MUTUAL EXISTENCE OF PRODUCT INTEGRALS
IN NORMED RINGS

BY

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ABSTRACT. Definitions and integrals are of the subdivision-refinement type, and functions are from \( R \times R \) to \( N \), where \( R \) denotes the set of real numbers and \( N \) denotes a ring which has a multiplicative identity element represented by 1 and a norm \(| \cdot |\) with respect to which \( N \) is complete and \(|1| = 1\). If \( G \) is a function from \( R \times R \) to \( N \), then \( G \in OM^* \) on \([a, b]\) only if (i) \( \prod_y^n (1 + G) \) exists for \( a < x < y < b \) and (ii) if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \([a, b]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \) and \( 0 < p < q < n \), then

\[
\prod_{x_p}^q (1 + G) - \prod_{i=p+1}^q (1 + G) < \epsilon;
\]

and \( G \in OM^0 \) on \([a, b]\) only if (i) \( \prod_x^n (1 + G) \) exists for \( a < x < y < b \) and (ii) the integral \( \int_a^b (1 + G - \prod(1 + G)) \) exists and is zero. Further, \( G \in OP^0 \) on \([a, b]\) only if there exist a subdivision \( D \) of \([a, b]\) and a number \( B \) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \) and \( 0 < p < q < n \), then \( |\prod_{i=p}^q (1 + G)| < B \).

If \( F \) and \( G \) are functions from \( R \times R \) to \( N \), \( F \in OP^0 \) on \([a, b]\), each of \( \lim_{x \to p} F(x, y) \) and \( \lim_{y \to p} F(x, y) \) exists and is zero for \( p \in [a, b] \), each of \( \lim_{x \to p} G(x, p) \), \( \lim_{y \to p} F(p, y) \), \( \lim_{x \to p} G(p, x) \) and \( \lim_{y \to p} G(x, p) \) exists for \( p \in [a, b] \), and \( G \) has bounded variation on \([a, b]\), then any two of the following statements imply the other:

1. \( F + G \in OM^* \) on \([a, b]\),
2. \( F \in OM^* \) on \([a, b]\), and
3. \( G \in OM^* \) on \([a, b]\).

In addition, with the same restrictions on \( F \) and \( G \), any two of the following statements imply the other:

1. \( F + G \in OM^0 \) on \([a, b]\),
2. \( F \in OM^0 \) on \([a, b]\), and
3. \( G \in OM^0 \) on \([a, b]\).


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All definitions are of the subdivision-refinement type, and functions are from \( R \times R \) to \( N \), where \( R \) denotes the set of real numbers and \( N \) denotes a ring which has a multiplicative identity element represented by 1 and a norm \(| \cdot |\) with respect to which \( N \) is complete and \(|1| = 1\). Functions are assumed to be defined only for elements \( \{x, y\} \) of \( R \times R \) such that \( x < y \).

If \( G \) is a function from \( R \times R \) to \( N \), then \( \int_a^b G \) exists only if there exists an element \( L \) of \( N \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \([a, b]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then \( |L - \sum_{i=1}^n G_i| < \epsilon \), where \( G_i = G(x_{i-1}, x_i) \). Similarly, \( \alpha \Pi^b(1 + G) \) exists only if there exists an element \( L \) of \( N \) such that, if \( \epsilon > 0 \), then there exists a subdivision \( D \) of \([a, b]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then \( |L - \sum_{i=1}^n (1 + G_i)| < \epsilon \).

The statements that \( G \) is bounded on \([a, b] \), \( G \in O^\circ \) on \([a, b]\) and \( G \in O^B \) on \([a, b]\) mean there exist a subdivision \( D \) of \([a, b]\) and a number \( B \) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D \), then

1. \( |G_i| < B \) for \( 1 \leq i \leq n \),
2. \( |\Pi_{p=q}^b (1 + G_i)| < B \) for \( 1 \leq p \leq q \leq n \), and
3. \( \sum_{i=1}^n |G_i| < B \),

respectively.

Let \( G(p, p^+), G(p^+, p), G(p^-, p) \) and \( G(p^-, p^-) \) represent

\[
\lim_{x \to p} G(x, y), \lim_{x \to p} G(x, y), \lim_{x \to -p} G(x, y) \text{ and } \lim_{x \to y} G(x, y),
\]

respectively. Now, \( G \in S^1 \) on \([a, b]\) only if \( G(p, p^+) \) exists and is zero for \( a \leq p < b \) and \( G(p^-, p^-) \) exists and is zero for \( a < p \leq b \); and \( G \in S^2 \) on \([a, b]\) only if \( G(p, p^+) \) exists for \( a \leq p < b \) and \( G(p^-, p) \) exists for \( a < p \leq b \). Further, \( G \in O^\circ \) on \([a, b]\) only if \( G(p, p^+) \) and \( G(p^+, p^-) \) exist for \( a \leq p < b \) and \( G(p^-, p) \) and \( G(p^-, p^-) \) exist for \( a < p \leq b \).

For additional background on product integration, the reader is referred to papers by P. R. Masani [10], J. S. MacNerney [9], B. W. Helton [2] and the author [7].

Suppose \( F \) and \( G \) are functions on \( R \times R \). If \( \int_a^b F \) exists and \( \int_a^b G \) exists, then it is easily shown that \( \int_a^b (F + G) \) exists. However, if \( x^\Pi^y (1 + F) \) and \( x^\Pi^y (1 + G) \) exist for \( a \leq x < y \leq b \), it does not necessarily follow that \( x^\Pi^y (1 + F + G) \) exists for \( a \leq x < y \leq b \). The purpose of this paper is to investigate the existence of such product integrals. In particular, with suitable restrictions on the functions involved, we interrelate the existence of \( x^\Pi^y (1 + F) \), \( x^\Pi^y (1 + G) \) and \( x^\Pi^y (1 + F + G) \). However, before stating our results, we need several additional definitions.

First, \( G \in O^A \) on \([a, b]\) only if \( \int_a^b G \) exists and \( \int_a^b |G - \int G| = 0 \). Second, \( G \in O^M \) on \([a, b]\) only if \( x^\Pi^y (1 + G) \) exists for \( a \leq x < y \leq b \) and \( \int_a^b |1 + G - \Pi (1 + G)| = 0 \). Third, \( G \in O^* \) on \([a, b]\) only if \( (1) x^\Pi^y (1 + G) \)
exists for \(a \leq x < y \leq b\), and (2) if \(\epsilon > 0\), then there exists a subdivision \(D\) of \([a, b]\) such that, if \(\{x_i\}_{i=0}^n\) is a refinement of \(D\) and \(0 \leq p < q \leq n\), then

\[
\left| x_p \prod_{i=p+1}^{x_q}(1 + G) - \prod_{i=p+1}^{q}(1 + G) \right| < \epsilon.
\]

We now state the main results of this paper.

Theorem 1. If \(F\) and \(G\) are functions from \(\mathbb{R} \times \mathbb{R}\) to \(N\), \(F\) is in \(\mathcal{O}P^0\) and \(S_1 \cap S_2\) on \([a, b]\) and \(G\) is in \(\mathcal{O}B^0\) and \(S_2\) on \([a, b]\), then any two of the following statements imply the other:

1. \(F + G \in \mathcal{O}M^*\) on \([a, b]\),
2. \(F \in \mathcal{O}M^*\) on \([a, b]\), and
3. \(G \in \mathcal{O}M^*\) on \([a, b]\).

Theorem 2. If \(F\) and \(G\) are functions from \(\mathbb{R} \times \mathbb{R}\) to \(N\), \(F\) is in \(\mathcal{O}P^0\) and \(S_1 \cap S_2\) on \([a, b]\) and \(G\) is in \(\mathcal{O}B^0\) and \(S_2\) on \([a, b]\), then any two of the following statements imply the other:

1. \(F + G \in \mathcal{O}M^0\) on \([a, b]\),
2. \(F \in \mathcal{O}M^0\) on \([a, b]\), and
3. \(G \in \mathcal{O}M^0\) on \([a, b]\).

Theorems 1 and 2 are not the same. A function can belong to \(\mathcal{O}M^*\) on \([a, b]\) without belonging to \(\mathcal{O}M^0\) on \([a, b]\). For example, if \(G \in \mathcal{O}B^0\) on \([a, b]\) and \(x \prod(1 + G)\) exists for \(a \leq x < y \leq b\), then \(G \in \mathcal{O}M^*\) on \([a, b]\) [7, Theorem 1]; but, it is possible to construct a function \(G\) such that \(G \in \mathcal{O}B^0\) on \([a, b]\), \(x \prod(1 + G)\) exists for \(a \leq x < y \leq b\) and \(G \notin \mathcal{O}M^0\) on \([a, b]\) [4, pp. 153–154]. However, if \(G\) is in \(\mathcal{O}M^0\) and \(\mathcal{O}P^0\) on \([a, b]\), then \(G \in \mathcal{O}M^*\) on \([a, b]\).

Theorem 2 is proved for functions from \(\mathbb{R} \times \mathbb{R}\) to \(\mathbb{R}\) in a previous paper by the author [6, Theorem 1, p. 101]. However, that proof relies heavily on the commutativity of \(\mathbb{R}\) and thus is not the same as the proof presented in this paper. In this presentation, the lack of commutativity is handled by using a series representation for products.

The classes \(\mathcal{O}M^*\) and \(\mathcal{O}M^0\) are not as restricted as may initially appear. As noted before, if \(G \in \mathcal{O}B^0\) on \([a, b]\) and \(x \prod(1 + G)\) exists for \(a \leq x < y \leq b\), then \(G \in \mathcal{O}M^*\) on \([a, b]\) [7, Theorem 1]. For another example, suppose

\[
F(x, y) = \begin{bmatrix} 0 & 0 \\ b(y) - h(x) & 0 \end{bmatrix}
\]

for \(a \leq x < y \leq b\), where \(h\) is a quasi-continuous function from \(\mathbb{R}\) to \(N\). Then, with a suitable norm, \(F\) is in \(\mathcal{O}P^0\), \(\mathcal{O}M^0\) and \(S_1 \cap S_2\) on \([a, b]\). Thus, \(F\)
satisfies the hypotheses of Theorems 1 and 2; however, it does not necessarily follow that \( F \in OB^o \) on \([a, b]\). With Theorems 1 and 2 and functions such as \( F \), it is possible to construct many functions in \( OM^* \) and \( OM^o \). A fundamental correspondence exists between sum and product integrals. In particular, if \( G \in OB^o \) on \([a, b]\), then \( \int_a^b G \) exists if and only if \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \) [7, Theorem 4], and \( G \in OA^o \) on \([a, b]\) if and only if \( G \in OM^o \) on \([a, b]\) [2, Theorem 3.4, p. 301]. If \( G \) is a function from \( R \times R \) to \( R \), then \( G \in OA^o \) on \([a, b]\) [8, p. 669]. Further, there exist other systems such that the existence of \( \int_a^b G \) is sufficient to imply that \( G \in OA^o \) on \([a, b]\) [1, Theorem 2, p. 155], [2, Theorem 4.1, p. 304]. Thus, with the preceding results, many functions in \( OM^* \) and \( OM^o \) can be obtained. In addition, if \( H \in OL^o \) on \([a, b]\) and \( G \in OB^o \) on \([a, b]\) and either \( \int_a^b G \) exists or \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \), then \( \int_a^b HG \) and \( \int_a^b GH \) exist and \( \prod_x^y (1 + HG) \) and \( \prod_x^y (1 + GH) \) exist for \( a \leq x < y \leq b \) [7, Theorem 5]. Therefore, there exist many functions to which the theorems of this paper apply.

We now establish Theorem 1. Several lemmas are needed.

**Lemma 1.** If \( H \) and \( G \) are functions from \( R \times R \) to \( N \), \( H \in OL^o \) on \([a, b]\), \( G \in OB^o \) on \([a, b]\) and either \( \int_a^b G \) exists or \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \), then \( \int_a^b HG \) and \( \int_a^b GH \) exist and \( \prod_x^y (1 + HG) \) and \( \prod_x^y (1 + GH) \) exist for \( a \leq x < y \leq b \) [7, Theorem 5].

**Lemma 2.** If \( f \) is a function from \( R \) to \( R \) such that \( (LR) \int_a^b (-df) f^{n-i} \) exists for \( i = 0, 1, \ldots, n \), then

\[
\sum_{i=0}^{n} (LR) \int_a^b (-df) f^{n-i} = f^{n+1}(a) - f^{n+1}(b).
\]

**Proof.** This result follows by applying the identity

\[
(r - s) \sum_{i=0}^{n} r^{n-i} s^i = r^{n+1} - s^{n+1}
\]

to the approximating sums of the integrals involved.

**Lemma 3.** If \( \{F_i\}_{i=m}^{n} \) and \( \{G_i\}_{i=m}^{n} \) are sequences of elements of \( N \), then

\[
\prod_{i=m}^{n} (1 + F_i + G_i) = \sum_{i=0}^{n+1-m} S_{imn},
\]

where

\[
S_{0pn} = \begin{cases} 
\prod_{j=p}^{n} (1 + F_j) & \text{if } 0 < p \leq n, \\
1 & \text{if } p > n,
\end{cases}
\]
and
\[
S_{ipn} = \begin{cases} 
\sum_{j=p}^{n} [\prod_{k=p}^{j-1} (1 + F_k)] G_j S_{i-1,i+1,n} & \text{if } 0 \leq p \leq n, \\
0 & \text{if } p > n 
\end{cases}
\]
for \( i = 1, 2, \ldots \).

Proof. This lemma can be established by induction.

Lemma 4. If \( F \) and \( G \) are functions from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \), \( F \in \mathcal{O}^0 \) on \( [a, b] \) and \( G \in \mathcal{O}^0 \) on \( [a, b] \), then there exist a subdivision \( D \) of \( [a, b] \), a number \( B \) and a positive nondecreasing function \( g \) defined on \( [a, b] \) such that, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \), \( j \) is a nonnegative integer and \( 0 < p < q \leq n \), then
\[
|S_{ipq}| \leq B^{j+1}[g(x_q) - g(x_{p-1})]/j!,
\]
where \( S_{ipq} \) is defined in Lemma 3.

Proof. Since \( F \in \mathcal{O}^0 \) on \( [a, b] \) and \( G \in \mathcal{O}^0 \) on \( [a, b] \), there exist a subdivision \( D \) of \( [a, b] \) and a number \( B \) such that, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \), then
\[
\begin{align*}
(1) \quad |\prod_{i=p}^{q} (1 + F_i)| &< B \quad \text{for } 0 \leq p < q \leq n, \\
(2) \quad \sum_{i=1}^{n} |G_i| &< B.
\end{align*}
\]
Let \( g \) be the function defined on \( [a, b] \) such that
\[
\begin{align*}
(1) \quad g(a) & = 1, \\
(2) \quad g(x) & = 1 + \max \{\sum_{j} |G_j| : j \text{ a refinement of } \{x\}_{i=0}^{p-1} \cup \{x\}, \; 0 < p \leq n \text{ and } x_{p-1} < x \leq x_p\}.
\end{align*}
\]
Thus, \( g \) is a positive nondecreasing function.

We use induction to establish the desired inequality. If \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and \( 0 < p \leq q \leq n \), then
\[
|S_{0pq}| = \left| \prod_{i=p}^{q} (1 + F_i) \right| \leq B.
\]
Thus, the inequality is true for \( j = 0 \).

Suppose the inequality holds for the nonnegative integer \( j \). That is, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and \( 0 < p \leq q \leq n \), then \((\dagger)\) holds.

We now establish that the inequality also holds for \( j + 1 \). Suppose \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D \) and \( 0 < p \leq q \leq n \). To simplify notation in the following manipulations, let
\[
f(v) = g(x_q) - g(v)
\]
for \( x_p \leq v \leq x_q \). Now,
\[ |S_{j+1, p, q}| = \left| \sum_{i=p}^{q} \prod_{k=p}^{i-1} (1 + F_k) G_i S_{j, i+1, q} \right| \]
\[ \leq B \sum_{i=p}^{q} |G_i| |S_{j, i+1, q}| \]
\[ \leq B \sum_{i=p}^{q} \{g(x_i) - g(x_{i-1})\} \{B^{i+1} [g(x_q) - g(x_i)] i/j!\} \]
\[ \leq B \left[ (R) \int_{x_{p-1}}^{x_q} dg (B^{i+1} [g(x_q) - g(x_i)] i/j!\right] \]
\[ = [B^{i+2}/j!] \left( (R) \int_{x_{p-1}}^{x_q} (-df)/j! \right) \]
\[ \leq [B^{i+2} / (j+1)!] \sum_{k=0}^{j} (LR) \int_{x_{p-1}}^{x_q} (-df)/j! - k/k! \]
\[ = [B^{i+2} / (j+1)!] [i+1^[q] - j+1^[q]] \quad \text{(Lemma 2)} \]

Thus, the inequality holds for \( j + 1 \). Hence, the inequality is valid for \( j = 0, 1, 2, \ldots \). Therefore, Lemma 4 is established.

**Lemma 5.** If \( F \) and \( G \) are functions from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \), \( F \in \mathcal{O}^\circ \) on \([a, b]\) and \( G \in \mathcal{O}^\circ \) on \([a, b]\), then \( F + G \in \mathcal{O}^\circ \) on \([a, b]\).

**Proof.** This lemma follows as a corollary to Lemmas 3 and 4.

Lemma 5 is established in a previous paper by the author for functions from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \) [5, Theorem 1 (1 \( -\to 2 \)), p. 378]. However, the proof presented there is different from the proof employed in this paper.

**Lemma 6.** If \( \{F_i\}_{i=m}^{n} \) and \( \{G_i\}_{i=m}^{n} \) are sequences of elements of \( \mathbb{N} \), then
\[ \prod_{i=m}^{n} (1 + F_i + G_i) = \prod_{i=m}^{n} (1 + F_i) \]
\[ + \sum_{i=m}^{n} \left[ \prod_{j=m}^{i-1} (1 + F_j) \right] G_i \left[ \prod_{j=i+1}^{n} (1 + F_j + G_j) \right]. \]

**Proof.** This lemma can be established by induction.

**Lemma 7.** If \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{N} \) and \( G \in \mathcal{O}^\circ \) on \([a, b]\), then the following statements are equivalent:
(1) \( \int_a^b G \) exists, and

(2) \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \).

Further, if \( G \in OB^o \) on \( [a, b] \) and (1) or (2) is true, then \( G \in OM^* \) on \( [a, b] \).

Proof. If \( G \in OB^o \) on \( [a, b] \) and \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \), then \( G \in OM^* \) on \( [a, b] \) [7, Theorem 1]. Also, if \( G \in OB^o \) on \( [a, b] \), then \( \int_a^b G \) exists if and only if \( \prod_x^y (1 + G) \) exists for \( a \leq x < y \leq b \) [7, Theorem 4]. Thus, the lemma follows.

We now establish Theorem 1.

Proof of Theorem 1 [(2), (3) \( \rightarrow \) (1)]. We initially establish that \( \sum_{i=0}^{\infty} P_i(x, y) \) converges uniformly and absolutely for \( a \leq x < y \leq b \), where

\[
P_0(x, y) = \prod_x^y (1 + F)
\]

and

\[
P_i(x, y) = (LR) \int_x^y \prod_x^u (1 + F) GP_{i-1}(u, y)
\]

for \( a \leq x < y \leq b \) and \( i = 1, 2, \ldots \). The existence of these integrals follows by applying Lemma 1.

From Lemma 4, there exist a subdivision \( D_1 \) of \( [a, b] \), a number \( B \) and a positive nondecreasing function \( g \) defined on \( [a, b] \) such that, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D_1 \), \( i \) is a nonnegative integer and \( 0 < p \leq q \leq n \), then

\[
|S_{ipq}| \leq B^{i+1} [g(x_q) - g(x_{p-1})]/i!
\]

and

\[
|S_{ipq}| \leq B^{i+1} [g(x_q) - g(x_{p-1})]/i!,
\]

where \( S_{ipq} \) is defined as in Lemma 3.

It follows from the result stated in the preceding paragraph that

\[
|P_i(x, y)| \leq B^{i+1} [g(y) - g(x)]/i!
\]

for \( a \leq x < y \leq b \) and \( i = 0, 1, 2, \ldots \). Therefore, \( \sum_{i=0}^{\infty} P_i \) converges uniformly and absolutely on \( [a, b] \).

Suppose \( a \leq x < y \leq b \). We now establish that \( \prod_x^y (1 + F + G) \) exists and is \( \sum_{i=0}^{\infty} P_i(x, y) \). Let \( \epsilon > 0 \).

There exists a positive integer \( N \) such that

\[
\sum_{i=N+1}^{\infty} B^{i+1} [g(b) - g(a)]/i! < \epsilon/3.
\]

Further, from the existence properties of the integrals involved, there exists a subdivision \( D_2 \) of \( [a, b] \) such that, if \( \{x_i\}_{i=0}^{n} \) is a refinement of \( D_2 \) and \( 0 < p < q \leq n \), then
Let $D$ denote a subdivision of $[x, y]$ which refines the intersection of $[x, y]$ and $D_1 \cup D_2$ and has at least $N + 1$ elements. Suppose $\{x_i\}_{i=0}^n$ is a refinement of $D$. Now,

\[
\left| \sum_{i=0}^{\infty} P_i(x, y) - \prod_{i=1}^{n} (1 + F_i + G_i) \right| = \left| \sum_{i=0}^{\infty} P_i(x, y) - \sum_{i=0}^{n} S_i \right| \leq \left| \sum_{i=0}^{N} P_i(x, y) - \sum_{i=0}^{n} S_i \right| + \left| \sum_{i=N+1}^{\infty} P_i(x, y) \right| + \sum_{i=N+1}^{n} S_i \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\]

Hence, $\prod_{i}^{x,y} (1 + F + G)$ exists and is $\sum_{i=0}^{\infty} P_i(x, y)$.

We now establish that $F + G \in OM^*$ on $[a, b]$. Since $x_i^{1/3}(1 + F + G)$ exists for $a < x < y < b$, it is only necessary to establish the approximation part of the definition. Let $\epsilon > 0$. Further, let $D_1, D_2$ and $N$ be defined as before.

Since $F$ is in $OM^*$, $OP^0$ and $S_2$ on $[a, b]$ and $G$ is in $OB^0$ and $S_2$ on $[a, b]$, there exists a subdivision $D_3$ of $[a, b]$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of $[a, b]$, $0 \leq p < q \leq n$ and $q - p \leq N$, then

\[
\left| \prod_{i=p+1}^{q} (1 + F + G) - \prod_{i=p+1}^{q} (1 + F_i + G_i) \right| < \epsilon.
\]

Let $D$ denote the subdivision $D_1 \cup D_2 \cup D_3$ of $[a, b]$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of $D$ and $0 \leq p < q \leq n$. If $q - p \leq N$, then the desired inequality follows immediately from the definition of $D_3$. If $q - p \geq N$, then

\[
\left| \prod_{i=p+1}^{q} (1 + F + G) - \prod_{i=p+1}^{q} (1 + F_i + G_i) \right| = \left| \sum_{i=0}^{\infty} P_i(x_p, x_q) - \sum_{i=0}^{p-q} S_{i, p+1, q} \right| \leq \left| \sum_{i=0}^{N} P_i(x_p, x_q) - \sum_{i=0}^{N} S_{i, p+1, q} \right| + \sum_{i=N+1}^{\infty} P_i(x_p, x_q) \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\]
Therefore, \( F + G \in OM^* \) on \([a, b]\). Thus, (2) and (3) imply (1).

Proof of Theorem 1 [(1), (2) \(\rightarrow\) (3)]. Since \( F \) and \( F + G \) are in \( OM^* \) and \( OP^o \) on \([a, b]\), the existence of

\[
(LR) \int_x^y \prod_x^y (1 + F)G \prod_x^y (1 + F + G)
\]

for \( a \leq x < y \leq b \) can be established by employing Lemma 6.

Since \( F \) and \( F + G \) are in \( S_1 \) on \([a, b]\), there exists a subdivision \( \{x_i\}_{i=0}^n \) of \([a, b]\) such that, if \( 1 \leq i \leq n \) and \( x_{i-1} < x < y < x_i \), then

\[
\left| \prod_x^y (1 + F) \right| < \frac{1}{2} \quad \text{and} \quad \left| \prod_x^y (1 + F + G) \right| < \frac{1}{2}.
\]

Suppose \( 1 \leq i \leq n \) and \( x_{i-1} < x < y < x_i \). Let \( J \) and \( K \) represent interval functions such that, if \( x \leq u < v \leq y \), then

\[
J(u, v) = \prod_x^y (1 + F) \quad \text{and} \quad K(u, v) = \prod_x^y (1 + F + G).
\]

Since \( J \) and \( K \) are in \( OL^o \) on \([a, b]\), it follows that \( J^{-1} \) and \( K^{-1} \) are also in \( OL^o \) on \([a, b]\). Thus, from Lemma 1 and the existence of the integral in the preceding paragraph, we have that \( \int_x^y G \) exists.

We have now established that, if \( 1 \leq i \leq n \) and \( x_{i-1} < x < y < x_i \), then \( \int_x^y G \) exists. From this and the fact that \( G \) is in \( OB^o \) and \( S_2 \) on \([a, b]\), it follows that \( \int_a^b G \) exists. Hence, \( G \in OM^* \) on \([a, b]\) by Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 1 [(1), (3) \(\rightarrow\) (2)]. It follows from Lemma 5 that \( F + G \in OP^o \) on \([a, b]\). Further, \( -G \in OM^* \) by Lemma 7. We have already established that (2) and (3) imply (1). Now, since \( F + G - G = F \), it follows that \( F \in OM^* \) on \([a, b]\). Thus, (1) and (3) imply (2).

The proof of Theorem 1 is now complete. We next establish Theorem 2.

One additional lemma is needed.

Lemma 8. If \( G \) is a function from \( R \times R \) to \( N \) and \( G \in OB^o \) on \([a, b]\), then the following statements are equivalent:

1. \( G \in OA^o \) on \([a, b]\), and
2. \( G \in OM^o \) on \([a, b]\) [2, Theorem 3.4, p. 301].

Proof of Theorem 2 [(2), (3) \(\rightarrow\) (1)]. Since \( F \) and \( G \) are in \( OP^o \) and \( OM^o \) on \([a, b]\), \( F \) and \( G \) are also in \( OM^* \) on \([a, b]\). Hence, it follows from Theorem 1 [(2), (3) \(\rightarrow\) (1)] that \( \prod_x^y (1 + F + G) \) exists for \( a \leq x < y \leq b \).

Thus, it is only necessary to show that \( \int_a^b [1 + F + G - \Pi(1 + F + G)] \) exists and is zero in order to establish that \( F + G \in OM^o \) on \([a, b]\). Let \( \epsilon > 0 \).

Since \( F \in OM^o \) on \([a, b]\), there exists a subdivision \( D_1 \) of \([a, b]\) such that, if \( \{x_i\}_{i=0}^n \) is a refinement of \( D_1 \), then
We know that $F$ is in $O_{P}^{o}$ and $O_{M}^{*}$ on $[a, b]$. Further, $F + G \in O_{P}^{o}$ on $[a, b]$ by Lemma 5 and $F + G \in O_{M}^{*}$ on $[a, b]$ by Theorem 1 [(2), (3) $\rightarrow$ (1)]. Now, since $G \in O_{B}^{o}$ on $[a, b]$, it follows by using Lemma 6 that

$$
(LR) \int_{x}^{y} \prod_{i=1}^{x} (1 + F + G)
$$

exists and is

$$
\prod_{i=1}^{y} (1 + F + G) - \prod_{i=1}^{x} (1 + F)
$$

for $a \leq x < y \leq b$.

Since $F$ and $F + G$ are in $S_{1}$ and $O_{M}^{*}$ on $[a, b]$, for each positive number $\beta$ there exists a subdivision $\{x_{i}\}_{i=0}^{n}$ of $[a, b]$ such that, if $1 \leq i \leq n$ and $x_{i-1} < x < y < x_{i}$, then

$$
\left| 1 - \prod_{i=1}^{y} (1 + F) \right| < \beta \quad \text{and} \quad \left| 1 - \prod_{i=1}^{y} (1 + F + G) \right| < \beta.
$$

By Lemma 8, $G \in O_{A}^{o}$ on $[a, b]$. Further, $F$ and $F + G$ are in $O_{P}^{o}$ on $[a, b]$ and $G \in O_{B}^{o}$ on $[a, b]$. From these facts, it follows that

$$
\int_{a}^{b} \left| G(x, y) - (LR) \int_{x}^{y} \prod_{i=1}^{x} (1 + F + G) \prod_{i=1}^{y} (1 + F + G) \right|
$$

exists and is zero. Thus, there exists a subdivision $D_{2}$ of $[a, b]$ such that, if $\{x_{i}\}_{i=0}^{n}$ is a refinement of $D_{2}$, then

$$
\sum_{i=1}^{n} \left| G_{i} - (LR) \int_{x_{i-1}}^{x_{i}} \prod_{i=1}^{x_{i}} (1 + F + G) \prod_{i=1}^{x_{i}} (1 + F + G) \right| < \frac{\epsilon}{2}.
$$

Let $D$ denote the subdivision $D_{1} \cup D_{2}$ of $[a, b]$. Suppose $\{x_{i}\}_{i=0}^{n}$ is a refinement of $D$. Now,

$$
\sum_{i=1}^{n} \left| 1 + F_{i} + G_{i} - \prod_{i=1}^{x_{i}} (1 + F + G) \right|
$$

$$
= \sum_{i=1}^{n} \left| 1 + F_{i} + G_{i} - \prod_{i=1}^{x_{i}} (1 + F) + (LR) \int_{x_{i-1}}^{x_{i}} \prod_{i=1}^{x_{i}} (1 + F) \prod_{i=1}^{x_{i}} (1 + F + G) \right|
$$

$$
\leq \sum_{i=1}^{n} \left| 1 + F_{i} - \prod_{i=1}^{x_{i}} (1 + F) \right| + \sum_{i=1}^{n} \left| G_{i} - (LR) \int_{x_{i-1}}^{x_{i}} \prod_{i=1}^{x_{i}} (1 + F) \prod_{i=1}^{x_{i}} (1 + F + G) \right|
$$

$$
< \epsilon/2 + \epsilon/2 = \epsilon.
$$
Therefore, \( F + G \in O \mathcal{M}^\circ \) on \([a, b]\). Thus, (2) and (3) imply (1).

Proof of Theorem 2 [(1), (2) \(\rightarrow\) (3)]. The proof of Theorem 2 [(1), (2) \(\rightarrow\) (3)] is similar to the proof of Theorem 1 [(1), (2) \(\rightarrow\) (3)]. The only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (2) imply (3).

Proof of Theorem 2 [(1), (3) \(\rightarrow\) (2)]. The proof of Theorem 2 [(1), (3) \(\rightarrow\) (2)] is similar to the proof of Theorem 1 [(1), (3) \(\rightarrow\) (2)]. As before, the only difference is that it is necessary to use Lemma 8 rather than Lemma 7. Thus, (1) and (3) imply (2).

The proof of Theorem 2 is now complete.

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