FIXED POINTS IN REPRESENTATIONS OF CATEGORIES

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ABSTRACT. Fixed points of endomorphisms of representations, i.e. functors into the category of sets, are investigated. A necessary and sufficient condition on a category $K$ is given for each of its indecomposable representations to have the fixed point property. The condition appears to be the same as that found by Isbell and Mitchell for Colim: $\text{Ab}^K \rightarrow \text{Ab}$ to be exact. A well-known theorem on mappings of Katětov and Kenyon is extended to transformations of functors.

Introduction. A representation of a category $K$, i.e. a covariant functor $F$ from $K$ to the category of sets, is said to have the fixed point property if each transformation $\tau: F \rightarrow F$ has a fixed point, i.e. $x \in FM$ with $\tau^M(x) = x$. The aim of the current paper is to express the fixed point property by algebraic means. Clearly, all representations of a category never have the fixed point property: if $F$ is any representation then the transformation of the sum $F \vee F$ which interchanges copies of $F$ has no fixed points. Therefore it seems natural to try to characterize those categories whose all indecomposable representations have the fixed point property; a representation $F$ is indecomposable if it is nontrivial, i.e. distinct from the constant functor to $\emptyset$, and cannot be expressed as $F = F_1 \vee F_2$ with $F_i$ nontrivial. We call these categories Brouwerian.

It is our pleasant duty to express our gratitude to Věra Trnková for the attention paid to our work, which she has also initiated by putting the problem of generalization of the theorem on mappings by Katětov and Kenyon (see below).

I. Properties of Brouwerian categories.

Theorem 1.1. Each indecomposable representation of a Brouwerian category is hereditarily indecomposable, i.e. each of its subfunctors is indecomposable. Equivalently, each two nontrivial subfunctors have a nontrivial intersection.

Proof. Assume the contrary. Then we have indecomposable representations $H, H_1, H_2$ with $H_1 \leq H, H_2 \leq H$ and $H_1 \cap H_2 = \emptyset$. The colimit of...
the diagram

\[
\begin{array}{ccc}
H_1 & \rightarrow & H \\
\downarrow & & \downarrow \\
H & \rightarrow & H \\
\downarrow & & \downarrow \\
H_2 & \rightarrow & H \\
\end{array}
\]

(*)

is a functor \( F \) which is indecomposable. Indeed, a representation is indecomposable iff its colimit is a singleton set \( 1 \). Now, denote by \( D : \mathcal{D} \rightarrow \text{Set}^\mathcal{K} \) the diagram (\( * \)) and define \( J : \mathcal{D} \times \mathcal{K} \rightarrow \text{Set} \) by \( J(d, X) = (Dd)X \). Then \( F = \text{Colim}_d \text{Id} \) and so \( \text{Colim} F = \text{Colim}_d \text{Colim}_X J(d, X) = \text{Colim}_d \text{Colim}_X (Dd)X \) and since \( H, H_1, H_2 \) are indecomposable, we have \( \text{Colim}_d \text{Colim}_X (Dd)X = \text{Colim}_d 1 = 1 \).

Because of the symmetry of (\( * \)), \( \{i_{n+4}\}_{n=0}^7 \) is also its direct bound (+ is the addition mod 8). Thus, there is a unique transformation \( \tau : F \rightarrow F \) with \( \tau \cdot i_n = i_{n+4} \). The transformation has no fixed points. Indeed, \( F \) consists of four copies of \( H \) glued as follows: the first one with the second one in \( H_1 \), the second one with the third one in \( H_2 \), the third one with the fourth one in \( H_1 \) and the fourth one with the first one in \( H_2 \). In particular, the \( n \)th copy of \( H \) is disjoint with the \( (n + 2) \)nd (+ is the addition mod 4). But \( \tau \) sends the \( n \)th copy isomorphically just onto the \( (n + 2) \)nd so that \( \tau \) cannot have fixed points. This concludes the proof.

The condition of the above theorem is not sufficient for a category to be Brouwerian, as will be seen from the characterization of Brouwerian categories and

**Proposition 1.2.** Given a category \( K \), the following conditions are equivalent.

(i) Each indecomposable representation of \( K \) is hereditarily indecomposable.

(ii) For each indecomposable representation \( F \) of \( K \) and each \( x \in FM, y \in FN \) there exist morphisms \( f: M \rightarrow X, g: N \rightarrow X \) with \( fx = gy \).
(iii) Each diagram (a) in $K$ can be embedded into a commutative one (b):

![Diagram](image)

**Proof.** (i) $\implies$ (iii). Let $f_1: M \to N_1, f_2: M \to N_2$ be given. Let $H_1 \subseteq \text{Hom}(M, -), H_1 X = \{h f_1; h: N_1 \to X\}$, analogously $H_2$. As $\text{Hom}(M, -)$ is indecomposable and $H_1, H_2$ are nontrivial, they are not disjoint and (iii) follows.

(iii) $\implies$ (ii). Fix $x \in FM$ and for each object $A$ denote $G_A = \{t \in FA; \text{ there exist } f: M \to X, g: A \to X \text{ with } f x = g t\}$ and $G_2 A = FA - G_1 A$. Then $f x = g t$ for some $g, f$; due to (iii), there exist $g', h'$ with $g' g = h' h$. Then $(g' t)x = (h' h)t$. Thus $ht \in G_1 B$. Therefore $G_1$ is a subfunctor of $F$. Clearly, $G_2$ is a subfunctor of $F$, too: let $h: A \to B, t \in G_2 A$. If $ht \in G_1 B$ then $f x = g t h$ for some $f, g$ and so $t \in G_1 A$ which is a contradiction. Therefore, $ht \in G_2 B$.

As $F$ is indecomposable and $G_1$ is nontrivial ($x \in G_1 M), F = G_1$.

(ii) $\implies$ (i). Let $F$ be an indecomposable representation of $K$, $G_1$ and $G_2$ its disjoint subfunctors. Then $G_1$ or $G_2$ is trivial because otherwise there are $x \in G_1 M, y \in G_2 N$ and, given $f: M \to X, g: N \to X$ with $f x = g y = z$ we have $z \in G_1 X \cap G_2 X$. This concludes the proof. 

The following theorem generalizes a theorem on mappings proved by Katětov [5] and Kenyon [6]: For each set $X$ and each mapping $f: X \to X$ without fixed points there exists a decomposition of $X$ into disjoint subsets $X_1, X_2, X_3$ such that $f(X_i) \cap X_i = \emptyset, i = 1, 2, 3$. Our paper was, in fact, initiated by Věra Trnková, who suggested that the above theorem should be generalized for functor-categories.

**Theorem 1.3.** A category is Brouwerian iff for each of its representations $F$ and each transformation $\tau: F \to F$ without fixed points there exists a decomposition $F = F_1 \lor F_2 \lor F_3$ such that $\tau F_i \cap F_i$ is trivial, $i = 1, 2, 3$.

**Proof.** Let $K$ be Brouwerian, $F: K \to \text{Set}$. Each element $x \in FM$ generates an indecomposable subfunctor $H_x \subseteq F, H_x N = \{y; y = f x \text{ for some } f: M \to N\}$. Therefore, $F$ is a disjoint union of its maximal indecomposable subfunctors, $F = \bigvee F_j$. If $\tau: F \to F$ then clearly for each $j \in J$ there is $t_j \in F_j$ with $\tau F_j \cap F_j t_j$. Apply the above theorem on the mapping $i \to t_i$: As $K$ is Brouwerian, if $\tau$ has no fixed points, clearly $i \neq t_i$ for all $i$, and so the
set $J$ can be decomposed into subsets $J_1$, $J_2$, $J_3$ as above. Put $F_1 = \bigvee_{j \in J_1} F_j$, analogously $F_2$, $F_3$.

Let $K$ not be Brouwerian. Let $F$ be an indecomposable representation of $K$ and let $\tau: F \to F$ have no fixed points. $F$ cannot be decomposed into $F_1$, $F_2$, $F_3$ with $\tau(F_i) \cap F_i$ trivial simply because two of the three functors would be trivial, as $F$ is indecomposable.

Theorem 1.4. Let $F$ be an indecomposable representation of a Brouwerian category. Then every collection $\tau_1, \ldots, \tau_n$ of endomorphisms of $F$ has a common fixed point, i.e. there exists $M$ and $x \in FM$ with $\tau_i^M(x) = x$, $i = 1, \ldots, n$.

Proof. For each $i = 1, \ldots, n$, let $F_i$ be the subfunctor of $F$ such that for every object $M$, $F_i M$ is just the set of all fixed points of $\tau_i^M$. By Theorem 1.1, two, and hence finitely many, nontrivial subfunctors of $F$ have always a nontrivial intersection. Thus $\bigcap F_i$ is nontrivial so that $\tau_1, \ldots, \tau_n$ have a common fixed point.

The preceding theorem can be regarded as a fixed point theorem for a multiple of transformations. The following theorem is a formulation of the fixed point property for a multivalued transformation.

Given a representation $F$ of a category $K$, by a multitransformation $\tau: F \to F$ we shall mean a partial nonvoid multivalued transformation, that is, a family $\{\tau^M\}$ such that

1. $M$ runs over all $K$-objects,
2. each $\tau^M$ is a partial multivalued mapping of $FM$ to $FM$, i.e., simply $\tau^M \subset FM \times FM$,
3. if $(x, y) \in \tau^M$ and $f: M \to N$ then $(fx, fy) \in \tau^N$,
4. some $\tau^M$ is nonvoid.

Note that $F$, equipped with a multitransformation, can be regarded as a functor into the category of graphs and compatible maps. In fact, $\tau^M$ is a graph on $FM$, and condition (3) ensures the compatibility of maps $Ff: FM \to FN$.

We shall need some graph-theoretical notions. Let $(X, R)$ be a graph. A sequence $a_0, a_1, \ldots, a_n$ of its vertices is called a chain from $a_0$ to $a_n$ with length $n$ if for each $i = 1, \ldots, n$ either $(a_{i-1}, a_i) \in R$ or $(a_i, a_{i-1}) \in R$. In the former case put $m_i = 1$, in the latter put $m_i = -1$; then $m = \sum_{i=1}^n m_i$ is called the characteristic of the chain. If $a_0 = a_n$, the chain is called a cycle. A chain with pairwise distinct vertices is regular if $m = n$, i.e., roughly speaking, if the direction of arrows $(a_{i-1}, a_i)$ is either always the same as in $R$ or the opposite one. Analogously regular cycle. Given
an infinite sequence $a_i$ of vertices, it is called a chain (a regular chain) if each subsequence $a_0, \ldots, a_n$ is a chain (regular chain).

Theorem 1.5. A category is Brouwerian iff every multitransformation of each of its indecomposable representations has a cycle with characteristic 1.

Proof. If a category fulfills the above condition then it is Brouwerian: if $\tau$ is a transformation then all its cycles with pairwise distinct vertices are regular and thus a cycle with characteristic 1 is just $x, x$ with $x$ a fixed point of $\tau$.

Let $K$ be a Brouwerian category, $\tau: F \to F$ a multitransformation of an indecomposable representation $F$ of $K$. We shall show that $\tau$ has a cycle with characteristic 1. Define a congruence $\sim$ on $F$ as follows: if $x, y \in FM$ then $x \sim_M y$ iff there is a chain with characteristic 0 from $x$ to $y$ in $F^M$. Clearly for $f: M \to N$ in $K$, $x \sim_M y$ implies $fx \sim_N fy$ and so we may define a factor functor of $F$ under $\sim$, $\sigma = F/\sim$. As a factor functor of an indecomposable functor, $\sigma$ is also indecomposable. For each $M$, let $\sigma^M$ be the quotient graph of $\sigma^M$ with respect to $\sim_M: \sigma^M = \{[x], [y]; (x, y) \in \sigma^M\}$ where $[\ ]$ denotes the congruence classes. Then $\sigma = [\sigma^M]$ is a multitransformation of $\sigma$. Moreover, $\sigma$ is single-valued and one-to-one, that is:

(a) for each $t \in \sigma^M$, $(t, u) \in \sigma^M$ for at most one $u \in GM$,
(b) for each $u \in \sigma^M$, $(t, u) \in \sigma^M$ for at most one $t \in \sigma^M$.

To prove (a), assume $(t, u_i) \in \sigma^M$ for $i = 1, 2$. Then we have some $(x_i, y_i) \in \sigma^M$ where $[x_i] = t$, $[y_i] = u_i$, $i = 1, 2$. As $[x_1] = [x_2]$, there is a chain $x_1 = a_0, a_1, \ldots, a_n = x_2$ with characteristic 0 in $\sigma^M$. Then $y_1$, $a_0, \ldots, a_n, y_2$ is a chain with characteristic $-1 + 0 + 1 = 0$ from $y_1$ to $y_2$, hence $[y_1] = [y_2]$, i.e. $u_1 = u_2$. (b) is analogous.

So we have proved that each component of $(\sigma^M)$ is either a regular cycle or a regular chain. There are two possibilities:

I. Some $(\sigma^M)$ contains a regular cycle, $a_0, \ldots, a_n, a_0$. Let $H$ be the subfunctor of $G$ generated by the set $\{a_0, \ldots, a_n\}$, let $\rho$ be the restriction of $\sigma$ to $H$. Then $\rho$ is a transformation with $\rho^{n+1} = 1$. By Theorem 1.1, $H$ is indecomposable, so that $\rho$ has a fixed point, i.e. there is $M$ and $t \in \sigma^M$ with $(t, t) \in \rho^M$. We have a chain $b_0, \ldots, b_m$ with $[b_0] = [b_m] = t$ in $\sigma^M$, $(b_m, b_0) \in \sigma^M$. Then $b_0, \ldots, b_m, b_0$ is a cycle with characteristic 1. That concludes the proof.

II. No $(\sigma^M)$ contains a cycle. Then components of any $(\sigma^M)$ are regular chains. Moreover, there is an object $Z$ such that $(\sigma^M)$ contains a regular chain of length $\geq 2$ (e.g., consider a chain $f_1x, f_1y, f_2y$ where $(x, y) \in \sigma^M$ is arbitrary, $f_1: M \to Z$, $f_2: M \to Z$ are chosen so...
that \( f_1x = f_2y \); see 1.2). Let \( H \) be a subfunctor of \( G \) generated by a regular chain of length \( \geq 2 \). Let \( \cong \) be a congruence on \( H, x \cong y \) iff there is a chain of an even length from \( x \) to \( y \) in \( \sigma^M \). Put \( H' = H/\cong \).

For each \( M \), let \( \sigma^M_1 \) be the quotient graph of \( \sigma^M \) with respect to \( \cong_M \). Then components of \( \sigma^M_1 \) are regular cycles of length 2 so that \( \sigma^M_1 \) is a transformation without fixed points. This is a contradiction because \( H' \) is clearly indecomposable; thus case II cannot occur. \( \square \)

II. Characterization. Recall that a category \( K \) is filtered [7] if

(1) for every pair \( M, N \) of objects there is an object \( Z \) with \( \text{Hom}(M, Z) \neq \emptyset \neq \text{Hom}(N, Z) \),

(2) for every pair \( f_1, f_2 : M \to N \) of morphisms there is \( h : N \to Z \) with \( hf_1 = hf_2 \).

We shall say that a category \( K \) is quasifiltered if it satisfies (1) above and

(2') for every pair \( f_1, f_2 : M \to N \) of morphisms there are morphisms \( h_0, h_1, \ldots, h_n : N \to Z \) such that

\[
\begin{align*}
    h_0f_1 & = h_1f_1, \\
    h_1f_2 & = h_2f_2, \\
    & \vdots \\
    h_{n-1}f_n & = h_nf_n, \\
    h_nf_{n+1} & = h_0f_{n+1},
\end{align*}
\]

where \( i, j \) are 1 or 2 and \( \sum_{i=1}^{n+1} (i - j) = 1 \).

Let us observe that a category has filtered (quasifiltered) components iff it fulfills (2) ((2')) and (iii) of 1.2. A filtered category is quasifiltered, of course. The converse fails to be true.

Example (Isbell and Mitchell [4]). The category of finite ordinals and order preserving injections is quasifiltered but not filtered.

Theorem 2.1 (The Characterization Theorem). A category is Brouwerian iff it has quasifiltered components.

Proof. Sufficiency. Let \( K \) have quasifiltered components.

Consider a transformation \( \tau : F \to F \) where \( F \) is an indecomposable representation of \( K \). We shall prove that \( \tau \) has a fixed point.

Choose an arbitrary object \( M \) with \( FM \neq \emptyset \) and choose \( a \in FM \). Applying Proposition 1.2 (ii) on \( a, \tau^M(a) \) we get \( K \)-morphisms \( f_1, f_2 : M \to N \)
with $f_1a = f_2 r^M(a)$. There exist $K$-morphisms $h_0, h_1, \cdots, h_n, h_{n+1} = h_0$; $N \rightarrow Z$ with $h_{i-1}/i = h_i/i_i$ where $|\Sigma(i_i - j_i)| = 1$. Denote

$$x_k = (r^M)k(a); \quad y_k = (h_0/1)x_k \quad \text{and} \quad p_k = \sum_{i=1}^{k} (i_i - j_i)$$

(for $k \leq n + 1$). The proof will be concluded when we show that for all $k \leq n + 1$,

$$x_k = (r^M)k(a); \quad y_k = (h_0/1)x_k \quad \text{and} \quad p_k = \sum_{i=1}^{k} (i_i - j_i)$$

Indeed, then $h_0 = h_{n+1}$ yields $y_n = y_{n+p}$ and since $p_{n+1} = \pm 1$, clearly $y_n$ is a fixed point of $r$ (recall that $r^Z(y_{n-1}) = y_n$ and $r^Z(y_n) = y_{n+1}$).

The proposition $(\dagger)$ holds for $k = 0$; let us prove it for $k + 1$ assuming that it holds for $k \leq n$. We have $h_{k}/i_{k+1} = h_{k+1}/i_{k+1}$ which we apply to $x_n$ if $j_{k+1} = 1$, or to $x_{n+1}$ if $j_{k+1} = 2$. Then the proof is very easy when we take into consideration that $p_{k+1} = p_k + i_{k+1} - j_{k+1}$ and that $r^Z(y_{n-1-p_k}) = y_{n-p_k}$ while $r^Z(y_{n+p_k}) = y_{n+1-p_k}$.

**Necessity.** If $K$ is a Brouwerian category then it satisfies (iii) of 1.2, so that to show that $K$ has quasifiltered components it suffices to prove $(2 \dagger)$. Let $f_1, f_2: M \rightarrow N$ be given. Define a multitransformation $r: \text{Hom}(M, -) \rightarrow \text{Hom}(M, -)$ by $rX = \{h_{f_1}, h_{f_2}, h: N \rightarrow X\}$. By 1.5 there is a cycle $a_0, a_1, \cdots, a_n, a_0$ with characteristic 1 in some $\text{Hom}(M, Z)$. For every $t = 1, \cdots, n$ either $(a_{t-1}, a_t) \in rZ$ or $(a_t, a_{t-1}) \in rZ$. Thus there are morphisms $h_0, \cdots, h_n: N \rightarrow Z$ and numbers $i_1, j_1, i_2, j_2, \cdots, i_{n+1}, j_{n+1} = 1, 2$ such that

$$a_0, a_1 = (h_0/i_{n+1}, h_0/i_{1}),$$
$$a_1, a_2 = (h_1/i_{1}, h_1/i_{2}),$$
$$a_{n-1}, a_n = (h_{n-1}/i_{n-1}, h_{n-1}/i_{n}),$$
$$a_n, a_0 = (h_{n}/i_{n}, h_{n}/i_{n+1}).$$

Now $(2 \dagger)$ follows immediately. The characteristic of the cycle $a_0, \cdots, a_n$, $a_0$ is

$$1 = |(i_1 - j_{n+1}) + (i_2 - j_1) + (i_2 - j_2) + \cdots + (i_n - j_{n-1}) + (i_{n+1} - j_n)|$$

$$= \left| \sum_{t=1}^{n+1} (i_t - j_t) \right|.$$
Brouwerian turns out to be the same as that for Colim: $\text{Ab}^K \to \text{Ab}$ to be exact (Isbell and Mitchell [4]); $\text{Ab}$ is the category of Abelian groups. In fact, Colim: $\text{Ab}^K \to \text{Ab}$ is exact iff the category $\text{aff} K$ has filtered components [4]. Here $\text{aff} K$ denotes the category with the same objects as $K$ such that morphisms from $M$ to $N$ in $\text{aff} K$ are those elements $\Sigma a_i f_i$ of the free Abelian group over $\text{Hom}_K (M, N)$ with $\Sigma a_i = 1$. The composition in $\text{aff} K$ is defined by $(\Sigma a_i f_i)(\Sigma \beta_j g_j) = \Sigma (a_i \beta_j) f_i g_j$. It follows easily by [4, Lemma 1] that $\text{aff} K$ has filtered components iff $K$ has quasifiltered ones. This enables us to formulate a proposition of [4] in terms of Brouwerian categories.

**Theorem 2.2.** Let each component $C$ of a category $K$ possess a weak terminal object, i.e. an object $T$ such that $\text{Hom}(M, T) \neq \emptyset$ for $M \in C$. Then $K$ is Brouwerian iff it has filtered components.

**Corollary 2.3** [3]. A monoid is Brouwerian iff it is filtered.

Let us note that 2.3 follows from [1] too.

It follows immediately from the characterization theorem that a preordered class is a Brouwerian category iff it has directed components. More generally,

**Proposition 2.4.** Let $K$ be a category such that for each object $M$ there is a natural number $n(M)$ which is bigger or equal to the number of morphisms from $M$ to any given object. Then $K$ is Brouwerian iff it has filtered components.

**Proof.** Let $K$ be a category which is Brouwerian and fulfills the above condition. Let $f, g: M \to N$. We have to find $k$ with $kf = kg$. As $K$ fulfills 1.2 (iii) the following defines a congruence $\sim$ on $\text{Hom}(M, -)$: if $p, q \in \text{Hom}(M, X)$ then $p \sim_X q$ iff $kp = kq$ for some $k$. Denote $F = \text{Hom}(M, -)/\sim$. Clearly, for any morphism $f$, the mapping $Ff$ is one-to-one. Moreover, for any object $X$, $|FX| \leq |\text{Hom}(M, X)| \leq n(M)$. Thus, we can choose among all $X$ with $\text{Hom}(M, X) \neq \emptyset$ such an object $C$ that $|FC|$ is maximal. Let $h: N \to C$. Notice that for each morphism $r$ with domain $C$, $Fr$ is a bijection. Due to 1.2 (iii), there are $f', g': C \to D$ with $f' h f = g' h g$. We shall show that $Ff' = Fg'$. Then we get $Ff' ((h f)) = Fg' ((h f))$, i.e. $Ff' f = Fg' f$, and so there is $k'$ with $k' g' h f = k' g' h f = k' g' f h g$; put $k = k' g' h$; then $kf = kg$.

To prove $Ff' = Fg'$, put $f_1 = f'$ and $f_2 = g'$ and let $h_0, \ldots, h_n$ be from condition (2'), i.e. $F(h f_{i+1}) = F(h f_i f_{i+1})$ where $|\Sigma_{t=1}^{n+1} (i_t - j_t)| = 1$. Then for the morphism $m = Ff_2 (Ff_1)^{-1}$ we get:
$Fh_0 = Fh_{n+1} = Fh_n \cdot m^{(i_{n+1} - i_n + 1)} = Fh_{n-1} \cdot m^{(i_{n+1} - i_n + 1)} \cdot m^{(i_n - i_m)}$

$= \ldots = Fh_0 \cdot \sum (i_i - i_i) = Fh_0 m^{\pm 1}.$

Therefore $Fh_0 = Fh_0 \cdot m$ and since $Fh_0$ is one-to-one, clearly $m = id$, i.e. $F_{/1} = F_{/2}$.

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