\frac{1}{2}} A_{k-1}^{-\left(\frac{\sigma}{s}\right)-\theta}.
\]

If the max in (16) is 1, then for \( a_{k-1} \) large enough \( |\phi_k(I) - \phi_{k-1}(I)| < \epsilon_k \).

Otherwise,

\[
|\phi_k(I) - \phi_{k-1}(I)| \leq K\delta(I)^{s-1}L_{k-2}^{-\left(\frac{\sigma}{s}\right)-\theta}(s-1)^{-\sigma(1+\eta)}.
\]

For \( \theta \) small and \( A_{k-1} \) large enough, this is smaller than \( \epsilon_k \). This proves that the functions \( \phi_n \) converge on each cube \( I \in \mathbb{R} \). Since the functions \( \phi_n \) are additive, they converge also for every \( I \in \mathbb{R} \). The limit function \( \phi \) is non-negative, finite and additive. If \( I \in \mathbb{R}, I \supset F \), there exists \( n \) such that \( I \supset F \) and so \( \phi(I) = \phi_n(I) = 1 \). For every cube \( I \subset W \) there exists \( n \) such that

\[
|\phi_n(I) - \phi(I)| < \delta(I)^{\rho}, \quad \frac{\phi(I)}{\delta(I)^{\rho}} < \frac{\phi_n(I) + \delta(I)^{\rho}}{\delta(I)^{\rho}} < k_n + 1 < k.
\]

So \( \phi, F, \rho \) satisfy the conditions of Lemma 2, and we have \( \rho - m^*E_T > 0 \).

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