TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 212, 1975

EQUIVARIANT HOMOLOGY THEORIES ON G-COMPLEXES

BY

STEPHEN J. WILLSON

ABSTRACT. A definition is given for a "cellular" equivariant homology theory on G-complexes. The definition is shown to generalize to G-complexes with prescribed isotropy subgroups. A ring I is introduced to deal with the general definition. One obtains a universal coefficient theorem and studies the universal coefficients.

1. **Introduction.** An equivariant homology theory is a functor analogous to an homology theory, satisfying analogues of the Eilenberg-Steenrod axioms, but defined for G-spaces and G-equivariant maps. A precise definition will be given later. The notion of such an equivariant homology (and cohomology) theory was introduced by Bredon [1] and [2] for finite groups and abstracted by C. N. Lee [8]. Definitions of singular equivariant homology theories have been given by Bröcker [3], Illman [5], [6], [7], and Willson [12]. Related questions of representability and obstruction theory have been discussed by Vaseekaran [11].

This paper is a systematic discussion of equivariant homology theories on a particularly nice class of spaces—the so-called "G-complexes", where G is merely assumed to be a topological group. These spaces share many properties with CW complexes, after which they are patterned. Matumoto [10] and Illman [5] have shown that all smooth G-manifolds, if G is a compact Lie group, admit the structure of a G-complex; hence the class of G-complexes is quite rich.

In §2 we define G-complexes and give a definition for certain equivariant homology groups for such spaces. The proof that the groups have the desired properties is relegated to an appendix, since the techniques are of use nowhere else in this paper.

It turns out that this definition of equivariant homology groups admits an easy extension to the study of G-spaces with specific restrictions on the allowed isotropy types. Roughly, if H is a list of certain "nice" subgroups of G, one can specialize one's attention to the study of G-complexes for which only the groups in H appear as isotropy subgroups. In effect, this is what one is doing when one studies "free" actions, "semi-free" actions, or "regular $O(n)$-manifolds".

Received by the editors May 1, 1974 and, in revised form, July 9, 1974.
AMS (MOS) subject classifications (1970). Primary 55B25; Secondary 57E15.

This generalization is presented in §§4 and 5. With each group G and list $\mathcal{H}$ of allowed subgroups of G, we associate a ring $\mathcal{I}$, which we call the isotropy ring. It is shown that the coefficient systems appropriate to equivariant homology theories are precisely left $\mathcal{I}$ modules, whereas the coefficient systems appropriate to equivariant cohomology theories are precisely right $\mathcal{I}$ modules.

Use of this ring $\mathcal{I}$ permits a considerable simplification of the theory. In §6 a universal coefficient theorem is obtained, which is dominated by the use of $\mathcal{I}$. It is seen that long exact sequences of homology groups result from short exact sequences of left $\mathcal{I}$ modules; the familiar Smith sequences are obtained in this manner.

The homology groups with coefficients in $\mathcal{I}$, viewed as a left $\mathcal{I}$ module, play a crucial role in §6. Hence it is desirable to obtain a geometric interpretation of these groups. Such an interpretation is presented in §7 for compact Lie groups G. For finite groups G, the result is particularly simple; ${}_G H_n(X; \mathcal{I})$ merely consists of all the nth homology groups of all the fixed point sets X^H for various H, appropriately combined.

It is easy to generalize these results to equivariant cohomology theories.

In a future paper we intend to discuss further the algebraic properties of the ring $\mathcal{I}$. In particular we shall be interested in the homological dimension of $\mathcal{I}$, in view of its role in simplifying the universal coefficient theorem.

This paper contains much of my doctoral dissertation, written at the University of Michigan under A. G. Wasserman. I wish to acknowledge my gratitude to him for suggesting the problem and working with me. Part of this research was done while the author held an NSF Graduate Fellowship.

2. **An equivariant homology theory for G-complexes.** In this section we recall the notion of a G-complex, where G is a topological group; and we give the basic definitions for the (cellular) equivariant homology theory we shall use in this paper.

If G is a topological group, a *G-space* is a topological space X together with a left action of G on X. If X and Y are G-spaces, a continuous map $f: X \longrightarrow Y$ is a *G-map* (or is *G-equivariant*) if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$. If X is a G-space and $H \subset G$, then X^H shall denote the set of points in X left fixed by each element of H. Two G-maps, $h, k: X \longrightarrow Y$ are *G-homotopic* if there exists a G-map $K: X \times I \longrightarrow Y$ (where I is the unit interval with the trivial G-action) such that $K|X \times \{0\} = h$ and $K|X \times \{1\} = k$.

Let D^n denote the standard n-disk with boundary $\dot{D}^n$. Let H be a closed subgroup of G, where G is a topological group. The space $G/H \times D^n$ is called a *G-cell* of *type H* and *dimension n*, or sometimes an n-cell; it is a G-space with G-action $g_1 \cdot (g_2H, s) = (g_1g_2H, s)$ for $g_1, g_2 \in G$ and $s \in D^n$. Note that the dimension of a G-cell need not equal its dimension as a topological space.

If X is a G-space and f: $G/H \times \dot{D}^n \longrightarrow X$ is a G-map (where $G/H \times \dot{D}^n \subset G/H \times D^n$) then we may obtain a new G-space Y by setting $Y = G/H \times D^n \cup_f X$; i.e., by identifying points on $G/H \times \dot{D}^n$ with their image in X. The G-space Y is said to be obtained from X by adjoining an n-cell.

Let Y be a G-space. A *relative G-complex* (X, Y) is a G-space X obtained inductively as follows: Let $\overline{X}^{-1} = Y$. Define $\overline{X}^i$ to be the result of adjoining arbitrarily many G-cells of arbitrary type but dimension i to $\overline{X}^{i-1}$; we give $\overline{X}^i$ the weak topology and the natural G-action. Let $X = \bigcup_i \overline{X}^i$, with the weak topology. We call $\overline{X}^i$ the *i-skeleton* of the pair (X, Y). A *G-complex* X is a relative G-complex $(X, \emptyset)$. Its i-skeleton is denoted X^i. A G-subcomplex A of a G-complex X is a G-complex such that (X, A) is a relative G-complex.

Proofs of elementary theorems about G-complexes tend to mimic exactly the proofs of corresponding theorems about CW complexes. The following two theorems may be proved in this manner, or one may consult Illman [5], or Willson [12] for proofs.

THEOREM 2.1 (G-HOMOTOPY EXTENSION PROPERTY). *Let A be a G-subcomplex of the G-complex X. Let Y be a G-space, and let I denote the unit interval with the trivial G-action. Any G-homotopy F: $A \times I \longrightarrow Y$ has the property that, if $F|A \times 0$ extends to a G-map f: $X \longrightarrow Y$, then F extends to a G-map $\overline{F}$: $X \times I \longrightarrow Y$ such that $\overline{F}|A \times I = F$ and $\overline{F}|X \times 0 = f$.*

DEFINITION. Let X and Y be G-complexes. A G-map f: $X \longrightarrow Y$ is *cellular* if $f(X^i) \subset Y^i$ for all i.

THEOREM 2.2 (CELLULAR APPROXIMATION THEOREM). *Let X and Y be G-complexes, and let $A \subset X$ be a G-subcomplex. Let f: $X \longrightarrow Y$ be a G-map which is cellular on A. Then f is G-homotopic relative to A to a cellular G-map f': $X \longrightarrow Y$.*

The appropriate notion of a coefficient system for equivariant homology theories was introduced by Bredon [2].

DEFINITION. A (*covariant*) *coefficient* system M for G is a function which assigns to each left homogeneous space G/H an abelian group $M(G/H)$ and to each G-map f: $G/H \longrightarrow G/K$ a homomorphism $M(f)$: $M(G/H) \longrightarrow M(G/K)$ such that

(1) $M(1_{G/H})$ is the identity map on $M(G/H)$, where $1_{G/H}$ denotes the identity map on G/H.

(2) $M(kf) = M(k)M(f)$ if f: $G/H \longrightarrow G/L$ and k: $G/L \longrightarrow G/K$.

(3) If f and k: $G/H \longrightarrow G/K$ are G-homotopic, then $M(f) = M(k)$.

Suppose X is a G-complex and A is a G-subcomplex. Suppose M is a coefficient system for G. We now define the nth equivariant homology group of (X, A) with coefficients in M, denoted ${}_GH_n(X, A; M)$.

For each integer $n \geqslant 0$ let the (equivariant) n-cells of X which are not cells of A be indexed by a set B_n; if $b \in B_n$, then the corresponding cell is $G/H_b \times D^n$ adjoined along the G-map f_b: $G/H_b \times \dot{D}^n \longrightarrow \bar{X}^{n-1}$. Observe that each cell $G/H_b \times D^n$ carries an orientation. We define ${}_GC_n(X, A; M) = \bigoplus_{b \in B_n} M(G/H_b)$. This is the group of (equivariant) cellular n-chains of (X, A).

We shall define for each n a map

$$\partial\colon {}_GC_n(X, A; M) \longrightarrow {}_GC_{n-1}(X, A; M).$$

This map will have the property that $\partial^2 = 0$, and we will define

$${}_GH_n(X, A; M) = H_n({}_GC_*(X, A; M), \partial);$$

i.e., ${}_GH_n(X, A; M)$ will be the homology of the resulting complex. If $A = \varnothing$, we shall write it instead as ${}_GH_n(X; M)$.

The definition of ∂ is somewhat involved. If $b \in B_n$ and $c \in B_{n-1}$ we shall actually define $\partial_{b,c}$: $M(G/H_b) \longrightarrow M(G/H_c)$ with the understanding that $\partial_{b,c}$ be the homomorphism ∂ restricted to the appropriate factors. Thus $\partial = \Sigma_{b \in B_n, c \in B_{n-1}} \partial_{b,c}$.

If $n = 0$, let $\partial_{b,c}$ be the zero map (by definition). If $n = 1$, then $D^n = [0, 1]$: let the image of $f_b | G/H_b \times \{1\}$ lie in $G/H_{c_1} \times D^0$ and let the image of $f_b | G/H_b \times \{0\}$ lie in $G/H_{c_0} \times D^0$. Then $f_b | G/H_b \times \dot{I}$ induces a G-map $f_{b,i}$: $G/H_b \longrightarrow G/H_{c_i}$ for $i = 0, 1$. Define for $s \in M(G/H_b)$, $\partial_{b,c}(s) = (-1)^{i+1}(M(f_{b,i}))s$ if $c = c_i$ and $\partial_{b,c}(s) = 0$ if $c \neq c_0$ and $c \neq c_1$. (If $c = c_0 = c_1$, then let $\partial_{b,c}(s) = M(f_{b,1})s - M(f_{b,0})s$.)

If $n \geqslant 2$, we observe that the composition

$$k\colon G/H_b \times D^n \xrightarrow{f_b} X^{n-1} \xrightarrow{p} (G/H_c \times D^{n-1})/(G/H_c \times D^{n-1})$$

is a well-defined G-map where p is the obvious projection. It is easily seen, via Theorem 2.1, that k may be G-homotoped to a map $\bar{k}$ such that the induced orbit map $\bar{k}/G$: $\dot{D}^n \longrightarrow D^{n-1}/\dot{D}^{n-1}$ is transverse regular at $0 \in D^{n-1}/\dot{D}^{n-1}$. Hence $\bar{k}^{-1}(G/H_c \times \{0\})$ is $G/H_b \times \{x_1, x_2, \ldots, x_m\}$ for some finite set of points x_i in $\dot{D}^n$. For each i, $i = 1, \ldots, m$, let $\epsilon_i = +1$ if $\bar{k}/G$ preserves orientation near x_i and let $\epsilon_i = -1$ if $\bar{k}/G$ reverses orientation near x_i; let k_i: $G/H_b \times \{x_i\} \longrightarrow G/H_c \times \{0\}$ be the G-map induced from $\bar{k}$. For $s \in M(G/H_b)$ define

$$\partial_{b,c}(s) = \sum_{i=1}^{m} \epsilon_i(M(k_i))(s).$$

Note that k_i may be viewed as a map from G/H_b to G/H_c, so that $(M(k_i))(s)$ is indeed an element of $M(G/H_c)$.

It can be shown (see Appendix) that the definition of ∂ is independent of

the choices of $\bar{k}$ and that $\partial^2 = 0$. Moreover, if f: $(X, A) \longrightarrow (Y, B)$ is a G-map, one may define a map ${}_GH_n(f)$: ${}_GH_n(X, A; M) \longrightarrow {}_GH_n(Y, B; M)$: one uses Theorem 2.2 to replace f by a cellular map, one restricts to appropriate cells, and one defines ${}_GH_n(f)$ by homotoping the appropriate orbit map to be transverse regular, and then performing a certain count, analogous to the definition of ∂. The details need not concern us here.

To state the theorem in the generality we shall need, we make the following definition:

DEFINITION. Let $\mathcal{C}$ be a category whose objects are certain pairs (X, A) of G-spaces, where $A \subset X$, and whose morphisms from (X, A) to (Y, B) consist of all G-equivariant maps f: $(X, A) \longrightarrow (Y, B)$. A *G-homology theory* on $\mathcal{C}$ is a sequence $h_0, h_1, \ldots$ of covariant functors from $\mathcal{C}$ to the category of abelian groups satisfying the following properties:

1. There is a natural transformation ∂: $h_i(X, A) \longrightarrow h_{i-1}(A, \varnothing)$.

2. *Exact sequence axiom.* If i: $A \longrightarrow X$ and j: $(X, \varnothing) \longrightarrow (X, A)$ are the inclusions, there is a natural exact sequence

$$\ldots \longrightarrow h_k(A, \varnothing) \xrightarrow{i} h_k(X, \varnothing) \xrightarrow{j} h_k(X, A) \xrightarrow{\partial} h_{k-1}(A, \varnothing)$$
$$\longrightarrow \ldots \longrightarrow h_0(X, A) \longrightarrow 0.$$

3. *Homotopy axiom.* If f and k: $(X, A) \longrightarrow (Y, B)$ are G-maps which are G-homotopic, then $h_i(f) = h_i(k)$ for each i as maps from $h_i(X, A)$ to $h_i(Y, B)$.

4. *Excision axiom.* If U is a G-subspace of A and $\bar{U} \subset \operatorname{int}(A)$, then for any i the map from $h_i(X - U, A - U)$ to $h_i(X, A)$ induced by the inclusion is an isomorphism of groups.

REMARK. We shall usually write $h_i(X)$ for $h_i(X, \varnothing)$. If there is no reference to $\mathcal{C}$, we shall mean that $\mathcal{C}$ is the category of all pairs of G-spaces.

DEFINITION. A G-homology theory on $\mathcal{C}$ is said to be *ordinary* if it satisfies the following *Dimension axiom*: $h_i(G/H) = 0$ for any $i \neq 0$ and any closed subgroup H of G such that G/H is an object in $\mathcal{C}$.

We now state the principal properties of the groups ${}_GH_n(X, A; M)$ defined above:

THEOREM 2.3. *Let $\mathcal{C}$ be the category of pairs of G-complexes (X, A). Let M be a (covariant) coefficient system. Then the functors which assign to n and (X, A) the group ${}_GH_n(X, A; M)$ form an ordinary G-homology theory on $\mathcal{C}$. In particular, if K is a closed subgroup of G, then ${}_GH_n(G/K; M) = 0$ if $n \neq 0$ and ${}_GH_0(G/K; M) = M(G/K)$; moreover, if f: $G/H \longrightarrow G/K$ is a G-map, then ${}_GH_0(f) = M(f)$.*

The proof appears in the Appendix.

3. **Generalized G-homology theories on G-complexes.** In standard homology theory there is a spectral sequence relating a generalized homology theory on a CW complex X to various singular homology theories on X. There is an analogous theorem for G-complexes and arbitrary G-homology theories.

THEOREM 3.1. *Let h_* be a G-homology theory. Let X be a finite G-complex and Y a G-subcomplex of X. For each integer i let $h_i(\cdot)$ denote the covariant coefficient system whose value at G/H is $h_i(G/H)$ and whose value at a map $f\colon G/H \longrightarrow G/K$ is the homomorphism $h_i(f)\colon h_i(G/H) \longrightarrow h_i(G/K)$. There is a natural spectral sequence whose $E^2_{p,q}$ term is $E^2_{p,q} = {}_GH_p(X, Y; h_q(\cdot))$ and which converges to a filtration of $h_*(X, Y)$.*

PROOF. The spectral sequence is that obtained from the exact couple

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "k"] \\ & C \arrow[ul, "g"] & \end{tikzcd}$$

where A and C are bigraded modules with $A_{p,q} = h_{p+q}(\overline{X}^p)$; $C_{p,q} = h_{p+q}(\overline{X}^p, \overline{X}^{p-1})$; and $k\colon A_{p,q} \longrightarrow C_{p,q}$, $g\colon C_{p,q} \longrightarrow A_{p-1,q}$, $f\colon A_{p,q} \longrightarrow A_{p+1,q-1}$ are all the obvious maps induced from inclusions or the ∂ map of a long exact sequence. Here, as usual, $\overline{X}^p = X^p \cup Y$, the p skeleton of the G-complex (X, Y). Further details are precisely as in Dyer [4]. Q.E.D.

COROLLARY 3.2. *Let h_* be a G-homology theory satisfying the Dimension axiom. Then for any finite G-complex pair (X, Y), $h_n(X, Y) = {}_GH_n(X, Y; h_0(\cdot))$. Moreover, the identification is natural in (X, Y).*

PROOF. $h_i(\cdot) = 0$ for $i \neq 0$. Hence the spectral sequence collapses, and $h_n(X, Y) = E^2_{n,0}$. Q.E.D.

COROLLARY 3.3. *Let h_* and k_* be G-homology theories. Suppose $\lambda\colon h_* \longrightarrow k_*$ is a natural transformation. Suppose for each n that $\lambda\colon h_n(\cdot) \longrightarrow k_n(\cdot)$ is an isomorphism. Then λ is a natural equivalence, and $h_n(X, Y) = k_n(X, Y)$ for any finite G-complex pair (X, Y).*

PROOF. By naturality, λ induces a map from the spectral sequence of $h_*(X, Y)$ to that of $k_*(X, Y)$. By hypothesis, the map is an isomorphism on the E^2 level. Hence it induces an isomorphism at all levels; hence the map $\lambda\colon h_n(X, Y) \longrightarrow k_n(X, Y)$ is an isomorphism. Q.E.D.

4. **The isotropy ring.** The ring $\mathcal{I}$ we shall describe in this section is useful in interpreting and defining coefficient systems and also in obtaining relations between them.

DEFINITION. Let G be a topological group and let $\mathcal{H} = \{H_1, \ldots, H_n,$

$\ldots\}$ be a collection of closed subgroups of G satisfying that if $i \neq j$, then H_i and H_j are not conjugate subgroups in G. Then $\mathcal{H}$ is called a *list of isotropy groups* or, more simply, a *list*; any H_i is referred to as an *allowed isotropy type.* If $\mathcal{H}$ contains one (and therefore precisely one) subgroup from every conjugacy class of closed subgroups of G, then $\mathcal{H}$ is called a *complete* list. If $H_i, H_j \in \mathcal{H}$, we denote by $\mathcal{M}(H_j, H_i)$ the set of G-homotopy classes of G-maps from G/H_i to G/H_j.

LEMMA 4.1. *If G is a topological group, then $\mathcal{M}(H_j, H_i)$ is the set of path components of $(G/H_j)^{H_i}$; i.e., of the fixed point set of G/H_j under the induced H_i action.*

PROOF. A G-map f: $G/H_i \longrightarrow G/H_j$ is uniquely determined by $f(H_i) = aH_j$, where $a^{-1}H_i a \subset H_j$. Hence the set of G-maps is in one-to-one correspondence with $(G/H_j)^{H_i}$. The lemma follows. Q.E.D.

COROLLARY 4.2. *If G is a compact Lie group, then $\mathcal{M}(H, K)$ is a finite set.*

PROOF. G/H is a compact smooth manifold on which G acts. Hence its fixed point set under the induced K action is a compact smooth manifold, which can have only finitely many components. Q.E.D.

DEFINITION. Let F be a commutative ring, G a topological group, $\mathcal{H}$ a list of isotropy groups for G. Then the *isotropy ring* $\mathcal{I}_G^{F,\mathcal{H}}$ is the free F module on $\bigcup_{H,K\in\mathcal{H}} \mathcal{M}(H, K)$. We shall write elements of $\mathcal{I}_G^{F,\mathcal{H}}$, by abuse of notation, as formal finite sums of terms of form ah where $a \in F$ and h is a G-map from some G/H to some G/K. It is, of course, understood that we are actually dealing with the G-homotopy class of h and not h itself. Notice that it is possible that $H = K$. When the context makes any of F, G, $\mathcal{H}$ clear, they may be omitted from the notation. Thus frequently we shall write $\mathcal{I}$ for $\mathcal{I}_G^{F,\mathcal{H}}$.

The above definition yields $\mathcal{I}$ as a free F-module. We shall impose a ring structure on $\mathcal{I}$ by composition of G-maps. Explicitly, we shall define a multiplication as follows: Let ϕ: $G/H \longrightarrow G/J$, ψ: $G/K \longrightarrow G/L$ be G-maps in $\mathcal{I}$; we shall define $(a\phi)(b\psi)$ to equal 0 if $L \neq H$ and to equal $ab(\phi\psi)$ if $L = H$. Clearly the G-homotopy class of $\phi\psi$ is determined by the G-homotopy class of ϕ and ψ; so our multiplication is well-defined. We extend the above operation over $\mathcal{I}$ in the unique manner so that right and left distributive laws hold.

DEFINITION. If $H \in \mathcal{H}$, let $1_{G/H}$ denote the identity map of G/H onto itself.

THEOREM 4.3. *$\mathcal{I}_G^{F,\mathcal{H}}$ is an associative ring. If $\mathcal{H}$ is a finite list, then $\mathcal{I}$ has the two-sided multiplicative unit $1 = \Sigma_{H\in\mathcal{H}}\ 1_{G/H}$. If in addition G is compact Lie, then $\mathcal{I}$ is a finitely generated free F-module.*

PROOF. The summation for 1 makes sense since $\mathcal{H}$ is a finite list. The last

assertion follows from Corollary 1.2. The rest is trivial. Q.E.D.

We now treat briefly the notion of a module over the ring $\mathcal{I}_G^{F,H}$. If H is infinite, then $\mathcal{I}$ has no multiplicative unit, and it is perhaps not quite clear what properties should correspond to the standard axiom that $1m = m$ for any m in a module. Hence we make the following special definition:

DEFINITION. A (left) *module* M over $\mathcal{I}$ is an abelian group M together with a map $\mathcal{I} \times M \longrightarrow M$, denoted by writing rm for the image of $(r, m) \in \mathcal{I} \times M$, so that

(1) $r(m_1 + m_2) = rm_1 + rm_2$;
(2) $(r_1 + r_2)m = r_1 m + r_2 m$;
(3) $(r_1 r_2)m = r_1(r_2 m)$;
(4) for each $m \in M$, there is a finite set K_m depending on m, so that $(\Sigma_{H \in K_m} 1_{G/H})m = m$.

(Here (1), (2) and (3) are assumed to hold for all $r, r_1, r_2 \in \mathcal{I}$ and all $m, m_1, m_2 \in M$.)

REMARK. The condition (4) is equivalent to the assertion that $M = \bigoplus_{H \in H} 1_{G/H} M$.

PROPOSITION 4.4. *Let $\mathcal{I}$ denote $\mathcal{I}_G^{F,H}$. If $H \in H$, then $\mathcal{I} 1_{G/H}$ is a projective left $\mathcal{I}$ module.*

PROOF. Let M and N be left $\mathcal{I}$ modules, let j: $M \longrightarrow N$ be an epimorphism, and let k: $\mathcal{I} 1_{G/H} \longrightarrow N$ be a homomorphism. Since j is surjective, there exists $m \in M$ so $j(m) = k(1_{G/H})$. Define l: $\mathcal{I} 1_{G/H} \longrightarrow M$ by $l(r) = r 1_{G/H} m$ for $r \in \mathcal{I} 1_{G/H}$. Then l is a homomorphism; and $jl = k$ since $jl(r) = j(r 1_{G/H} m) = r 1_{G/H} j(m) = r 1_{G/H} k(1_{G/H}) = k(r)$. Q.E.D.

COROLLARY 4.5. *$\mathcal{I}$ is a projective left $\mathcal{I}$ module.*

PROOF. $\mathcal{I} = \bigoplus_{H \in H} \mathcal{I} 1_{G/H}$ as a left $\mathcal{I}$ module.Q.E.D.

COROLLARY 4.6. *Any left $\mathcal{I}$ module M admits a resolution by projective left $\mathcal{I}$ modules.*

PROOF. By condition (4) in the definition of a left $\mathcal{I}$ module, any element m in M lies in the image of the projective module $\bigoplus_{H \in K_m} \mathcal{I} 1_{G/H}$. Q.E.D.

REMARK. It is easy to see that many other properties associated with modules of rings with multiplicative unit are still true for modules over $\mathcal{I}$. We shall use such properties, for the most part, without comment.

EXAMPLE 1. Let p be a positive integer, $G = Z_p$, $F = Z$, $H = \{G, (e)\}$. Let a generator for G be a. Then $\mathcal{I}_G^{F,H}$ is generated as a ring by $1_{G/e}$, $1_{G/G}$, g, q where g: $G/e \longrightarrow G/e$ is the G-map $g(e) = a(e)$ and q: $G/e \longrightarrow G/G$ is the collapsing map. The most general element in $\mathcal{I}$ is of the form

$$c_0 1_{G/e} + c_1 g + c_2 g^2 + \cdots + c_{p-1} g^{p-1} + c_p q + c_{p+1} 1_{G/G}$$

where the c_i are all integers. The multiplication table includes such statements as $g^p = 1_{G/e}$; $gq = 0$; $qg = q$; $g^i g^j = g^{i+j \bmod p}$; $g 1_{G/G} = 0$; $1_{G/G} q = q$.

EXAMPLE 2. Let $G = O(n)$, $\mathcal{H} = \{O(n), O(n-1), \ldots, O(2), O(1), O(0) = e\}$ where each $O(i)$ is embedded in $O(n)$ in the natural manner; i.e., the square i by i matrix A is identified with the square n by n matrix

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)$$

where I is the identity matrix. The corresponding isotropy ring is called that of *regular $O(n)$-manifolds.*

It is not hard to see that $\mathcal{M}(O(k), O(k))$ has two elements if $k < n$, one if $k = n$; and $\mathcal{M}(O(k), O(l))$ has just one element if $l < k$. Hence $\mathcal{I}$ is generated as a ring by the maps $1_k; \tau_k$: $O(n)/O(k) \longrightarrow O(n)/O(k)$ where $\tau_k^2 = 1_k$ for $k = 0, \ldots, n-1$; 1_n: $O(n)/O(n) \longrightarrow O(n)/O(n)$; and p_k: $O(n)/O(k) \longrightarrow O(n)/O(k+1)$ for $k = 0, \ldots, n-1$. The only interesting multiplication is $p_k \tau_k = p_k$.

5. **The isotropy ring and cellular homology.** The definition of $\mathcal{I}$ is motivated by the following theorem:

THEOREM 5.1. *Let G be a topological group. Let $\mathcal{H}$ be a complete list. Let $F = Z$. There is a one-to-one correspondence between (isomorphism classes of) covariant coefficient systems and (isomorphism classes of) left $\mathcal{I}_G^{F,\mathcal{H}}$ modules.*

PROOF. Choose, once and for all, for each closed subgroup H of G an element $a_H \in G$ so that $a_H^{-1} H a_H = H_i$ for some $H_i \in \mathcal{H}$. Such an a_H exists since $\mathcal{H}$ is complete. If $H \in \mathcal{H}$, choose $a_H = e$. The choices of the a_H yield for each H a G-homeomorphism λ_H: $G/H \longrightarrow G/H_i$ defined by $\lambda_H(H) = a_H H_i$.

Let M be a left $\mathcal{I}$ module. We define a covariant coefficient system $\overline{M}$ as follows: If $a_H^{-1} H a_H = H_i$ then define $\overline{M}(G/H) = 1_{G/H_i} M$. If f: $G/H \longrightarrow G/K$ is a G-map and $a_H^{-1} H a_H = H_i$, $a_K^{-1} K a_K = H_j$, define $\overline{M}(f)$: $1_{G/H_i} M \longrightarrow 1_{G/H_j} M$ by $\overline{M}(f)(1_{G/H_i} m) = (\lambda_K f \lambda_H^{-1}) 1_{G/H_i} m$ for $m \in M$. Since $\lambda_K f \lambda_H^{-1}$: $G/H_i \longrightarrow G/H_j$ is a G-map, $\overline{M}(f)$ is well-defined. It is now immediate that $\overline{M}$ is a coefficient system.

Conversely, suppose that N is a covariant coefficient system. We construct a left $\mathcal{I}$ module $\widetilde{N}$ as follows: Let $\widetilde{N} = \bigoplus_{H_i \in \mathcal{H}} N(G/H_i)$ as an abelian group. We define the module structure on $\widetilde{N}$ by $(af)n = a((N(f))(n))$ if $K = H$ and $(af)n = 0$ if $K \neq H$, where f: $G/H \longrightarrow G/J$, $n \in N(G/K)$, and $a \in Z$; we then extend this definition linearly. Note that if $K = H$ then $(af)n \in N(G/J)$ as desired.

It is now a simple matter to verify that the correspondence given above is one-to-one. Q.E.D.

REMARK. The correspondence in the theorem is not natural in G: it is affected by the choices of the elements a_H.

REMARK. A correspondence between *right* $\mathcal{I}$ modules and "contravariant coefficient systems"—defined in Bredon [2] and appropriate to equivariant cohomology theories—may be made analogously.

It follows from the above theorems that if (X, Y) is a relative G-complex, $\mathcal{H}$ is a complete list, and M is a left $\mathcal{I}_G^{Z,\mathcal{H}}$ module, then we may define ${}_GH_n(X, Y; M)$ merely by interpreting M as a coefficient system.

In fact, considerably more may be said, and we will obtain uses for $\mathcal{I}_G^{F,\mathcal{H}}$ even if $\mathcal{H}$ is not a complete list.

DEFINITION. Let X be a G-complex. Let $\mathcal{H}$ be a list. We say *X has type $\mathcal{H}$* if for every G-cell $G/K \times D^i$ of any dimension in X, we have K conjugate to an element of $\mathcal{H}$. Evidently, we may always assume that in fact $K \in \mathcal{H}$.

Thus, if $\mathcal{H}$ is a complete list, every G-complex X has type $\mathcal{H}$. If $\mathcal{H}$ is not a complete list, a G-complex X which has type $\mathcal{H}$ has only certain kinds of cells permitted.

The significance of $\mathcal{I}_G^{F,\mathcal{H}}$ for general $\mathcal{H}$ is that only information in a coefficient system M associated with $\mathcal{I}_G^{F,\mathcal{H}}$ is required to compute ${}_GH_n(X, Y; M)$ when X is a G-complex of type $\mathcal{H}$. More precisely, suppose $\mathcal{H}$ is an arbitrary list, X is a G-complex of type $\mathcal{H}$, Y is a G-subcomplex of X, and M is a left $\mathcal{I}_G^{F,\mathcal{H}}$ module. For each $n \geqslant 0$ let B_n be the set of n-cells in X but not in Y. If $b \in B_n$, let $H_b \in \mathcal{H}$ be the type of the n-cell b. Define the group ${}_G\hat{C}_n(X, Y; M) = \bigoplus_{b \in B_n} 1_{G/H_b}M$. Define $\partial\colon {}_G\hat{C}_n(X, Y; M) \longrightarrow {}_G\hat{C}_{n-1}(X, Y; M)$ by the formulas in §2. Then we obtain a chain complex since $\partial^2 = 0$, and we may define ${}_G\hat{H}_n(X, Y; M)$ to be the nth homology group of this chain complex.

We should like to conclude that ${}_G\hat{H}_n(X, Y; M)$ is in the appropriate sense an equivariant homology theory.

THEOREM 5.2. *Let G be a topological group and $\mathcal{H}$ be a list for G. Let $\mathcal{C}$ be the category of pairs (X, Y) where X is a G-complex of type $\mathcal{H}$ and Y is a G-subcomplex of X. Let M be a left $\mathcal{I}_G^{F,\mathcal{H}}$ module, where F is arbitrary. Then the functors defined by sending (X, Y) to ${}_G\hat{H}_n(X, Y; M)$ for $n \geqslant 0$ form an ordinary equivariant homology theory on $\mathcal{C}$.*

PROOF. The theorem would follow from the corresponding properties of ${}_GH_n(X, Y; N)$ provided there were a coefficient system N (in the sense of §2) so that ${}_G\hat{H}_n(X, Y; M) = {}_GH_n(X, Y; N)$ for all n and (X, Y). Thus we reduce the proof to the following lemma:

LEMMA 5.3. *Let $\mathcal{L}$ be a complete list and $\mathcal{H}$ a subset of $\mathcal{L}$. If N is a left $\mathcal{I}_G^{F,\mathcal{L}}$ module, there is a unique left $\mathcal{I}_G^{F,\mathcal{H}}$ module $\tilde{N}$ defined by setting $1_{G/H}\tilde{N} =$*

$1_{G/H}N$ *for* $H \in \mathcal{H}$. *If* M *is a left* $\mathcal{I}_G^{F,\mathcal{H}}$ *module, there exists at least one left* $\mathcal{I}_G^{F,\mathcal{L}}$ *module* N *so* $\widetilde{N} = M$.

PROOF. The first assertion is obvious. To see the second assertion, it suffices, by an application of Zorn's Lemma, to show that if $H \in \mathcal{L} - \mathcal{H}$, then there is an $\mathcal{I}_G^{F,\mathcal{H}\cup\{H\}}$ module N whose restriction to $\mathcal{I}_G^{F,\mathcal{H}}$ is just M.

Let $A = \bigcup_{K\in\mathcal{H}} \mathcal{M}(H, K)$. If $f\colon G/K \longrightarrow G/H$ is in A, denote $d(f) = K$. Define $\overline{N} = \bigoplus_{f\in A} 1_{G/d(f)}M$. The desired module N will involve the quotient of $\overline{N}$ by a subgroup B, which we now define.

Whenever there is a commutative diagram

$$\begin{tikzcd} G/K_1 \arrow[r, "f_1"'] \arrow[d, "k"'] & G/H \\ G/K_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

for $K_1, K_2 \in \mathcal{H}$, and whenever $m \in 1_{G/K_1}M$, we obtain an element of $\overline{N}$ whose f_1 component is m, whose f_2 component is $-km$, and whose other components are zero. (If $f_1 = f_2$, we let the f_1 component be $(m - km)$ and all other components be zero.) Let B be the subgroup of $\overline{N}$ generated by all such elements for all such commutative diagrams.

Define N as an F-module by $1_{G/K}N = 1_{G/K}M$ if $K \in \mathcal{H}$, $1_{G/H}N = \overline{N}/B$. If $f\colon G/K \longrightarrow G/L$ is a G-map, we must define a corresponding operation in N. The definition is obvious if both K and L are in $\mathcal{H}$. There are three other cases:

(i) $K = L = H$. For this case, f is invertible. If $k\colon G/J \longrightarrow G/H$ is in A, and $m \in \overline{N}/B$, then fm is the element whose k component is the $f^{-1}k$ component of m; i.e., $(fm)_k = (m)_{f^{-1}k}$.

(ii) $L = H$, $K \in \mathcal{H}$. For this case, if $m \in 1_{G/K}M = 1_{G/K}N$, we define fm to have the value m in the component corresponding to f and 0 in the other components.

(iii) $K = H$, $L \in \mathcal{H}$. For this case, if $m \in \overline{N}/B$, we represent m in $\overline{N}$ and let the component of m corresponding to $k\colon G/J \longrightarrow G/H$ be $(m)_k \in 1_{G/J}M$. Define $fm = \Sigma_k(fk)(m)_k$. Observe that $fk\colon G/J \longrightarrow G/L$ so that $fm \in 1_{G/L}M$. To see this is well-defined, we verify that any element of B is taken to zero. But if

$$\begin{tikzcd} G/K_1 \arrow[r, "f_1"'] \arrow[d, "p"'] & G/H \\ G/K_2 \arrow[ur, "f_2"'] & \end{tikzcd}$$

commutes and $(m)_{f_2} = -p(m)_{f_1}$ and all other components are 0, then $fm =$

$(ff_2)(m)_{f_2} + (ff_1)(m)_{f_1} = (ff_2)(-p(m)_{f_1}) + (ff_2p)(m)_{f_1} = 0.$

There remains to verify that the module axioms hold for N. The verification is a tedious consideration of numerous cases, each of which is very easy. Q.E.D.

Notation. It is clear from Theorem 5.2 that if $\mathcal{H}$ is a complete list, then ${}_G\hat{H}_n(X, Y; M) = {}_GH_n(X, Y; M)$. Henceforth we shall omit the circumflex in the notation: if $\mathcal{H}$ is an arbitrary list and M is an $\mathcal{I}_G^{F,\mathcal{H}}$ module, we shall write ${}_GH_n(X, Y; M)$ for ${}_G\hat{H}_n(X, Y; M)$ and ${}_GC_n(X, Y; M)$ for ${}_G\hat{C}_n(X, Y; M)$.

We note the following corollary, which we shall use extensively:

COROLLARY 5.4. *Let $\mathcal{H}$ be a list, $\mathcal{I} = \mathcal{I}_G^{F,\mathcal{H}}$, M be a left $\mathcal{I}$ module. Let X be a G-complex of type $\mathcal{H}$ and A a G-subcomplex of X. The chain complex $\{{}_GC_n(X, A; M), \partial\}$ is naturally chain isomorphic with the chain complex $\{{}_GC_n(X, A; \mathcal{I}) \otimes_{\mathcal{I}} M, \partial \otimes \mathrm{id}\}$ where* id *is the identity map.*

PROOF. $1_{G/H}M = 1_{G/H}\mathcal{I} \otimes_{\mathcal{I}} M$. Q.E.D.

6. **Applications.** In this section we note some easy consequences of the algebraic formulation in §5. Roughly, if $\mathcal{H}$ is a list, the ring $\mathcal{I} = \mathcal{I}_G^{F,\mathcal{H}}$ may itself be regarded as a left $\mathcal{I}$ module and hence as a coefficient system; and this fact leads us to some long exact sequences and a universal coefficient theorem.

The following proposition is the fundamental lemma which makes everything work.

PROPOSITION 6.1. *Let G be a topological group, $\mathcal{H}$ a list. Let X be a G-complex of type $\mathcal{H}$ and Y be a G-subcomplex of X. Let $\mathcal{I} = \mathcal{I}_G^{F,\mathcal{H}}$. Then for each n, ${}_GC_n(X, Y; \mathcal{I})$ is a projective right $\mathcal{I}$ module.*

PROOF. Let B_n be the set of n-cells of (X, Y); i.e., the set of cells f: $G/H_f \times D^n \longrightarrow \overline{X}^n$ which are not cells of Y. Then ${}_GC_n(X, Y; \mathcal{I}) = \bigoplus_{f\in B_n} 1_{G/H_f}\mathcal{I}$. But $1_{G/H_f}\mathcal{I}$ is a projective right $\mathcal{I}$ module by an argument like the proof of Proposition 4.4. Hence ${}_GC_n(X, Y; \mathcal{I})$ is projective. Q.E.D.

THEOREM 6.2. *Let $\mathcal{H}$ be a list, X a G-complex of type $\mathcal{H}$, Y a G-subcomplex of X, $\mathcal{I} = \mathcal{I}_G^{F,\mathcal{H}}$. Suppose $0 \longrightarrow A \xrightarrow{h} B \xrightarrow{k} C \longrightarrow 0$ is an exact sequence of left $\mathcal{I}$ modules. Then there is a natural long exact sequence*

$$\cdots \longrightarrow {}_GH_n(X, Y; A) \xrightarrow{h_*} {}_GH_n(X, Y; B)$$

$$\xrightarrow{k_*} {}_GH_n(X, Y; C) \longrightarrow {}_GH_{n-1}(X, Y; A) \longrightarrow \cdots$$

in which the maps h_* and k_* are induced by h and k respectively.

PROOF. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & {}_GC_n(X, Y; \mathcal{I}) \otimes A & \to & {}_GC_n(X, Y; \mathcal{I}) \otimes B & \to & {}_GC_n(X, Y; \mathcal{I}) \otimes C & \to & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \to & {}_GC_{n-1}(X, Y; \mathcal{I}) \otimes A & \to & {}_GC_{n-1}(X, Y; \mathcal{I}) \otimes B & \to & {}_GC_{n-1}(X, Y; \mathcal{I}) \otimes C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Since ${}_GC_n(X, Y; \mathcal{I})$ is a projective right $\mathcal{I}$ module by Proposition 6.1, the rows are exact. In the familiar manner a diagram chase yields a long exact sequence in homology. By Corollary 5.4, the theorem is proved. Q.E.D.

EXAMPLE (SMITH THEORY). Let $G = Z_p$ for p prime; $\mathcal{H} = \{e, Z_p\}$, a complete list; $F = Z_p$. The ring $\mathcal{I}$ was described in Example 1 of §4. Using the same notation, let $\tau = 1_{G/e} - g$, $\rho_i = \tau^i$; $\bar{\rho}_i = \tau^{p-i}$. There is for each i a short exact sequence of left $\mathcal{I}$ modules

$$0 \to \mathcal{I}1_{G/G} \oplus \mathcal{I}\bar{\rho}_i \xrightarrow{h} \mathcal{I}1_{G/e} \xrightarrow{k} \mathcal{I}\rho_i \to 0$$

where k is right multiplication by ρ_i and $h(a, b) = aq + b$ for $a \in \mathcal{I}1_{G/G}$ and $b \in \mathcal{I}\bar{\rho}_i$. Hence there is a long exact sequence

$$\cdots \to {}_GH_n(X; \mathcal{I}1_{G/G}) \oplus {}_GH_n(X; \mathcal{I}\bar{\rho}_i) \to {}_GH_n(X; \mathcal{I}1_{G/e})$$

$$\to {}_GH_n(X; \mathcal{I}\rho_i) \to \cdots.$$

We shall see that ${}_GH_n(X; \mathcal{I}1_{G/G}) = H_n(X^{Z_p}; Z_p)$ and ${}_GH_n(X; \mathcal{I}1_{G/e}) = H_n(X; Z_p)$; this long exact sequence is merely the Smith long exact sequence.

THEOREM 6.3 (UNIVERSAL COEFFICIENT THEOREM). *Let $\mathcal{H}$ be a list, X be a G-complex of type $\mathcal{H}$, and A be a G-subcomplex. Suppose $\mathcal{I} = \mathcal{I}_G^{F,\mathcal{H}}$ and M is a left $\mathcal{I}$ module. Then there is a first quadrant spectral sequence for which $E^2_{p,q} = \mathrm{Tor}_p^{\mathcal{I}}({}_GH_q(X, A; \mathcal{I}), M)$ and which converges to a filtration of ${}_GH_*(X, A; M)$.*

PROOF. We note that ${}_GH_q(X, A; \mathcal{I})$ is a right $\mathcal{I}$ module since ${}_GC_q(X, A; \mathcal{I})$ is a right $\mathcal{I}$ module and ∂ is a map of right $\mathcal{I}$ modules. Hence the formation of $\mathrm{Tor}_p^{\mathcal{I}}({}_GH_q(X, A; \mathcal{I}), M)$ makes sense.

To prove the theorem, we obtain a projective resolution $\to Y_r \to Y_{r-1} \to \cdots \to Y_0 \to M \to 0$ of M by projective left $\mathcal{I}$ modules. We then obtain a bicomplex $K_{p,q} = {}_GC_p(X, A; \mathcal{I}) \otimes_{\mathcal{I}} Y_q$ with the natural boundary maps. Consideration of the two spectral sequences for this bicomplex yields the theorem: one of them has the $E^2_{p,q}$ term indicated in the theorem; the other is degenerate with limit ${}_GH_*(X, A; M)$. Further details may be found in Mac Lane [9]. Q.E.D.

Bredon [2] obtains, for equivariant cohomology of finite groups, a theorem which is similar to Theorem 6.3.

7. **Identification of** ${}_GH_*(X, A; I)$. We assume that G is a compact Lie group and H is an arbitrary list. We have seen in Theorem 6.3 that ${}_GH_*(X, A; I)$ plays a central role in the study of the G-homology theories on (X, A). In this section we interpret ${}_GH_*(X, A; I)$ geometrically.

Suppose $K \subset H \subset G$ are closed subgroups. Let $N(K) = \{g \in G: gKg^{-1} = K\}$ denote the normalizer of K in G, and $C(K) = \{g \in G: gkg^{-1} = k \text{ for all } k \in K\}$ the centralizer of K in G. Recall that $(G/H)^K$ is a closed submanifold of G/H with a natural $N(K)$ action.

We observe that $eH \in (G/H)^K$. Moreover the orbit of eH under the $N(K)$ action is $N(K)/N(K) \cap H$. Similarly the orbit of eH under the $C(K)$ action is $C(K)/C(K) \cap H$. Let $(N(K)/N(K) \cap H)_0$ denote the component containing eH, and use a similar notation for $C(K)$.

LEMMA 7.1. *The component C of $(G/H)^K$ which includes eH is precisely* $(N(K)/N(K) \cap H)_0 = (C(K)/C(K) \cap H)_0$.

PROOF. Let $\tau(M)_x$ denote the tangent space to a manifold M at a point x. There is an exact sequence

$$0 \longrightarrow \tau(H)_e \xrightarrow{di} \tau(G)_e \xrightarrow{dp} \tau(G/H)_{eH} \longrightarrow 0$$

where i is the inclusion and p is the projection. Yet if the adjoint representation on G is denoted Ad G, it is clear that di: $(\text{Ad } H)|K \longrightarrow (\text{Ad } G)|K$ is a K-map of K representations, and the induced K representation on $\tau(G/H)_{eH}$ is merely the differential of the left K action on G/H. Hence

$$0 \longrightarrow (\text{Ad } H)|K \longrightarrow (\text{Ad } G)|K \longrightarrow \tau(G/H)_{eH}|K \longrightarrow 0$$

is an exact sequence of K representations, which splits since K is compact.

Since the dimension of C is the dimension of $(\tau(G/H)_{eH})^k$, we obtain that $\dim C = \dim(\text{Ad } G)^K - \dim(\text{Ad } H)^K$ where $\dim C$ denotes the dimension of C.

Yet $(\text{Ad } G)^K = \tau(C(K))_e$. To see this, we observe immediately the inclusion $(\text{Ad } G)^K \supset \tau(C(K))_e$. If $X \in \tau(G)_e$, then the curve $\exp_e(tX)$ is a one-parameter subgroup of G; hence so is $k \exp_e(tX)k^{-1}$ for any $k \in K$, with tangent vector $(\text{Ad } k)(X)$ at e. If $X \in (\text{Ad } G)^K$, then it follows by uniqueness of one-parameter subgroups that $k \exp_e(tX)k^{-1} = \exp_e(tX)$ for all t and k. Hence $\exp_e(tX) \in C(K)$, and $X \in \tau(C(K))_e$.

Thus $\dim C = \dim C(K) - \dim(C(K) \cap H) = \dim(C(K)/C(K) \cap H)$. Since $(C(K)/C(K) \cap H)_0 \subset (N(K)/N(K) \cap H)_0 \subset C$ and the manifolds on the ends have the same dimension, our result follows. Q.E.D.

COROLLARY 7.2. *Let $aH \in (G/H)^K$. Then the component of $(G/H)^K$ containing aH lies in the image of $N(K)$ under the map which takes $N(K)$ into $(G/H)^K$ by sending $n \in N(K)$ to naH.*

PROOF. If $aH \in (G/H)^K$ then $K \subset aHa^{-1}$; so $aHa^{-1} \in (G/aHa^{-1})^K$. We apply the lemma to this latter space. Q.E.D.

THEOREM 7.3. *Let G be a compact Lie group and K a closed subgroup of G. Let $\mathcal{H}$ be a list containing K. Then for all i and all pairs (X, Y) of G-complexes of type $\mathcal{H}$, there is a natural equivalence between the G-homology theories ${}_GH_i(X, Y; \mathcal{I}_G^{F,\mathcal{H}} 1_{G/K})$ and $H_i(X^K/(N(K)/K)_0, Y^K/(N(K)/K)_0; F)$.*

Here $X^K/(N(K)/K)_0$ denotes the orbit space of X^K under the action by the identity component of the Lie group $N(K)/K$.

PROOF. To prove the theorem we first observe that the latter is a G-homology theory. Moreover, $H_i((G/H)^K/(N(K)/K)_0; F) = 0$ for $i \neq 0$ by Corollary 7.2. Hence it is an ordinary G-homology theory. Finally, we note that

$$\begin{aligned} {}_GH_0(G/H; \mathcal{I}_G^{F,\mathcal{H}} 1_{G/K}) &= 1_{G/H} \mathcal{I} 1_{G/K} = H_0((G/H)^K; F) \\ &= H_0((G/H)^K/(N(K)/K)_0; F) \end{aligned}$$

where the identification is clearly natural in the subgroup H. By Corollary 3.2 of §3 we obtain the theorem. Q.E.D.

COROLLARY 7.4. *Let G be a finite group and $\mathcal{H}$ be a list for G. Let X be a finite G-complex of type $\mathcal{H}$ and Y be a G-subcomplex. Then ${}_GH_i(X, Y; \mathcal{I}) = \bigoplus_{K \in \mathcal{H}} H_i(X^K, Y^K; F)$.*

COROLLARY 7.5. *Let G be a finite group and $\mathcal{H}$ be a complete list for G. Let X and Y be finite G-complexes and $f: X \longrightarrow Y$ be a G-map. Suppose for each i and for each subgroup K of G the map from $H_i(X^K; Z)$ to $H_i(Y^K; Z)$ induced by f is an isomorphism. Then for any equivariant homology theory h_* the map $h_i(f)$: $h_i(X) \longrightarrow h_i(Y)$ is an isomorphism.*

PROOF. By Corollary 7.4, the map ${}_GH_i(f; \mathcal{I})$: ${}_GH_i(X; \mathcal{I}) \longrightarrow {}_GH_i(Y; \mathcal{I})$ is an isomorphism for each i. By the universal coefficient theorem, it follows that ${}_GH_i(f; M)$: ${}_GH_i(X; M) \longrightarrow {}_GH_i(Y; M)$ is an isomorphism for any coefficient system M and any integer i. By Theorem 3.1, the result now follows. Q.E.D.

Appendix. In this appendix we sketch the proof of Theorem 2.3. The proof relies on properties of the singular equivariant homology theories described in Illman [7] or Willson [12]. In particular, we use the fact that they are indeed equivariant homology theories. For purposes of this appendix, we shall denote the singular homology groups of (X, A) by ${}_G\hat{H}_n(X, A; M)$ and retain the notation ${}_GH_n(X, A; M)$ for our cellular theory. Recall that $\bar{X}^n$ denotes $X^n \cup A$, the relative n-skeleton of the G-complex (X, A).

Step 1. There is a natural isomorphism ${}_GC_n(X, A; M) \cong {}_G\hat{H}_n(\bar{X}^n, \bar{X}^{n-1}; M)$.

Step 2. Define $\hat{\partial}\colon {}_G\hat{H}_n(\bar{X}^n, \bar{X}^{n-1}; M) \longrightarrow {}_G\hat{H}_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}; M)$ by the composition

$$ {}_G\hat{H}_n(\bar{X}^n, \bar{X}^{n-1}; M) \xrightarrow{\partial} {}_G\hat{H}_{n-1}(\bar{X}^{n-1}; M) \longrightarrow {}_G\hat{H}_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}; M). $$

Then $\hat{\partial}^2 = 0$ and ${}_G\hat{H}_n(X, A; M)$ is the nth homology group of the chain complex $\{{}_G\hat{H}_n(\bar{X}^n, \bar{X}^{n-1}; M), \hat{\partial}\}$. The proofs of both the above steps are exactly analogous to the proofs of the corresponding theorems about homology of CW complexes. By our cellular definition, it therefore suffices to show the following diagram commutes:

$$ \begin{array}{ccc} {}_GC_n(X, A; M) & \xrightarrow{\approx} & {}_G\hat{H}_n(\bar{X}^n, \bar{X}^{n-1}; M) \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\hat{\partial}} \\ {}_GC_{n-1}(X, A; M) & \xrightarrow{\approx} & {}_G\hat{H}_{n-1}(\bar{X}^{n-1}, \bar{X}^{n-2}; M) \end{array} $$

where ∂ is the map of §2.

The general case will ultimately be reduced to the following lemma.

LEMMA. *Assume $n \geqslant 1$. Let $f\colon S^n \longrightarrow S^n$ be a continuous map (with trivial G-action), and $t\colon G/H \longrightarrow G/K$ be a G-map. Let $p\colon G/K \times S^n \longrightarrow (G/K \times S^n)/(G/K \times x)$ be the projection, where $x \in S^n$. Let h be the composition*

$$ G/H \times S^n \xrightarrow{t\times f} G/K \times S^n \xrightarrow{p} \frac{G/K \times S^n}{G/K \times x} = \frac{G/K \times D^n}{G/K \times \dot{D}^n}. $$

If $c \in M(G/H) = {}_G\hat{H}_n(G/H \times S^n; M)$, then $({}_G\hat{H}_n(h))(c) = (\deg f)(M(t)c)$ where $\deg f$ is the degree of the map f.

PROOF. It is not hard to see that $M(G/H) = {}_G\hat{H}_n(G/H \times S^n; M)$ and $M(G/K) = {}_G\hat{H}_n((G/K \times D^n)/(G/K \times \dot{D}^n); M)$; so the above claim makes sense. Then it is easy to verify that $({}_G\hat{H}_n(t \times f; M)(c)) = (\deg f)(M(t)c)$ and that ${}_G\hat{H}_n(p; M)$ is an isomorphism. Q.E.D.

For the general case, we denote by $[X, Y]_G$ the set of G-homotopy classes of G-maps from X to Y, and by $[X, Y]$ the set of (nonequivariant) homotopy classes of maps from X to Y. Then

$$ \begin{aligned} &[G/H \times \dot{D}^n, (G/K \times D^{n-1})/(G/K \times \dot{D}^{n-1})]_G \\ &\quad = [G/H \times S^{n-1}, (G/K \times S^{n-1})/(G/K \times x)]_G \\ &\quad = [S^{n-1}, ((G/K)^H \times S^{n-1})/((G/K)^H \times x)] = \pi_{n-1}((G/K)^{H+} \wedge S^{n-1}). \end{aligned} $$

Here $(G/K)^H$ is the fixed point set of G/K under the obvious H action; x is a point of S^{n-1}; and $(G/K)^{H+}$ is $(G/K)^H$ with an extra point added disjointly from $(G/K)^H$. But $(G/K)^H$ is a C^∞ manifold since G is compact Lie and K is closed.

A familiar argument shows then that $\pi_{n-1}((G/K)^{H+} \wedge S^{n-1}) = \Omega_0((G/K)^{H+})$ if $n - 1 \geqslant 2$. The map from Ω_0 to π_{n-1} is given by the Pontryagin construction; the map is an isomorphism by transversality arguments.

Hence, if f: $G/H \times D^n \longrightarrow (G/K \times D^{n-1})/(G/K \times \dot{D}^{n-1})$ is a G-map, to obtain the induced map on G-homology, it suffices to view f as an element of $\Omega_0((G/K)^{H+})$. But the generators of $\Omega_0((G/K)^{H+})$ correspond precisely to maps of the form given in the lemma, on which we can already compute the map induced on G-homology. By the lemma, the formula for ∂: ${}_GC_n(X, Y; M) \longrightarrow {}_GC_{n-1}(X, Y; M)$ given in §2 holds if $n \geqslant 3$. That the formula for ∂ holds if $n = 2$ is an exercise in one-manifolds. For $n < 2$, the verification that the formula holds is trivial. Q.E.D.

REFERENCES

1. G. E. Bredon, *Equivariant cohomology theories*, Bull. Amer. Math. Soc. **73** (1967), 266–268. MR **34** #6762.

2. ———, *Equivariant cohomology theories*, Lecture Notes in Math., no. 34, Springer-Verlag, Berlin and New York, 1967. MR **35** #4914.

3. Th. Bröcker, *Singuläre Definition der äquivarianter Bredon Homologie*, Manuscripta Math. **5** (1971), 91–102. MR **45** #2695.

4. Eldon Dyer, *Cohomology theories*, Math. Lecture Notes Series, Benjamin, New York, 1969. MR **42** #3780.

5. Sören Illman, *Equivariant algebraic topology*, Thesis, Princeton University, Princeton, N. J., 1972.

6. ———, *Equivariant singular homology and cohomology for actions of compact Lie groups*, Proc. Conf. on Transformation Groups, Lecture Notes in Math., no. 298, Springer-Verlag, Berlin and New York, 1972.

7. ———, *Equivariant singular homology and cohomology*, Bull. Amer. Math. Soc. **79** (1973), 188–192. MR **46** #6340.

8. C. N. Lee, *Equivariant homology theories*, Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 1968, pp. 237–244. MR **40** #3538.

9. Saunders Mac Lane, *Homology*, Die Grundlehren der math. Wissenschaften, Band 114, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #122.

10. T. Matumoto, *Equivariant K-theory and Fredholm operators*, J. Fac. Sci. Univ. Tokyo Sect IA Math. **18** (1971), 109–125. MR **44** #7538.

11. C. Vaseekaran, *Equivariant homotopy theory*, Bull. Amer. Math. Soc. **80** (1974), 322–324.

12. S. J. Willson, *Equivariant homology theories*, Thesis, University of Michigan, Ann Arbor, Mich., 1973.

13. ———, *Equivariant maps between representation spheres*, Pacific J. Math. **56** (1975).

DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50010