ON SOME REAL HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

BY

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ABSTRACT. A principal circle bundle over a real hypersurface of a complex projective space $\mathbb{CP}^n$ can be regarded as a hypersurface of an odd-dimensional sphere. From this standpoint we can establish a method to translate conditions imposed on a hypersurface of $\mathbb{CP}^n$ into those imposed on a hypersurface of $S^{2n+1}$. Some fundamental relations between the second fundamental tensor of a hypersurface of $\mathbb{CP}^n$ and that of a hypersurface of $S^{2n+1}$ are given.

Introduction. As is well known a sphere $S^{2n+1}$ of dimension $2n + 1$ is a principal circle bundle over a complex projective space $\mathbb{CP}^n$ and the Riemannian structure on $\mathbb{CP}^n$ is given by the submersion $\pi: S^{2n+1} \to \mathbb{CP}^n$ [4], [7]. This suggests that fundamental properties of a submersion would be applied to the study of real submanifolds of a complex projective space. In fact, H. B. Lawson [2] has made one step in this direction. His idea is to construct a principal circle bundle $\widetilde{M}^{2n}$ over a real hypersurface $M^{2n-1}$ of $\mathbb{CP}^n$ in such a way that $\widetilde{M}^{2n}$ is a hypersurface of $S^{2n+1}$ and then to compare the length of the second fundamental tensors of $M^{2n-1}$ and $\widetilde{M}^{2n}$. Thus we can apply theorems on hypersurfaces of $S^{2n+1}$.

In this paper, using Lawson's method, we prove a theorem which characterizes some remarkable classes of real hypersurfaces of $\mathbb{CP}^n$. First of all, in §1, we state a lemma for a hypersurface of a Riemannian manifold of constant curvature for the later use. In §2, we recall fundamental formulas of a submersion which are obtained in [4], [7] and those established between the second fundamental tensors of $M$ and $\widetilde{M}$. In §3, we give some identities which are valid in a real hypersurface of $\mathbb{CP}^n$. After these preparations, we show, in §4, a geometric meaning of the commutativity of the second fundamental tensor of $M$ in $\mathbb{CP}^n$ and a fundamental tensor of the submersion $\pi: \widetilde{M} \to M$.
manifold $\tilde{M}$ into $\tilde{M}$. The Riemannian metric $\tilde{g}$ of $\tilde{M}$ is naturally induced from $\tilde{G}$ in such a way that $\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{G}(i(\tilde{X}), i(\tilde{Y}))$, where $\tilde{X}, \tilde{Y}$ are vector fields on $\tilde{M}$ and we denote by the same letter $i$ the differential of the immersion. For an arbitrary point $\tilde{x} \in \tilde{M}$, we choose a unit normal vector and extend it to a field $\tilde{N}$. The Riemannian connections $\tilde{D}$ in $\tilde{M}$ and $\tilde{\nabla}$ in $\tilde{M}$ are related by the following formulas:

\begin{align}
(1.1) \quad & \tilde{D}_i(\tilde{X}) \tilde{Y} = i(\tilde{\nabla}_X \tilde{Y}) + \tilde{g}(\tilde{H}X, \tilde{Y})\tilde{N}, \\
(1.2) \quad & \tilde{D}_i(\tilde{X})\tilde{N} = -i(\tilde{H}X),
\end{align}

where $\tilde{H}$ is the second fundamental tensor of $\tilde{M}$ in $\tilde{M}$.

The mean curvature $\tilde{m}$ of $\tilde{M}$ in $\tilde{M}$ is defined by

\begin{equation}
(1.3) \quad \tilde{m} \tilde{\mu} = \text{trace } \tilde{H}.
\end{equation}

Let $\tilde{R}$ and $\tilde{R}$ be curvature tensors of $\tilde{M}$ and of $\tilde{M}$ respectively, then we have the following Gauss and Mainardi-Codazzi equations:

\begin{align}
(1.4) \quad & \tilde{g}(\tilde{R}(i(\tilde{X}), i(\tilde{Y}))(\tilde{Z}), i(\tilde{W})) = \tilde{g}(R(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) - \tilde{g}(\tilde{H}\tilde{Y}, \tilde{Z})\tilde{g}(\tilde{H}X, \tilde{W}) \\
& \quad + \tilde{g}(\tilde{H}X, \tilde{Z})\tilde{g}(\tilde{H}Y, \tilde{W}), \\
(1.5) \quad & \tilde{g}(\tilde{R}(i(\tilde{X}), i(\tilde{Y}))(\tilde{Z}), \tilde{N}) = \tilde{g}(\tilde{\nabla}\tilde{X} \tilde{H}, \tilde{Z}) - \tilde{g}(\tilde{\nabla}\tilde{Y} \tilde{H}, \tilde{X}, \tilde{Z}),
\end{align}

where $\tilde{X}, \tilde{Y}, \tilde{Z}$ and $\tilde{W}$ are vector fields on $\tilde{M}$.

If the ambient manifold is of constant curvature $k$, the curvature tensor $\tilde{R}$ has the form

\begin{equation}
(1.6) \quad \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = k(\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y})
\end{equation}

for vector fields $\tilde{X}, \tilde{Y}$ and $\tilde{Z}$ on $\tilde{M}$. Consequently we have

\begin{align}
(1.7) \quad & \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = k(\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X} - \tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}) + \tilde{g}(\tilde{H}\tilde{Y}, \tilde{Z})\tilde{H}X - \tilde{g}(\tilde{H}X, \tilde{Z})\tilde{H}Y, \\
(1.8) \quad & (\tilde{\nabla}\tilde{X} \tilde{H})\tilde{Y} = (\tilde{\nabla}\tilde{Y} \tilde{H})\tilde{X}.
\end{align}

We assume that $\tilde{M}$ has constant mean curvature, that is, $\text{trace } \tilde{H} = \text{const}$.

Let $\{\tilde{E}_1, \ldots, \tilde{E}_m\}$ be an orthonormal basis in $T_x(\tilde{M})$ and extend them to vector fields in a normal neighborhood of $\tilde{x}$ by parallel translation along geodesics with respect to the Riemannian connection of $\tilde{M}$. Then we have $\nabla_{\tilde{E}_i} = 0$ ($i = 1, \ldots, m$) at $\tilde{x}$. Since $\tilde{H}$ and $\nabla_{\tilde{E}_i} \tilde{H}$ are both symmetric linear transformations on $T(\tilde{M})$, we get, by using (1.8)

\begin{align}
\tilde{g}
\left( \sum_{i=1}^{m} (\tilde{\nabla}_{\tilde{E}_i} \tilde{H})\tilde{E}_i, \tilde{X} \right)
= \sum_{i=1}^{m} \tilde{g}(\tilde{E}_i, (\tilde{\nabla}_{\tilde{E}_i} \tilde{H})\tilde{X})
= \sum_{i=1}^{m} \tilde{g}(\tilde{E}_i, (\tilde{\nabla}_{\tilde{X}} \tilde{H})\tilde{E}_i)
= \text{trace } (\tilde{\nabla}_{\tilde{X}} \tilde{H}) = \tilde{X}(\text{trace } \tilde{H}) = 0,
\end{align}
which implies that

(1.9) \[ \sum_{i=1}^{m} (\nabla_{\bar{E}_i} \bar{H}) \bar{E}_i = 0. \]

Thus we have

(1.10) \[ \sum_{i=1}^{m} (\nabla_{\bar{X}} (\nabla_{\bar{E}_i} \bar{H})) \bar{E}_i = 0 \quad \text{at} \, \bar{x}. \]

Now we prove the

**Lemma 1.1.** Let \( \bar{M} \) be a hypersurface of a Riemannian manifold of constant curvature \( k \). If the second fundamental tensor \( \bar{H} \) satisfies for a constant \( \alpha \),

(1.11) \[ \bar{H}^2 \bar{X} = \alpha \bar{H} \bar{X} + k \bar{X}, \quad \bar{X} \in \mathcal{T}(\bar{M}) \]

then we have \( \nabla \bar{H} = 0 \).

**Proof.** Since \( \bar{H} \) is a symmetric operator and (1.7), (1.8) are valid, we have

\[
(\nabla_{\bar{X}} (\nabla_{\bar{Y}} \bar{H}) - \nabla_{\bar{Y}} (\nabla_{\bar{X}} \bar{H}) - \nabla_{[\bar{X}, \bar{Y}]} \bar{H}) \bar{Z} = \bar{R}(\bar{X}, \bar{Y}) \bar{H} \bar{Z} - \bar{H}(\bar{R}(\bar{X}, \bar{Y}) \bar{Z})
\]

\[
= k (\bar{g}(\bar{Y}, \bar{H} \bar{Z}) \bar{X} - \bar{g}(\bar{X}, \bar{H} \bar{Z}) \bar{Y}) + \bar{g}(\bar{H} \bar{Y}, \bar{H} \bar{Z}) \bar{H} \bar{X} - \bar{g}(\bar{H} \bar{X}, \bar{H} \bar{Z}) \bar{H} \bar{Y}
\]

\[- k (\bar{g}(\bar{Y}, \bar{Z}) \bar{H} \bar{X} - \bar{g}(\bar{X}, \bar{Z}) \bar{H} \bar{Y}) - \bar{g}(\bar{H} \bar{Y}, \bar{Z}) \bar{H}^2 \bar{X} + \bar{g}(\bar{H} \bar{X}, \bar{Z}) \bar{H}^2 \bar{Y} = 0.
\]

Let \( \{\bar{E}_1, \ldots, \bar{E}_m\} \) be an orthonormal basis which is chosen as above and \( \bar{X} \) be a tangent vector at \( \bar{x} \). Extend \( \bar{X} \) to a vector field in a normal neighborhood of \( \bar{x} \) by parallel translation along geodesics, then \( \nabla \bar{X} = 0 \) at \( \bar{x} \). In the last equation we replace \( \bar{Y} \) and \( \bar{Z} \) by \( \bar{E}_i \) and sum over \( i \). Then we have, from (1.8) and (1.10),

(1.12) \[ \sum_{i=1}^{m} (\nabla_{\bar{E}_i} (\nabla_{\bar{X}} \bar{H})) \bar{E}_i = \sum_{i=1}^{m} (\nabla_{\bar{E}_i} (\nabla_{\bar{E}_i} \bar{H})) \bar{X} = 0 \quad \text{at} \, \bar{x}, \]

because from (1.11) we know that \( \bar{M} \) has constant mean curvature. Furthermore (1.11) implies that \( \text{trace} \, \bar{H}^2 = \alpha \text{trace} \, \bar{H} + mk = \text{const.} \) Differentiating this covariantly, we have

\[ \frac{1}{2} \bar{Y} \bar{X} (\text{trace} \, \bar{H}^2) = \text{trace} (\nabla_{\bar{Y}} (\nabla_{\bar{X}} \bar{H})) \bar{H} + \text{trace} (\nabla_{\bar{Y}} \bar{H} (\nabla_{\bar{X}} \bar{H})) = 0, \]

from which, at \( \bar{x} \),

\[ \text{trace} \, (\nabla_{\bar{X}} \bar{H})^2 = - \text{trace} (\nabla_{\bar{X}} (\nabla_{\bar{Y}} \bar{H}) \bar{H}) = - \sum_{i=1}^{m} \bar{g}((\nabla_{\bar{X}} (\nabla_{\bar{Y}} \bar{H})) \bar{E}_i, \bar{H} \bar{E}_i). \]

Thus we have

\[ \bar{g}(\nabla \bar{H}, \nabla \bar{H}) = \sum_{i=1}^{m} \text{trace} (\nabla_{\bar{E}_i} \bar{H})^2 = - \sum_{i,j=1}^{m} \bar{g}((\nabla_{\bar{E}_i} (\nabla_{\bar{E}_j} \bar{H})) \bar{E}_i, \bar{H} \bar{E}_j) = 0, \]

because of (1.12). This completes the proof.
2. Submersion and immersion. Let $\tilde{M}$ and $M$ be differentiable manifolds of dimension $n + 1$ and $n$ respectively and assume that there exists a differentiable mapping $\pi$ of $\tilde{M}$ onto $M$ which has maximum rank, that is, each differential map $\pi_\ast$ of $\pi$ is onto. Hence, for each $x \in M$, $\pi^{-1}(x)$ is a 1-dimensional submanifold of $\tilde{M}$, which is called the fibre over $x$. We suppose that every fibre is connected. A vector field on $\tilde{M}$ is called vertical if it is always tangent to fibres, horizontal if always orthogonal to fibres; we use corresponding terminology for individual vectors. Thus $\tilde{X} \in T_{\tilde{x}}(\tilde{M})$ decomposes as $\tilde{X}^V + \tilde{X}^H$, where $\tilde{X}^V$ and $\tilde{X}^H$ denote respectively vertical part and horizontal part of $\tilde{X}$.

We assume that the mapping $\pi$ is a Riemannian submersion, that is, there are given in $\tilde{M}$ a vertical vector field $\tilde{V}$ and a Riemannian metric $\bar{g}$ of $\tilde{M}$ satisfying the condition that $\tilde{V}$ is a unit Killing vector field with respect to the Riemannian metric $\bar{g}$. Then a Riemannian metric $g$ can be defined on $M$ by

\begin{equation}
(2.1) \quad g(X, Y)(x) = \bar{g}(X^L, Y^L)(\pi(x)),
\end{equation}

where $\tilde{x}$ is an arbitrary point of $\tilde{M}$ such that $\pi(\tilde{x}) = x$ and $X^L$, $Y^L$ are the lifts of $X$, $Y \in T_x(M)$ respectively. Hence we have

\begin{equation}
(2.2) \quad g(X, Y)^L = \bar{g}(X^L, Y^L).
\end{equation}

The fundamental tensor $F$ of the submersion $\pi$ is a skew-symmetric tensor of type (1,1) on $M$ and is related to covariant differentiation $\nabla_{\tilde{V}}$ and $\nabla_{\bar{V}}$ in $\tilde{M}$ and $M$, respectively, by the following formulas:

\begin{equation}
(2.3) \quad \nabla_{\bar{V}}X^L = (\nabla_Y X)^L + \bar{g}(F^L, X^L)\bar{V} = (\nabla_Y X)^L + g(FY, X)^L \bar{V},
\end{equation}

\begin{equation}
(2.4) \quad \nabla_{\bar{V}}X^L = \nabla_{X^L} \bar{V} = -F^L X^L.
\end{equation}

Now we consider two Riemannian submersions $\tilde{\pi}: \tilde{M} \to M'$ and $\pi: \tilde{M} \to M$ with 1-dimensional fibres and suppose that $\tilde{M}$ is a hypersurface of $\tilde{M}$ which respects the submersion $\tilde{\pi}$. That is, suppose that there are immersions $\tilde{i}: \tilde{M} \to \tilde{M}$ and $i: M \to M'$ such that the diagram

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{i} & \tilde{M} \\
\pi \downarrow & & \tilde{\pi} \\
M & \xrightarrow{i} & M'
\end{array}
$$

commutes and the immersion $\tilde{i}$ is a diffeomorphism on the fibres. The commutativity implies that for the unit vertical vector field $\bar{V}$ of $\tilde{M}$, $\tilde{i}(\bar{V})$ is also the unit vertical vector field of $\tilde{M}$ and that for any tangent vector field $X$ to $M$, $i(X)^L = \tilde{i}(X^L)$. Furthermore, for a field of unit normal vector $N$ to $M$ defined in a neigh-
borhood of \( x \in \hat{M} \), the lift \( N^L \) is a field of unit normal vectors to \( \hat{M} \) defined in a tubular neighborhood of \( \bar{x} \), where \( \bar{x} \) is an arbitrary point on a fibre over \( x \).

We denote by \( \overline{D}, \overline{V}, D \) and \( \overline{V} \) the Riemannian connections of \( \hat{M}, \bar{M}, M' \) and \( M \) respectively. By means of (1.1), (2.3) and (2.4), we have

\[
\overline{D}_{i(x^L)} \overline{V}(Y^L) = \overline{\iota}((D_X Y^L) + \overline{\mathbf{g}}(H_{XL}, Y^L)N^L)
\]

\[
= \overline{\iota}((\nabla_X Y^L + \overline{\mathbf{g}}(F^L X^L, Y^L)\overline{V}) + \overline{\mathbf{g}}(H^L X^L, Y^L)N^L),
\]

\[
\overline{D}_{i(x^L)} \overline{V}(Y^L) = \overline{\iota}(D_X Y^L) + \overline{\mathbf{g}}(H, X^L)N^L.
\]

Using the above two equations and Gauss equation (1.1) and comparing the vertical parts and horizontal parts, we have

\[
(2.5) \quad \overline{\mathbf{g}}(H^L X^L, Y^L) = g(HX, Y^L),
\]

\[
(2.6) \quad 'F(X) = \overline{\iota}(FX) - g(HV, X^L)N^L,
\]

where '\( F \) is the fundamental tensor of the submersion \( \pi \). Thus the transforms '\( F(X) \) and '\( F X \) of \( i(X) \) and \( N \) by '\( F \) can be written in the form:

\[
(2.7) \quad 'F(X) = i(FX) + u(X)N,
\]

\[
(2.8) \quad 'FN = - u(U),
\]

\( u(X) = g(U, X) \). Moreover the following identities are known [1].

\[
(2.9) \quad \overline{\mathbf{g}}(H^L V, X^L) = - g(U, X)^L, \]

\[
(2.10) \quad \overline{\mathbf{g}}(HV, \overline{V}) = 0,
\]

\[
(2.11) \quad \text{trace } H = (\text{trace } H)^L.
\]

**Lemma 2.1.** If the second fundamental tensor \( H \) of the hypersurface \( M \) is parallel, the second fundamental tensor \( H \) of \( M \) and the fundamental tensor \( F \) of the submersion \( \pi \) commutes.

**Proof.** Differentiating (2.5) covariantly in the direction of \( \overline{V} \) and making use of the fact that \( g(HX, Y) \circ \pi \) is invariant along the fibre, we get

\[
\overline{V}(g(HX, Y) \circ \pi) = \overline{V}(\overline{\mathbf{g}}(H^L X^L, Y^L)) = \overline{\mathbf{g}}(H\nabla_{\overline{V}} X^L, Y^L) + \overline{\mathbf{g}}(H^L X^L, \nabla_{\overline{V}} Y^L)
\]

\[
= - \overline{\mathbf{g}}(HF^L X^L, Y^L) - \overline{\mathbf{g}}(H^L X^L, F^L Y^L)
\]

\[
= - g(HFX, Y)^L + g(FHX, Y)^L = 0,
\]

where we have used (2.4) and the skew-symmetric property of \( F \). This completes the proof.
3. Real hypersurfaces of a complex projective space. Let \( S^{n+2} \) be an odd-dimensional unit sphere in an \((n + 3)\)-dimensional Euclidean space \( E^{n+3} = C^{(n+3)/2} \) and \( \mathcal{J} \) the natural almost complex structure on \( C^{(n+3)/2} \). The image \( \nabla = \mathcal{J}\nabla \) of the outward unit normal vector \( \nabla \) to \( S^{n+2} \) by \( \mathcal{J} \) defines a tangent vector field on \( S^{n+2} \) and the integral curves of \( \nabla \) are great circles \( S^1 \) in \( S^{n+2} \) which are the fibres of the standard fibration \( \mathcal{F} \),

\[
S^1 \to S^{n+2} \mathcal{F} \to CP^{(n+1)/2}
\]

onto complex projective space. The usual Riemannian structure on \( CP^{(n+1)/2} \) is characterized by the fact that \( \mathcal{F} \) is a submersion.

Let \( M^n \) be a real hypersurface of a complex projective space \( CP^{(n+1)/2} \). Then the principal circle bundle \( \tilde{M}^{n+1} \) over \( M^n \) is a hypersurface of \( S^{n+2} \) and the natural immersion \( \tilde{M}^{n+1} \) into \( S^{n+2} \) respects the submersion \( \mathcal{F} \). Thus \( S^{n+2} \) and \( CP^{(n+1)/2} \) are in the same situations as \( M \) and \( M' \) respectively, so we continue to use the same notations as those in §2. In the sequel, we always assume that the hypersurface is connected.

In \( S^{n+2} \) we have the family of products \( M_{p,q} = S^p \times S^q \), where \( p + q = n + 1 \). By choosing the spheres to lie in complex subspaces we get fibrations

\[
S^1 \to M_{2p+1, 2q+1} \to M_{p,q}^c,
\]

compatible with (3.1), where \( p + q = (n - 1)/2 \). In the special case \( p = 0 \), the hypersurface is a homogeneous, positively curved manifold diffeomorphic to the sphere.

The almost complex structure \( J \) of \( CP^{(n+1)/2} \) is nothing but the fundamental tensor of the submersion \( \mathcal{F} \), that is,

\[
J^L \vec{X} = -D_{\vec{X}} \nabla, \quad \vec{X} \in T(S^{n+2}).
\]

From the discussions of §2, the transform \( Ji(X) \) of \( i(X) \) by \( J \), can be put

\[
Ji(X) = i(FX) + g(U, X)N
\]

and we know that \( F, U \) and \( g \) define the induced almost contact metric structure \( \phi \) on \( M \). Hence we have, for any \( X \in T(M) \),

\[
F^2 X = -X + g(U, X)U, \quad g(U, U) = 1, \quad FU = 0.
\]

Differentiating (3.3) covariantly and making use of the fact that the almost complex structure \( J \) of \( CP^{(n+1)/2} \) is covariant constant, we have easily
(3.7) \( (\nabla_Y F)X = u(X)HY - g(HX, Y)U, \)

(3.8) \( \nabla_Y U = FHY. \)

**Lemma 3.1.** \( g(\bar{H}V, \bar{H}V) = 1. \)

**Proof.** Let \( \bar{x} \) be an arbitrary point of \( M \) and \( \{E_1, \ldots, E_n\} \) be an orthonormal basis at \( T_{\bar{x}}(M) \). We choose an orthonormal basis \( \{\bar{E}_1, \ldots, \bar{E}_{n+1}\} \) at \( T_{\bar{x}}(\bar{M}) \) in such a way that \( \bar{E}_i = F_i (i = 1, 2, \ldots, n) \) and \( \bar{E}_{n+1} = \bar{V} \). Then, we have

\[
\sum_{a=1}^{n+1} g(\bar{H}V, \bar{E}_a)g(\bar{H}V, \bar{E}_a) = \sum_{i=1}^n g(\bar{H}V, E^L_i)g(\bar{H}V, E^L_i)
\]

\[
= \sum_{i=1}^n g(U, E_i)g(U, E_i) = g(U, U) = 1,
\]

because of (2.9), (2.10) and (3.5).

4. Real hypersurface satisfying a certain commutative condition. In the following we assume that a real hypersurface \( M^n \) of a complex projective space \( CP^{(n+1)/2} \) satisfies the commutative condition

(4.1) \( FH = HF. \)

By virtue of Lemma 2.1 if, as a hypersurface of \( S^{n+2} \), the principal circle bundle \( \bar{M}^{n+1} \) over \( M^n \) has the parallel second fundamental tensor, then \( M \) satisfies (4.1) and \( M^{p,q}_c \) is an example. In this section we discuss the converse problem, that is, we want to prove that \( M^{p,q}_c \) is the only hypersurface of \( CP^{(n+1)/2} \) which satisfies (4.1).

We recall the structure equations of a hypersurface of a complex projective space \( CP^{(n+1)/2} \) of the maximal sectional curvature 4:

\[
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY
\]

\[
- 2g(FX, Y)FZ + g(HY, Z)HX - g(HX, Z)HY,
\]

(4.3) \( (\nabla_X H)Y - (\nabla_Y H)X = g(U, X)FY - g(U, Y)FX - 2g(FX, Y)U, \)

where \( R \) denotes the curvature tensor of the hypersurface. So we have

(4.4) \( g((\nabla_X H)Y, U) - g((\nabla_Y H)X, U) = -2g(FX, Y), \)

because of (3.5) and (3.6). From (4.1) we easily see that \( U \) is an eigenvector of \( H \), that is,

(4.5) \( HU = \alpha U, \quad \alpha = g(HU, U). \)

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Differentiating (4.5) covariantly and making use of (3.8) and (4.1), we have
\[ g((\nabla_X H)Y, U) + g(H^2 FX, Y) = (X\alpha)g(U, Y) + \alpha g(HFX, Y). \]
Forming a similar equation by interchanging \(X\) and \(Y\) in the last equation and using (4.4), we get
\[ (4.6) - 2g(FX, Y) + 2g(H^2 FX, Y) = (X\alpha)g(U, Y) - (Y\alpha)g(U, X) + 2\alpha g(HFX, Y). \]
In (4.6) if we replace \(X\) by \(U\), we obtain \(Y\alpha = (U\alpha)g(U, Y)\) and substituting this into (4.6) yields \(FH^2 X - \alpha FHX - FX = 0\). Transforming this by \(F\) and making use of (3.4), we have
\[ (4.7) H^2 X - \alpha HX - X + g(U, X)U = 0. \]
We prove the
*Lemma 4.1.* If a hypersurface \(M^n\) of \(CP^{(n+1)/2}\) satisfies (4.1), the eigenvalue \(\alpha\) is constant.

**Proof.** From the above discussions we have \(\text{grad } \alpha = \beta U\), \(\beta = \langle V, \text{grad } \alpha \rangle\). Differentiating this covariantly, we get \(\nabla_X \text{grad } \alpha = (X\beta)U + \beta FHX\), from which
\[ (4.8) (Y\beta)g(U, X) - (X\beta)g(U, Y) = 2\beta g(FHX, Y), \]
because of the fact that \(g(\nabla_X \text{grad } \alpha, Y) = g(\nabla_Y \text{grad } \alpha, X)\).
Replacing \(X\) by \(U\) and making use of (3.5), (3.6), we get \(Y\beta = (U\beta)g(U, Y)\).
Substituting this into (4.8), we get \(\beta g(FHX, Y) = 0\). Now let \(x\) be a point of \(M^n\) where \(\beta(x) \neq 0\). Then the last equation shows that \(FH = 0\) at \(x\). Hence, from (4.6), \(FX = 0\). But \(F\) has the maximal rank; this is a contradiction. Thus we know that at every point of \(M^n\), \(\beta = 0\). Hence \(\alpha\) is constant.

*Lemma 4.2.* If the second fundamental tensor \(H\) of the hypersurface \(M^n\) in \(CP^{(n+1)/2}\) satisfies (4.7), the second fundamental tensor \(\overline{H}\) of \(\overline{M}^{n+1}\) in \(S^{n+2}\) satisfies
\[ (4.9) \overline{H}^2 \overline{X} = \alpha \overline{H}X + \overline{X}, \]
for any \(\overline{X} \in T(\overline{M}^{n+1})\).

**Proof.** Let \(X\) be a tangent vector of \(M^n\) and first compute \(\overline{H}^2 X^L - \alpha \overline{H}X^L - X^L\) at \(x \in \overline{M}^{n+1}\). Since any tangent vector \(\overline{Y}\) of \(\overline{M}^{n+1}\) can be written in the form \(\overline{Y} = \overline{Y}^H + \overline{Y}^V = Y^L + \overline{g}(\overline{Y}, \overline{V})\overline{V}\), at \(x\), where \(Y\) is a tangent vector of \(M^n\) at \(\pi(x)\), we have
\[ g(\overline{H}^2 X^L - \alpha \overline{H}X^L - X^L, \overline{Y}) = g(\overline{H}^2 X^L - \alpha \overline{H}X^L - X^L, Y^L) \]
\[ + \overline{g}(\overline{H}^2 X^L - \alpha \overline{H}X^L, \overline{V})\overline{g}(\overline{Y}, \overline{V}). \]
Since (4.5) implies that $g(HX, U) = \alpha g(U, X)$, it follows from (2.9) that $\bar{g}(\bar{H}(HX)^L, \bar{V}) = -\alpha g(U, X)^L$.

On the other hand, (2.5) and the relation $g(HX, Y)^L = \bar{g}(HX^L, Y^L)$ show that

$$\bar{H}X^L = (HX)^L + \bar{g}(\bar{H}X^L, \bar{V})\bar{V} = (HX)^L - g(X, U)^L\bar{V}.$$  

Hence

$$\bar{H}^2X^L = (\bar{H}^2X)^L - \alpha g(X, U)^L\bar{V} - g(X, U)^L\bar{H}\bar{V}.$$  

Thus we have

$$\bar{H}^2X^L - \alpha \bar{H}X^L - X^L = (\bar{H}^2X - \alpha \bar{H}X - X)^L - g(X, U)^L\bar{H}\bar{V},$$  

and consequently

$$\bar{g}(\bar{H}^2X^L - \alpha \bar{H}X^L - X^L, \bar{Y})$$  

$$= g(\bar{H}^2X - \alpha \bar{H}X - X + g(X, U)U, Y)^L = 0,$$

because of (2.10) and (4.7).

Next we consider $\bar{H}^2\bar{V} - \alpha \bar{H}\bar{V} - \bar{V}$. For any $\bar{V} \in T_x(M^{n+1})$, we get

$$\bar{g}(\bar{H}^2\bar{V} - \alpha \bar{H}\bar{V} - \bar{V}, \bar{Y}) = \bar{g}(\bar{H}^2\bar{V} - \alpha \bar{H}\bar{V} - \bar{V}, Y^L + \bar{g}(\bar{V}, \bar{Y})\bar{V})$$  

$$= \bar{g}(\bar{H}^2\bar{V}, Y^L) - \alpha \bar{g}(\bar{H}\bar{V}, Y^L),$$

because of (2.10) and Lemma 3.1.

Making use of (4.12), we have

$$\bar{g}(\bar{H}^2\bar{V} - \alpha \bar{H}\bar{V} - \bar{V}, \bar{Y}) = -\alpha g(U, Y)^L + \alpha g(U, Y)^L = 0.$$  

Combining (4.14) and (4.15), we have (4.9). This completes the proof.

As a consequence of Lemmas 1.1, 2.1 and 4.2, we have

**Theorem 4.3.** Let $M^n$ be a hypersurface of a complex projective space $\mathbb{C}P^{(n+1)/2}$ and $\pi: M^{n+1} \to M^n$ the submersion which is compatible with the fibration $S^1 \to S^{n+2} \to \mathbb{C}P^{(n+1)/2}$. In order that the second fundamental tensor $H$ of $M^n$ commute with the fundamental tensor $F$ of the submersion $\pi$, it is necessary and sufficient that the second fundamental tensor $\bar{H}$ of $\bar{M}^{n+1}$ is parallel.

From this theorem and theorems in Ryan's papers [5], [6], we have

**Theorem 4.4.** $M_{p,q}^c$ are the only complete hypersurfaces of a complex projective space in which the second fundamental tensor $H$ commutes with the fundamental tensor $F$ of the submersion $\pi$.  

Since in [3] we proved that the induced almost contact structure of a hypersurface of a Kaehlerian manifold is normal if and only if $H$ commutes with $F$, we have

**Corollary 4.5.** $M_{p,q}^c$ is the only normal almost contact hypersurface of a complex projective space.

**BIBLIOGRAPHY**


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