TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 211, 1975

ENDOMORPHISM RINGS AND DIRECT SUMS OF TORSION FREE ABELIAN GROUPS

BY

D. M. ARNOLD[1] AND E. L. LADY

ABSTRACT. Properties of abelian groups related to a given finite rank torsion free abelian group A are analyzed in terms of $\mathrm{End}(A)$, the endomorphism ring of A. This point of view gives rise to generalizations of some classical theorems by R. Baer and examples of pathological direct sum decompositions of finite rank torsion free abelian groups.

An abelian group G is *A-free* if G is isomorphic to a direct sum of copies of an abelian group A and *A-projective* if G is a direct summand of an A-free group. Define $S_A(G)$ to be the subgroup of G generated by $\{f(A) \mid f \in \mathrm{Hom}_Z(A, G)\}$.

The following properties are considered for a finite rank torsion free abelian group A.

(I) Every A-projective group is A-free.

(II) Every exact sequence $0 \to B \to G \to C \to 0$ of abelian groups, where $S_A(G) + B = G$ and C is A-projective, is split exact.

(III) Every subgroup B of an A-free group with $S_A(B) = B$ is A-free.

(IV) Every subgroup B of an A-projective group with $S_A(B) = B$ is A-projective.

Theorem 1. (a) (I) *is true iff every projective right* $\mathrm{End}(A)$*-module is free.*

(b) (II) *is true iff* $IA \neq A$ *for all proper right ideals* I *of* $\mathrm{End}(A)$.

(c) (II) *and* (III) *are true iff* $\mathrm{End}(A)$ *is right principal and has no zero-divisors.*

(d) (II) *and* (IV) *are true iff* $\mathrm{End}(A)$ *is right hereditary.*

Conditions (I)–(IV) are all true if $\mathrm{End}(A)$ is a principal ideal domain. Examples of groups with this property include:

Received by the editors January 11, 1974.

AMS (MOS) subject classifications (1970). Primary 20K15.

Key words and phrases. Strongly indecomposable groups, Krull-Schmidt property, quasi-isomorphism.

[1] Preparation of manuscript supported, in part, by NSF Grant GP-40872.

Copyright © 1975, American Mathematical Society

(i) $\operatorname{rank}(A) = 1$;

(ii) $\operatorname{rank}(A) = 2$ and the typeset of A has more than 3 elements (Beaumont-Pierce [4]);

(iii) A is indecomposable of finite rank and Z/pZ-dimension $(A/pA) \leq 1$ for all primes p (Murley [13]); and

(iv) rigid groups (Fuchs [9, p. 124]).

Suppose that I is a right ideal of $\operatorname{End}(A)$ with $IA = A$. If $\operatorname{End}(A)$ is commutative, right principal, or right hereditary then $I = \operatorname{End}(A)$. Furthermore, if A is *strongly indecomposable* (i.e., $nA \subset B \oplus C \subset A$ for some $0 \neq n \in Z$ implies that $B = 0$ or $C = 0$) or $Q \otimes_Z \operatorname{End}(A)$ is semisimple then $\operatorname{End}(A)/I$ is finite. On the other hand, there does exist a finite rank torsion free abelian group A and a two-sided ideal I of $\operatorname{End}(A)$ with $IA = A$ and $\operatorname{End}(A)/I$ infinite.

§4 includes some cancellation and exchange properties for certain classes of finite rank torsion free abelian groups. Most notably, we have

Theorem 2. (a) *If $A \oplus H_1 \simeq A \oplus H$, and if A and H have no quasi-summands in common then $H \simeq H_1$.*

(b) *If $A \oplus K = B \oplus C$, where A and K have no quasi-summands in common, then $A \oplus K = B_1 \oplus C_1 \oplus K$ for some $B_1 \subset B$, $C_1 \subset C$.*

Finally, properties (I)–(IV) are examined, for the case that $\operatorname{rank}(A) = 2$, in §5.

Basic references are Fuchs [8] and [9] for general properties of abelian groups; Reid [15] for quasi-endomorphisms and quasi-decompositions of torsion free abelian groups; and Lambek [12] or Bass [3] for general properties of rings and modules.

1. A-projective groups. The following general homological remarks are essential to the proofs in this section. Let $\mathcal{G}$ be the category of abelian groups, A an abelian group, R the endomorphism ring of A and $\mathfrak{M}_R$ the category of right R-modules. A left exact functor $H: \mathcal{G} \to \mathfrak{M}_R$ is defined by letting $H(G) = \operatorname{Hom}_Z(A, G)$ and $H(f)(g) = fg$ where $f: G \to G'$ is an abelian group homomorphism. Moreover, there is a right exact functor $T: \mathfrak{M}_R \to \mathcal{G}$ given by $T(M) = M \otimes_R A$, regarding A as a left R-module, and $T(f) = f \otimes 1$ for a homomorphism $f: M \to M'$ of right R-modules.

There is a natural transformation θ from TH to the identity functor on $\mathcal{G}$, where $\theta_G: \operatorname{Hom}_Z(A, G) \otimes_R A \to G$ is defined by $\theta_G(f \otimes a) = f(a)$ for $a \in A$, $f \in \operatorname{Hom}_Z(A, G)$, and G an abelian group. On the other hand, there is a natural transformation ϕ from the identity functor on $\mathfrak{M}_R$ to HT where

$\phi_M: M \to \mathrm{Hom}_Z(A, M \otimes_R A)$ is given by letting $\phi_M(x)(a) = x \otimes a$. (These functors and transformations are analogous to those used in defining a Morita equivalence, e.g., see Bass [3].)

Theorem 1.1. *Let A be a torsion free abelian group of finite rank. The functor H from the category of A-projective groups to the category of projective right* End(A)*-modules is a category equivalence with inverse T.*

Proof. Let $\mathcal{P}_A$ be the category of A-projective groups and $\mathcal{P}_R$ the category of projective right R-modules, where $R = \mathrm{End}(A)$. In view of the preceding remarks, it suffices to prove that: (i) H sends A-free groups to free R-modules; (ii) T sends free R-modules to A-free groups; and (iii) $\theta_G: TH(G) \to G$ and $\phi_M: M \to HT(M)$ are isomorphisms for all objects G and M of $\mathcal{P}_A$ and $\mathcal{P}_R$, respectively.

It is easy to verify that $\theta_A: TH(A) \to A$ and $\phi_R: R \to HT(R)$ are isomorphisms (regard R as a right R-module). Clearly T preserves direct sums. Furthermore, $H(\Sigma \oplus G_i) = \mathrm{Hom}_Z(A, \Sigma \oplus G_i) \simeq \Sigma \oplus \mathrm{Hom}_Z(A, G_i)$: since A has finite rank, the image of A under a homomorphism into $\Sigma \oplus G_i$ is contained in a subgroup generated by finitely many G_i. Consequently, if G is A-free then $H(G)$ is a free R-module and if M is a free R-module then $T(M)$ is A-free. In these two cases θ_G and ϕ_M are isomorphisms. Since H and T commute with direct sum decompositions, θ_G and ϕ_M are isomorphisms if G is A-projective and M is R-projective.

Remark. Warfield [16] proved Theorem 1.1 for the case that $\mathrm{rank}(A) = 1$.

Corollary 1.2. *Let A be a torsion free abelian group of finite rank. Then* (I) *is true iff every projective right* End(A)*-module is free.*

Call a set $\mathcal{F} = \{A_i\}_{i \in I}$ of abelian groups *semirigid* if there is a partial ordering on the index set, I, such that $i \leq j$ iff $\mathrm{Hom}_Z(A_i, A_j) \neq 0$. Define $\mathcal{F}_\Sigma$ to be the class of groups isomorphic to direct sums of groups in $\mathcal{F}$.

Corollary 1.3. *Let $\mathcal{F} = \{A_i\}_{i \in I}$ be a semirigid set of torsion free abelian groups of finite rank such that* (I) *is true for all A in $\mathcal{F}$. Then any two direct sum decompositions of an $\mathcal{F}_\Sigma$ group G have isomorphic refinements. In particular, every summand of G is an $\mathcal{F}_\Sigma$ group.*

Proof. Note that G is a direct sum of countable groups. Thus if $B \oplus C = G$ then B is a direct sum of countable groups (e.g., see Fuchs [8, p. 49]). So assume that B is a countable summand of G. Now $G = \Sigma_{i \in I} \bigoplus G(i)$, where $G(i)$ is either 0 or a direct sum of groups isomorphic to A_i. But $B = \Sigma_{i \in I} \bigoplus B(i)$, where each $B(i)$ is either 0 or a summand of $G(i)$ (Charles [5]).

By Corollary 1.2, each $B(i)$ is an A_i-free group. The corollary now follows from the observation that two A_i-free groups are isomorphic iff they have the same cardinal number of summands isomorphic to A_i.

Remark. Corollary 1.3 includes the classical Baer-Kulikov-Kaplansky theorem (e.g., see Fuchs [9, p. 114]) and a theorem by Murley [14] as special cases.

2. Splitting and quasi-splitting exact sequences.

Theorem 2.1. *Let A be a finite rank torsion free abelian group. Then* (II) *is true iff $IA \neq A$ for all proper right ideals I of* $\operatorname{End}(A)$.

Proof. ($\Leftarrow$) It is sufficient to prove that every exact sequence $0 \to B \to G \xrightarrow{\pi} A \to 0$ of abelian groups with $S_A(G) + B = G$ is split exact.

Define $I = \{\pi h | h: A \to G\}$, a right ideal of R. Then clearly $IA = A$. By assumption, $I = \operatorname{End}(A)$ so there is some $h: A \to G$ with $\pi h = 1$. Consequently, (II) is true for A.

($\Rightarrow$) Let I be a right ideal of $R = \operatorname{End}(A)$ with $IA = A$ and choose a free right R-module P with an epimorphism $\pi: P \to I$. The map $\mu: I \otimes_R A \to A$ induced by $\mu(x \otimes a) = xa$ is epic since $IA = A$. Since $T(-)$ is right exact, $T(\pi): T(P) \to T(I)$ is epic so $\sigma = \mu T(\pi): P \otimes_R A \to A$ is epic. But $P \otimes_R A$ is an abelian group with $S_A(P \otimes_R A) = P \otimes_R A$ so that $\sigma: P \otimes_R A \to A$ splits. In particular, $H(\sigma): HT(P) \to H(A)$ is epic. Let $i: I \to H(A)$ be the inclusion map. Then $i\pi = H(\sigma)\phi_P$, where $\phi_P: P \to HT(P)$ is an isomorphism (Theorem 1.1). Consequently, i is epic and $I = H(A) = R$.

Theorem 2.2. *Suppose that A is a finite rank torsion free abelian group and that either A is strongly indecomposable, $Q \otimes_Z$* $\operatorname{End}(A)$ *is semisimple or* $\operatorname{End}(A)$ *is commutative. If I is a right ideal of* $\operatorname{End}(A)$ *with $IA = A$ then* $\operatorname{End}(A)/I$ *is finite.*

Proof. We prove that if $IA = A$ then I contains a monomorphism f. Since monomorphisms are units in $Q \otimes_Z \operatorname{End}(A)$, there is a $g \in \operatorname{End}(A)$ with $fg = n$ for some nonzero integer n. Thus $n\operatorname{End}(A) \subset I \subset \operatorname{End}(A)$ and $\mathit{End}(A)/I$ is finite (e.g., $\operatorname{End}(A)$ is torsion free of finite rank so $\operatorname{End}(A)/n\operatorname{End}(A)$ is bounded and contained in a finite direct sum of copies of Q/Z).

If A is strongly indecomposable then every $0 \neq f \in \operatorname{End}(A)$ is monic or nilpotent (Reid [15]). Moreover, every nil ideal of $\operatorname{End}(A)$ is nilpotent (since $Q \otimes_Z \operatorname{End}(A)$ is artinian) so I must contain a monomorphism.

Assume that $\operatorname{End}(A)$ is commutative and $IA = A$ for some ideal I of $\operatorname{End}(A)$. Then $Q \otimes_Z \operatorname{End}(A)$ is commutative, $Q \otimes_Z A$ is a finitely generated

$Q \otimes_Z \mathrm{End}(A)$-module and $(Q \otimes_Z I)(Q \otimes_Z A) = Q \otimes_Z A$. Consequently, $(1+y)(Q \otimes_Z A) = 0$ for some $y \in Q \otimes_Z I$ (Kaplansky [10, Theorem 76]). But this means that $y = -1$ and it follows that I contains an integer.

Finally, if $Q \otimes_Z \mathrm{End}(A)$ is semisimple then $Q \otimes_Z I$ is principal and the theorem follows easily.

Corollary 2.3. *Suppose that A is a finite rank torsion free abelian group such that either A is strongly indecomposable, $Q \otimes_Z \mathrm{End}(A)$ is semisimple or $\mathrm{End}(A)$ is commutative. If*

$$(*) \qquad 0 \to B \to G \xrightarrow{\pi} C \to 0$$

is an exact sequence of torsion free abelian groups such that $S_A(G) + B = G$ and C is a finite rank A-projective group then $(*)$ is a quasi-split sequence (i.e., there is $\delta\colon C \to G$ with $\pi\delta = n$ for some $0 \neq n \in Z$).

Proof. Let $C = A$ and $I = \{\pi h\colon h \in \mathrm{Hom}_Z(A, G)\}$, a right ideal of $\mathrm{End}(A)$ with $IA = A$. By 2.2, $n\mathrm{End}(A) \subseteq I \subseteq \mathrm{End}(A)$ for some $0 \neq n \in Z$. Consequently, $n = \pi\delta$ for some $\delta \in \mathrm{Hom}_Z(A, G)$. The corollary now follows in case C is A-projective.

Corollary 2.4. *If A is a finite rank torsion free abelian group and if $\mathrm{End}(A)$ is commutative or right principal then* (II) *is true for A.*

Proof. By 2.1 it suffices to prove that if I is a right ideal of A with $IA = A$ then $I = \mathrm{End}(A) = R$. If R is right principal then $I = fR$ and $f(A) = A$ for some $f \in I$. Thus f is a unit in R and $I = R$.

Assume that R is commutative. By Theorem 2.2, R/I is finite, say $nR \subset I \subset R$ for some $0 \neq n \in Z$. Let $\bar{R} = R/nR$, $\bar{I} = I/nR$ and $\bar{A} = A/nA$. Then $\bar{R}$ is commutative, $\bar{A}$ is a finitely generated $\bar{R}$-module and $\bar{I}\bar{A} = \bar{A}$. Once again, $(\bar{1} + \bar{y})(\bar{A}) = 0$ for some $y \in I$. Thus $(1+y)(A) \subseteq nA$ so that $1 + y \in nR$ and $1 \in I + nR = I$. This proves that $R = I$.

Remark. If $\mathrm{End}(A)$ is a nonprincipal Dedekind domain then A satisfies (II) but not (I). Corollary 2.4, for the case that $\mathrm{rank}(A) = 1$, is a well-known theorem of Baer (Fuchs [9, p. 114]).

Example 2.5. Let $A = B \oplus C$ be a finite rank torsion free abelian group with $S_B(C) = C$ and $S_C(B) = 0$ (e.g., $B = Z$ and C has no free summands). Then there is a two-sided ideal I of $\mathrm{End}(A)$ such that $IA = A$ and $\mathrm{End}(A)/I$ is infinite.

Proof. Let $I = \{f \in \mathrm{End}(A)\colon f(C) = 0\}$, a two-sided ideal of $\mathrm{End}(A)$ since C is fully invariant. Then $A = S_B(A) = IA$ and $Z\pi_C \subset \mathrm{End}(A)/I$ where $\pi_C\colon A \to C$ is a projection map.

Remark. The following question remains open: Is it true that if A is strongly indecomposable and I is a right ideal of $\mathrm{End}(A)$ with $IA = A$ then $I = \mathrm{End}(A)$? The answer is yes if $\mathrm{rank}(A) \leq 2$ (see §5).

3. Subgroups of A-free groups. Let $\mathcal{S}_A$ be the category of subgroups B of A-free groups with $S_A(B) = B$ and $\mathcal{S}_R$ the category of submodules of free right R-modules, where $R = \mathrm{End}(A)$. Note that $\mathcal{S}_A$ is closed under direct sums and summands. We write $t(G)$ for the torsion subgroup of an abelian group G.

It is well known that if A and G are torsion free abelian groups then $G \otimes_Z A$ is torsion free. But if G is a right R-module, where $R = \mathrm{End}(A)$, then $G \otimes_R A$ may not be torsion free (the examples are somewhat complicated, require that $\mathrm{End}(A)$ be a non-Dedekind domain such that A is not a flat $\mathrm{End}(A)$-module and will be contained in a forthcoming paper).

Theorem 3.1. *Assume that A is a finite rank torsion free abelian group, $R = \mathrm{End}(A)$ and that $Q \otimes_Z R$ is semisimple. Then UH is naturally equivalent to the identity functor on $\mathcal{S}_A$, where $H(-) = \mathrm{Hom}_Z(A, -)$ and $U(-) = (- \otimes_R A)/t(- \otimes_R A)$.*

Proof. Let M be a submodule of the free right R-module L. Since $Q \otimes_Z R$ is semisimple, $Q \otimes_Z A$ is a flat $Q \otimes_Z R$-module. The inclusion $i: M \longrightarrow L$ induces a monomorphism $h: Q \otimes_Z M \otimes_R A \longrightarrow Q \otimes_Z L \otimes_R A$. Moreover, there are monomorphisms $f: U(M) \longrightarrow Q \otimes_Z M \otimes_R A$ and $g: U(L) \longrightarrow Q \otimes_Z L \otimes_R A$ induced by $f(m \otimes a) = 1 \otimes m \otimes a$ and $g(l \otimes a) = 1 \otimes l \otimes a$, respectively. But $gi^* = hf$, where $i^*: U(M) \longrightarrow U(L)$ is the map induced by i. Consequently, i^* is monic and $U(M) \in \mathcal{S}_A$ since $U(L)$ is A-free (Theorem 1.1).

Let B be a subgroup of an A-free group G with $S_A(B) = B$. Then θ_B: $UH(B) \longrightarrow B$ is epic (since $S_A(B) = B$). Furthermore, $\theta_G UH(i) = i\theta_B$ where $i: B \longrightarrow G$ and θ_B is monic since θ_G is an isomorphism. Thus UH is naturally equivalent to the identity functor on $\mathcal{S}_A$.

Remark. If $Q \otimes_Z \mathrm{End}(A)$ is semisimple then A is a flat $\mathrm{End}(A)$-module iff the torsion subgroup of $I \otimes_R A$ is 0 for all finitely generated right ideals I of $\mathrm{End}(A) = R$. In particular, if $\mathrm{End}(A)$ is right hereditary then A is flat.

Corollary 3.2. *Let A be a finite rank torsion free abelian group.*

(a) (II) *and* (III) *are true iff* $\mathrm{End}(A)$ *is right principal with no zero-divisors.*

(b) (II) *and* (IV) *are true iff* $\mathrm{End}(A)$ *is right hereditary.*

Proof. The proof of (a) is similar to the proof of (b).

(b) ($\Leftarrow$) If $\mathrm{End}(A)$ is right hereditary then $Q \otimes_Z \mathrm{End}(A)$ is right hereditary and artinian, hence semisimple (a consequence of Nakayama's lemma). Thus (IV) is a consequence of Theorem 3.1.

To prove (II) suppose that I is a right ideal of $\mathrm{End}(A)$ with $IA = A$. Since I is projective, $I \simeq HT(I)$. But $T(I) \simeq IA = A$ so $I \simeq \mathrm{End}(A)$, i.e., I is right principal. Now apply Corollary 2.4 to see that (II) is true.

(b) ($\Longrightarrow$) Let I be a nonzero right ideal of $R = \mathrm{End}(A)$ and choose a free R-module P with an epimorphism Π: $P \longrightarrow I$. Since $T(\Pi)$: $T(P) \longrightarrow T(I)$ is epic, there is an epimorphism σ: $T(P) \longrightarrow IA$. By (IV), IA is A-projective and, by (II), σ is split. Thus $H(\sigma) : HT(P) \longrightarrow H(IA)$ is epic. Let i: $I \longrightarrow H(IA)$ be the inclusion map. Since $i\Pi = H(\sigma)\phi_P$ and ϕ_P is an isomorphism, i is epic and $I \simeq H(IA)$. But IA is A-projective so I is a projective right R-module.

Remark. Special cases of Corollary 3.2 include a classical theorem by Baer-Kolettis (see Fuchs [9, p. 114]) and a theorem of Murley [14].

4. Direct sum decompositions and cancellation. As a further application of the equivalence in Theorem 1.1, we prove an exchange theorem with some interesting applications.

Theorem 4.1. *Let G be a finite rank torsion free abelian group and $G = A \oplus K = B \oplus C$. If no quasi-summand of A is quasi-isomorphic to any quasi-summand of K then there exist decompositions $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2$ so that $G = B_1 \oplus C_1 \oplus K = A \oplus B_2 \oplus C_2$.*

Proof. Let $H = \mathrm{Hom}_Z(G, -)$, $R = \mathrm{End}(G)$, and let N be the nil radical of R. If X and Y are summands of G then, by Theorem 1.1, $X \simeq Y$ iff $H(X) \simeq H(Y)$. Also $H(X) \simeq H(Y)$ iff $\overline{X} \simeq \overline{Y}$ where $\overline{X} = H(X)/H(X)N$ (e.g., see Bass [3, p. 90]). Furthermore $\overline{X}$ and $\overline{Y}$ are finitely generated (in fact, principal) right ideals in R/N. Conversely, if $\overline{W}$ is any summand of $\overline{X}$ then there exists a summand P of $H(X)$ such that $\overline{W} = P/PN$ (e.g., Bass [3, p. 88–90]). Thus $\overline{W} = H(W)/H(W)N$, where W corresponds to P via Theorem 1.1 (in fact, $W = PG$).

Furthermore, X and Y are quasi-isomorphic iff $Q \otimes_Z \overline{X} \simeq Q \otimes_Z \overline{Y}$ as right ideals in $Q \otimes_Z (R/N)$: given an isomorphism ϕ: $Q \otimes_Z \overline{X} \simeq Q \otimes_Z \overline{Y}$ with inverse ψ, there exists an integer $n \neq 0$ such that $n\phi(\overline{X}) \subseteq \overline{Y}$ and $n\psi(\overline{Y}) \subseteq \overline{X}$. Thus $n\phi$ and $n\psi$ induce maps between $H(X)$ and $H(Y)$ and so, via Theorem 1.1, there are maps ϕ': $X \longrightarrow Y$ and ψ': $Y \longrightarrow X$ such that $\phi'\psi' = n = \psi'\phi'$. Therefore, X and Y are quasi-isomorphic. The other direction is similar but easier.

Applying the preceding remarks to the decomposition $G = A \oplus K = B \oplus C$ we get $\overline{G} = \overline{A} + \overline{K} = \overline{B} + \overline{C}$. Since $Q \otimes_Z (R/N) = Q \otimes_Z \overline{G}$ is a finite dimensional algebra with trivial nil radical, it is semisimple. The minimal quasi-summands of A and K correspond to the simple submodules of $Q \otimes_Z \overline{A}$

and $Q \otimes_Z \overline{K}$. Since any nontrivial homomorphism between simple modules is an isomorphism, it follows from the hypothesis that there are no nontrivial homomorphisms from $Q \otimes_Z \overline{A}$ to $Q \otimes_Z \overline{K}$ or conversely, and consequently there are no nontrivial homomorphisms between $\overline{A}$ and $\overline{K}$ in either direction. Thus $\overline{A}$ and $\overline{K}$ are fully invariant submodules of $\overline{G}$, so that $\overline{G} = \overline{B}_1 \oplus \overline{B}_2 \oplus \overline{C}_1 \oplus \overline{C}_2$ with $\overline{B}_1 \oplus \overline{B}_2 = \overline{B}$, $\overline{C}_1 \oplus \overline{C}_2 = \overline{C}$, $\overline{B}_1 \oplus \overline{C}_1 = \overline{A}$, and $\overline{B}_2 \oplus \overline{C}_2 = \overline{K}$. We can lift these decompositions of $\overline{B}$ and $\overline{C}$ to get $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2$ for suitable B_i and C_i. Furthermore, $B_1 \oplus C_1 \simeq A$, $B_2 \oplus C_2 \simeq K$ and $H(G) = H(B_1) + H(C_1) + H(K) + N = H(A) + H(B_2) + H(C_2) + N$. By Nakayama's lemma, the N in this equation can be omitted. Since, for any summand X of G, we have $X = H(X)G$, it follows that $G = B_1 + C_1 + K = A + B_2 + C_2$. Adding up ranks guarantees that these sums must be direct.

As a special case, we have the following:

Corollary 4.2. *If $A \oplus K \simeq A \oplus K_1$ and no quasi-summand of A is quasi-isomorphic to any quasi-summand of K then $K \simeq K_1$.*

Corollary 4.3. *Let $F = \{A_i\}_{i \in I}$ be a set of mutally non-quasi-isomorphic strongly indecomposable finite rank groups. Let F_Σ be the class of finite direct sums of groups in F_Σ and let $G = B \oplus C \in F_\Sigma$. Then*

(a) *$B = \Sigma \oplus B(i)$, where each $B(i)$ is an A_i-projective group.*

(b) *If each A_i has property* (I) *then B is an F_Σ group and G has the Krull-Schmidt property.*

In particular, Corollary 4.3 applies to the case $F = \mathcal{E}$, the class of finite rank indecomposable groups A with the Z/pZ-dimension of $A/pA \leq 1$ for all primes p [14]. It then follows that $\mathcal{E}_\Sigma$ groups have the Krull-Schmidt property.

The same basic pattern can be used to prove a theorem slightly different from Theorem 4.1:

Theorem 4.4. *If G is a finite rank torsion free abelian group and $G = A \oplus K = B \oplus C$, where no quasi-summand of A is quasi-isomorphic to any quasi-summand of C, then $G = A \oplus B_1 \oplus C$ for some $B_1 \subseteq B$.*

Proof. Using the notation and techniques of Theorem 4.1, $\mathrm{Hom}_R(\overline{A}, \overline{C}) = 0$, so that $\overline{A} \subseteq \overline{B}$. Thus $\overline{B} = \overline{A} \oplus \overline{B}_1$ and the conclusion follows as in Theorem 4.1.

The following theorem may be deduced from the recent proof by Lady [11] that every torsion free abelian group of finite rank has (up to isomorphism) at most a finite number of summands. This proof used some deep results of classical ring theory. On the other hand, the following arguments are group theoretic in nature.

Theorem 4.5. *Let A be a torsion free abelian group of finite rank such that* End(A) *has no zero-divisors and is right principal. Then for any torsion free finite rank abelian group H there are, up to isomorphism, only finitely many groups H_1 such that $A \oplus H \simeq A \oplus H_1$.*

Proof. Let $G = A \oplus H = A_1 \oplus H_1$ with $A \simeq A_1$ and let α and α_1 be the projections of G onto A and A_1 with kernels H and H_1, respectively.

Assume $A \cap H_1 \neq 0$. Then the restriction of α_1 to A is not monic. But $A \simeq A_1$ and nonzero endomorphisms of A are monic, so it follows that $\alpha_1(A) = 0$. Thus $A \subseteq H_1$ and $H_1 = A \oplus D$ where $D = H \cap H_1$. Then $G = A_1 \oplus A \oplus D$, so that $H \simeq A_1 \oplus D \simeq A \oplus D = H_1$. Similarly, we dispose of the case $A_1 \cap H \neq 0$.

Now suppose $G = A \oplus H = A_1 \oplus H_1$ with $A \simeq A_1$, $A \cap H_1 = 0$, $A_1 \cap H = 0$, and suppose that H and H_1 are not isomorphic, Choose an A-free subgroup $K \subseteq H$ of maximal rank such that K is a quasi-summand of H. Then there is a pure subgroup $C \subseteq H$ such that $K \oplus C$ has finite index in H. Since A has property (III) and $(S_A(H) + C)/C$ is quasi-isomorphic to $H/C \simeq K$, $(S_A(H) + C)/C$ is A-free. By property (II) there is an A-free subgroup B of H with $S_A(H) + C = B \oplus C$ (and $B \simeq K$). Then $S_A(H) = B \oplus S_A(C)$ (since S_A commutes with direct sums).

Let η and η_1 be the projections of G onto H and H_1 with kernels A and A_1, respectively. Since $A \cap H_1 = A_1 \cap H = 0$, η and η_1 are monomorphisms and $\eta_1(H)$ has finite index in H_1. Now $\eta_1(B)$ is A-free, $\eta_1(B) \cap C_1 = 0$ and $\eta_1(B) \oplus C_1$ has finite index in H_1, where C_1 is the pure subgroup of H_1 generated by $\eta_1(C)$. The previous argument now applies to give an A-free subgroup B_1 of $S_A(H_1)$ with $S_A(H_1) = B_1 \oplus S_A(C_1)$. Necessarily, $B_1 \simeq B$.

There is a nonzero integer n such that $nH \subseteq B \oplus C \subseteq H$. Now $nH_1 = n\eta_1(G) = \eta_1(nA \oplus nH) \subseteq \eta_1(nA \oplus B \oplus C) \subseteq S_A(H_1) + \eta_1(C)$. If $S_A(H_1) + \eta_1(C) = B_1 \oplus \eta_1(C)$ then the proof would be complete. For n is independent of H_1, and $B_1 \oplus \eta_1(C) \simeq B \oplus C$, so $H_1 \simeq H'$ where $n(B \oplus C) \subseteq H' \subseteq B \oplus C$. Since $(B \oplus C)/n(B \oplus C)$ is a finite group, there are only finitely many such H'.

To show that $S_A(H_1) + \eta_1(C) = B_1 \oplus \eta_1(C)$, we need to show that $S_A(H_1) \subseteq B_1 \oplus \eta_1(C)$. Since $S_A(H_1) = B_1 \oplus S_A(C_1)$, this amounts to showing that

$S_A(C_1) \subseteq \eta_1(C)$. We claim first that $S_A(C_1) \subseteq H$. Otherwise there is a map $f\colon A \to C_1$ such that $\alpha f \neq 0$ and αf is a monomorphism (since $\alpha f\colon A \to A$). It readily follows that $f(A)$ is a quasi-summand of C_1, so that $B \oplus \alpha f(A)$ is an A-free quasi-summand of H, contradicting the maximality of K. Thus $S_A(C_1) \subseteq H$, and so $S_A(C_1) \subseteq S_A(H) \subseteq B \oplus C$ and $S_A(C_1) = \eta_1(S_A(C_1)) \subseteq (\eta_1(B) \oplus \eta_1(C)) \cap C_1 = \eta_1(C)$. This completes the proof.

Fuchs-Loonstra [9, p. 138] showed that for any $n \geq 2$ there is a torsion free abelian group G of rank 3 such that $G = A_i \oplus H_i$, $1 \leq i \leq n$ where $\{A_1, \ldots, A_n\}$ is a set of isomorphic groups of rank 1; $H_i \approx H_j$ iff $i = j$; and each H_i is quasi-isomorphic to $A_i \oplus B$ for some rank 1 group B such that type (A_i) and type (B) are incomparable. The following corollary indicates that these examples were well chosen.

Corollary 4.6 (J. Cook [6]). *Assume that G is a torsion free abelian group of rank 3.*

(a) *Suppose that $G = A_1 \oplus H_1 = A_2 \oplus H_2$, where A_1 and A_2 are rank 1 groups, and that H_1 and H_2 are not isomorphic. Then A_1 and A_2 are isomorphic and H_2 (hence H_1) is quasi-isomorphic to $A_1 \oplus B_1$, where B_1 is a rank 1 group such that type (A_1) and type (B_1) are not comparable.*

(b) *G has at most a finite number of nonequivalent direct sum decompositions.*

Proof. (a) By Corollary 4.2, H_2 must be quasi-isomorphic to $A_1 \oplus B_1$ for some B_1. If type (A_1) and type (B_1) are comparable, then $H_2 \simeq A_1 \oplus B_1$ (Beaumont-Pierce [4]) so that G is completely decomposable and $H_1 \simeq H_2$, contradicting the assumptions. Consequently, type (A_1) and type (B_1) are not comparable. As in the proof of Theorem 4.5, $A_1 \cap H_2 = A_2 \cap H_1 = 0$, so it follows that A_1 and A_2 are quasi-isomorphic, hence isomorphic, since both A_1 and A_2 are rank 1 groups.

(b) is now a consequence of (a) and Theorem 4.5.

5. Groups of rank 2. A torsion free abelian group is *homogeneous* if any two pure rank 1 subgroups are isomorphic.

Corollary 5.1. *Let A be a strongly indecomposable torsion free abelian group of rank 2.*

(a) (II) *is true for A.*

(b) *If A is not homogeneous then every finite rank A-projective group is A-free.*

Proof. (a) In this case, End(A) is commutative (e.g., Beaumont-Pierce [4] or Reid [15]), so Corollary 2.4 applies.

(b) It is known that $Q \otimes_Z$ End(A) is either Q or the ring of 2×2 triangular matrices over Q with equal diagonal elements (see Beaumont-Pierce [4] or Reid [15]). In the first case, End(A) is a principal ideal domain, so (I) is true by Corollary 1.2.

In the second case $S = \text{End}(A)/N$ is a subring of Q and thus is a principal ideal domain, where N is the nil radical. Thus finitely generated projective S-modules are free so finitely generated projective End(A)-modules are free (see proof of Theorem 4.1). Now apply Corollary 1.2.

Theorem 5.2. *Let R be a nonprincipal ring of integers in a quadratic algebraic number field, and let I be a nonprincipal ideal in R. Then there exists a strongly indecomposable rank 2 group A such that* End(A) $\simeq R$. *The subgroup IA is a summand of $A \oplus A$ but is not A-free. If n is the order of I in the ideal class group of R, then $[A] - [IA]$ is an element of order exactly n in the Krull-Schmidt-Grothendieck group of the category of finite rank torsion free abelian groups.*

Proof. There exists a rank 2 group A with End(A) $\simeq R$ by [17]. Since R is an integral domain, A is strongly indecomposable. By Theorem 1.1, $B = IA$ is A-projective. (Note that $IA \simeq I \otimes_R A$, since I is projective.) Also $B \oplus \cdots \oplus B \simeq A \oplus \cdots \oplus A$ (n summands), since $I \oplus \cdots \oplus I \simeq R \oplus \cdots \oplus R$ (n summands). Now we claim that $[A]$ and $[B]$ are distinct elements of the Krull-Schmidt-Grothendieck group. Otherwise, it would be true that $A \oplus L \simeq B \oplus L$ for some finite rank torsion free group L. Applying $\text{Hom}_Z(A, *)$ yields an isomorphism $R \oplus M \simeq I \oplus M$ where M is a finite rank torsion free R-module. Now write $M = P \oplus S$, where P is projective and S has no projective summands. Since R is a Dedekind domain, this implies that $\text{Hom}_R(S, I \oplus P) = 0$, so the above isomorphism maps S onto itself and thus induces an isomorphism between $R \oplus P$ and $I \oplus P$. Since R and I represent different elements in the ideal class group, this is in contradiction to the well-known structure theory for finitely generated projective modules over Dedekind domains. Thus we have shown that $[A] - [B]$ is a nontrivial element in the Krull-Schmidt-Grothendieck group with order dividing n. Now if r is a proper divisor of n then $I \oplus \cdots \oplus I$ (r summands) is isomorphic to $R \oplus \cdots \oplus R \oplus I^r$. It follows that $B \oplus \cdots \oplus B \simeq A \oplus \cdots \oplus A \oplus I^rA$. Since the argument above shows that $[I^rA] \neq [A]$, we see that $r([A] - [B]) \neq 0$.

By using a theorem of Zassenhaus [17] or the well-known theorem of Corner [7], we can construct such examples where R is any algebraic number ring. It follows that the Krull-Schmidt-Grothendieck group for the category of finite rank torsion free abelian groups contains a copy of the class group of every algebraic number field (as announced by Arnold [1]).

Example 5.3. There is a homogeneous strongly indecomposable group A of rank 2 and a subgroup B of A such that $S_A(B) = B$, A/B is finite and B is not A-projective (in particular, B and A are not isomorphic). Furthermore, every finite rank A-projective group is A-free.

Proof. Let $R = Z + 2Zi$, a subring of the ring of Gaussian integers. By Zassenhaus [17], there is a rank 2 group A with $\mathrm{End}(A) = R$. Let I be the ideal of R generated by $\{2, 2i\}$, a nonprincipal nonprojective ideal of R. Then $B = IA$ has the desired properties. The latter statement follows from the observation that every invertible ideal of R is principal and every finitely generated projective R-module is a direct sum of ideals of R.

REFERENCES

1. D. M. Arnold, *Grothendieck and Whitehead groups of torsion free abelian groups*, Bull. Amer. Math. Soc. 79 (1973), 723–724. MR 48 #4149.

2. R. Baer, *Abelian groups without elements of finite order*, Duke Math. J. 3 (1937), 68–122.

3. H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR 40 #2736.

4. R. A. Beaumont and R. S. Pierce, *Torsion free groups of rank two*, Mem. Amer. Math. Soc. No. 38 (1961). MR 24 #A162.

5. B. Charles, *Sous-groupes fonctoriels et topologies*, Studies in Abelian Groups (Sympos., Montpellier, 1967), Springer, Berlin, 1968, pp. 75–92. MR 39 #1547.

6. J. Cook, Thesis, Univ. of Illinois, 1971.

7. A. L. S. Corner, *Every countable reduced torsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) 13 (1963), 687–710. MR 27 #3704.

8. L. Fuchs, *Infinite abelian groups*. Vol. I, Pure and Appl. Math., vol. 36, Academic Press, New York and London, 1970. MR 41 #333.

9. ———, *Infinite Abelian groups*. Vol. II, Academic Press, New York, 1973.

10. I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR 40 #7234.

11. L. Lady, *Summands of finite rank torsion free abelian groups*, J. Algebra 32 (1974), 51–52.

12. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR 34 #5857.

13. C. E. Murley, *The classification of certain classes of torsion free abelian groups*, Pacific J. Math. 40 (1972), 647–665. MR 48 #441.

14. ———, *Direct products and sums of torsion-free abelian groups*, Proc. Amer. Math. Soc. 38 (1973), 235–241. MR 47 #362.

15. J. D. Reid, *On quasi-decomposition of torsion free abelian groups*, Proc. Amer. Math. Soc. 13 (1962), 550–554. MR 25 #1209.

16. R. B. Warfield, Jr., *Homomorphisms and duality of torsion-free groups*, Math. Z. 107 (1968), 189–200. MR 38 #5923.

17. H. J. Zassenhaus, *Orders as endomorphism rings of modules of the same rank*, J. London Math. Soc. 42 (1967), 180–182. MR 34 #5876.

DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NEW MEXICO 88003

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66045