ON SOME REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A principal circle bundle over a real hypersurface of a complex projective space $\mathbb{C}P^n$ can be regarded as a hypersurface of an odd-dimensional sphere. From this standpoint we can establish a method to translate conditions imposed on a hypersurface of $\mathbb{C}P^n$ into those imposed on a hypersurface of $S^{2n+1}$. Some fundamental relations between the second fundamental tensor of a hypersurface of $\mathbb{C}P^n$ and that of a hypersurface of $S^{2n+1}$ are given.

Introduction. As is well known a sphere $S^{2n+1}$ of dimension $2n + 1$ is a principal circle bundle over a complex projective space $\mathbb{C}P^n$ and the Riemannian structure on $\mathbb{C}P^n$ is given by the submersion $\pi: S^{2n+1} \to \mathbb{C}P^n$ [4], [7]. This suggests that fundamental properties of a submersion would be applied to the study of real submanifolds of a complex projective space. In fact, H. B. Lawson [2] has made one step in this direction. His idea is to construct a principal circle bundle $\widetilde{M}^{2n}$ over a real hypersurface $M^{2n-1}$ of $\mathbb{C}P^n$ in such a way that $\widetilde{M}^{2n}$ is a hypersurface of $S^{2n+1}$ and then to compare the length of the second fundamental tensors of $M^{2n-1}$ and $\widetilde{M}^{2n}$. Thus we can apply theorems on hypersurfaces of $S^{2n+1}$.

In this paper, using Lawson’s method, we prove a theorem which characterizes some remarkable classes of real hypersurfaces of $\mathbb{C}P^n$. First of all, in §1, we state a lemma for a hypersurface of a Riemannian manifold of constant curvature for the later use. In §2, we recall fundamental formulas of a submersion which are obtained in [4], [7] and those established between the second fundamental tensors of $M$ and $\widetilde{M}$. In §3, we give some identities which are valid in a real hypersurface of $\mathbb{C}P^n$. After these preparations, we show, in §4, a geometric meaning of the commutativity of the second fundamental tensor of $M$ in $\mathbb{C}P^n$ and a fundamental tensor of the submersion $\pi: \widetilde{M} \to M$.

1. Hypersurfaces of a Riemannian manifold of constant curvature. Let $\widetilde{M}$ be an $(m + 1)$-dimensional Riemannian manifold with a Riemannian metric $\tilde{G}$ and $\iota: \widetilde{M} \to \widetilde{M}$ be an isometric immersion of an $m$-dimensional differentiable
manifold $\tilde{M}$ into $\tilde{M}$. The Riemannian metric $\bar{g}$ of $\tilde{M}$ is naturally induced from $\bar{g}$ in such a way that $\bar{g}(\bar{X}, \bar{Y}) = \bar{g}(i(\bar{X}), i(\bar{Y}))$, where $\bar{X}, \bar{Y}$ are vector fields on $\tilde{M}$ and we denote by the same letter $i$ the differential of the immersion. For an arbitrary point $x \in \tilde{M}$, we choose a unit normal vector and extend it to a field $\bar{N}$. The Riemannian connections $\bar{D}$ in $\tilde{M}$ and $\bar{\nabla}$ in $\tilde{M}$ are related by the following formulas:

\begin{align*}
\bar{D}_{i(\bar{X})} i(\bar{Y}) &= i(\bar{\nabla}_{\bar{X}} \bar{Y}) + \bar{g}(\bar{H} \bar{X}, \bar{Y}) \bar{N}, \\
\bar{D}_{i(\bar{X})} \bar{N} &= -i(\bar{H} \bar{X}),
\end{align*}

where $\bar{H}$ is the second fundamental tensor of $\tilde{M}$ in $\tilde{M}$.

The mean curvature $\mu$ of $\tilde{M}$ in $\tilde{M}$ is defined by

\begin{equation}
\mu = \text{trace } \bar{H}.
\end{equation}

Let $\bar{R}$ and $\bar{\nabla}$ be curvature tensors of $\tilde{M}$ and of $\tilde{M}$ respectively, then we have the following Gauss and Mainardi-Codazzi equations:

\begin{align*}
\bar{\nabla} \bar{R}(i(\bar{X}), i(\bar{Y})) &\equiv (\bar{Z}, i(\bar{W})) = \bar{g}(\bar{R}(\bar{X}, \bar{Y}) \bar{Z}, \bar{W}) - \bar{g}(\bar{H} \bar{Y}, \bar{Z}) \bar{g}(\bar{H} \bar{X}, \bar{W}) \\
&\phantom{=} + \bar{g}(\bar{H} \bar{X}, \bar{Z}) \bar{g}(\bar{H} \bar{Y}, \bar{W}), \\
\bar{\nabla} \bar{R}(i(\bar{X}), i(\bar{Y})) &\equiv (\bar{Z}, i(\bar{W})) = \bar{g}(\bar{\nabla}_{\bar{X}} \bar{H} \bar{Y}, \bar{Z}) - \bar{g}(\bar{\nabla}_{\bar{Y}} \bar{H} \bar{X}, \bar{Z}),
\end{align*}

where $\bar{X}, \bar{Y}, \bar{Z}$ and $\bar{W}$ are vector fields on $\tilde{M}$.

If the ambient manifold is of constant curvature $k$, the curvature tensor $\bar{R}$ has the form

\begin{equation}
\bar{R}(\bar{X}, \bar{Y}) \bar{Z} = k(\bar{g}(\bar{Y}, \bar{Z}) \bar{X} - \bar{g}(\bar{X}, \bar{Z}) \bar{Y})
\end{equation}

for vector fields $\bar{X}, \bar{Y}$ and $\bar{Z}$ on $\tilde{M}$. Consequently we have

\begin{align*}
\bar{R}(\bar{X}, \bar{Y}) \bar{Z} &= k(\bar{g}(\bar{Y}, \bar{Z}) \bar{X} - \bar{g}(\bar{X}, \bar{Z}) \bar{Y}) + \bar{g}(\bar{H} \bar{Y}, \bar{Z}) \bar{H} \bar{X} - \bar{g}(\bar{H} \bar{X}, \bar{Z}) \bar{H} \bar{Y}, \\
(\bar{\nabla}_{\bar{X}} \bar{H}) \bar{Y} &= (\bar{\nabla}_{\bar{Y}} \bar{H}) \bar{X}.
\end{align*}

We assume that $\tilde{M}$ has constant mean curvature, that is, $\text{trace } \bar{H} = \text{const}$.

Let $\{\bar{E}_1, \ldots, \bar{E}_m\}$ be an orthonormal basis in $T_x(\tilde{M})$ and extend them to vector fields in a normal neighborhood of $\bar{x}$ by parallel translation along geodesics with respect to the Riemannian connection of $\tilde{M}$. Then we have $\bar{\nabla}_{\bar{E}_i} = 0$ ($i = 1, \ldots, m$) at $\bar{x}$. Since $\bar{H}$ and $\bar{\nabla}_{\bar{E}_i} \bar{H}$ are both symmetric linear transformations on $T(\tilde{M})$, we get, by using (1.8)

\begin{align*}
\bar{g} \left( \sum_{i=1}^{m} (\bar{\nabla}_{\bar{E}_i} \bar{H}) \bar{E}_i, \bar{X} \right) &= \sum_{i=1}^{m} \bar{g}(\bar{E}_i, (\bar{\nabla}_{\bar{E}_i} \bar{H}) \bar{X} = \sum_{i=1}^{m} \bar{g}(\bar{E}_i, (\bar{\nabla}_{\bar{X}} \bar{H}) \bar{E}_i) \\
&= \text{trace}(\bar{\nabla}_{\bar{X}} \bar{H}) = \bar{X}(\text{trace } \bar{H}) = 0,
\end{align*}
which implies that
\[(1.9) \quad \sum_{i=1}^{m} (\nabla_{E_i} H)E_i = 0.\]
Thus we have
\[(1.10) \quad \sum_{i=1}^{m} (\nabla_{X}(\nabla_{E_i} H))E_i = 0 \quad \text{at } \bar{x}.\]
Now we prove the

**Lemma 1.1.** Let \(\bar{M}\) be a hypersurface of a Riemannian manifold of constant curvature \(k\). If the second fundamental tensor \(H\) satisfies for a constant \(\alpha\),
\[(1.11) \quad H^2X = \alpha HX + kX, \quad X \in (\bar{M})\]
then we have \(\nabla H = 0\).

**Proof.** Since \(H\) is a symmetric operator and (1.7), (1.8) are valid, we have
\[
(\nabla_{X}(\nabla_{Y} H) - \nabla_{Y}(\nabla_{X} H) - \nabla_{[X,Y]} H)\bar{Z} = R(X, Y)HZ - H(R(X, Y)\bar{Z})
\]
\[= k \left( g(Y, HZ)X - g(X, HZ)Y \right) + g(HY, HZ)HX - g(HX, HZ)HY
\]
\[- k \left( g(Y, \bar{Z})HX - g(X, \bar{Z})HY \right) - g(HY, \bar{Z})H^2X + g(HX, \bar{Z})H^2Y = 0.\]

Let \((\bar{E}_1, \ldots, \bar{E}_m)\) be an orthonormal basis which is chosen as above and \(\bar{X}\) be a tangent vector at \(\bar{x}\). Extend \(\bar{X}\) to a vector field in a normal neighborhood of \(\bar{x}\) by parallel translation along geodesics, then \(\nabla \bar{X} = 0\) at \(\bar{x}\). In the last equation we replace \(Y\) and \(Z\) by \(\bar{E}_i\) and sum over \(i\). Then we have, from (1.8) and (1.10),
\[
(1.12) \quad \sum_{i=1}^{m} (\nabla_{E_i} (\nabla_{X} H))E_i = \sum_{i=1}^{m} (\nabla_{E_i} (\nabla_{E_i} H))\bar{X} = 0 \quad \text{at } \bar{x},
\]
because from (1.11) we know that \(\bar{M}\) has constant mean curvature. Furthermore (1.11) implies that trace \(H^2 = \alpha \text{ trace } H + mk = \text{const.} \) Differentiating this covariantly, we have
\[
\frac{1}{2} \nabla \bar{X} (\text{trace } H^2) = \text{trace } (\nabla_{X}(\nabla_{X} H))H + \text{trace } (\nabla_{Y} H)(\nabla_{X} H) = 0,
\]
from which, at \(\bar{x}\),
\[
\text{trace } (\nabla_{X} H)^2 = - \text{trace } (\nabla_{Y} H)(\nabla_{X} H) = - \sum_{i=1}^{m} g((\nabla_{E_i}(\nabla_{E_i} H))E_i, HE_i).
\]
Thus we have
\[
\bar{g}(\nabla H, \nabla H) = \sum_{i=1}^{m} \text{trace } (\nabla_{E_i} H)^2 = - \sum_{i,j=1}^{m} g((\nabla_{E_i}(\nabla_{E_j} H))E_i, HE_i) = 0,
\]
because of (1.12). This completes the proof.
2. Submersion and immersion. Let $\bar{M}$ and $M$ be differentiable manifolds of dimension $n + 1$ and $n$ respectively and assume that there exists a differentiable mapping $\pi$ of $\bar{M}$ onto $M$ which has maximum rank, that is, each differential map $\pi_*$ of $\pi$ is onto. Hence, for each $x \in M$, $\pi^{-1}(x)$ is a 1-dimensional submanifold of $\bar{M}$, which is called the fibre over $x$. We suppose that every fibre is connected. A vector field on $\bar{M}$ is called vertical if it is always tangent to fibres, horizontal if always orthogonal to fibres; we use corresponding terminology for individual vectors. Thus $\bar{X} \in T_x(\bar{M})$ decomposes as $\bar{X}^V + \bar{X}^H$, where $\bar{X}^V$ and $\bar{X}^H$ denote respectively vertical part and horizontal part of $\bar{X}$.

We assume that the mapping $\pi$ is a Riemannian submersion, that is, there are given in $\bar{M}$ a vertical vector field $\bar{V}$ and a Riemannian metric $\bar{g}$ of $\bar{M}$ satisfying the condition that $\bar{V}$ is a unit Killing vector field with respect to the Riemannian metric $\bar{g}$. Then a Riemannian metric $g$ can be defined on $M$ by

$$g(X, Y)(x) = \bar{g}(X^L, Y^L)(\pi(x)),$$

where $x$ is an arbitrary point of $\bar{M}$ such that $\pi(x) = x$ and $X^L$, $Y^L$ are the lifts of $X$, $Y \in T_x(M)$ respectively. Hence we have

$$g(X, Y)^L = \bar{g}(X^L, Y^L).$$

The fundamental tensor $F$ of the submersion $\pi$ is a skew-symmetric tensor of type $(1,1)$ on $M$ and is related to covariant differentiation $\nabla$ and $\nabla$ in $\bar{M}$ and $M$, respectively, by the following formulas:

$$\nabla_{Y^L}X^L = (\nabla_Y X)^L + \bar{g}(F^L Y^L, X^L)\bar{V} = (\nabla_Y X)^L + g(FY, X)^L \bar{V},$$

$$\nabla_Y X^L = \nabla_{X^L} \bar{V} = -F^L X^L.$$
borhood of $x \in M$, the lift $N^L$ is a field of unit normal vectors to $\tilde{M}$ defined in a tubular neighborhood of $\tilde{x}$, where $\tilde{x}$ is an arbitrary point on a fibre over $x$.

We denote by $\tilde{D}, \tilde{V}, D$ and $V$ the Riemannian connections of $\tilde{M}, \tilde{M}, M'$ and $M$ respectively. By means of (1.1), (2.3) and (2.4), we have

$$\tilde{D}_{\tilde{v}(x^L)} \tilde{v}(y^L) = \tilde{v}(X^L Y^L + g(HX^L, Y^L) N^L)$$

$$= \tilde{v}(\left(\nabla_X Y^L + g(F^L X^L, Y^L) V + g(HX^L, Y^L) N^L \right),$$

$$\tilde{D}_{\tilde{v}(x^L)} \tilde{v}(\nabla) = \tilde{v}(X^L \nabla + g(H, X^L) N^L).$$

Using the above two equations and Gauss equation (1.1) and comparing the vertical parts and horizontal parts, we have

(2.5) $g(HX^L, Y^L) = g(HX, Y^L),$

(2.6) $'F_i(X) = g(HV, X^L, Y^L),$

where $'F$ is the fundamental tensor of the submersion $\pi$. Thus the transforms $'F_i(X)$ and $'F_N$ of $i(X)$ and $N$ by $'F$ can be written in the form:

(2.7) $'F_i(X) = i(FX) + u(X) N,$

(2.8) $'F_N = - i(U),$

$u(X) = g(U, X).$ Moreover the following identities are known [1].

(2.9) $g(HV, X^L) = - g(U, X) L,$

(2.10) $\tilde{g}(HV, V) = 0,$

(2.11) $\text{trace } H = (\text{trace } F)^L.$

**Lemma 2.1.** If the second fundamental tensor $H$ of the hypersurface $\tilde{M}$ is parallel, the second fundamental tensor $H$ of $M$ and the fundamental tensor $F$ of the submersion $\pi$ commutes.

**Proof.** Differentiating (2.5) covariantly in the direction of $\nabla$ and making use of the fact that $g(HX, Y) \circ \pi$ is invariant along the fibre, we get

$$\tilde{v}(g(HX, Y) \circ \pi) = \tilde{v}(g(HX^L, Y^L))) = \tilde{g}(H\nabla_Y X^L, Y^L) + \tilde{g}(H X^L, \nabla_Y Y^L)$$

$$= - g(H F^L X^L, Y^L) - g(HX^L, F^L Y^L)$$

$$= - g(H FX, Y^L) + g(FHX, Y^L) = 0,$$

where we have used (2.4) and the skew-symmetric property of $F$. This completes the proof.
3. Real hypersurfaces of a complex projective space. Let $S^{n+2}$ be an odd-dimensional unit sphere in an $(n+3)$-dimensional Euclidean space $E^{n+3} = C^{(n+3)/2}$ and $\mathcal{J}$ the natural almost complex structure on $C^{(n+3)/2}$. The image $\mathcal{V} = \mathcal{J}N$ of the outward unit normal vector $N$ to $S^{n+2}$ by $\mathcal{J}$ defines a tangent vector field on $S^{n+2}$ and the integral curves of $\mathcal{V}$ are great circles $S^1$ in $S^{n+2}$ which are the fibres of the standard fibration $\pi$.

$$S^1 \rightarrow S^{n+2} \xrightarrow{\mathcal{J}} CP(n+1)/2$$

onto complex projective space. The usual Riemannian structure on $CP(n+1)/2$ is characterized by the fact that $\pi$ is a submersion.

Let $M^n$ be a real hypersurface of a complex projective space $CP(n+1)/2$. Then the principal circle bundle $\tilde{M}^{n+1}$ over $M^n$ is a hypersurface of $S^{n+2}$ and the natural immersion $\tilde{M}^{n+1}$ into $S^{n+2}$ respects the submersion $\pi$. Thus $S^{n+2}$ and $CP(n+1)/2$ are in the same situations as $M$ and $M'$ respectively, so we continue to use the same notations as those in §2. In the sequel, we always assume that the hypersurface is connected.

In $S^{n+2}$ we have the family of products $M_{p,q} = S^p \times S^q$, where $p + q = n + 1$. By choosing the spheres to lie in complex subspaces we get fibrations

$$S^1 \rightarrow M_{2p+1, 2q+1} \rightarrow M_{p,q},$$

compatible with (3.1), where $p + q = (n - 1)/2$. In the special case $p = 0$, the hypersurface is a homogeneous, positively curved manifold diffeomorphic to the sphere.

The almost complex structure $J$ of $CP(n+1)/2$ is nothing but the fundamental tensor of the submersion $\pi$, that is,

$$J^L \mathcal{X} = - \bar{D}_X \mathcal{V}, \quad \mathcal{X} \in T(S^{n+2}).$$

From the discussions of §2, the transform $Ji(X)$ of $i(X)$ by $J$, can be put

$$Ji(X) = i(FX) + g(U, X)N$$

and we know that $F$, $U$ and $g$ define the induced almost contact metric structure on $M$. Hence we have, for any $X \in T(M)$,

$$F^2 X = - X + g(U, X)U,$$

$$g(U, U) = 1,$$

$$FU = 0.$$
(3.7) \( (\nabla_Y F)X = u(X)HY - g(HX, Y)U, \)
(3.8) \( \nabla_Y U = FHY. \)

**Lemma 3.1.** \( g(HV, HV) = 1. \)

**Proof.** Let \( x \) be an arbitrary point of \( M \) and \( \{E_1, \cdots, E_n\} \) be an orthonormal basis at \( T_x(M) \). We choose an orthonormal basis \( \{E_1, \cdots, E_{n+1}\} \) at \( T_x(M) \) in such a way that \( W_1 = F(f) = 1, 2, \cdots, n \) and \( E_{n+1} = V \). Then, we have

\[
\begin{align*}
\sum_{a=1}^{n+1} g(HV, E_a)g(HV, E_a) &= \sum_{i=1}^{n} g(E_i, E_i)g(U, U) = g(U, U) = 1,
\end{align*}
\]

because of (2.9), (2.10) and (3.5).

4. Real hypersurface satisfying a certain commutative condition. In the following we assume that a real hypersurface \( M^n \) of a complex projective space \( CP^{(n+1)/2} \) satisfies the commutative condition

(4.1) \( FH = HF. \)

By virtue of Lemma 2.1 if, as a hypersurface of \( S^{n+2} \), the principal circle bundle \( \tilde{M}^{n+1} \) over \( M^n \) has the parallel second fundamental tensor, then \( M \) satisfies (4.1) and \( M^{s,q} c \) is an example. In this section we discuss the converse problem, that is, we want to prove that \( M^{s,q} c \) is the only hypersurface of \( CP^{(n+1)/2} \) which satisfies (4.1).

We recall the structure equations of a hypersurface of a complex projective space \( CP^{(n+1)/2} \) of the maximal sectional curvature 4:

(4.2) \[ R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY - 2g(FX, Y)FZ + g(HY, Z)HX - g(HX, Z)HY, \]
(4.3) \( (\nabla_X H)Y - (\nabla_Y H)X = g(U, X)FY - g(U, Y)FX - 2g(FX, Y)U, \)

where \( R \) denotes the curvature tensor of the hypersurface. So we have

(4.4) \[ g((\nabla_X H)Y, U) - g((\nabla_Y H)X, U) = -2g(FX, Y), \]

because of (3.5) and (3.6). From (4.1) we easily see that \( U \) is an eigenvector of \( H \), that is,

(4.5) \[ HU = \alpha U, \quad \alpha = g(HU, U). \]
Differentiating (4.5) covariantly and making use of (3.8) and (4.1), we have
\[ g(\nabla_X H Y, U) + g(H^2 F X, Y) = (X \alpha) g(U, Y) + \alpha g(H F X, Y). \]

Forming a similar equation by interchanging \( X \) and \( Y \) in the last equation and using (4.4), we get
\[ (4.6) - 2g(F X, Y) + 2g(H^2 F X, Y) = (X \alpha) g(U, Y) - (Y \alpha) g(U, X) + 2 \alpha g(H F X, Y). \]

In (4.6) if we replace \( X \) by \( U \), we obtain \( Y \alpha = (U \alpha) g(U, Y) \) and substituting this into (4.6) yields \( F H^2 X - \alpha F H X - F X = 0 \). Transforming this by \( F \) and making use of (3.4), we have
\[ (4.7) H^2 X - \alpha H X - X + g(U, X) U = 0. \]

We prove the

**Lemma 4.1.** If a hypersurface \( M^n \) of \( CP^{(n+1)/2} \) satisfies (4.1), the eigenvalue \( \alpha \) is constant.

**Proof.** From the above discussions we have \( \nabla_X \text{grad} \, \alpha = \beta U, \beta = \alpha \nabla \alpha \). Differentiating this covariantly, we get \( \nabla_X \text{grad} \, \alpha = (X \beta) U + \beta F H X \), from which
\[ (4.8) (Y \beta) g(U, X) - (X \beta) g(U, Y) = 2 \beta g(H F X, Y), \]
because of the fact that \( g(\nabla_X \text{grad} \, \alpha, Y) = g(\nabla_Y \text{grad} \, \alpha, X) \).

Replacing \( X \) by \( U \) and making use of (3.5), (3.6), we get \( Y \beta = (U \beta) g(U, Y) \). Substituting this into (4.8), we get \( \beta g(F H X, Y) = 0 \). Now let \( x \) be a point of \( M^n \) where \( \beta(x) \neq 0 \). Then the last equation shows that \( F H = 0 \) at \( x \). Hence, from (4.6), \( F X = 0 \). But \( F \) has the maximal rank; this is a contradiction. Thus we know that at every point of \( M^n \), \( \beta = 0 \). Hence \( \alpha \) is constant.

**Lemma 4.2.** If the second fundamental tensor \( H \) of the hypersurface \( M^n \) in \( CP^{(n+1)/2} \) satisfies (4.7), the second fundamental tensor \( \overline{H} \) of \( \overline{M}^n+1 \) in \( S^{n+2} \) satisfies
\[ (4.9) \overline{H}^2 \overline{X} = \alpha \overline{H} \overline{X} + \overline{X}, \]
for any \( \overline{X} \in T(M^{n+1}) \).

**Proof.** Let \( X \) be a tangent vector of \( M^n \) and first compute \( \overline{H}^2 X^L - \alpha \overline{H} X^L - X^L \) at \( \overline{x} \in \overline{M}^{n+1} \). Since any tangent vector \( \overline{Y} \) of \( \overline{M}^{n+1} \) can be written in the form \( \overline{Y} = \overline{Y}^H + \overline{Y}^V = Y^L + \overline{g}(\overline{Y}, \overline{V}) \overline{V} \), at \( \overline{x} \), where \( Y \) is a tangent vector of \( M^n \) at \( \pi(\overline{x}) \), we have
\[ \overline{g}(\overline{H}^2 X^L - \alpha \overline{H} X^L - X^L, \overline{Y}) = \overline{g}(\overline{H}^2 X^L - \alpha \overline{H} X^L - X^L, Y^L) \]
\[ + \overline{g}(\overline{H}^2 X^L - \alpha \overline{H} X^L, \overline{V}) \overline{g}(\overline{Y}, \overline{V}). \]

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Since (4.5) implies that $g(HX, U) = \alpha g(U, X)$, it follows from (2.9) that
\[\overline{g}(\overline{H}(HX)^L, \overline{V}) = -\alpha g(U, X)^L.\]

On the other hand, (2.5) and the relation $g(HX, Y)^L = \overline{g}((HX)^L, Y^L)$ show that
\[g(H(HX)L, V) = -\alpha g(U, X)^L.\]

Hence
\[\overline{H}X^L = (HX)^L + \overline{g}(\overline{H}X^L, \overline{V})\overline{V} = (HX)^L - g(X, U)^L \overline{V}.\]

Thus we have
\[\overline{H}X^L = (HX)^L - \alpha g(X, U)^L \overline{V} - g(X, U)^L \overline{H} \overline{V}.\]

Therefore
\[\overline{H}X^L = (H^2X)^L - \alpha HX^L - X^L = (H^2X - \alpha HX - X)^L - g(X, U)^L \overline{H} \overline{V},\]
and consequently
\[\overline{g}(\overline{H}^2X^L - \alpha \overline{H}X^L - X^L, \overline{V})\]
\[= g(H^2X - \alpha HX - X + g(X, U)U, Y)^L = 0,\]
because of (2.10) and (4.7).

Next we consider $\overline{H}^2 \overline{V} - \alpha \overline{H} \overline{V} - \overline{V}$. For any $\overline{V} \in T_x(M^{n+1})$, we get
\[\overline{g}(\overline{H}^2 \overline{V} - \alpha \overline{H} \overline{V} - \overline{V}, \overline{V}) = \overline{g}(\overline{H}^2 \overline{V} - \alpha \overline{H} \overline{V} - \overline{V}, Y^L + \overline{g}(\overline{V}, \overline{V})\overline{V})\]
\[= \overline{g}(\overline{H}^2 \overline{V}, Y^L) - \alpha \overline{g}(\overline{H} \overline{V}, Y^L),\]
because of (2.10) and Lemma 3.1.

Making use of (4.12), we have
\[\overline{g}(\overline{H}^2 \overline{V} - \alpha \overline{H} \overline{V} - \overline{V}, \overline{V}) = -\alpha g(U, Y)^L + \alpha g(U, Y)^L = 0.\]

Combining (4.14) and (4.15), we have (4.9). This completes the proof.

As a consequence of Lemmas 1.1, 2.1 and 4.2, we have

**Theorem 4.3.** Let $M^n$ be a hypersurface of a complex projective space $CP^{(n+1)/2}$ and $\pi: \overline{M}^{n+1} \rightarrow M^n$ the submersion which is compatible with the fibration $S^1 \rightarrow S^{n+1} \rightarrow CP^{(n+1)/2}$. In order that the second fundamental tensor $H$ of $M^n$ commute with the fundamental tensor $F$ of the submersion $\pi$, it is necessary and sufficient that the second fundamental tensor $\overline{H}$ of $\overline{M}^{n+1}$ is parallel.

From this theorem and theorems in Ryan's papers [5], [6], we have

**Theorem 4.4.** $M_{p,q}$ are the only complete hypersurfaces of a complex projective space in which the second fundamental tensor $H$ commutes with the fundamental tensor $F$ of the submersion $\pi$. 
Since in [3] we proved that the induced almost contact structure of a hypersurface of a Kaehlerian manifold is normal if and only if $H$ commutes with $F$, we have

**Corollary 4.5.** $M^c_{p,q}$ is the only normal almost contact hypersurface of a complex projective space.

**BIBLIOGRAPHY**


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