THE WEDDERBURN PRINCIPAL THEOREM FOR
GENERALIZED ALTERNATIVE ALGEBRAS I

BY

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ABSTRACT. A generalized alternative ring $R$ is a nonassociative ring in which the identities

$$(wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x;$$

$$(w, x, yz) = y(w, x, z) - (w, x, y)z; \text{ and } (x, x, x) \text{ are identically zero. Let } A \text{ be a}$$

finite-dimensional algebra of this type over a field $F$ of characteristic $\neq 2, 3$. Then it is established that (1) $A$ cannot be a nodal algebra, and (2) the standard Wedderburn principal theorem is valid for $A$.

1. Preliminaries. Let $R$ be a nonassociative ring. For $x, y, z$ in $R$ we denote by $(x, y, z)$ the associator $(x, y, z) = x(yz) - (xy)z$ and by $(x, y)$ the commutator $(x, y) = xy - yx$.

In [3] Kleinfeld has defined a generalized alternative ring $I$ to be a nonassociative ring $R$ such that for all $w, x, y, z$ in $R$ the following identities are satisfied:

$$(1) \quad (wx, y, z) + (w, x, (y, z)) = w(x, y, z) + (w, y, z)x,$$

$$(2) \quad ((w, x), y, z) + (w, x, yz) = y(w, x, z) + (w, x, y)z,$$

$$(3) \quad (x, x, x) = 0.$$

In particular, these identities are satisfied by any alternative ring, that is any ring which satisfies the identities $(x, x, y) = 0 = (y, x, x)$. Conversely, from [3] and [8] it is known that if $R$ is a ring of this type with characteristic $\neq 2, 3$, then $R$ is alternative if $R$ is prime and contains an idempotent $e \neq 1$. Also, from [3] we have that $R$ is alternative if whenever $x, y, z$ are contained in a subring of $R$ generated by two elements and $(x, y, z)^2 = 0$, then $(x, y, z) = 0$.

Throughout this work we shall assume $A$ to be a finite-dimensional generalized alternative algebra $I$ over a field $F$ of characteristic $\neq 2, 3$. We note that from [9] it is then known that if $A$ is a nilalgebra, $A$ is nilpotent.

In addition to the above defining identities, we shall also need to make use of the following:

$$(4) \quad (w, xy, z) - (w, x, yz) + (w, x, y)z - (w, y, z)x = 0,$$

$$(5) \quad (w, xy, z) - (xw, y, z) + w(x, y, z) - y(w, x, z) = 0,$$
Identities (4), (5) and (6) are established in [8], while (7) can be found in [9].

2. Nodal algebras. If $A$ is an algebra over a field $F$ of characteristic $\neq 2$, we can construct a new algebra $A^+$ over $F$, where the vector space operations are the same as those in $A$ but multiplication is defined by the (commutative) product $x \circ y = \frac{1}{2}(xy + yx)$. A Jordan algebra is a commutative algebra which satisfies the identity $(x, y, x^2) = 0$.

**Lemma 1.** If $A$ is a generalized alternative algebra I over a field $F$ of characteristic $\neq 2, 3$, then $A^+$ is a Jordan algebra.

**Proof.** From [3] we have $(x, y, x^2) = 2(x, y, x)x$, $2x(x, y, x) = (x^2, y, x)$, and $(x, y, x)x = x(x, y, x)$, whence

$$(x, y, x^2) = (x^2, y, x).$$

Next letting $z = y = x$ in (4) we obtain $(w, x^2, x) - (w, x, x^2) = 0$, whence

$$(y, x^2, x) = (y, x, x^2).$$

Now using (7), (8), and (9) we have

$$0 = (x, x^2, y) - (x^2, x, y) + (x, y, x^2) - (y, x, x^2) - (y, x^2, x) - (y^2, x) + (x^2, y) - (x, y^2, x) - (y, x, x^2) - (y, x^2, x)$$

$$= 4((x \circ y) \circ x^2 - x \circ (y \circ x^2)).$$

Thus $(x, y, x^2) = 0$ in $A^+$, and so $A^+$ is a Jordan algebra.

Let $A$ be a finite-dimensional power-associative algebra with unity element over a field $F$. If every $x$ in $A$ is of the form $x = ax + n$ with $a$ in $F$ and $n$ nilpotent, and if the set $N$ of nilpotent elements of $A$ does not form a subalgebra of $A$, then $A$ is called a nodal algebra.

Let $A$ be a nodal generalized alternative algebra $I$. Since the Jordan algebra $A^+$ cannot be a nodal algebra [2], $A^+ = F1 + N^+$ where $N^+$ is a nilideal of $A^+$; that is $A = F1 + N$ where $N$ is a subspace of $A$ consisting of all nilpotent elements of $A$, and $x \circ y$ is in $N$ for all $x$ in $N, y$ in $A$. We denote by $N \circ N$ the subspace of $N$ generated by all elements of the form $x \circ y$ with $x, y$ in $N$.

**Lemma 2.** If $A$ is a nodal generalized alternative algebra $I$ over a field $F$ of characteristic $\neq 2, 3$, then $(N \circ N)N \subseteq N$ and $N(N \circ N) \subseteq N$.

**Proof.** From [3] we have $(x, y, x)^2 = 0$, whence $(x, y, x)$ is in $N$ for all $x, y$ in $A$. Continuing now as in the proof of Lemma 2 in [1], we have $(x \circ y)x = \frac{1}{2}(x, y, x) + (yx) \circ x$ is in $N$ for $x$ in $N$. Linearization of $(x \circ y)x$
then gives \((x \circ y)z + (z \circ y)x\) is in \(N\) for \(x, z\) in \(N\). Taking \(z = y\), this in turn yields 
\((x \circ y)y + y^2x\) is in \(N\), whence \(y^2x\) is in \(N\) for \(x, y\) in \(N\). Linearization of \(y^2x\) now gives \(2(y \circ z)x\), hence \((y \circ z)x\) is in \(N\) for \(x, y, z\) in \(N\). Since this implies \(x(y \circ z) = 2((y \circ z) \circ x) - (y \circ z)x\) is also in \(N\), we have shown \((N \circ N)N\) and \(N(N \circ N)\) are contained in \(N\).

**Lemma 3.** If \(A\) is a nodal generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\), then \((x, x, y)\) and \((y, x, x)\) are in \(N\) for all \(x, y\) in \(A\).

**Proof.** It is clear we may assume that \(x, y\) are in \(N\). Let \(xy = \alpha 1 + n\). Then (7) gives 
\[2(x, x, yx) = (x^2, x, y) = x^3y - x^2(xy) = x^3y - \alpha x^2 - x^2n\] is in \(N \circ N + (N \circ N)N\), which by Lemmas 1 and 2 is contained in \(N\). Thus \((x, x, yx)\) is in \(N\). Next from (6) and (3) we have 
\[2(x, x, y) = x \circ (x, x, y)\] is in \(N\), whence \((x, x, xy)\) is also in \(N\). If we take \(y = x\) in (6) and apply (3), we obtain
\[(10) (x, x, xy) = x(x, x, y)\).
Then using (10) we have 
\[x(x(x, x, y)) = x(x, x, xy) = x(x, x, \alpha 1 + n) = x(x, x, n) = (x, x, xn)\] is in \(N\). Hence \((x, x, (x, x, y)) = x^2(x, x, y) - x(x, x, y)\) is in \(N\), since by Lemmas 1 and 2 \(x^2(x, x, y)\) is in \(N\), whence \((x, x, xy)\) is also in \(N\). If we take \(y = x\) in (6) and apply (3), we obtain
\[x(xix,x,y)) = x(x, x, xw) = (x, x, xw)\] is in \(N\). Hence \((x, x, (x, x, y)) = x^2(x, x, y) - x(x, x, y)\) is in \(N\), since by Lemmas 1 and 2 \(x^2(x, x, y)\) is in \(A\). Then using (6) we have \(N\) contains
\[2(x, x, (x, x, y^2)) = 2(x, x, y \circ u) = 2(y \circ (x, x, u) + u \circ (x, x, y)) = 2(y \circ (x, x, u) + u^2).
This implies \(2u^2\), hence \(u^2\), is in \(N\), since \(2y \circ (x, x, u)\) is in \(N\) by Lemma 1. Thus \((x, x, y) = u\) is itself in \(N\). Lastly, linearization of (3) gives \((y, x, x) = -(x, y, x) - (x, x, y)\) is in \(N\), since as in the proof of Lemma 2 we know \((x, y, x)\) to be in \(N\).

**Lemma 4.** If \(A\) is a nodal generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\), then 
\[(N \circ N)N \subseteq N\] and \(N((N \circ N)N) \subseteq N\).

**Proof.** Let \(x, y, z\) be in \(N\). Then from (1) we have \((x^2, y, z) = -(x, x, (y, z)) + 2x \circ (x, y, z)\), whence \((x^2, y, z)\) is in \(N\) by Lemmas 1 and 3. Since \(x^2(yz)\) is in \(N\) by Lemmas 1 and 2, this in turn implies \((x^2y)z = (x^2, y, z) + x^2(yz)\) is in \(N\). Linearization of \((x^2y)z\) now yields \(2((w \circ x)y)z\), hence \(((w \circ x)y)z\) is in \(N\) for \(w, x, y, z\) in \(N\). Since this implies \(z((w \circ x)y) = 2((w \circ x)y) \circ z - ((w \circ x)y)z\) is also in \(N\), we have proven \((N \circ N)N)N\) and \(N((N \circ N)N)\) to be contained in \(N\).

**Lemma 5.** If \(A\) is a nodal generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\), then \(K = N \circ N + (N \circ N)N + ((N \circ N)N)N\) is an ideal of \(A\) contained in \(N\).
PROOF. That $K$ is contained in $N$ follows directly from Lemmas 1, 2, and 4. Take $x = \alpha 1 + n$ in $A$, $k$ in $K$. Then $kx = \alpha k + kn$ and $xk = \alpha k + nk = \alpha k + 2n \circ k - kn$. Thus $AK$ and $KA$ are both contained in $K + KN$, and so to show $K$ is an ideal of $A$ it is sufficient to show $((N \circ N)N)N$, hence $KN$, is contained in $K$.

Let $u, v, w, x, y, z$ be in $N$. Then taking $w = u \circ v$, from (2) we obtain

\[ ((x(u \circ v))y)z + y(((u \circ v)x)z) = 2((x \circ (u \circ v))y)z - (((u \circ v)x)y)z) \]

this gives

(i) $((u \circ v)x)z \equiv -(((u \circ v)x)z)u \mod K$.

Now from (1) we obtain $(((u \circ v)x)z)z = (((u \circ v)x)z)z = (((u \circ v)x)z)z = (((u \circ v)x)z)z$. Using (i) this gives

(ii) $(((u \circ v)x)z)z \equiv -(((u \circ v)x)z)u \mod K$.

Next taking $w = u \circ v$, from (1) we obtain

\[ w(x(z(u \circ v))) - w((w(u \circ v))z) = (w(u \circ v))z + (w(u \circ v))z = \]

(is in $K$. Letting $w = v$ this gives

(iii) $y(x(z(u \circ v))) \equiv y(((u \circ v)x)z)z + (y(((u \circ v)x)z))z \equiv 0 \mod K$.

Noting that $nk + kn = 2n \circ k$ implies $nk \equiv -kn \mod N \circ N$ and so that also $N(N \circ N), N(N(N \circ N)), (N(N \circ N))N, N((N \circ N)N)$ are in $K$, from (iii) we now have

$0 \equiv y(x(z(u \circ v))) - y(((u \circ v)x)z)z + (y(((u \circ v)x)z))z \equiv -((x(z(u \circ v)))y + ((x(u \circ v))z)y + (((u \circ v)x)z)y)x - (((u \circ v)x)y)z) \equiv -(((((u \circ v)z)x)y + (((u \circ v)x)y)z + (((u \circ v)x)y)z) \mod K$. 
That is

\[(iv) - (((u \circ v)x)z)x)y - (((u \circ v)x)z)x)y + (((u \circ v)y)z)x - (((u \circ v)y)z)x \equiv 0 \mod K.\]

Finally, applying (ii) to (iv) we have \(0 \equiv (((u \circ v)x)z)x)y - (((u \circ v)x)z)x)y + (((u \circ v)y)z)x - (((u \circ v)y)z)x \mod K.\) Thus \(((N \circ N)N)N\) is contained in \(K\), and so it follows that \(K\) is an ideal of \(A\) contained in \(N\).

**Lemma 6.** There are no nodal generalized alternative algebras \(I\) over fields of characteristic \(\neq 2, 3\) such that \(n^2 = 0\) for each \(n\) in \(N\).

**Proof.** Suppose that \(A\) is a nodal generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\) such that \(n^2 = 0\) for each \(n\) in \(N\). We first note that for \(x, y\) in \(N\) we have \(0 = (x + y)^2 = xy + yx\) implies \(xy = -yx\). Let \(xy = a1 + n = -yx\). Then taking \(w = x\) and \(z = y\) in (1) we have

\[
0 = (x^2)y - x^2y^2 - x((xy)y) + x(xy^2) - ((xy)y)x + (xy^2)x
\]

\[
= -x((xy)y) - ((xy)y)x = -x((a1 + n)y) - ((a1 + n)y)x
\]

\[
= -axy - x(ny) - ayx - (ny)x = -2x \circ (ny).
\]

Thus

\[(v) x \circ (ny) = 0.\]

Next taking \(w\) and \(y\) as \(x, x\) and \(z\) as \(y\) in (1) we have

\[
0 = ((xy)x)y - (xy)(yx) + x(y(yx)) - x((yx)y) - x^2y^2 + x(xy)
\]

\[
= ((xy)x)y + (xy)(xy) - x(y(xy)) + x(xy)y - (xy)(xy)
\]

\[
= ((a1 + n)x)y + (a1 + n)^2 - x(y(a1 + n)) + x((a1 + n)y) + (x(a1 + n)y)
\]

\[
= axy + (nx)y + a^2y + 2an + n^2 - axy - x(ny) + axy + x(ny) + axy + (nx)y
\]

\[
= a^2 + 2an + 2axy + 2x(ny) = 3a^2 + 4an + 2(nx).
\]

Thus

\[(vi) 3a^2 + 4an + 2x(ny) = 0.\]

Now taking \(w = y\) and \(z = x\) in (2) we have

\[
0 = ((xy)y)x - (xy)(yx) + y(x(yx)) + y((yx)x) - y(yx^2) - (y(xy)x)
\]

\[
= ((xy)y)x + (xy)(xy) - y(x(xy)) - y((xy)x) - (y(xy)x)
\]

\[
= ((a1 + n)y)x + (a1 + n)^2 - y(x(a1 + n)) - y((a1 + n)x) - (y(a1 + n)x)
\]

\[
= axy + (nx)x + a^2y + 2an + n^2 - axy - x(nx) - ayx - y(nx) - ayx - (ny)x
\]

\[
= a^2x + 2an + 2axy + 2xy + 2(nx) = 3a^2 + 4an + 2(nx).
\]

Thus

\[(vii) 3a^2 + 4an + 2(nx)x = 0.\]

Finally, adding (vi) and (vii) and using (v) we obtain \(0 = 6a^2 + 8an + 4x \circ (ny) = 6a^2 + 8an\). But then \(6a^2 = 0\) implies \(a = 0\), that is \(xy\) is in \(N\) for every \(x, y\).
in $N$. Since this means the set $N$ of nilpotent elements of $A$ is a subalgebra of $A$, $A$ cannot be a nodal algebra.

**Theorem 1.** There are no nodal generalized alternative algebras I over fields of characteristic $\neq 2, 3$.

**Proof.** Suppose that $A$ is a nodal generalized alternative algebra I over a field $F$ of characteristic $\neq 2, 3$. Then $A$ has a homomorphic image which is a simple nodal algebra, and so we can assume $A$ itself to be simple. Now by Lemma 5, since the ideal $K = N \cap N + (N \cap N)N + (N \cap N)N$ of $A$ is contained in $N$, it must be zero. In particular, $N \cap N = 0$, and so $n^2 = 0$ for each $n$ in $N$. But then, by Lemma 6, $A$ cannot be a nodal algebra.

3. Wedderburn principal theorem.

**Lemma 7.** Let $A$ be a generalized alternative algebra I. If $B$ is an ideal of $A$, then $AB^2 + B^2A + B^2$ and $B^* = B^2 + A(BB^2) + (B^2B)A$ are ideals of $A$ with $B^* \subseteq B^2$.

**Proof.** Take $a_i$ in $A$, $b_j$ in $B$ for $i = 1, 2; j = 1, 2, 3$. Then from (1) we have $a_i(b_1, b_2, a_2) + (a_1, b_2, a_2)b_1 = (a_1b_1, b_2, a_2) + (a_1, b_1, (b_2, a_2))$, whence $A(B^2A) \subseteq AB^2 + B^2A + B^2$. Also from (1) we have $(b_1, b_2, a_1)b_2 + b_1(a_2, b_2, a_1) = (a_1b_1, b_2, a_2) + (a_1, b_2, (b_1, a_2, a_1))$, whence $(B^2A)A \subseteq B^2A + B^2$. Now using (2), in symmetric fashion one obtains that $(AB^2)A \subseteq AB^2 + B^2A + B^2$ and $A(AB^2) \subseteq AB^2 + B^2$. Thus $AB^2 + B^2A + B^2$ is an ideal of $A$.

To show $B^*$ is an ideal of $A$, we first note that from (1) we have $(b_1b_2, a_1, b_2) + (b_1, b_2, (a_1, b_2)) = b_1(b_2, a_1, b_3) + (b_1, a_1, b_3)b_2$ or $((b_1b_2)a_1)b_3$ is in $B^3$. Symmetrically from (2) one also has $b_1(a_1(b_2b_3))$ is in $B^3$. Hence

\[(\text{viii)} \quad (B^2A)B, B(AB^2) \subseteq B^3.\]

Now (5) gives $a_1(b_1, b_2, b_3) = (b_1, b_2a_1, b_3) - (b_2b_1, a_1, b_3) + b_1(b_2, a_1, b_3)$, and using (viii) this implies $A(B^2B) \subseteq B^3 + A(BB^2) \subseteq B^*$. Symmetrically (4) and (viii) imply $(BB^2)A \subseteq B^3 + (B^2B)A \subseteq B^*$. Thus we have shown $AB^3, B^3A \subseteq B^*$. Next, letting $z = b_2b_3$, (2) yields $a_1(a_2, b_1, b_2b_3) + (a_2, b_1, a_1)(b_2b_3) = ((a_2, b_1), a_1, b_2b_3) + (a_2, b_1, a_1(b_2b_3))$, whence using (viii) and that $AB^3 \subseteq B^*$ we have $A(AB^2B) \subseteq B^*$. Then using our earlier calculations that $A(B^2B) \subseteq B^3 + A(BB^2)$ and $AB^3 \subseteq B^*$, this in turn gives $A(AB^2B) \subseteq AB^3 + A(BB^2) \subseteq B^*$. Still letting $z = b_2b_3$, (4) now yields $a_1(a, b_1, b_2b_3a_2 = (a_1, a_2b_1, b_2b_3) - (a_1, a_2, (b_2b_3)a_1) + (a_1, a_2, b_1)(b_2b_3)$, whence using $B^3A, AB^3, A(A(B^2B)) \subseteq B^*$ we have $A(ABB^2) \subseteq B^*$. Thus we have shown $A(ABB^2), (AB^2B)A \subseteq B^*$. In symmetric fashion using (1) and (5) one also has $((B^2B)A)A, A((B^2B)A) \subseteq B^*$; and this completes the proof that $B^*$ is an ideal of $A$. 

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Finally, from (1) we have \((b_1b_2, b_3, a_1) + (b_1, b_2, (b_3, a_1)) = b_1(b_2, b_3, a_1) + (b_1, b_3, a_1)b_2\) or \((B^2B)A \subseteq B^2\). Symmetrically from (2) we have \(A(BB^2) \subseteq B^2\), and thus \(B^* \subseteq B^2\).

For any nonassociative algebra \(A\) one can obtain a descending chain of subalgebras \(A^{(0)} \supseteq A^{(1)} \supseteq \cdots \supseteq A^{(n)} \supseteq \cdots\) by defining inductively \(A^{(0)} = A,\)

\[A^{(i+1)} = (A^{(i)})^2.\]

The algebra \(A\) is called solvable in case \(A^{(n)} = 0\) for some \(n\).

Let \(A\) be a generalized alternative algebra \(I\). If \(B\) is any ideal of \(A\), we define \(B^{(i)}\) inductively by \(B^{(0)} = B, B^{(i+1)} = A(B^{(i)}B + B^{(i)}A + B^{(i)}B^2)\). Then by Lemma 7 we obtain a descending chain \(B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(m)} \supseteq \cdots\) of ideals of \(A\) which we call a Penico sequence. As in [7], we shall call \(B\) Penico solvable in case there is some \(m\) for which \(B^{(m)} = 0\).

**Lemma 8.** Let \(A\) be a generalized alternative algebra \(I\). If \(B\) is an ideal of \(A\), then \(B\) is solvable if and only if \(B\) is Penico solvable.

**Proof.** If \(B\) is Penico solvable, then \(B\) is solvable since \(B^{(i)} \subseteq B^{(i+1)}\). To see that \(B\) solvable implies \(B\) is Penico solvable, suppose \(B^{(2)} \subseteq B^* \subseteq B^2 \subseteq B^{(1)}\). Then, as in the proof of Theorem 3 in [7], by induction one has \(B^{(2k)} \subseteq B^{(k)}\), since \(B^{(2k+1)} = (B^{(2k)})^2 \subseteq (B^{(2k)})^{(1)} \subseteq (B^{(k)})^{(1)} = B^{(k+1)}\). Hence if \(B\) is solvable, then \(B^{(2k)} \subseteq B^{(k)} = 0\) for some \(k\), and \(B\) is Penico solvable. Thus to prove \(B\) solvable implies \(B\) is Penico solvable, it is sufficient to prove


To do this, since by Lemma 7 \(B^*\) is itself an ideal of \(A\), it is in turn sufficient to show \((AB^2 + B^2 A + B^2)^2 \subseteq B^*\). Now \(B\) an ideal of \(A\) gives \(B^2 B^2, B^2 (B^2 A), (B^2 A) B^2, (B^2 A) B^2 \subseteq B^3 \subseteq B^*\). Also, using (viii) from the proof of Lemma 7, we have \((B^2 A) (B^2 A) (B^2 A) (B^2 A) (B^2 A) (B^2 A) (B^2 A) \subseteq B^3 \subseteq B^*\) and \((AB^2) (AB^2) \subseteq B(AB^2) \subseteq B^3 \subseteq B^*\). Lastly, taking \(a\) in \(A\), \(b_i\) in \(B\), from (1) we obtain

\[b_1(b_2, b_3, a) + (b_1, b_3, a)b_2 = (b_1 b_2, b_3, a) + (b_1, b_2, (b_3, a)),\]

whence again using (viii) we have \(B^2 A \subseteq B^3 + (B^2 A) B + (B^2 A) A \subseteq B^3 + (B^2 B) A \subseteq B^*\). But then \((AB^2) (AB^2) \subseteq B(AB^2) \subseteq B^*\). We now have shown \((AB^2 + B^2 A + B^2)^2 \subseteq B^*\), and so our proof is complete.

Let \(A\) be a finite-dimensional generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\). We define the radical \(N\) of \(A\) to be the maximal nilideal (= solvable = nilpotent [9]) of \(A\), and we call \(A\) semisimple in case \(N = 0\). If, in addition, \(A\) is semisimple over every scalar extension of the base field \(F\), then \(A\) is said to be separable. We note too that \(A/N\), as is the case for any power-associative algebra, is semisimple.

**Theorem 2.** Let \(A\) be a finite-dimensional generalized alternative algebra \(I\) over a field \(F\) of characteristic \(\neq 2, 3\). If \(A\) is semisimple, then \(A\) has a unity
element and is the direct sum of simple algebras.

Proof. The proof is the same as that of Theorem 9 in [4].

Theorem 3 (Wedderburn principal theorem). Let $A$ be a finite-dimensional generalized alternative algebra over a field $F$ of characteristic $\neq 2, 3$. If $A/N$ is separable, then $A = S + N$ (vector space direct sum) where $S$ is a subalgebra of $A$ such that $S \cong A/N$.

Proof. If $A$ has dimension one, then since either $N = 0$ or $N = A$ the theorem is clearly true. We make an induction on the dimension of $A$ and assume the theorem to be true for all algebras of lesser dimension.

Now, as in the proof of Theorem 2.4 in [9], it is possible to make the following standard reductions. First one may assume $N$ not to properly contain any ideals of $A$. Then using Theorem 1.3 in [9] and our Lemma 8, one can reduce to the case $N^2 = 0$, and hence to the case $F$ is an algebraically closed field. Next, using our Theorem 2 and the fact from [3] that $A_1$ and $A_0$ are subalgebras in the Albert decomposition for $A$ relative to an idempotent $e$, we can use Theorem 2.1 in [5] to assume $A$ has a unity element and that $A/N$ is simple. As a final reduction we note, as in the proof of Theorem 2.2 of [5], that if there exists a primitive idempotent $e$ such that our theorem is true for the ideal of $A$ generated by the subspace $A_{e/2}$ in the Albert decomposition of $A$, then it is valid for $A$ as well.

We suppose first that $I$ is the only idempotent in $A/N$. Then since we are assuming the field $F$ to be algebraically closed, and since by Theorem 1 there are no nodal generalized alternative algebras over fields of characteristic $\neq 2, 3$, we must have $A/N = Fl$. Now by Lemma 2.1 in [5], $1$ lifts to an idempotent $e$ in $A$. But then we have $Fe$ a subalgebra of $A$ such that $Fe \cong A/N$, and our theorem is proven. Thus we may assume that $A/N$, hence $A$, contains an idempotent $e \neq 1$. Furthermore, since $A$ is finite-dimensional, we can take $e$ to be primitive. Now by Theorem 1 in [8], $I = (e, e, A)$ is an ideal of $A$ such that $I^2 = 0$. Hence, since we are assuming $N$ not to properly contain any ideals of $A$, either $I = 0$ or $I = N$.

Suppose that $I = 0$. Then, as in the proof of the corollary in [8], $A$ has a Peirce decomposition relative to $e$. Let $H = A_{10}A_{01} + A_{10} + A_{01} + A_{01}A_{10}$. As in the proof of Theorem 2 in [8], $H$ is an ideal of $A$. In particular, $H$ must be the ideal of $A$ generated by $A_{e/2} = A_{10} + A_{01}$. Now if $H$ is a proper ideal of $A$, then our induction assumption implies that the theorem is true for $H$. But then our final reduction applies, and so we may conclude that the theorem is true for $A$ itself. On the other hand, if $H = A$, then $A_{11} = A_{10}A_{01}$ and $A_{00} = A_{01}A_{10}$. Take $w_{ij}, x_{ij}, y_{ij}, z_{ij}$ in $A_{ij}$ for $i, j = 0, 1$. Then using the fact established in [3] that
the Peirce subspaces of a generalized alternative algebra $I$ multiply the same as for an alternative algebra, from (4) we obtain
\[
(w_{11}, x_{11}, z_{10}y_{01}) = (w_{11}, x_{11}y_{01}, z_{10}) + (w_{11}, x_{11}, y_{01})z_{10} - (w_{11}, y_{01}, z_{10})x_{11} = 0,
\]
and
\[
(w_{00}, x_{00}, z_{01}y_{10}) = (w_{00}, x_{00}y_{10}, z_{01}) + (w_{00}, x_{00}, y_{10})z_{01} - (w_{00}, y_{10}, z_{01})x_{00} = 0.
\]
Hence $A_{11}$ and $A_{00}$ are associative subalgebras of $A$. But then it follows from the proof of Theorem 2 in [8] and the proof of Theorem 3 in [3] that $A$ is an alternative algebra. Since in this case the theorem is known from [6] to be valid for $A$, our induction is complete.

We consider now the other possibility, namely $I = N$, and take $k = (e, e, x) \neq 0$. For the Albert decomposition of $A$, we have from [3] that $A_{1/2}A_{1/2} \subseteq A_{1/2}$ for $i = 0, 1$. In particular, this says that $N = (e, e, A) \subseteq A_{1/2}$. Now if $H$ is the ideal in $A$ generated by the subspace $A_{1/2}$, then $H = A_{1/2} + (A_{1/2})^2$. To see this, take $x_{1/2}, y_{1/2}, z_{1/2}$ in $A_{1/2}$ for $i = 0, 1$. Then for $i = 0, 1$ we have
\[
(x_{1/2}y_{1/2})z_{1/2} = (x_{1/2}, y_{1/2}, z_{1/2}) + x_{1/2}(y_{1/2}z_{1/2})
\]
\[
= (x_{1/2}, y_{1/2} + z_{1/2}, y_{1/2} + z_{1/2}) - (x_{1/2}, y_{1/2}, y_{1/2})
\]
\[- (x_{1/2}, z_{1/2}, z_{1/2}) - (x_{1/2}, z_{1/2}, y_{1/2}) + x_{1/2}(y_{1/2}z_{1/2}).
\]
But by Theorem 3 in [3] our assumption that $A/N$ is simple implies $A/N$ is alternative, that is $(a, a, b)$ and $(b, a, a)$ are in $N$ for all $a, b$ in $A$, so we have shown that $(x_{1/2}y_{1/2})z_{1/2}$ is in $N + (A_{1/2})^2 \subseteq A_{1/2} + (A_{1/2})^2$. Similarly one has $z_{1/2}(x_{1/2}y_{1/2})$ is in $A_{1/2} + (A_{1/2})^2$ for $i = 0, 1$. Since the cases for $i = 1/2$ are immediate if one writes $x_{1/2}y_{1/2} = a_{1/2} + a_{1/2} + a_{0}$ with $a_{i}$ in $A_{i}$, we have established $H = A_{1/2} + (A_{1/2})^2$ as claimed. Now by Theorem 1 in [8] $Hk = 0$, but from the proof of that same Theorem 1 $ek = \frac{1}{2}k \neq 0$. Hence $e$ is not in $H$. But then $H$ is a proper ideal of $A$, and so by the induction assumption the theorem is true for $H$. Our final reduction now applies to complete the induction and the proof of the theorem.

BIBLIOGRAPHY


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