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SMOOTH LOCALLY CONVEX SPACES[1]

BY

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ABSTRACT. The main theorem is

Let E be a separable (real) Fréchet space with a nonseparable strong dual. Then E is not strongly D_F^1-smooth.

It follows that if X is uncountable, locally compact, σ-compact, metric space, then $C(X)$ (with the topology of compact convergence) does not have a class of seminorms which generate its topology and are Fréchet differentiable (away from their null-spaces).

1. **Preliminaries.** Throughout this paper, TLS will denote the class of all (Hausdorff) topological linear spaces over the real field $\mathbf{R}$. If X is a topological space, $\mathcal{O}(X)$ will denote the class of all open subsets of X. $L_1(E, F) = L(E, F)$ will denote the set of all continuous linear maps from E into F, where $E, F \in$ TLS. We define by induction $L_p(E, F) = L(E, L_{p-1}(E, F))$. Each $L_p(E, F)$ is given the topology of uniform convergence on bounded subsets of E. It will be convenient in a later proof to identify $L_p(E, F)$ with the space $\widetilde{L}_p(E, F)$ of multilinear maps u from E^p into F, which satisfy the following "continuity" condition:

For each $m \in \{1, \ldots, p\}$, for each sequence $\{y_1, \ldots, y_{m-1}\}$ of points in E, for each sequence $\{B_{m+1}, \ldots, B_p\}$ of bounded subsets of E and for each 0-neighbourhood V in F, there exists a 0-neighbourhood U in E such that

$$u(\{y_1\} \times \cdots \times \{y_{m-1}\} \times U \times B_{m+1} \times \cdots \times B_p) \subset V.$$

A basic 0-neighbourhood in $\widetilde{L}_p(E, F)$ is a set of the form $\{u \in \widetilde{L}_p(E, F) : u(B_1 \times \cdots \times B_p) \subset W\}$, where the B_i are bounded subsets of E and W is a 0-neighbourhood in F.

Let $f : U \longrightarrow V$, where $U \in \mathcal{O}(E)$, $V \in \mathcal{O}(F)$, $E \in$ TLS and $F \in$ TLS. Then f is *Fréchet differentiable at* $x \in U$, if there exists $u \in L(E, F)$, such that for each bounded subset B of E and for each 0-neighbourhood W in F, there exists $\delta > 0$ such that $f(x + th) - f(x) - u(th) \in tW$, whenever $h \in B$ and $|t| \leqslant \delta$.

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The mapping u is then uniquely determined and is denoted by $f'(x)$. If f is Fréchet differentiable at each $x \in U$, then f is *Fréchet differentiable.* The map $f' : U \longrightarrow L(E, F)$ defined by $x \longrightarrow f'(x)$ is called the *Fréchet derivative* of f. Higher order derivatives are defined in the natural way. The nth order Fréchet derivative is denoted by $f^{(n)}$ and is a map from U into $L_n(E, F)$. In case E and F are normed spaces, the Fréchet derivative coincides with the standard definition of derivative in normed spaces [5, p. 149]. Note that a Fréchet differentiable mapping need not be continuous. In fact, we have the following result, the proof of which appears in [12, p. 18]:

Let E be a nonnormed quasi-barrelled locally convex space with strong dual E'. Then the evaluation mapping $\mathrm{ev}: E \times E' \longrightarrow \mathbf{R}$ *defined by* $\mathrm{ev}(x, x') = \langle x, x' \rangle$ *has Fréchet derivatives of all orders and each derivative is continuous, but* ev *is not continuous.*

Let $E, F \in$ TLS be separated by their duals, $U \in \mathcal{O}(E)$ and $V \in \mathcal{O}(F)$. Then $D_F^k(U, V)$ will denote the class of all continuous mappings from U into V, which are k-times Fréchet differentiable ($k \in \{1, 2, \ldots, \infty\}$). The Fréchet derivative has the *composition property*: if $f \in D_F^k(U, V)$ and $g \in D_F^k(V, W)$, then $g \circ f \in D_F^k(U, W)$ [2, p. 234], [13, p. 7].

If we replace "bounded set" by "sequentially compact set" in the definition of the Fréchet derivative, we get a weaker derivative, known as the *Hadamard derivative.* $D_H^k(U, V)$ will denote the class of all continuous mappings from U into V, which are k-times Hadamard differentiable ($k \in \{1, 2, \ldots, \infty\}$). The Hadamard derivative also has the composition property. In normed spaces, the Hadamard derivative coincides with the quasi-derivative [5, p. 157], [3, p. 91]. Finally, if we replace "bounded set" by "finite set", we get the *Gâteaux derivative.* For a detailed discussion of the differential calculus in topological linear spaces, see [2] and [3].

2. **Smoothness and density character.** In this section we prove the result announced in the abstract. Suppose p is a seminorm on a linear space. Its *null-space* N_p is the set $\{x : p(x) = 0\}$. LCS will denote the class of (Hausdorff) locally convex spaces over $\mathbf{R}$. We say $E \in$ LCS is *strongly D_F^k-smooth* ($k \in \{1, 2, \ldots, \infty\}$), if there exists a collection $P(E)$ of continuous seminorms on E which generate the topology on E and satisfy $p \in D_F^k(E \backslash N_p, \mathbf{R})$, for each $p \in P(E)$. Similarly, we define *strongly D_H^k-smooth* spaces. This definition was first given (in the abstract setting of S-categories) in [10] and was also studied in [11] and [12].

2.1 *Let $E \in LCS$. Then E is strongly D_F^k-smooth* ($k \in \{1, 2, \ldots, \infty\}$) *if* *and only if E has a 0-neighbourhood base $\mathfrak{N}$ consisting of absolutely convex,*[2] *closed sets such that if $U \in \mathfrak{N}$ and p is the gauge of U, then $p \in D_F^k(E \backslash N_p, \mathbf{R})$. The analogous statement for the Hadamard derivative also holds.*

PROOF. The sufficiency of the condition is obvious, so we have only to show its necessity. Consider $\phi_1: \mathbf{R} \longrightarrow \mathbf{R}$ defined by $\phi_1(x) = \exp(-1/x)$, for $x > 0$ and $\phi_1(x) = 0$, for $x \leqslant 0$. Put $\phi_2(x) = \phi_1(x-1) \cdot \phi_1(2-x)$ and $\phi_3(x) = \int_1^x \phi_2(t)\,dt / \int_1^2 \phi_2(t)\,dt$. Finally, put $\phi(x) = 1 - \phi_3(|x|)$. Clearly $\phi \in C^\infty(\mathbf{R}, \mathbf{R})$, and ϕ is convex downwards for $|x| \leqslant 3/2$. Also $\phi(3/2) = \phi(-3/2) = \frac{1}{2}$.

Since E is strongly D_F^k-smooth, it has a class, $P(E)$, of seminorms which generate its topology and satisfy $p \in D_F^k(E \backslash N_p, \mathbf{R})$, for each $p \in P(E)$. We can suppose $p \in P(E)$ and $\alpha > 0$ imply $\alpha p \in P(E)$. For each finite subset $\{p_1, \ldots, p_n\} \subset P(E)$, consider the set $U = \{x \in E : \prod_{i=1}^n \phi(p_i(x)) \geqslant \frac{1}{2}\}$.

2.2. LEMMA. *U is an absolutely convex, closed 0-neighbourhood in E and the collection of all such U (as $\{p_1, \ldots, p_n\}$ varies over all finite subsets of $P(E)$) is a 0-neighbourhood base for E.*

PROOF OF LEMMA 2.2. Clearly U is closed and balanced. To show U is convex, let $x, y \in U$ and $0 \leqslant \alpha \leqslant 1$. Thus $\phi(p_1(x)) \cdots \phi(p_n(x)) \geqslant \frac{1}{2}$ and so $p_i(x) \leqslant 3/2$ for each $i = 1, \ldots, n$. Similarly, for y. Then

$$\begin{aligned}
\prod_{i=1}^n \phi(p_i[\alpha x + (1-\alpha)y]) &\geqslant \prod_{i=1}^n \phi(\alpha p_i(x) + (1-\alpha)p_i(y)) \\
&\geqslant \prod_{i=1}^n [\alpha\phi(p_i(x)) + (1-\alpha)\phi(p_i(y))] \\
&\geqslant \prod_{i=1}^n \phi^\alpha(p_i(x)) \cdot \phi^{1-\alpha}(p_i(y)) \\
&\geqslant (\tfrac{1}{2})^\alpha \cdot (\tfrac{1}{2})^{1-\alpha} = \tfrac{1}{2}.
\end{aligned}$$

Finally, the fact that U is a 0-neighbourhood and the collection of all such U forms a 0-neighbourhood base follows by verifying that

$$\left\{x : \sup_{i=1,\ldots,n} p_i(x) \leqslant 1\right\} \subset U \subset \left\{x : \sup_{i=1,\ldots,n} p_i(x) \leqslant 2\right\}.$$

2.3. LEMMA. *Let $q : \mathbf{R}^n \longrightarrow \mathbf{R}$ be the gauge of the absolutely convex, closed 0-neighbourhood $W = \{x = (x_1, \ldots, x_n) : \prod_{i=1}^n \phi(x_i) \geqslant \frac{1}{2}\}$. Then $q \in C^\infty(\mathbf{R}^n \backslash \{0\}, \mathbf{R})$.*

PROOF OF LEMMA 2.3. We give the proof for $n = 2$ only. Let $(x, y) \neq$

[2] "Absolutely convex" means convex and balanced.

$(0, 0)$. Then $q(x, y)$ is the unique $z > 0$ such that $\phi(x/z) \cdot \phi(y/z) = \frac{1}{2}$. Consider ψ: $\mathbf{R}^3 \longrightarrow \mathbf{R}$ defined by $\psi(x, y, z) = \phi(x/z) \cdot \phi(y/z) - \frac{1}{2}$. If $U = \{(x, y, z): (x, y) \neq (0, 0) \text{ and } z > 0\}$, then $\psi \in C^\infty(U, \mathbf{R})$.

Suppose $(a, b, c) \in U$ and $\psi(a, b, c) = 0$. We show $D_3\psi(a, b, c) \neq 0$. Now $D_3\psi(x, y, z) = (-y/z^2) \cdot \phi(x/z) \cdot \phi'(y/z) + (-x/z^2) \cdot \phi(y/z) \cdot \phi'(x/z)$. Thus $D_3\psi(a, b, c) > 0$, because each term in the sum is $\geqslant 0$ and, since $\phi(a/c) \cdot \phi(b/c) = \frac{1}{2}$, at least one is > 0.

Thus, by the implicit function theorem, $q \in C^\infty(\mathbf{R}^2\backslash\{0\}, \mathbf{R})$.

2.4. LEMMA. *Let p be the gauge of $U = \{x \in E: \Pi_{i=1}^n \phi(p_i(x)) \geqslant \frac{1}{2}\}$. Then $p \in D_F^k(E\backslash N_p, \mathbf{R})$.*

PROOF OF LEMMA 2.4. Suppose $p(x) > 0$. Choose β such that $0 < \beta < p(x)$. Put $I = \{1, 2, \ldots, n\}$ and $J = \{i \in I: p_i(x) = 0\}$. J may be empty. Let m be the number of elements in $I\backslash J$. Put $W = \{y \in E : p(y) > \beta, p_i(y) < \beta, \text{ for } i \in J \text{ and } p_i(y) > 0, \text{ for } i \in I\backslash J\}$.

W is an open set containing x. Let $y \in W$. Then

$$\begin{aligned} p(y) &= \inf\left\{\lambda > 0: \prod_{i\in I} \phi(p_i(y/\lambda)) \geqslant \tfrac{1}{2}\right\} \\ &= \inf\left\{\lambda > \beta: \prod_{i\in I} \phi(p_i(y/\lambda)) \geqslant \tfrac{1}{2}\right\} \\ &= \inf\left\{\lambda > \beta: \prod_{i\in I\backslash J} \phi(p_i(y/\lambda)) \geqslant \tfrac{1}{2}\right\} \\ &= q(A(y)), \end{aligned}$$

where $A : E \longrightarrow \mathbf{R}^m$ is defined by $A(z) = (p_i(z))_{i\in I\backslash J}$ and q is the seminorm for $\mathbf{R}^m$ given in Lemma 2.3. Now $A \mid W \in D_F^k(W, \mathbf{R}^m\backslash\{0\})$ and $q \in C^\infty(\mathbf{R}^n\backslash\{0\}, \mathbf{R})$. Hence $p \mid W \in D_F^k(W, \mathbf{R})$ and the result follows.

This completes the proof of 2.1 for the Fréchet derivative. The proof for the Hadamard derivative is the same. In fact, if we define strongly S-smooth spaces, where S is an S-category, as in [10], [11] or [12], then, with the same proof, 2.1 holds with D_F^k replaced by S.

Let $E \in$ TLS. We define the *neighbourhood base character* of E to be the minimal cardinal belonging to the set of cardinals of 0-neighbourhood bases for E. We denote it by bas(E). It is clear that we may assume the cardinality of N in 2.1 is bas(E).

Now let X be a topological space. We define the *density character* of X to be the minimal cardinal belonging to the set of cardinals of dense subsets of X. We denote it by dense(X). We will need the following simple properties of density character.

2.5. *Let X and Y be topological spaces.*

(i) *If $U \in \mathcal{O}(X)$, then* $\operatorname{dens}(U) \leqslant \operatorname{dens}(X)$.

(ii) *Let $f: X \longrightarrow Y$ be a continuous mapping. Then* $\operatorname{dens}(f(X)) \leqslant \operatorname{dens}(X)$.

Now let C be a closed convex subset of $E \in$ LCS. A point x_0 on the boundary of C is called a *support point* of C if there is a nonzero continuous linear functional u such that $\sup_{y \in C} uy = ux_0$. u is called a *support functional* of C. u is a *normalised support functional* if $\sup_{y \in C} uy = ux_0 = 1$.

Let p be a continuous seminorm on E and $C = \{x \in E: p(x) \leqslant 1\}$. Let $p(x_0) = 1$. If p is a Gâteaux differentiable at x_0, then $p'(x_0)$ is the unique normalised support functional to C at x_0 [6, p. 349].

2.6 (Phelps [14, p. 397]). *If C is a closed convex set with nonempty interior in the complete locally convex space E, then the support functionals of C are dense (in the strong topology) among those continuous linear functionals in E', which are bounded above on C.*

2.7 (Asplund and Rockafellar [1, p. 459]). *Let $E \in LCS$ and p be a continuous seminorm on E. Suppose p is Fréchet differentiable on $A \subset E$. Then $p' : A \longrightarrow E'$ is continuous, when E' has the strong topology.*

Now we can prove our main result.

2.8. *Let E be a complete locally convex space, with strong dual E', such that* $\operatorname{dens}(E') > \operatorname{bas}(E) \cdot \operatorname{dens}(E)$. *Then E is not strongly D^1_F-smooth.*

Proof. Suppose E is strongly D^1_F-smooth. We show $\operatorname{dens}(E') \leqslant \operatorname{bas}(E) \cdot \operatorname{dens}(E)$. Let $N = \{U_\alpha\}_{\alpha \in A}$ be the 0-neighbourhood base for E given by 2.1 (for the case $k = 1$) and such that the cardinality of A is $\operatorname{bas}(E)$. For each $\alpha \in A$, put $E'_\alpha = \{u \in E' : u$ is bounded on $U_\alpha\}$. Then $E' = \bigcup_{\alpha \in A} E'_\alpha$, since a linear functional is continuous if and only if it is bounded on some 0-neighbourhood.

Let p be the gauge of some U_α. By 2.7, $p': E \backslash W_p \longrightarrow E'$ is continuous. Define $\mu : E \backslash W_p \longrightarrow E'$ by $\mu(x) = p(x) \cdot p'(x)$. Then μ is continuous. Thus $\operatorname{dens}(\mu(E \backslash W_p)) \leqslant \operatorname{dens}(E)$. But $\mu(E \backslash W_p)$ is the set of all support functionals to U_α and so is dense in E'_α, by 2.6. Thus $\operatorname{dens}(E'_\alpha) \leqslant \operatorname{dens}(E)$ and so $\operatorname{dens}(E') \leqslant \operatorname{bas}(E) \cdot \operatorname{dens}(E)$.

2.9. Corollary. *Let E be a separable Fréchet space with a nonseparable strong dual. Then E is not strongly D^1_F-smooth.*

2.8 generalises the result of Kadec [16] and Restrepo [15] to locally convex spaces. Stronger versions of the Kadec-Restrepo result have been obtained in Banach spaces by Leach and Whitfield [7] and Leduc [8].

Of course, we cannot omit the hypothesis in 2.9 that E be metrizable. For let $E = \mathbf{R}^{\mathbf{R}}$ (product of $\mathbf{R}$ copies of $\mathbf{R}$, with the product topology). Then E is a complete separable locally convex space, which is clearly strongly D_F^1-smooth. But the strong dual of $\mathbf{R}^{\mathbf{R}}$ is $\mathbf{R}^{(\mathbf{R})}$ (locally convex direct sum), which is not separable.

We give a class of locally convex spaces satisfying the hypotheses of 2.9. Let X be an uncountable, σ-compact, locally compact, metric space. Let $C(X)$ be the real linear space of all continuous, complex- or real-valued functions on X, with the topology of compact convergence. Then $C(X)$ is a separable Fréchet space with a nonseparable strong dual and, consequently, is not strongly D_F^1-smooth. Note that $C(X)$ is strongly D_H^1-smooth, however, since it is separable [10], [11].

3. **Smooth locally convex direct sums.** First we need a result about the differentiability of functions defined on a strict inductive limit. A topological inductive limit of the form $E[T] = \Sigma_\alpha E_\alpha[T_\alpha]$, where each $E_\alpha[T_\alpha]$ is a locally convex space, is said to be *strict* if $E_\alpha \subset E_\beta$, for $\alpha < \beta$, and if the topology induced by T_β on the subspace E_α of E_β is equal to T_α [6. p. 222]. The next result was given in [10]. However, the proof given there contains a mistake in the induction step.

3.1. *Let $E[T] = \Sigma_\alpha E_\alpha[T_\alpha]$ be a strict inductive limit with the property that a subset $B \subset E$ is bounded if and only if B is contained in some E_α and is bounded there. Let $f: E \longrightarrow F$, where $f \in LCS$, be a continuous mapping. Let $U \in O(E)$. Then $f \in D_F^k(U, F)$ $(k \in \{1, 2, \ldots, \infty\})$ if and only if $f|E_\alpha \in D_F^k(U \cap E_\alpha, F)$, for each α.*

Proof. The necessity is obvious. For the sufficiency, we prove, by induction, that $f \in D_F^k(U, F)$ $(k \in \{1, 2, \ldots\})$ and, for each $x \in U$, $f^{(k)}(x) \cdot (y_1, \ldots, y_k) = (f|E_\alpha)^{(k)}(x) \cdot (y_1, \ldots, y_k)$, where $x, y_1, \ldots, y_k \in E_\alpha$.

Thus suppose first that $f|E_\alpha \in D_F^1(U \cap E_\alpha, F)$ for each α. Let $x \in U$. We define a map $u_1: E \longrightarrow F$ as follows. Given $y \in E$, there exists α such that $x, y \in E_\alpha$. Then define $u_1(y) = (f|E_\alpha)'(x) \cdot y$. The value of $u_1(y)$ is independent of the choice of α. For suppose $x, y \in E_\beta$ also. Choose γ such that $\gamma \geqslant a$ and $\gamma \geqslant \beta$. Then $(f|E_\alpha)'(x) \cdot y = (f|E_\gamma)'(x) \cdot y = (f|E_\beta)'(x) \cdot y$. Also u_1 is linear and is continuous, since $u_1|E_\alpha$ is continuous, for each α [6, p. 217].

We show that $u_1 = f'(x)$. Let B be a bounded subset of E. Then there exists an α such that $B \subset E_\alpha$ and is bounded there. Also $x \in E_\beta$, for some β. Now choose γ such that $\gamma \geqslant \beta$ and $\gamma \geqslant \alpha$. Then $x \in E_\gamma$ and $B \subset E_\gamma$. Also, B is bounded in E_γ, since the topology induced by E_γ on E_α is the original topology T_α on E_α.

Now let W be a 0-neighbourhood in F. Then the existence of $(f|E_\gamma)'(x)$ gives the existence of $\delta > 0$ such that $f(x + th) - f(x) - u_1 \cdot th \in tW$, whenever $|t| \leqslant \delta$ and $h \in B$. That is, $f'(x) = u_1$. Thus $f \in D_F^1(U, F)$ and the proposition is true for $k = 1$.

Now suppose the proposition is true for some k. Let $f \mid E_\alpha \in D_F^{k+1}(U \cap E_\alpha, F)$, for each α, and $x \in U$. We define a map $u_{k+1}: E^{k+1} \longrightarrow F$ as follows: $u_{k+1}(y_1, \ldots, y_{k+1}) = (f|E_\alpha)^{(k+1)}(x) \cdot (y_1, \ldots, y_{k+1})$, where α is chosen so that $x, y_1, \ldots, y_{k+1} \in E_\alpha$.

First we have to show u_{k+1} is well defined. Thus suppose we also have $x, y_1, \ldots, y_{k+1} \in E_\beta$. Choose γ so that $\gamma \geqslant \alpha$ and $\gamma \geqslant \beta$. Then

$$\begin{aligned}(f|E_\alpha)^{(k+1)}(x) \cdot (y_1, \ldots, y_{k+1}) &= (f|E_\gamma)^{(k+1)}(x) \cdot (y_1, \ldots, y_{k+1}) \\ &= (f|E_\beta)^{(k+1)}(x) \cdot (y_1, \ldots, y_{k+1}).\end{aligned}$$

Next we show $u_{k+1} \in \widetilde{\mathcal{L}}_{k+1}(E, F)$. Clearly u_{k+1} is multilinear. As for the continuity property, let $m \in \{1, \ldots, k+1\}$, $y_1, \ldots, y_{m-1}, B_{m+1}, \ldots, B_{k+1}$ and V, an absolutely convex 0-neighbourhood in F, be given. Let α be given. Choose $\beta \geqslant \alpha$ such that $x, y_1, \ldots, y_{m-1} \in E_\beta$ and $B_{m+1}, \ldots, B_{k+1} \subset E_\beta$. Now since $(f|E_\beta)^{(k+1)}(x) \in \widetilde{\mathcal{L}}_{k+1}(E_\beta, F)$, there exists a 0-neighbourhood W_β in E_β such that

$$(f|E_\beta)^{(k+1)}(x) \cdot (\{y_1\} \times \cdots \times \{y_{m-1}\} \times W_\beta \times B_{m+1} \times \cdots \times B_{k+1}) \subset V.$$

Then $W_\alpha = W_\beta \cap E_\alpha$ is a 0-neighbourhood in E_α and

$$u_{k+1}(\{y_1\} \times \cdots \times \{y_{m-1}\} \times W_\alpha \times B_{m+1} \times \cdots \times B_{k+1}) \subset V.$$

Put $W = \Gamma_\alpha W_\alpha$. That is, W is the absolutely convex cover of the W_α. Then W is a 0-neighbourhood in E and, since u_{k+1} is multilinear,

$$u_{k+1}(\{y_1\} \times \cdots \times \{y_{m-1}\} \times W \times B_{m+1} \times \cdots \times B_{k+1}) \subset V.$$

Thus $u_{k+1} \in \widetilde{\mathcal{L}}_{k+1}(E, F)$.

Now we show $u_{k+1} = f^{(k+1)}(x)$. Let $\{u \in \widetilde{\mathcal{L}}_k(E, F) : u(B_1 \times \cdots \times B_k) \subset W\}$ be a 0-neighbourhood in $\widetilde{\mathcal{L}}_k(E, F)$ and B a bounded subset of E. Choose α such that $x \in E_\alpha$ and $B, B_1, \ldots, B_k \subset E_\alpha$. By the existence of $(f|E_\alpha)^{(k+1)}(x)$, there exists $\delta > 0$ such that

$$\begin{aligned}(f|E_\alpha)^{(k)}(x + th) \cdot (h_1, \ldots, h_k) &- (f|E_\alpha)^{(k)}(x) \cdot (h_1, \ldots, h_k) \\ &- (f|E_\alpha)^{(k+1)}(x) \cdot (th, h_1, \ldots, h_k) \in tW,\end{aligned}$$

whenever $h \in B$, $h_1 \in B_1, \ldots, h_k \in B_k$ and $|t| \leqslant \delta$. Thus, by the inductive hypothesis,

$$f^{(k)}(x + th) \cdot (h_1, \ldots, h_k) - f^{(k)}(x) \cdot (h_1, \ldots, h_k)$$

$$- u_{k+1}(th, h_1, \ldots, h_k) \in tW,$$

whenever $h \in B$, $h_1 \in B_1, \ldots, h_k \in B_k$ and $|t| \leqslant \delta$. Thus $f^{(k+1)}(x)$ exists and $f^{(k+1)}(x) = u_{k+1}$. Hence $f \in D_F^{k+1}(U, F)$. This completes the proof of 3.1.

If E satisfies the conditions of 3.1 and also has the property that a subset of $K \subset E$ is sequentially compact if and only if K is contained in some E_α and is sequentially compact there, then there is a result analogous to 3.1 for the Hadamard derivative. In particular, we have the following result.

3.2. *Let $E = \bigoplus_{\alpha \in A} E_\alpha$ be the locally convex direct sum of the locally convex spaces E_α. Let $f : E \longrightarrow F$ be a continuous mapping, where $F \in LCS$. Let $U \in \mathcal{O}(E)$. Then $f \in D_F^k(U, F)$ (resp. D_H^k) ($k \in \{1, 2, \ldots, \infty\}$) if and only if $f | \bigoplus_{i=1}^n E_{\alpha_i} \in D_F^k(U \cap \bigoplus_{i=1}^n E_{\alpha_i}, F)$ (resp. D_H^k) for each finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset A$.*

Now let $E = \bigoplus_{n \in \mathbf{N}} E_n$ be the countable locally convex direct sum of the locally convex spaces $\{E_n\}_{n \in \mathbf{N}}$ and I_n the canonical injection from E_n into E (where $\mathbf{N} = \{1, 2, \ldots\}$). The absolutely convex covers $\Gamma_{n \in \mathbf{N}} I_n(V_n)$ form a 0-neighbourhood base for E, as V_n ranges over a 0-neighbourhood base for E_n. Our final theorem (3.3) will show that the countable locally convex direct sum of smooth spaces is smooth. First we introduce another type of smoothness.

Let $E \in$ TLS be separated by its dual. We say E is *D_F^k-smooth* ($k \in \{1, 2, \ldots, \infty\}$) if, given $V \in \mathcal{O}(E)$ and $a \in V$, there exists $f \in D_F^k(E, \mathbf{R})$ such that $f(a) > 0$, $f \geqslant 0$ and $\{x \in E: f(x) > 0\} \subset V$. Similarly we define *$D_H^k$-smooth* spaces. This concept was first given (in the abstract setting of $\mathcal{S}$-categories) by Bonic and Frampton [4] for Banach spaces and later studied in topological linear spaces in [10], [11] and [12]. In [10] we showed that if S is an arbitrary $\mathcal{S}$-category (e.g. D_F^k) and $E \in$ LCS is strongly S-smooth, then E is S-smooth. It is not known if the converse is true, although some partial converses are known. For example, combining the results of Leach and Whitfield [7] and Restrepo [15], if E is a separable, D_F^1-smooth Banach space, then E is strongly D_F^1-smooth. See also [9].

3.3. *Let $E = \bigoplus_{n \in \mathbf{N}} E_n$ be the countable locally convex direct sum of the locally convex spaces $\{E_n\}_{n \in \mathbf{N}}$. Then:*

(i) *E is D_F^k-smooth (resp. D_H^k-smooth) ($k \in \{1, 2, \ldots, \infty\}$) if and only if each E_n is D_F^k-smooth (resp. D_H^k-smooth).*

(ii) *E is strongly D_F^k-smooth (resp. strongly D_H^k-smooth) if and only if each E_n is strongly D_F^k-smooth (resp. strongly D_H^k-smooth).*

PROOF. We give the proofs for the Fréchet derivative only. The Hadamard case is similar.

(i) The necessity is obvious. For the converse, it suffices to verify the smoothness condition for $V = \Gamma_{n\in\mathbf{N}} I_n(V_n)$, a basic 0-neighbourhood in E, and $a = 0$. For each $n \in \mathbf{N}$, there exists $f_n \in D_F^k(E_n, \mathbf{R})$ such that $f_n \geqslant 0, f_n(0) = 1$ and $\{ x_n \in E_n\colon f_n(x_n) > 0\} \subset V_n$. Define $A\colon E \longrightarrow l^2$ by $x \longrightarrow (1 - f_n(2^n \cdot x_n))_{n\in\mathbf{N}}$, where $x = (x_n)_{n\in\mathbf{N}} \in E$. Choose $\psi \in C^\infty(l^2, \mathbf{R})$ such that $\psi \geqslant 0$, $\psi(0) > 0$ and $\psi(x) = 0$, if $\|x\|_2 \geqslant 1$. Then define $f\colon E \longrightarrow \mathbf{R}$ by $f = \psi \circ A$.

Now $f \geqslant 0$ and $f(0) = \psi(0) > 0$. Also $\{x\colon f(x) > 0\} \subset V$. For let $f(x) > 0$. Hence $\|A(x)\|_2 < 1$. That is, $1 - f_n(2^n \cdot x_n) < 1$ and so $2^n \cdot x_n \in V_n$, for each $n \in \mathbf{N}$. Thus $x = \Sigma_{n\in\mathbf{N}} 2^{-n} \cdot I_n(2^n \cdot x_n) \in V$.

Finally, we show $f \in D_F^k(E, \mathbf{R})$. For this it suffices to show $A \in D_F^k(E, l^2)$ and hence, by 3.2, $A | \bigoplus_{n=1}^m E_n \in D_F^k(\bigoplus_{n=1}^m E_n, l^2)$, for each $m \in \mathbf{N}$. However, this is clear.

(ii) The necessity is obvious. Conversely, let each E_n be strongly D_F^k-smooth. Consequently, by 2.1, each E_n has a 0-neighbourhood base $\mathcal{N}_n$ consisting of absolutely convex sets such that if $V_n \in \mathcal{N}_n$ and p_n is the gauge of V_n, then $p_n \in D_F^k(E_n \backslash W_{p_n}, \mathbf{R})$.

Consider a basic 0-neighbourhood $V = \Gamma_{n\in\mathbf{N}} I_n(V_n)$ in E, where each V_n has the above property. Put $U = \{x = (x_n)_{n\in\mathbf{N}}\colon \Pi_{n\in\mathbf{N}}\ \phi(p_n(2^{n+1} \cdot x_n)) \geqslant 1/2\}$, where ϕ is the function in the proof of 2.1 Then U is absolutely convex. Also U is a 0-neighbourhood. For let $W = \Gamma_{n\in\mathbf{N}} I_n(2^{-n-1} \cdot V_n)$ and $x \in W$. Then $p_n(x_n) \leqslant 2^{-n-1}$ and, consequently $\phi(p_n(2^{n+1} x_n)) = 1$ for each $n \in \mathbf{N}$. Thus $x \in U$.

The collection of all such U forms a 0-neighbourhood base for E. For let $x \in U$. Then $\phi(p_n(2^{n+1} x_n)) \neq 0$, and so $p_n(2^{n+1} x_n) < 2$ for each $n \in \mathbf{N}$. Thus $x \in V$.

Finally, we show that if p is the gauge of U, then $p \in D_F^k(E \backslash W_p, \mathbf{R})$. By 3.2, it suffices to show that $p | \bigoplus_{n=1}^m E_n \in D_F^k(\bigoplus_{n=1}^m E_n \backslash W_p, \mathbf{R})$ for each $m \in \mathbf{N}$. Put $F = \bigoplus_{n=1}^m E_n$. Let q_n be the continuous seminorm on F defined by $q_n(x) = p_n(2^{n+1} x_n)$, where $x = (x_n) \in F$. Then $q_n \in D_F^k(F \backslash W_{q_n}, \mathbf{R})$. Also $p|F$ is the gauge of $U \cap F = \{x \in F\colon \Pi_{n=1}^m \phi(q_n(x)) \geqslant 1/2\}$. Consequently, by 2.4, $p|F \in D_F^k(F \backslash W_p, \mathbf{R})$.

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