SINGULARITY SUBSCHEMES AND GENERIC PROJECTIONS

BY

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ABSTRACT. Corresponding to a morphism \( f: V \rightarrow W \) of algebraic varieties (such that \( \dim(V) \leq \dim(W) \)), we construct a family of subschemes \( S^{(q)}(f) \subset V \). When \( V \) and \( W \) are nonsingular, the \( S^{(q)} \), \( q > 1 \), induce a filtration of the set of closed points \( x \in V \) such that the tangent space map \( df_x: T(V) \rightarrow T(W) \) has rank \( = \dim(V) - 1 \). We prove that if \( V \) is a suitably embedded nonsingular projective variety and \( \pi: V \rightarrow \mathbb{P}^m \) is a generic projection, then the \( S^{(q)}(f) \) and certain fibre products of several of the \( S^{(q)}(f) \) are either empty or smooth and of the smallest possible dimension, except in cases where \( q + 1 \) is divisible by the characteristic of the ground field. We apply this result to describe explicitly the ring homomorphisms \( \pi_*: \mathcal{O}_{\mathbb{P}^m(\pi)} \rightarrow \mathcal{O}_{\pi(V)} \) and (when \( m > r + 1 \)) to study the local structure of the image \( V' = \pi(V) \subset \mathbb{P}^m \).

0. Introduction. Let \( k \) be an algebraically closed field, and let \( V \) be a nonsingular projective variety over \( k \). As in [11], we consider an embedding \( V \subset \mathbb{P}^n \) (projective \( n \)-space) and a finite morphism \( \pi: V \rightarrow \mathbb{P}^m \), where \( \dim(V) \leq m \leq 2 \cdot \dim(V) \), induced by projection from an \((n - m - 1)\)-subspace \( L \subset \mathbb{P}^n \). We study certain properties of generic projections, i.e. properties which \( \pi \) has when \( L \) corresponds to a point of a suitable dense open subset of the Grassmann variety \( G(n, n - m - 1) \).

When \( m > \dim(V) \), one objective is to describe the structure of the local rings \( \mathcal{O}_{\pi'(y), y} \), where \( y \in V' = \pi(V) \subset \mathbb{P}^m \). (If \( (A, m) \) is a local ring, then \( \hat{A} \) is its completion in the \( m \)-adic topology.) Using our main results, which are stated in §1, this can be done in the case that the map of Zariski tangent spaces \( d\pi_x: T_V \rightarrow T_{\mathbb{P}^m} \) has rank \( \geq \dim(V) - 1 \) for all \( x \in \pi^{-1}(y) \). Details of this are given in §13. If \( m = \dim(V) \), the corresponding problem is to describe the local structure of the branch locus \( \pi(S_1(\pi)) \subset \mathbb{P}^m \). (\( S_1(\pi) \) is defined in §2.) I have obtained some results concerning this problem which will appear in another paper. (See also [14].) The results of the present paper were announced in [13]. I would

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like to thank the referee for suggesting several ways to improve this paper.

Our techniques involve studying the higher order singularity subschemes of the morphism \( \pi: V \to \mathbb{P}^m \). Some properties of these subschemes are described in §1; the definition is given in §3.

All rings and algebras will be assumed to be commutative with identity; all schemes and morphisms of schemes will be assumed to be separated. The field \( k \) may always be assumed to be algebraically closed, but of arbitrary characteristic. A variety is a reduced and irreducible scheme of finite type over \( k \).

1. Statement of the main results. Let \( f: X \to Y \) be a morphism of varieties (over the algebraically closed field \( k \)). In this section we will assume that \( X \) is nonsingular. When \( i \geq 1 \), let \( S_i(f) \subset X \) be the closed subset whose closed points are all \( x \in X \) such that the map of Zariski tangent spaces \( (df)_x: T(X)_x \to T(Y)_f(x) \) has rank \( \leq \dim(X) - i \).

If \( \dim(X) < \dim(Y) \), one can study the points where \( (df)_x \) has rank = \( \dim(X) - 1 \). To do this, we define closed subschemes \( S^{(q)}_1(f) \subset X - S_2(f) \), for \( q \geq 0 \), called higher order singularity subschemes. (See §3.) \( S^{(q)}_1(f) \) is defined in terms of a certain Fitting ideal of the \( O_X \)-algebra \( P^d_{X/Y} \) of relative principal parts. If \( x \) is a closed point of \( X - S_2(f) \) and \( y = f(x) \), then \( x \in S^{(q)}_1(f) \) if and only if \( \dim_k O_{X,Y}/m_y O_x \geq q + 1 \), where \( (O_x, m_x) = O_{X,x} \) is an algebra over \( (O_y, m_y) = O_{Y,y} \) via the usual homomorphism \( f^*: O_y \to O_x \). The \( S^{(q)}_1(f) \) are closely related to the Thom-Boardman singularities \( S_1, \ldots, S_q \) of a map of differentiable manifolds. (See [16], [17, §12], [5], [1], or [7].) Thom-Boardman singularities in algebraic geometry have been studied in [6] and [8].

With \( f \) as above and \( d \geq 1 \), let \( \Sigma_d(f) \subset X \times_Y \cdots \times_Y X \) (\( d \)-fold fibre product) be the complement of the union of all diagonals. If \( q_1, \ldots, q_d \) are nonnegative integers, we regard \( S^{(q_1)}_1(f) \times_Y \cdots \times_Y S^{(q_d)}_1(f) \) as a subscheme of \( X \times_Y \cdots \times_Y X \).

**Definition 1.1.** \( \Sigma_d(f; q_1, \ldots, q_d) = \Sigma_d(f) \cap (S^{(q_1)}_1(f) \times_Y \cdots \times_Y S^{(q_d)}_1(f)) \).

One can regard \( \Sigma_d(f; q_1, \ldots, q_d) \) as consisting of all \( d \)-tuples \( (x_1, \ldots, x_d) \) of distinct closed points of \( V \) such that (a) \( x_j \in S^{(q_j)}_1(f) \), \( j = 1, \ldots, d \), and (b) \( f(x_1) = \cdots = f(x_d) \).

**Theorem A.** Let \( V \) be a nonsingular projective variety over \( k \), and let \( r \leq m \leq 2r \), where \( r = \dim(V) \). Then there is an embedding \( V \subset \mathbb{P}^n \), for some \( n \), such that if \( \pi: V \to \mathbb{P}^m \) is a generic projection, then the following statements hold for every \( d \geq 1 \) and every \( d \)-tuple \( (q_1, \ldots, q_d) \) of nonnegative integers.

- (i) \( \Sigma_d(\pi; q_1, \ldots, q_d) \) is either empty or of pure dimension \( dr - (d - 1)m - \Sigma_{j=1}^d q_j(m - r + 1) \).
- (ii) If \( \text{char}(k) \nmid (q_j + 1) \) for all \( j \), then \( \Sigma_d(\pi; q_1, \ldots, q_d) \) is smooth (over \( k \)).
(iii) In general, the dimension of the nonsmooth locus of \( \Sigma_d(\pi; q_1, \ldots, q_d) \) is \( \leq \text{card}(Q) - 1 \), where \( Q = \{ j | \text{char}(k) \text{ divides } q_j + 1 \} \).

Theorem 10.5 and Corollary 10.6 give a more detailed version of (iii). Non-smooth points do occur for some generic projections of varieties over fields of positive characteristic. The simplest example is a pinch point of a generic projection \( \pi: V^2 \rightarrow P^3 \) when \( \text{char}(k) = 2 \). The results of [12] imply that there exist surfaces whose generic projections actually exhibit pinch points.

**Corollary 1.2.** Let \( \pi: V \rightarrow P^m \) be as in Theorem A.

(i) For every \( q \geq 0 \), \( S_1^{(q)}(\pi) \) is either empty or of pure codimension \( q(m - r + 1) \) in \( V \).

(ii) If \( \text{char}(k) \nmid (q + 1) \), then \( S_1^{(q)}(\pi) \) is smooth over \( k \).

(iii) If \( \text{char}(k) \mid (q + 1) \), then \( S_1^{(q)}(\pi) \) is smooth except at finitely many points of \( S_1^{(q)}(\pi) - S_1^{(q+1)}(\pi) \).

When \( dr - (d - 1)m - \Sigma_{j=1}^d q_j(m - r + 1) < 0 \), conclusion (i) of Theorem A implies that \( \Sigma_d(\pi; q_1, \ldots, q_d) = \emptyset \). When \( m \geq r + 1 \), this implies that \( \Sigma_d(\pi) = \emptyset \) when \( d > m/(m - r) \). If \( m = r \), it also follows that only finitely many \( \Sigma_d(\pi) \) are nonempty. Specifically, \( \Sigma_d(\pi) = \emptyset \) when \( d > \text{degree}(V) \).

**Corollary 1.3.** Let \( \pi: V \rightarrow P^m \) be as in Theorem A. Then only finitely many of the subschemes \( \Sigma_d(\pi; q_1, \ldots, q_d) \) are nonempty. If either \( \text{char}(k) = 0 \) or \( \text{char}(k) > (m + 1)/(m - r + 1) \), then all nonempty \( \Sigma_d(\pi; q_1, \ldots, q_d) \) are smooth. (Thus, for a given value of \( r = \dim(V) \), nonsmooth \( \Sigma_d(\pi; q_1, \ldots, q_d) \) will occur only when \( \text{char}(k) \) assumes finitely many values.)

**Proof.** We observe that \( S_1^{(q)}(\pi) = \emptyset \) when \( q > r/(m - r + 1) \). Therefore, if \( \text{char}(k) > 1 + r/(m - r + 1) = (m + 1)/(m - r + 1) \), then \( S_1^{(q)}(\pi) = \emptyset \) for all \( q \) such that \( \text{char}(k) \mid (q + 1) \). Q.E.D.

Theorem A is proved in §§10 and 11. Preliminary results needed in the proof are proved in §§5 through 9. When \( \text{char}(k) = 0 \), the proof of Theorem A is shorter and more conceptual than in the general case. This is because it is possible to apply Theorem 2.9, which is an algebraic version of Sard’s Theorem. The proof of the characteristic 0 case of Theorem A uses nothing from §§7 and 11.

**Theorem B.** Let \( f: X \rightarrow Y \) and \( f': X' \rightarrow Y' \) be finite morphisms of nonsingular varieties over \( k \), where \( \dim(X') = \dim(X) \leq \dim(Y) = \dim(Y') \). Let \( q_1, \ldots, q_d \) be nonnegative integers, and let \( x_1, \ldots, x_d \) and \( x'_1, \ldots, x'_d \) be distinct closed points of \( X \) and \( X' \) respectively. Assume

(a) \( x_j \in S_1^{(q)}(f) - S_1^{(q+1)}(f) \) and \( x'_j \in S_1^{(q)}(f') - S_1^{(q+1)}(f') \), \( j = 1, \ldots, d \);
(b) \( S^{(q_1)}(f) \times_Y \cdots \times_Y S^{(q_d)}(f') \times_Y \cdots \times_Y S^{(q_1)}(f') \)
are smooth and of pure dimension \( d - (d - 1)m - \Sigma_{j=1}^m q_j(m - r + 1) \) at 
\((x_1, \ldots, x_d)\) and \((x'_1, \ldots, x'_d)\) respectively.

Then there are isomorphisms \( \psi: \mathcal{O}_{Y,y} \cong \mathcal{O}_{X,x_j} \) and \( \phi: \mathcal{O}_{X,x_j} \cong \mathcal{O}_{X',x'_j} \),
\( j = 1, \ldots, d, \) such that \( f^* \circ f'^* = \psi^{-1} \circ \phi^{-1} \) for \( j = 1, \ldots, d, \) where \( f: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x_j} \) is induced by \( f \) and \( f'^*: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X',x'_j} \) is similarly induced by \( f' \).

In the case that \( \{x_1, \ldots, x_d\} = f^{-1}(y) \), one can consider the scheme
theoretic fibre \( \text{Spec}(A_y) \), where \( A_y = \mathcal{O}_{X,f^{-1}(y)}/f^*(m_y) \cdot \mathcal{O}_{X,f^{-1}(y)} \). (The
homomorphism \( f^*: \mathcal{O}_y \rightarrow \mathcal{O}_{X,f^{-1}(y)} \) is induced by \( f \). Note that \( A_y \) is an
Artinian ring.) The conclusion of Theorem B can be interpreted as saying that
the homomorphism
\[
f^*: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,f^{-1}(y)} = \prod_{j=1}^d \mathcal{O}_{X,x_j}
\]
depends only on \( A_y \), up to automorphisms of the domain and range of \( f^* \).

Theorem B is proved in §12. Specifically, Theorem 12.1 shows how to
identify \( \mathcal{O}_{Y,y} \) and \( \mathcal{O}_{X,x_j} \), \( j = 1, \ldots, d, \) with formal power series rings in a way
which makes it possible to give a very explicit description of \( f^* \). We call this
description a canonical form of \( f^* \).

Theorem B is relevant to the study of generic projections because a generic
projection \( \pi: V \rightarrow \mathbb{P}^m \) satisfies the hypotheses of Theorem B under the circumstances
described by Theorem A.

Remark 1.4. Theorem 3 of [13] contains a statement which would seem
to imply that if \( \pi: V \rightarrow \mathbb{P}^m \) is a generic projection and \( d \geq 2 \), then all nonempty
\( \Sigma_d(a; q_1, \ldots, q_d) \) are smooth. However, this is not the case; counterexamples
can be given with \( r = m = 2 \) and \( \text{char}(k) = 2 \).

2. First-order singularity subschemes. In this section we define and study
subschemas \( S(f) \subset X \) corresponding to certain morphisms \( f: X \rightarrow Y \). The
definition is given in terms of Fitting ideals; we begin by recalling the definition
of these ideals.

Let \( A \) be a ring, \( M \) a finitely presented \( A \)-module. Consider a presentation:
\[
(*) \quad A^m \xrightarrow{\psi} A^r \rightarrow M \rightarrow 0,
\]
where \( \psi(e_j) = \Sigma_{k=1}^r a_{jk} f_k \) relative to standard bases \( \{e_1, \ldots, e_m\} \) and
\( \{f_1, \ldots, f_r\} \). The \( i \)th Fitting ideal \( \Delta_i(M) \subset A \) is defined to be the ideal generated
by the \((r - i) \times (r - i)\) minors of the matrix \((a_{jk})\) [resp. the unit ideal] if \( 0 \leq i < r \) [resp. \( i \geq r \)]. It is known (see [3]) that the \( \Delta_i(M) \) are independent of the
choice of the presentation (\( * \)). We note that \( \Delta_i(M) \subset \Delta_{i+1}(M) \) for all \( i \). Verification
of the following lemma is also straightforward.
Lemma 2.1. If \((A, \mathfrak{m})\) is local and a minimal generating set for \(M\) contains \(r\) elements, then \(\Delta_{r-1}(M) \neq A\), and \(\Delta_i(M) \subseteq \mathfrak{m}^{r-1}\) when \(0 \leq i \leq r - 1\).

Since the Fitting ideals of a module are independent of the presentation used to compute them, we can define Fitting ideals of coherent sheaves on a Noetherian scheme.

Proposition 2.2. Let \(X\) be a Noetherian scheme, \(F\) a coherent sheaf of modules on \(X\). For each \(i \geq 0\) there is a unique sheaf of ideals \(\Delta_i(F) \subseteq \mathcal{O}_X\) such that \(\Gamma(U, \Delta_i(F)) = \Delta_i(\Gamma(U, F))\) for every affine open \(U \subseteq X\). If \(x \in X\), then \(\Delta_i(F)_x = \Delta_i(F_x)\).

We call \(\Delta_i(F)\) the \(i\)th Fitting ideal of \(F\).

We now fix a morphism \(f: X \to Y\) of Noetherian schemes. It will be assumed that the sheaf \(\Omega^1_{X/Y}\) of relative Kähler differentials is coherent.

Definition 2.3. Let \(i > 1\). \(S_i(f) \subseteq X\) is the closed subscheme defined by \(\Delta_{i-1}(\Omega^1_{X/Y})\). We call the \(S_i(f)\) first-order singularity subschemes.

We write \(S_i\) for \(S_i(f)\) when the context is clear. By using properties of Fitting ideals, one sees that \(S_i \supseteq S_{i+1}\).

Proposition 2.4. If \(x \in X\), then \(x \in S_i\) if and only if \(\text{dim}_k(x \Omega^1_{X/Y}(x)) \geq i\). Equality holds when \(x \in S_i - S_{i+1}\). If \(X\) and \(Y\) are varieties over an algebraically closed field \(k\), and \(x\) is a closed point, then \(x \in S_i\) if and only if the map of Zariski tangent spaces \(df_x: T_{X,x} \to T_{Y,f(x)}\) has rank \(\leq r - i\), where \(r = \text{dim}_k(T_{X,x})\).

Proposition 2.5. \(S_{i+1} \subseteq \text{Sing}(S_i)\).

Proof. To establish either result, we apply Lemma 2.1 with \((A, \mathfrak{m}) = \mathcal{O}_{X,x}\) and \(M = (\Omega^1_{X/Y})_x\). In particular, if \(j > i\) and \(x \in S_j - S_{j+1}\), then \(\Delta_{j-1}(M) \subseteq m^{j-1+1}\). By considering the exact sequence of vector spaces

\[
(f^*\Omega^1_{Y/k})(x) \to \Omega^1_{X/k}(x) \to \Omega^1_{X/Y}(x) \to 0,
\]

one proves the last statement of Proposition 2.4. Q.E.D.

Proposition 2.6. Let \(g: X \to Z\) and \(h: Y \to Z\) be morphisms of Noetherian schemes such that \(g = h \circ f\). Let \(f': X' \to Y'\) be obtained by base extension, where \(X' = X \times_Z Z'\) and \(Y' = Y \times_Z Z'\). \((Z'\) is a scheme over \(Z)\) Then \(S_i(f') = S_i(f) \times_Z Z'\) for all \(i\).

Proof. It is well known that \(\Omega^1_{X'/Y'} = v^*(\Omega^1_{X/Y})\), where \(v: X' \to X\) is projection to the first factor. We now obtain the conclusion by applying the following lemma.

Lemma 2.7. Let \(v: X' \to X\) be a morphism of Noetherian schemes, and
let $E$ be a coherent sheaf on $X$. Then $\Delta_i(v^*E) = v^*(\Delta_i(E))$ for all $i$.

**Proof.** After reducing to the case where $X = \text{Spec}(A)$, $X' = \text{Spec}(B)$ and $E = \mathcal{M}$, we apply $\otimes_A B$ to the exact sequence $(*)$. Q.E.D.

**Proposition 2.8.** Let $x \in X$, $y = f(x) \in Y$. Let $g, h$ be as in Proposition 2.6, and assume that $(\Omega^1_{X/Z})_x$ and $(\Omega^1_{Y/Z})_y$ are free modules of rank $r$ and $m$ respectively. If either $m \geq r$ or $r - i \leq m \leq r$ holds, then every component of $S_x(f)$ which contains $x$ has codimension $(in X) \leq i(m - r + i)$.

**Proof.** Let $z = g(x) = h(y) \in Z$ and set $A = \mathcal{O}_{Z,z}$, $B = \mathcal{O}_{Y,y}$, and $C = \mathcal{O}_{X,x}$. We have an exact sequence of $C$-modules:

$$0 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$  

The first two terms are free of rank $m$ and $r$ respectively, so that $\Delta_{i-1}(\Omega^1_{X/Y})_x = \Delta_{i-1}(\Omega^1_{C/B})$ is generated by the $(r - i + 1) \times (r - i + 1)$ minors of an $m \times r$ matrix. By Theorem 3 of Eagon and Northcott [2], every minimal prime ideal of $\Delta_{i-1}(\Omega^1_{C/B})$ has height $\leq i(m - r + i)$. Q.E.D.

The following result is probably well known, but I have included the proof because it is a direct application of the concepts introduced above.

**Theorem 2.9.** Let $f: X^r \to Y^m$ be a morphism of nonsingular varieties over an algebraically closed field of characteristic 0. If $r \geq m$, then either:

(a) $f(X)$ is contained in a proper closed subset of $Y$, or
(b) there is a dense open subset of $Y$ consisting of closed points $y$ such that $f^{-1}(y)$ is smooth, nonempty, and of pure dimension $r - m$.

**Proof.** If $x$ is a closed point of $X$, then every component of $f^{-1}(f(x))$ has dimension $\geq r - m$, and the Zariski tangent space $T_{f^{-1}(f(x))}x$ is isomorphic to the kernel of the linear map $df_\cdot: T_{X,x} \to T_{Y,f(x)}$. Therefore, $f^{-1}(f(x))$ is smooth and of dimension $r - m$ at $x$ if and only if $x \not\in S_{r-m+1}(f)$.

Let $Z$ be an irreducible component of $S_{r-m+1}(f)$. It must be shown that $\dim(f(Z)) < \dim(Y)$. Assume that this is false, so that $g = f|_Z$ is dominating. $T_{Z,x} \subset T_{X,x}$ for all closed points $x \in X$. Therefore, $dg_\cdot: T_{Z,x} \to T_{Y,g(x)}$ has rank $\leq m - 1$ for every closed point $x \in X$, so that $\dim_{k(x)}(\Omega^1_{Z/Y})_x \geq n - m + 1$ for every point $x$ (closed or not) of $Z$, where $n = \dim(Z)$. Taking $x$ to be the generic point of $Z$, we obtain

$$\dim_{k(Z)}(\Omega^1_{k(Z)/k(Y)}) \geq \dim(Z) - \dim(Y) = \text{tr} \deg_k(Y) k(Z).$$

(Note that $k(Y) \subset k(Z)$ since $g: Z \to Y$ is dominating.) We have thus obtained a contradiction. (Since $\text{char}(k) = 0$, the field extension must be separable.) Q.E.D.

**Example 2.10.** Let $\text{char}(k) = 2$, and let $f: A^2 \to A^1$ send $(a_1, a_2) \to$
For each $a \in k$, $f^{-1}(a)$ is the curve $(X_1 + a)^2 + X_2^3 = 0$, where $a^2 = a$. Each of these curves has a cusp, so that the conclusion of Theorem 2.9 fails. We note that $S_2(f)$ is the line $X_2 = 0$. Therefore, $g = f|_{S_2(f)}$; $S_2(f) \to A^1$ is surjective and induces a purely inseparable field extension of degree 2.

3. Higher order singularity subschemes. We again fix a morphism $f: X \to Y$ of Noetherian schemes such that $\Omega^1_{X/Y}$ is coherent. Let $q \geq 0$, and let $P^q_{X/Y}$ be the $O_X$-algebra of $q$th order relative principal parts corresponding to $f$. We recall that $P^q_{X/Y} = p_1 \cdot (\mathcal{O}_S/J^q + 1)$, where $S = X \times_Y X$, $J \subset \mathcal{O}_S$ is the sheaf of ideals defining the diagonal, and $p_1: S \to X$ is the projection to the first factor. The $O_X$-algebra structure $O_X \to P^q_{X/Y}$ is deduced from $p_1$. As an $O_X$-module, $P^q_{X/Y}$ has a finite filtration with factors $p_1 \cdot (J/J^{q+1})$. Since $p_1 \cdot (J/J^2) = \Omega^1_{X/Y}$, $P^q_{X/Y}$ is coherent.

**Definition 3.1.** Let $q \geq 0$. We define the $q$th order singularity subscheme $S^{(q)}(f) \subset X$ to be the closed subscheme of $X - S_2(f)$ corresponding to the sheaf of ideals $\Delta_q(P^q_{X/Y}) \subset O_X$.

Since $P^0_{X/Y} = O_X$ and $P^1_{X/Y} \cong O_X \oplus \Omega^1_{X/Y}$, we have $S^{(0)}(f) = X - S_2(f)$ and $S^{(1)}(f) = S_1(f) - S_2(f)$. The geometric significance of the remaining $S^{(q)}(f)$ will be explained by Proposition 3.5 below.

**Proposition 3.2.** Assume that $x \in X - S_2(f)$. Then $x \in S^{(q)}(f)$ if and only if $\dim_{O_X}(P^q_{X/Y}(x)) > q + 1$. In particular $S^{(q)}(f) \supseteq S^{(q+1)}(f)$.

**Proof.** Apply Lemma 2.1 with $A = O_{X,x}$ and $M = (P^q_{X/Y})_x$. The last statement follows because there is a surjection $P^{q+1}_{X/Y} \to P^q_{X/Y}$. Q.E.D.

**Proposition 3.3.** Let $g: X \to Z$ and $h: Y \to Z$ be morphisms of Noetherian schemes such that $g = h \circ f$. Let $u: Z' \to Z$ be a morphism, and let $f': X' \to Y'$ be obtained by base extension, where $X' = X \times_Z Z'$ and $Y' = Y \times_Z Z'$. Then $S^{(q)}(f') = S^{(q)}(f) \times_Z Z'$ for all $q$.

**Proof.** By Proposition 16.4.5 of [4], $P^q_{X'/Y'} = v^*(P^q_{X/Y})$, where $v: X' = X \times_Z Z' \to X$ is projection to the first factor. One completes the proof by applying Lemma 2.7. Q.E.D.

If $A$ is a ring and $B$ is an $A$-algebra, then $P^q_{\text{Spec}(B)/\text{Spec}(A)}$ is the sheaf on $\text{Spec}(B)$ corresponding to the $B$-algebra $P^q_{B/A} = (B \otimes_A B)/I^{q+1}$, where $I$ is the kernel of the multiplication map $\mu: B \otimes_A B \to B$. The $B$-algebra structure is given by $j: B \to P^q_{B/A}$, where $j(b) = b \otimes 1$. Another interesting homomorphism is $d^q = d^q_A: B \to P^q_{B/A}$, where $d^q(b) = 1 \otimes b$.

**Lemma 3.4.** Let $k$ be a field, $(A, m_A)$ and $(B, m_B)$ local rings with residue fields $\cong k$, and $g: A \to B$ a local homomorphism which induces an isomorphism of residue fields. There is a $k$-algebra isomorphism $\phi: k \otimes_B P^q_{B/A} \to^\cong$.  

$B/(g(m_A)B + m_B^{q+1})$ such that $\phi(1 \otimes d^q b) = \delta$ (residue class of $b$) for all $b \in B$.

**Proof.** Associativity of the tensor product implies that $k \otimes_B (B \otimes_A B) \cong B/(g(m_A)B)$. (Observe that $k \otimes_A B = B/(g(m_A)B)$.) Thus, the residue class map $B/(g(m_A)B) \to k$ is identified with $\text{id}_k \otimes \mu : k \otimes_B (B \otimes_A B) \to k \otimes_B B = k$.

Therefore,

$$[k \otimes_B (B \otimes_A B)]/(\text{Ker}(\text{id}_k \otimes \mu))^q + 1 \cong B/(g(m_A)B + m_B^{q+1}).$$

It is easily shown that this implies the desired conclusion. Q.E.D.

**Proposition 3.5.** Let $q \geq 1$, and assume that $X$ and $Y$ are varieties over the algebraically closed field $k$. Let $x$ be a closed point of $X$. Then $x \in S_x \cap f(y)$ if and only if (i) $\dim_k m_x/(f^*(m_y)O_x + m_x^{q+1}) = 1$ and (ii) $\dim_k m_x/(f^*(m_y)O_x + m_x^{q+1}) \geq q + 1$, where $(O_x, m_x) = O_{X,x}$, $y = f(x)$, $(O_y, m_y) = O_{Y,y}$, and $f^*: O_y \to O_x$ is induced by $f$.

**Proof.** By Proposition 2.4, (i) holds if and only if $x \in S_1 \cap S_2 = S_1(1)$, so we assume that (i) holds. Thus, the maximal ideal of $O_x/(f^*(m_y)O_x + m_x^{q+1})$ is principal, so that (ii) holds if and only if $\dim_k m_x/(f^*(m_y)O_x + m_x^{q+1}) \geq q + 1$.

By Lemma 3.4 and Proposition 3.2, this holds if and only if $x \in S_1(1)^q(f)$. (Note that $(P_{X/Y})_x = P_{0X/k0Y}$; see [4, Corollaire 16.4.16] or [15, Proposition A 1.2].)

**Remark 3.6.** If $X$ and $Y$ are varieties over $k$, then the higher order singularity subschemes are interesting only in the case that $\dim(X) \leq \dim(Y)$. Specifically, if $\dim(Y) < \dim(X) - 2$, then $X - S_2(f) = \emptyset$. If $\dim(Y) = \dim(X) - 1$, then (i) and (ii) hold at every closed point $x \in X - S_2(f)$ for every $q \geq 0$.

If $B$ is a $k$-algebra, then a $k$-linear map $D: B \to B$ is a differential operator if it is of the form $D = \phi \circ d^q_B$ for some $q \geq 0$, where $\phi: P^q_{B/k} \to B$ is a $B$-module homomorphism. The smallest $q$ for which such a factorization exists is the order of $D$.

**Lemma 3.7.** Let $X$ be a nonsingular variety over $k$, $x \in X$ a closed point, and $(O_x, m_x) = O_{X,x}$. Let $\{t_1, \ldots, t_r\} \subset m_x$ be a minimal generating set. Then

(i) $P^q_{X/\text{Spec}(k)}$ is a locally free sheaf;

(ii) $(P^q_{X/\text{Spec}(k)})_x = P^q_{0X/k}$ is freely generated by the monomials of degree $\leq q$ in $t_1, \ldots, t_r$, where $\xi_i = d^q_{0X/k} = t_i - t_1$;

(iii) corresponding to the $r$-tuples $(i_1, \ldots, i_r)$ of nonnegative integers such that $0 \leq i_1 + \cdots + i_r \leq q$ there are differential operators $D_{i_1 \cdots i_r}: O_x \to O_x$ of order $\leq q$ such that

(a) $d^q_{0X/k} b = \sum_{i_1 \cdots i_r} (D_{i_1 \cdots i_r} b) \xi_i^{i_1} \cdots \xi_r^{i_r}$ for all $b \in O_x$,

(b) $D_{i_1 \cdots i_r}(t_1^{n_1} \cdots t_r^{n_r}) = \binom{n_1}{i_1} \cdots \binom{n_r}{i_r} t_1^{n_1 - i_1} \cdots t_r^{n_r - i_r}$. 

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whenever \( n_v \geq 0 \) for all \( v \), subject to the convention that \( (\cdot)_i = 0 \) when \( n < i \);  
(c) more generally, if \( N \geq q \) and \( b \equiv t_1^{n_1} \cdots t_r^{n_r} \pmod{m_x^N} \), then

\[
D_{i_1} \cdots D_{i_r} b = \left( \begin{array}{c} n_1 \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n_r \\ i_r \end{array} \right) t_1^{n_1-i_1} \cdots t_r^{n_r-i_r} \pmod{m_x^{N-q}}.
\]

(We identify \( b \in \mathcal{O}_X \) with its image under \( d\mathcal{O}_x/k: \mathcal{O}_x \to \mathcal{P}^{\mathcal{O}_x/k} \); thus \( d\mathcal{O}_x/k b - b \) is represented by \( 1 \otimes b - b \otimes 1 \in \mathcal{O}_x \otimes_k \mathcal{O}_x \).) Proposition 17.12.4 of \([4, \text{Chapitre IV}]\) implies that \( \mathcal{P}^{\mathcal{O}_x/\text{Spec}(k)} \) is locally free. (More generally, \( \mathcal{P}^{\mathcal{O}_x/Z} \) is locally free if \( f: X \to Z \) is a smooth morphism.) Alternately, Lemma 3.4 can be used to show that \( \text{dim}_{k(x)} \mathcal{P}^{\mathcal{O}_x/\text{Spec}(k)}(x) \) is constant. Statements (ii), (iii)(a) and (iii)(b) are immediate consequences of (i), Lemma 3.4, and Nakayama’s lemma. (See also \([4, \text{Chapitre IV, Théorème 16.11.2}]\).) To establish (iii)(c) we observe that if \( D: B \to B \) has order \( \leq q \) and \( I \) is an ideal in \( B \), then \( D^{m} \subset I^{m-q} \) for all \( n \geq q \). (See \([4, \text{Section 16.8}]\) or \([15, \text{Proposition A 1.1}]\).)

**Lemma 3.8.** Let \( f: X^r \to Y^m \) (\( r < m \)) be a morphism of nonsingular varieties over \( k \), and let the remaining hypotheses and notation be as in Proposition 3.5. Let \( \{s_1, \ldots, s_m\} \subset m_y \) and \( \{t_1, \ldots, t_r\} \subset m_x \) be minimal generating sets such that \( f^*(s_i) = t_i \) for \( i = 1, \ldots, r-1 \). Then the ideal \( \Delta_q(\mathcal{P}^{\mathcal{O}_x/\text{spec}(k)}(x) = \Delta_q(\mathcal{P}^{\mathcal{O}_x/\mathcal{O}_y}) \subset \mathcal{O}_x \) is generated by the \( q(m-r+1) \) elements \( D^i f^*(s_i) \), \( 1 \leq i \leq q \) and \( r \leq l \leq m \), where \( D^i(\cdot): \mathcal{O}_x \to \mathcal{O}_x \) is the differential operator \( D_0 \cdots D_l \) described in Lemma 3.7.

Using this result and the Krull altitude theorem \([9, \text{Theorem 9.3}]\), we obtain

**Corollary 3.9.** If \( f: X^r \to Y^m \) is as above, then every irreducible component of \( S^1_q(f) \) has dimension \( \geq r - q(m-r+1) \), or else \( S^1_q(f) = \emptyset \).

**Proof of Lemma 3.8.** By Proposition 16.4.18 of \([4]\), there is an \( \mathcal{O}_x \)-algebra homomorphism \( w: \mathcal{P}^{\mathcal{O}_x/k} \to \mathcal{P}^{\mathcal{O}_y/\mathcal{O}_y} \) such that the ideal \( \text{Ker}(w) \subset \mathcal{P}^{\mathcal{O}_x/k} \) is generated by \( \{d^q f^*(a) - f^*(a) | a \in \mathcal{O}_y \} \), where \( d^q = d\mathcal{O}_x/k. \) If \( \xi_1, \ldots, \xi_r \) are as in Lemma 3.7, then \( \mathcal{P}^{\mathcal{O}_x/k} = E \oplus F \), where \( E \) is freely generated by \( 1, \xi_1, \ldots, \xi_r \), and \( F \) is freely generated by the monomials in \( \xi_1, \ldots, \xi_r \) which are not powers of \( \xi_i \). Since \( \xi_i = d^q f^*(s_i) - f^*(s_i) \) for \( i = 1, \ldots, r-1 \), it follows that \( E \subset \text{Ker}(w) \), and \( w \) induces a surjection \( w: F \to \mathcal{P}^{\mathcal{O}_y/\mathcal{O}_y} \). Suppose now that \( \eta_1, \ldots, \eta_N \) generate \( \text{Ker}(w) \), where \( \eta_i = \sum_{i=0}^q \beta_i \xi_i^q \). Then \( \{\beta_i\} \) will generate \( \Delta_q(\mathcal{P}^{\mathcal{O}_x/\mathcal{O}_y}) \). (Observe that \( \text{rank}(F) = \text{dim}_{k}(k \otimes_{\mathcal{O}_x} \mathcal{P}^{\mathcal{O}_x/\mathcal{O}_y}) \).) Therefore, we need to find a nice set of generators for \( \text{Ker}(w) \).

By Lemma 3.7, \( \text{Ker}(w) \) is generated as \( \mathcal{O}_x \)-module by the elements \( (d^q f^*(a) - f^*(a))\xi_1^{i_1} \cdots \xi_r^{i_r} \), where \( a \in \mathcal{O}_y \) and \( 0 \leq i_1 + \cdots + i_r \leq q \). It is not hard to see that we only need to let \( a \) run through the monomials of degree \( \leq q \) in \( s_1, \ldots, s_m \). To calculate \( \Delta_q(\mathcal{P}^{\mathcal{O}_x/\mathcal{O}_y}) \), we must therefore expand the
elements \( d^q f^*(a) - f^*(a) \) as polynomials in \( \xi_1, \ldots, \xi_r \) and extract the coefficients of \( 1, \xi_r, \ldots, \xi^q_r \). Since \( d^q \) is a ring homomorphism, monomials in \( s_1, \ldots, s_m \) of degree \( \geq 2 \) will give rise to coefficients which are polynomials in the coefficients occurring in the expansions of \( d^q f^*(s_i) - f^*(s_i) \), \( l = r, \ldots, m \). Now

\[
d^q f^*(s_i) - f^*(s_i) = \sum_{l=1}^q D^{(l)} f^*(s_i) \xi^l_r + \text{(other terms of degree } \geq 1)\]

We conclude that \( \{D^{(l)} f^*(s_i) \mid 1 \leq i \leq q \text{ and } r \leq l \leq m \} \) generates \( \Delta_q(P_{\mathcal{X}/A}^q) \).

Q.E.D.

We now consider certain morphisms of product varieties. Specifically, let \( X, Y, \) and \( Z \) be nonsingular varieties over \( k \), and let \( f: X \times_k Z \to Y \times_k Z \) be a morphism such that \( g = h \circ f \), where \( g \) and \( h \) are the projections of \( X \times_k Z \) and \( Y \times_k Z \) to \( Z \). Let \( (x, z) \) be a closed point of \( X \times_k Z \), and let \( (y, z) = f(x, z) \). Let \( (R, m_R), (A, m_A) \) and \( (B, m_B) \) be the local rings \( \mathcal{O}_{X \times_k Z, (x, z)} \), \( \mathcal{O}_{Y \times_k Z, (y, z)} \) and \( \mathcal{O}_{X \times_k Z, (x, z)} \) respectively. Then \( f^*: A \to B \) is an \( R \)-algebra homomorphism. In the following proposition, we regard \( \mathcal{O}_y \) and \( \mathcal{O}_x \) as subrings of \( A \) and \( B \) respectively.

**Proposition 3.10.** With the notation and assumptions as above, let

\( \{s_1, \ldots, s_m \} \subset m_A \) and \( \{t_1, \ldots, t_r \} \subset m_B \) be minimal generating sets such that \( f^*(s_j) = t_j \) for \( j = 1, \ldots, r - 1 \). If \( q \geq 0 \), then there exist differential operators \( D^{(l)}: B \to B \), \( l = 1, \ldots, q \), which are also \( \mathcal{O}_A \)-module homomorphisms, such that

1. \( (i) \) if \( b = t_1^x \cdots t_r^x \mod m_B^{q+1} \), then \( D^{(l)}(b) = t_1^{x \Delta^q} \cdots t_r^{x \Delta^q} \mod m_B^{q+1} \);

2. \( (ii) \) \( \Delta^q(P_{B/A}^q) \) is generated by the \( q^2(m-r+1) \) elements \( D^{(l)} f^*(s_j) \), \( 1 \leq l \leq q \) and \( 1 \leq j \leq m \).

**Proof.** Since \( P_{B/A}^q = (P_{X \times_k Z, (x, z)}^q, P_{B/A}^q) \) is a free \( B \)-module. (The projection to the second factor is a smooth morphism.) We proceed as in the proof of Lemma 3.7 to establish the existence of the differential operators \( D^{(l)}: B \to B \) which satisfy (i) and are also \( \mathcal{O}_A \)-linear. (In fact, \( d_{B/R}^q: B \to P_{B/R}^q \) is an \( R \)-algebra homomorphism.)

In order to show that \( \Delta^q(P_{B/A}^q) \) has the desired generating set, we represent \( P_{B/A}^q \) as a homomorphic image of \( P_{B/R}^q \) and proceed as in the proof of Lemma 3.8. Q.E.D.

**4. Consequences of smoothness.** The main result of this section is the following theorem, which can be shown to be equivalent to the case \( d = 1 \) of Theorem B.

**Theorem 4.1.** Let \( f: V^r \to W^m \) be a finite morphism of nonsingular varieties over \( k \), where \( m \geq r \). Let \( x \in S_1^q(f) - S_1^{q+1}(f) \), and assume that \( S_1^q(f) \) is smooth at \( x \) and locally of codimension \( q(m-r+1) \) in \( V \). Then
there are isomorphisms $\phi: \mathfrak{g}_{V,x} \xrightarrow{\sim} B = k[[t_1, \ldots, t_r]]$ and $\psi: \mathfrak{g}_{W,(x)} \xrightarrow{\sim} A = k[[s_1, \ldots, s_m]]$ such that $g = \phi \circ f^* \circ \psi^{-1}: A \to B$ satisfies

\begin{align*}
(i) g(s_i) &= t_i, \quad i = 1, \ldots, r - 1; \\
(ii)(a) g(s_r) &= t_r^{q+1} + \sum_{j=1}^{q-1} s_q^j(m-r)+t_r^j \text{ if char}(k)(q+1); \\
(ii)(b) g(s_r^q) &= t_r^{q+1} + \sum_{j=1}^{q-1} s_q^j(m-r)+t_r^j \text{ if char}(k)(q+1), \text{ and } q(m-r + 1) < r \text{ in this case}; \\
(iii) g(s_{r+i}) &= \sum_{j=1}^{q-j} s_q^j(i-1)+t_r^j, \quad i = 1, \ldots, m-r.
\end{align*}

Before giving the proof, we will discuss certain algebras of principal parts.

**Lemma 4.2** [4, Proposition 16.4.20]. Let $A$ be a ring, $R$ an $A$-algebra (commutative with identity), and let $B = R/I$, where $I$ is an ideal in $R$. Then there is a surjective $B$-algebra homomorphism $\nu: B \otimes_R P^q_{R/A} \to P^q_{B/A}$ such that $\text{Ker} (\nu)$ is generated (as an ideal) by $\{1 \otimes dqx | x \in I\}$, where $dq = dR/A: R \to P^q_{R/A}$.

**Lemma 4.3.** Let $(A, m_A)$ be a Noetherian local ring, $R$ an $A$-algebra which is finitely generated as $A$-module, and $(B, m_B) = R_m$, where $m$ is a maximal ideal in $R$. Then $P^q_{B/A} = B \otimes_B P^q_{B/A}$.

**Proof.** $B$ is a localization of $R$, so that $P^q_{B/A} = B \otimes_R P^q_{R/A}$. Therefore, it will suffice to show that $P^q_{B/A} = B \otimes_R P^q_{R/A}$. Let $R$ have the $m_A$-adic topology. Since $R = \hat{A} \otimes_A R$, $P^q_{B/A} = \hat{R} \otimes_R P^q_{R/A}$; hence it is enough to prove that $P^q_{B/A} = \hat{B} \otimes_R P^q_{R/A}$. Since $R$ is semilocal, $\hat{B} = \hat{R}/I$, where $I$ is generated by idempotent elements. By Lemma 4.2, it will suffice to show that $1 \otimes dqx = 0$ for every idempotent element $x \in I$, where $dq = dR/A$. For such an $x$, $(dqx - x)^{q+1} = 0$ [notation as in Lemma 3.7] and $x = x^{q+1}$, so that $dqx = dqx^{q+1} = \sum_{i=1}^{q+1} n_i dqx^{q+1-i}$ for suitable $n_i \in \mathbb{Z}$. Therefore $1 \otimes dqx = 0$. Q.E.D.

**Corollary 4.4.** $\Delta_q(P^q_{B/A}) = \Delta_q(P^q_{B/A}) \cdot \hat{B}$. (The notation is as above.)

The corollary is an immediate consequence of Lemma 4.3 and the affine case of Lemma 2.7. In the case that $B$ is regular and $\Delta_q(P^q_{B/A})$ is generated by a subset of a regular system of parameters, we conclude that $\Delta_q(P^q_{B/A})$ is generated by a subset of a regular system of parameters. Thus, Theorem 4.1 is a direct consequence of the following lemma.

**Lemma 4.5.** Let $k$ be a field; let $(A, m_A)$ and $(B, m_B)$ be formal power series rings over $k$ in $m$ and $r$ variables respectively, where $m \geq r$. Let $g: A \to B$ be a local $k$-algebra homomorphism which makes $B$ a finite $A$-module, and assume that

\begin{align*}
(a) \dim_k m_B/(g(m_A)B + m_B^2) &= 1, \\
(b) \dim_k B/g(m_A)B &= q + 1,
\end{align*}
(c) \( \Delta_q(P_{B/A}^{q}) \) is generated by \( q(m - r + 1) \) elements of a regular system of parameters in \( B \).

Then there exist minimal generating sets \( \{s_1, \ldots, s_m\} \subset m_A \) and \( \{t_1, \ldots, t_r\} \subset m_B \) such that \( g: A \to B \) satisfies (i), (ii), and (iii) of Theorem 4.1.

The proof is given below, following some preliminary material.

If \( M \) is a \( B \)-module, let \( \tilde{M} = M/(\bigcap_{n=1}^{m}m_B^nM) \). We have \( M = \tilde{M} \) if and only if \( M \) is separated in its \( m_B \)-adic topology. In general, \( \tilde{M} \) is separated in its \( m_B \)-adic topology; we call it the separated module associated to \( M \). If \( f: M \to N \) is a homomorphism of \( B \)-modules, then there is a homomorphism \( \tilde{f}: \tilde{M} \to \tilde{N} \).

**Lemma 4.6.** Consider an exact sequence of \( B \)-modules \( M \to N \to P \to 0 \), where \( P \) is separated. Then the corresponding sequence of separated modules \( \tilde{M} \to \tilde{N} \to \tilde{P} \to 0 \) is also exact. (In particular, the hypotheses are satisfied if \( P \) is finitely generated.)

Let \( \tilde{d}^q = \nu \circ \tilde{d}_{B/k}^q: B \to P_{B/k}^q \), where \( \nu: P_{B/k}^q \to \tilde{P}_{B/k}^q \) is the canonical map onto the associated separated module. Then every differential operator \( D: B \to B \) of order \( \leq q \) is of the form \( D = \phi \circ \tilde{d}^q \) for some \( B \)-module homomorphism \( \phi: \tilde{P}_{B/k}^q \to B \). (Note that \( B \) is separated in the \( m_B \)-adic topology.) \( \tilde{P}_{B/k}^q \) is a \( B \)-algebra because \( \bigcap_{n=1}^{m}m_B^nP_{B/k}^q \) is an ideal in \( P_{B/k}^q \). Finally, \( k \otimes_B \tilde{P}_{B/k}^q \cong B/m_B^{q+1} \). (See Lemma 3.4.) Theorem 30.6 of [9] therefore implies that \( \tilde{P}_{B/k}^q \) is a finitely generated \( B \)-module.

**Lemma 4.7.** Let \( t_1, \ldots, t_r \) generate \( m_B \), so that \( B = k[[t_1, \ldots, t_r]] \).

(i) If \( i_1 + \cdots + i_r \leq q \), then there is a unique differential operator \( D = D_{i_1} \cdots D_{i_r}: B \to B \) such that
\[
D(d_{i_1}^{1} \cdots d_{i_r}^{r}) = (d_{i_1}^{1}) \cdots (d_{i_r}^{r})d_{1}^{1-i_1} \cdots d_{r}^{r-i_r}.
\]

(ii) \( \tilde{P}_{B/k}^q \) is a free \( B \)-module, of rank \( = \dim_k B/m_B^{q+1} \).

**Proof.** We define a map \( D: B \to B \) on monomials by the above formula and extend it linearly to arbitrary power series. One shows that if \( b_1, \ldots, b_{q+1}, t \) are arbitrary elements of \( B \), then
\[
\sum_H (-1)^{\text{card}(H)} \left( \prod_{i \in H} b_i \right) D \left( \prod_{i \in H} b_i \right) t = 0.
\]

(\( H \) runs through all subsets of \( \{1, \ldots, q+1\} \).) This identity implies that \( D \) is a differential operator. Hence, we deduce a \( B \)-module homomorphism \( \lambda: \tilde{P}_{B/k}^q \to B \) such that \( D = \lambda \circ \tilde{d}^q \).

Let \( \xi_i = \tilde{d}^q t_i - t_i, \ i = 1, \ldots, r. \) Then \( \tilde{P}_{B/k}^q \) is generated as \( B \)-module by the monomials of degree \( \leq q \) in \( \xi_1, \ldots, \xi_r. \) If \( \lambda \) is the map constructed above, then
singularity subschemes and generic projections

\[ \lambda(\xi_1^l \cdots \xi_r^l) = 1, \]

\[ \lambda(\xi_1^l \cdots \xi_r^l) \in m_B \quad \text{if } j_v \neq i_v \text{ for some } v. \]

This shows that the monomials of degree \( \leq q \) in \( \xi_1, \ldots, \xi_r \) satisfy no \( B \)-linear relations. Q.E.D.

**Proof of Lemma 4.5.** We start with (minimal) generating sets \( \{s_1, \ldots, s_m\} \subset m_A \) and \( \{t_1, \ldots, t_r\} \subset m_B \), which will be modified as the proof proceeds. By (a), we may assume that \( g(s_i) = t_i \) for \( i = 1, \ldots, r - 1 \) and that \( t_r \notin g(m_A)B \).

By (b), we may assume that the monomial \( t_r^{q+1} \) occurs with nonzero coefficient in the power series expansion of \( g(s_i) \) and does not occur in the power series expansions of \( g(s_{r+1}), \ldots, g(s_m) \). Finally, we may assume that \( g(s_i) \in m_B^2 \), \( i = r, \ldots, m \). [If this is not the case, add \( A \)-linear combinations of \( s_1, \ldots, s_{r-1} \) to \( s_i \).] We then obtain

\[ g(s_i) = \sum_{j=1}^{\infty} b_{ij} t_r^j, \quad i = r, \ldots, m; \]

the \( b_{ij} \) are elements of \( k[[t_1, \ldots, t_{r-1}]] \) such that \( b_{ij} \) is a nonunit if \( j \leq q \), and \( b_{i,q+1} \) is a unit if and only if \( i = r \). We will assume that \( b_{r,q+1} \) has constant term = 1.

We now observe that there is a surjective \( B \)-algebra homomorphism \( w: P^q_{B/k} \to P^q_{B/A} \) such that \( \text{Ker}(w) \) is generated by \( \{\overline{D}(g(a))|a \in A\} \). (See [4, Proposition 16.4.18] and Lemma 4.6.) Exactly as in the proof of Lemma 3.8 one shows that \( \Delta_\lambda(P^q_{B/A}) \) is generated by the elements \( D^{(\lambda)}(g(s_i)) \), where \( 1 \leq \lambda \leq q \), \( r \leq i \leq m \), and \( D^{(\lambda)} = D_{0 \cdots 0\lambda}: B \to B \) is defined as in Lemma 4.7. From (1) we obtain the congruences:

\[ D^{(q)}(g(s_i)) \equiv (q + 1)t_r + b_{rq} \pmod{m_B^2}. \]

Now, (c) implies that \( \{D^{(\lambda)}(g(s_i))|1 \leq \lambda \leq q \text{ and } r \leq i \leq m\} \) is a subset of a system of parameters in \( B \). If \( \text{char}(k)(q + 1) \), then \( (q + 1)t_r = 0 \) and we conclude that \( \{b_{i\lambda}|1 \leq \lambda \leq q \text{ and } r \leq i \leq m\} \) is a subset of a system of parameters in \( k[[t_1, \ldots, t_{r-1}]] \). If \( \text{char}(k) \not| (q + 1) \), we draw the "weaker" conclusion that \( \{b_{i\lambda}|(i, \lambda) \neq (r, q)\} \) is a subset of a system of parameters.

Let \( (S, m_S) = k[[s_1, \ldots, s_m]] \subset A \). Then \( B/g(m_S)B \) has a basis consisting of the residues of \( 1, t_r, \ldots, t_r^q \). By Theorem 30.6 of [9], we conclude that \( B \) is generated as \( S \)-module by \( 1, t_r, \ldots, t_r^q \). After a possible change of variables, we obtain

\[ g(s_r) = t_r^{q+1} + \sum_{j=1}^{q} g(a_{rj})t_r^j. \]
Moreover $a_{rq} \in m_S$, $j = 1, \ldots, q$. We may also assume

$$a_{rq} = 0 \text{ if } \text{char}(k)\uparrow(q + 1).$$

[If this does not hold, replace $t_r$ with $t_r - (q + 1)^{-1}g(a_{rq})$ in the minimal generating set of $m_B$.] Finally, we may assume

$$g(s_i) = \sum_{j=1}^{q} g(a_{ij})t_r^j, \quad i = r + 1, \ldots, m, \text{ where all } a_{ij} \in m_S. \quad (5)$$

Let each of the nonunits $g(a_{ij}) \in B$ be expanded as power series in $t_1, \ldots, t_r$. In this way, (3) and (5) give power series expansions of $g(s_r), \ldots, g(s_m)$. If we compare these expansions with (1), then we obtain

$$g(a_{ij}) \equiv b_{ij} \pmod{m_B^2} \quad \text{when } r \leq i \leq m \text{ and } 1 \leq j \leq q. \quad (6)$$

Since $\{b_{ij}\}$ (possibly excluding $(i, j) = (r, q)$) is a subset of a system of parameters in $k[[t_1, \ldots, t_{r-1}]]$, it follows that we may change variables and assume that $\{a_{ij}\}$ (possibly excluding $(i, j) = (r, q)$) is a subset of $\{s_1, \ldots, s_{r-1}\}$. This implies that (i), (ii), and (iii) hold. Q.E.D.

**Remark 4.8.** If $q(m - r + 1) = r$ and $\text{char}(k)\uparrow(q + 1)$, then there are no points which satisfy the hypotheses of Theorem 4.1. This is because there are not enough parameters to serve as "coefficients" in conclusions (ii) and (iii). In the following example, we will demonstrate the nonexistence of smooth points more directly in the case $r = 2, m = 3, q = 1$, and $\text{char}(k) = 2$.

**Example 4.9.** Let $\text{char}(k) = 2$, and consider the $k$-algebra homomorphism $g: A \rightarrow B$, where $A = k[[s_1, s_2, s_3]], B = k[[t_1, t_2]]$, and

(a) $g(s_1) = t_1$,
(b) $g(s_i) = \sum_{j=1}^{m} b_{ij}t_j^i, \quad i = 2, 3$, where $b_{ij} \in k[[t_1]]$.

Assume that $b_{21}$ and $b_{31}$ are nonunits but that some $b_{ij}$ is a unit, so that $2 \leq \dim_k B/g(m_A)B < \infty$. Then $\Delta_1(P_B/A) = \Delta_0(\Omega^1_B/A)$ is generated by $\partial g(s_2)/\partial t_2$ and $\partial g(s_3)/\partial t_2$; moreover, $\partial g(s_i)/\partial t_2 = b_{i1} + b_{i3}t_2^2 + \cdots$. Thus, $\Delta_1(P_B/A)$ cannot be generated by two elements of a regular system of parameters.

The following proposition gives an explicit description of the cotangent space of $S^1_{q}(f)$ in Theorem 4.1. It also shows that the converse of Lemma 4.5 is true.

**Proposition 4.10.** Let $A = k[[s_1, \ldots, s_m]], B = k[[t_1, \ldots, t_r]],$ and let $g: A \rightarrow B$ satisfy (i), (ii), and (iii) of Theorem 4.1. Then $(\Delta_q(P^1_B/A) + m_B^2)/m_B^2$ is the subspace of $m_B/m_B^2$ having the basis:

$$\{\bar{t}_i | 1 \leq i \leq q(m - r + 1) - 1 \} \cup \{\bar{t}_r\} \quad \text{if } \text{char}(k)\uparrow(q + 1),$$

$$\{\bar{t}_i | 1 \leq i \leq q(m - r + 1) \} \quad \text{if } \text{char}(k)\uparrow(q + 1).$$

(The bars denote residue class modulo $m_B^2$.)
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Proof. We use the notation of the proof of Lemma 4.5. Thus, \( \Delta_q \) is generated by \( \{ D(x)g(s_i) \mid 1 \leq \lambda \leq q \text{ and } r \leq i \leq m \} \). If \( 1 \leq \nu \leq r - 1 \), then 
\[ D(x)g(s_i) = \left( x^{i-1} \right) \nu = \lambda \] (where \( \lambda = 0 \) if \( \lambda > f \)). Furthermore, \( D(q)g(t_{r+1}^*) = (q + 1)g(t_r) \). 
Therefore, the set of initial forms of the elements \( D(x)g(s_i) \) is
\[ \{ t_i \mid 1 \leq i \leq q(m - r + 1) - 1 \} \cup \{ (q + 1)t_r \} \text{ if char}(k) \not\equiv (q + 1), \]
\[ \{ t_i \mid 1 \leq i \leq q(m - r + 1) \} \text{ if char}(k) \equiv (q + 1). \]
This implies that the desired conclusions hold. Q.E.D.

5. Some properties of fibre products. In this section we study some properties of fibre products of the singularity subschemes of a morphism of varieties.

All of these are consequences of elementary properties of fibre products of schemes of finite type over the algebraically closed field \( k \).

Proposition 5.1. Let \( f_j: W_j \to Y \), \( j = 1, \ldots, d \), be morphisms over \( k \), where \( Y \) is a nonsingular variety and \( W_1, \ldots, W_d \) are of finite type over \( k \) and of pure dimension. If \( W_1 \times_Y \cdots \times_Y W_d \neq \emptyset \) and \( Z \) is an irreducible component of \( W_1 \times_Y \cdots \times_Y W_d \), then
\[ \dim(Z) \geq \sum_{j=1}^d \dim(W_j) - (d - 1) \cdot \dim(Y). \]

Proof. Consider the morphism \( f_1 \times \cdots \times f_d: W_1 \times \cdots \times W_d \to Y \times \cdots \times Y \) of \( d \)-fold products over \( \text{Spec}(k) \), and let \( \Delta \subset Y \times \cdots \times Y \) be the \( d \)-fold diagonal. Then
\[ W_1 \times_Y \cdots \times_Y W_d = (f_1 \times \cdots \times f_d)^{-1}(\Delta). \]
Since \( Y \) is nonsingular, the sheaf of ideals defining \( \Delta \) as a closed subscheme of \( Y \times \cdots \times Y \) is generated locally by \( (d - 1) \cdot \dim(Y) \) elements at every point. Hence \( (f_1 \times \cdots \times f_d)^{-1}(\Delta) \) has the same property as a closed subscheme of \( W_1 \times \cdots \times W_d \). By Theorem 9.3 of [9], the codimension is \( \leq (d - 1) \cdot \dim(Y) \). Q.E.D.

Let \( f: X \to Y \) be a morphism of nonsingular varieties. As in § 1, we form fibre products \( S^{(1)}_1(f) \times_Y \cdots \times_Y S^{(d)}_d(f) \) by using the restrictions of \( f \), which send \( S^{(q)}_j(f) \to Y \), \( j = 1, \ldots, d \). We recall that \( \Sigma_d(f) \) is the complement in \( X \times_Y \cdots \times_Y X \) of the union of all diagonals. The closed points of \( \Sigma_d(f) \) will be called \( d \)-fold points of \( f \). We also recall that
\[ \Sigma_d(f; q_1, \ldots, q_d) = \Sigma_d(f) \cap (S^{(1)}_1(f) \times_Y \cdots \times_Y S^{(d)}_d(f)). \]
(In particular, \( \Sigma_d(f; q_1, \ldots, q_d) \) is an open subset of \( S^{(1)}_1(f) \times_Y \cdots \times_Y S^{(d)}_d(f) \)).
Corollary 5.2. If \( Z \) is an irreducible component of \( \Sigma_d(f; q_1, \ldots, q_d) \) and \( \Sigma_d(f; q_1, \ldots, q_d) \neq \emptyset \), then \( \dim(Z) \geq dr - (d-1)m - \sum_{j=1}^d q_j(m-r+1) \), where \( r = \dim(X) \) and \( m = \dim(Y) \). In particular, every irreducible component of \( \Sigma_d(f) \) has dimension \( \geq dr - (d-1)m \).

Proof. The conclusion follows from Proposition 5.1, because every irreducible component of \( S_1^{(q_1)}(f) \) has dimension \( \geq r - q_j(m-r+1) \). (See Corollary 3.9.) Q.E.D.

Remark 5.3. The result of Corollary 5.2 is, of course, interesting only when \( m \geq r \). We observe that this result could be considered the “easy half” of conclusion (i) of Theorem A.

Proposition 5.4. With the assumptions of Proposition 5.1, assume further that \( W_1, \ldots, W_d \), and \( Y \) are smooth over \( k \). Let \( x_1, \ldots, x_d \) be closed points of \( W_1, \ldots, W_d \) respectively, and assume that \( f_j(x_1) = \cdots = f_j(x_d) = y \in Y \). Let \( (0_y, m_y) = 0_{Y,y}, (0_{W_j}, m_{x_j}) = 0_{W_j,x_j} \), and let \( \rho_j: m_y/m_y^2 \to m_{x_j}/m_{x_j}^2 \) be induced by \( f_j \), \( j = 1, \ldots, d \). Then \( W_1 \times_Y \cdots \times_Y W_d \) is smooth and of the smallest possible dimension at \( (x_1, \ldots, x_d) \) if and only if \( \dim_k(N_1 + \cdots + N_d) = \dim_k(N_1) + \cdots + \dim_k(N_d) \), where \( N_j = \ker(\rho_j) \subset m_y/m_y^2 \), \( j = 1, \ldots, d \).

Proof. Let \( \xi = (x_1, \ldots, x_d) \in W_1 \times \cdots \times W_d \) and \( \eta = (y, \ldots, y) \in Y \times \cdots \times Y \). (These \( d \)-fold products are taken over \( \text{Spec}(k) \).) Then \( m_\eta/m_\eta^2 \cong \sum_{j=1}^d \bigoplus m_y/m_y^2 \), and if \( J \subset 0_\eta \) defines the \( d \)-fold diagonal \( \Delta \), then

\[
\left( J + m_\eta^2 \right)/m_\eta^2 \cong \left\{ (s_1, \ldots, s_d) \in \sum_{j=1}^d \bigoplus m_y/m_y^2 \left| s_1 + \cdots + s_d = 0 \right. \right\}.
\]

On the other hand, \( f_1 \times \cdots \times f_d \) induces the linear map \( \rho: m_\eta/m_\eta^2 \to m_\xi/m_\xi^2 = \sum_{j=1}^d \bigoplus m_{x_j}/m_{x_j}^2 \), where \( \rho(s_1, \ldots, s_d) = (\rho_1(s_1), \ldots, \rho_d(s_d)) \).

\( (f_1 \times \cdots \times f_d)^{-1}(\Delta) \) is smooth at \( (x_1, \ldots, x_d) \) if and only if

\[
\ker(\rho) \cap \left( J + m_\eta^2 \right)/m_\eta^2 = (0).
\]

This holds if and only if \( \dim_k(N_1 + \cdots + N_d) = \dim_k(N_1) + \cdots + \dim_k(N_d) \). Q.E.D.

Corollary 5.5. Let \( f: X^r \to Y^m \) be a morphism of nonsingular varieties, and let \( x_1, \ldots, x_d \) be distinct closed points of \( X \) such that \( f(x_1) = \cdots = f(x_d) = y \in Y \). Then \( \Sigma_d(f) \) is smooth and of dimension \( dr - (d-1)m \) at \( (x_1, \ldots, x_d) \) if and only if

\[
\dim_k\left( \bigcap_{j=1}^d \text{Im}((df)_{x_j}) \right) = \sum_{j=1}^d \dim_k(\text{Im}((df)_{x_j})) - (d-1)m,
\]

where \( (df)_{x_j}: T(X)_{x_j} \to T(Y)_y \) is the map of Zariski tangent spaces induced by \( f \).
Corollary 5.6. With the assumptions and notation of Corollary 5.5, assume also that \( x_i \in S^{(q_1)}(f) \), \( j = 1, \ldots, d \). Let \( I_j \subset O_{x_j} \) be the ideal \( \Delta_q(\mathbb{P}^N_{x_j}/x_j) \). Let \( f \) induce \( \lambda: m_y/m_y^2 \to m_{x_1}/m_{x_1}^2 \) and \( \rho_j: m_y/m_y^2 \to m_{x_j}/(m_{x_j}^2 + I_j) \), \( j = 1, \ldots, d \). Let \( N_j = \text{Ker}(\rho_j) \), \( j = 1, \ldots, d \). If \( \Sigma_\lambda(f; 0, q_2, \ldots, q_d) \) is smooth and of the smallest possible dimension at \((x_1, \ldots, x_d)\), then \( \Sigma_\lambda(f; q_1, \ldots, q_d) \) is smooth and of the smallest possible dimension at \((x_1, \ldots, x_d)\) if and only if there exist elements \( t_1, \ldots, t_q \) of \( I_1 \) such that \( \{t_i\} \), \( 1 \leq i \leq q_1(m \to r + 1) \), is \( m_{x_1}/m_{x_1}^2 \) is linearly independent modulo the subspace \( N_2 + \cdots + N_d \subset m_{x_1}/m_{x_1}^2 \).

Proof. Each of the statements which must be shown to be equivalent implies that \( S^{(q_1)}(f) \) is smooth at \( x_1 \). Assume that this is true. Since \( \Sigma_\lambda(f; 0, q_2, \ldots, q_d) \) is smooth at \((x_1, \ldots, x_d)\), Proposition 5.4 implies that \( \text{Ker}(\lambda) \cap (N_2 + \cdots + N_d) = (0) \). Therefore, we must prove that \( \Sigma_\lambda(f; q_1, \ldots, q_d) \) is smooth and of the smallest possible dimension at \((x_1, \ldots, x_d)\) if and only if \( \text{Ker}(\nu) \cap \lambda(N_2 + \cdots + N_d) = (0) \), where \( \nu: m_{x_1}/m_{x_1}^2 \to m_{x_1}/(m_{x_1}^2 + I_j) \) is the canonical surjection. The equivalence of these two statements follows from Proposition 5.4 because \( \rho_1 = \nu \circ \lambda \).

6. A lemma about projective embeddings. In this section we prove the existence of projective embeddings having certain properties.

Definition 6.1. Let \( V \) be a closed subvariety of \( \mathbb{P}^N = \text{Proj} \ k[T_0, \ldots, T_N] \), and let \( H \subset \mathbb{P}^N \) be a hyperplane. Let \( \mathbb{P}^N - H \) be identified with \( \text{Spec} \ k[T_1, \ldots, T_n] \) (after a linear change of variables). A linear coordinate function on \( U = V - V \cap H \) is an element of \( \mathbb{F}(U, \mathcal{O}_U) \) induced by a linear combination of the elements \( 1, T_1, \ldots, T_n \).

Consider an affine variety \( U \) (over \( k \)) and points \( x_1, \ldots, x_d \in U \). Let \( b_1, \ldots, b_d \) be positive integers. For \( j = 1, \ldots, d \) we have ring homomorphisms \( \phi_j: \Gamma(U, \mathcal{O}_U) \to O_{x_j}/m_{x_j}^{b_j} \), where \( (O_{x_j}, m_{x_j}) \) is the local ring \( O_{U,x_j} \). Therefore, we have a ring homomorphism \( \phi: \Gamma(U, \mathcal{O}_U) \to A \), where \( A \) is the Artinian ring \( \Pi_{j=1}^d O_{x_j}/m_{x_j}^{b_j} \). It is well known that \( \phi \) is surjective.

Definition 6.2. Let \( U \) and \( \phi: \Gamma(U, \mathcal{O}_U) \to A \) be as above. Let \( E \) be a subset of \( \Gamma(U, \mathcal{O}_U) \). We say that \( A \) is spanned by \( E \) if \( A \) is spanned (as \( k \)-vector space) by \( \phi(E) \).

Lemma 6.3. Let \( b_1, \ldots, b_d \) be positive integers and let \( V \) be a projective variety over \( k \). There exists a projective embedding \( V \subset \mathbb{P}^N \) (for some \( N \)) such that if \( x_1, \ldots, x_d \) are closed points of \( V \) and \( H \subset \mathbb{P}^N \) is a hyperplane such that \( x_j \in U = V - V \cap H \) for \( j = 1, \ldots, d \), then \( \Pi_{j=1}^d O_{x_j}/m_{x_j}^{b_j} \) is spanned by the linear coordinate functions on \( U \), where \( (O_{x_j}, m_{x_j}) = \mathcal{O}_{V,x_j} \), \( j = 1, \ldots, d \).

Proof. Let an embedding \( V \subset \mathbb{P}^n \) be given. We reduce to the case \( V = \)
Lemma 6.4. If \( g \geq \left( \sum_{j=1}^{d} b_j \right) - 1 \), then the conclusion of Lemma 6.3 holds for the embedding \( \sigma_g: \mathbb{P}^n \hookrightarrow \mathbb{P}^N \).

Proof. Given a hyperplane \( H \subset \mathbb{P}^N \) such that \( \sigma_g(x_j) \notin H \) for \( j = 1, \ldots, d \), we change \( H \) so that \( \sigma_g^{-1}(H) \) is a hyperplane in \( \mathbb{P}^n \) counted \( g \) times. Such a change multiplies the linear coordinate functions by an element which is a unit in each \( \mathcal{O}_{x_j} \).

We now choose a basis \( \{ T_0, \ldots, T_n \} \) for \( \Gamma(\mathbb{P}^n, \mathcal{O}(1)) \) so that \( \sigma_g^{-1}(H) \) is the hypersurface \( T_0^g = 0 \); we identify \( \mathbb{P}^n - \sigma_g^{-1}(H) \) with \( \text{Spec}(B) \), where \( B = k[T_1, \ldots, T_n] \). Let \( m_i \subset B \) be the maximal ideal corresponding to \( x_i \), so that \( \prod_{j=1}^{d} \mathcal{O}_{x_j}/m_i^{b_j} \cong \prod_{j=1}^{d} B/m_i^{b_j} \).

We choose an \((n-2)\)-dimensional space \( A \) of the hyperplane \( T_0 = 0 \) which does not meet any of the lines \( x_i \), and we change the coordinates so that this subspace has the equations \( T_0 = T_1 = 0 \). Thus, if \( x_j \) has homogeneous coordinates \( (1, c_{j1}, \ldots, c_{jn}) \), it follows that \( c_{11}, \ldots, c_{1d} \) are distinct. (Check this by projecting from \( A \).) For each \( i \), we set \( h_i(T_1) = \Pi_{i \neq j} (T_1 - c_{ij})^{b_j} \). Then \( h_i \notin m_i \), but \( h_i \in m_i^{b_i} \) if \( i \neq j \). Multiplying \( h_i \) by the monomials of degree \( \leq b_i - 1 \) in \( T_1 - c_{11}, \ldots, T_n - c_{jn} \), we obtain polynomials of degree \( \leq g \) lying in \( m_i^{b_i} \) for \( i \neq j \) whose residue classes span \( B/m_i^{b_i} \). This completes the proof. Q.E.D.

Corollary 6.5. Let \( x_1, \ldots, x_d \) be distinct closed points of \( \mathbb{P}^n \). Assume that \( g \geq 2d - 1 \). Then the tangent spaces \( t_{\sigma_g(p_n), \sigma_g(x_j)} \), \( j = 1, \ldots, d \), span a linear subspace of \( \mathbb{P}^N \) of the largest possible dimension, namely \( d(n + 1) - 1 \).

Remark 6.6. This corollary is a stronger version of Corollary 1 of [11, §2].

7. The \( d \)-fold points of a generic projection. Let \( V' \) be a nonsingular quasi-projective variety over \( k \). When a projective embedding \( V' \subset \mathbb{P}^n \) and a closed point \( x \in V \) are given, \( t_{V', x} \) will denote the \( r \)-subspace of \( \mathbb{P}^n \) tangent to \( V \) at \( x \). If \( L \subset \mathbb{P}^n \) is an \((n - m - 1)\)-subspace with \( L \cap \overline{V} = \emptyset \) [so that \( m \geq r \)], we will denote by \( \pi_L: \mathbb{P}^n - L \rightarrow \mathbb{P}^m \) the projection from \( L \) and by \( \pi: V \rightarrow \mathbb{P}^m \) the restriction of \( \pi_L \) to \( V \). We will denote by \( \pi_L(t_{V', x}) \) the linear subspace \( \pi_L(t_{V', x} - L) \subset \mathbb{P}^m \).

Proposition 7.1. Let \( V' \) be a nonsingular subvariety of \( \mathbb{P}^n \), \( L \subset \mathbb{P}^n \) an \((n - m - 1)\)-subspace with \( L \cap \overline{V} = \emptyset \). Let \( x \in V \) be a closed point, set \( y = \pi(x) \in \mathbb{P}^m \), and let \( p: m_y/m_y^2 \rightarrow m_x/m_x^2 \) be induced by \( \pi^*: \mathcal{O}_{p_m, y} \rightarrow \mathcal{O}_{p_m, x} \). The following statements are equivalent (where \( i \) is any positive integer):
(a) $\dim_k(\text{Ker}(\rho)) \geq m - r + i$.
(b) $x \in S_i(\pi)$.
(c) $\dim(L \cap t_{V,x}) \geq i - 1$.
(d) $\dim(\pi_L(t_{V,x})) \leq r - i$.

**Proof.** (a) $\iff$ (b) is a consequence of Proposition 2.4. To prove (c) $\iff$ (d), observe that the closure (in $\mathbb{P}^n$) of $\pi_L^{-1}(\pi_L(x))$ is the subspace spanned by $L$ and $x$, so that the closure of $\pi_L^{-1}(\pi_L(x)) \cap t_{V,x}$ is the subspace spanned by $x$ and $L \cap t_{V,x}$. Since $\pi_L^{-1}(\pi_L(x)) \cap t_{V,x}$ is a typical fibre of $\pi_L(t_{V,x} - L)$, the equivalence of (c) and (d) is now easily established.

In proving (c) $\iff$ (a), we assume that equality holds in (c), and we select homogeneous coordinates $T_0, \ldots, T_n$ on $\mathbb{P}^n$ such that:

(i) $L$ is the zero locus of $\{T_0, \ldots, T_m\}$,
(ii) $x = (1, 0, \ldots, 0)$,
(iii) $t_{V,x}$ is the zero locus of $\{T_{r+1} - 1, \ldots, T_{n-1}\}$.

We set $t_0 = 1$ and work with affine coordinates $t_1, \ldots, t_n$ in an affine open neighborhood of $x \in V$. By (i), (ii), and (iii), it follows that $\text{coker}(\rho) \cong m_x/(m_x^2 + \pi^*(m_y) \cdot 0_x)$ is spanned by the residues of $t_{n-i+1}, \ldots, t_n$ and that these elements are linearly independent. Hence, equality holds in (a). Q.E.D.

**Proposition 7.2.** Let $V \subset \mathbb{P}^n$, $L \subset \mathbb{P}^n$, and $\pi: V \to \mathbb{P}^m$ be as before; let $x_1, \ldots, x_d$ be distinct closed points of $V$ such that $\pi(x_1) = \cdots = \pi(x_d) = y \in \mathbb{P}^m$ so that $(x_1, \ldots, x_d) \in \Sigma_d(\pi)$. Then $\Sigma_d(\pi)$ is smooth (over $k$) at $(x_1, \ldots, x_d)$ and locally of dimension $d(r + 1) - (d - 1)m$ if and only if

$$\dim\left(\bigcap_{j=1}^d \pi_L(t_{V,x_j})\right) = \sum_{j=1}^d \dim(\pi_L(t_{V,x_j})) - (d - 1)m.$$

**Proof.** It is well known that there is a bijective map

$\alpha: \{\text{Vector subspaces of } T(\mathbb{P}^n)_y\} \to \{\text{Subspaces of } \mathbb{P}^n \text{ containing } y\}$

which preserves dimension and intersections. By choosing coordinates as in the preceding proof, one shows that $\alpha$ sends $(d\pi)_x(T(V)_x)$ to $\pi_L(t_{V,x})$. Therefore, the desired conclusion follows from Corollary 5.5. Q.E.D.

We set $d_* = \lfloor m/(m - r) \rfloor + 1$ in the case $m > r$ and $d_* = r + 1$ in the case $m = r$. Consider the following property of a projective embedding $V \subset \mathbb{P}^n$.

(a) If $d \leq d_*$, and $x_1, \ldots, x_d$ are distinct closed points of $V$, then the tangent spaces $t_{V,x_1}, \ldots, t_{V,x_d}$ span a subspace of $\mathbb{P}^n$ of the largest possible dimension, viz. $d(r + 1) - 1$.

**Lemma 7.3.** Every nonsingular quasi-projective variety has a projective embedding which satisfies (a).
The proof uses the techniques of §6 or of [11, §2].

In the statement of the following proposition, we identify $\Sigma_d(\pi)$ with a subscheme of $V \times \cdots \times V$ (d-fold product over Spec$k$).

**Proposition 7.4.** Let $V^*$ be a nonsingular subvariety of $P^n$, and assume that (a) holds. If $\pi: V \rightarrow P^n$ is induced by projection from a generic $(n - m - 1)$-subspace $L \subset P^n$, then for all sequences $i_1, \ldots, i_d$ of nonnegative integers, $\Sigma_d(\pi) \cap (S_{i_1}(\pi) \times \cdots \times S_{i_d}(\pi))$ is either empty or of pure dimension $d - (d - 1)m - \sum_{j=1}^{d} i_j(m - r + i_j)$. (We set $S_0 = V$.) In particular, if this integer is negative, then $\Sigma_d \cap (S_{i_1} \times \cdots \times S_{i_d}) = \emptyset$.

**Corollary 7.5.** If $\pi: V \rightarrow P^n$ is a generic projection, then for every $i \geq 0$, $S_1(\pi)$ is either empty or of pure codimension $i(m - r + i)$ in $V$.

**Proof of Proposition 7.4.** I claim that it will suffice to show that the conclusion holds for the (finitely many) $d$-tuples $(i_1, \ldots, i_d)$ such that $d < d_*$ and $i_j < r$ for $j = 1, \ldots, d$. The restriction $i_j < r$ involves no loss of generality because $S_1(f) = \emptyset$ when $i > r$ for any morphism $f: V \rightarrow P^n$. If $m > r$, then this special case of the conclusion (more precisely the case $d = d_*$ and $i_1 = \cdots = i_d = 0$) implies that $\text{card}(\pi^{-1}(\pi(x))) \leq d_* - 1$ for all $x \in V$. This will imply that $\Sigma_d(\pi) = \emptyset$ whenever $d > d_*$. If $m = r$, it is only necessary to prove the result when $i_j > 0$ for $j = 1, \ldots, d$. (This is because $\pi$ may be assumed to be finite.) Thus, if $d < d'$ and $\Sigma_d(\pi) \rightarrow \Sigma_{d'}(\pi)$ sends $(x_1, \ldots, x_d) \rightarrow (x_1, \ldots, x_{d'})$, the fibres are finite. If $i_{d+1} = \cdots = i_d = 0$, then the result for $(i_1, \ldots, i_d, 0, \ldots, 0)$ follows from the result for $(i_1, \ldots, i_d)$. Therefore, if $m = r$ and we have proved the result for the case $d = d_* = r + 1$, we conclude that $\Sigma_{r+1}(\pi) \cap (S_1(\pi) \times \cdots \times S_1(\pi)) = \emptyset$. This implies that the result holds for all $d$-tuples $(i_1, \ldots, i_d)$.

Therefore, we may fix $d < d_*$ and $(i_1, \ldots, i_d)$ and then show that there is a dense open subset of the Grassmann variety $G(n, n - m - 1)$ whose points correspond to linear subspaces $L \subset P^n$ such that $\pi = \pi_L|V$ satisfies the conclusions of the proposition relative to this particular $d$-tuple. (Observe that the intersection of finitely many dense open subsets is still a dense open subset.) We recall that $\dim G(n, n - m - 1) = (m + 1)(n - m)$. Let $G_0 \subset G$ consist of all $(n - m - 1)$-subspaces $L \subset P^n$ such that $L \cap \tilde{V} = \emptyset$.

Let $U_d \subset V \times \cdots \times V$ (d copies) consist of all $d$-tuples of distinct points, and let $W \subset U_d \times G_0$ consist of all $(x_1, \ldots, x_d, L)$ such that

(i) $\dim(L \cap M(x_1, \ldots, x_d)) \geq d - 2$, where $M(x_1, \ldots, x_d)$ is the $(d - 1)$-subspace of $P^n$ spanned by $x_1, \ldots, x_d$;

(ii) $\dim(L \cap \tau_v(x_j)) \geq l_j - 1$ for $j = 1, \ldots, d$.

As in [11], we see that (i) is equivalent to saying that $\pi(x_1) = \cdots = \pi(x_d)$, or equivalently that $(x_1, \ldots, x_d) \in \Sigma_d(\pi)$. Proposition 7.1 implies that (ii) is
equivalent to saying that \( x \in S_j(\pi) \) for \( j = 1, \ldots, d \). By looking at suitable incidence correspondences, one shows that \( W \) is closed in \( U \times G_0 \).

Let \( p: W \to U_d \) send \((x_1, \ldots, x_d, L) \to (x_1, \ldots, x_d)\). By Lemma 7.9 below (applied in the case that \( q = n - m - 1 \) and \( r_1 = \cdots = r_d = r \)):

\[
\dim(p^{-1}(x_1, \ldots, x_d)) = (m + 1)(n - m) - \sum_{j=1}^{d} i_j(m - r + i_j) - (d - 1)m,
\]

so that \( \dim(W) = \dim(G(n, n - m - 1)) + dr - (d - 1)m - \sum_{j=1}^{d} i_j(m - r + i_j) \).

Thus, if \( L \) is chosen to lie in a dense open subset of \( G_0 \), the dimension of the fibre of \( W \) above \( L \) will be \( \dim(W) - \dim(G_0) \), or else the fibre will be empty. In other words,

\[
\dim(S_d \cap (S_{i_1} \times \cdots \times S_{i_d})) = dr - (d - 1)m - \sum_{j=1}^{d} i_j(m - r + i_j). \quad \text{Q.E.D.}
\]

**Theorem 7.6.** Let \( V^* \subseteq \mathbb{P}^n \) be as in Proposition 7.4. If \( \pi: V^* \to \mathbb{P}^m \) is a generic projection (where \( r < m < 2r \)), then for every \( d \geq 0 \), \( \Sigma_d(\pi) \) is either empty or smooth (over \( k \)) and of pure dimension \( dr - (d - 1)m \).

**Remark 7.7.** Statement (iii) of Theorem 1 of [11] is equivalent to smoothness of \( \Sigma_d(\pi) \) at points \((x_1, \ldots, x_d)\) such that \( x_j \notin S_j(\pi) \) for \( j = 1, \ldots, d \). The proof of Theorem 7.6 is similar to the proof of Lemma 7 of [11]; the essential difference is that Corollary 7.10 must now be used to calculate the dimension of the fibres of \( F_0 \to G_0 \).

**Remark 7.8.** When \( m = r \) it is necessary to prove smoothness only at points \((x_1, \ldots, x_d)\) such that \( x_j \in S_j(\pi) \) for all \( j \). In fact, if \( x_d \notin S_j(\pi) \), a straightforward application of Proposition 7.2 shows that \((x_1, \ldots, x_d)\) is a smooth point of \( \Sigma_d(\pi) \) if and only if \((x_1, \ldots, x_{d-1})\) is a smooth point of \( \Sigma_{d-1}(\pi) \). (The essential observation is that \( \pi_L(t_{V,x_d}) = \mathbb{P}^r \) because \( x_d \notin S_1(\pi) \)).

**Proof of Theorem 7.6.** We must show that if \( \pi: V \to \mathbb{P}^m \) is a generic projection, then for all \( d \)-tuples \((i_1, \ldots, i_d)\) of nonnegative integers, there do not exist points \((x_1, \ldots, x_d)\) such that (i) \((x_1, \ldots, x_d)\) is a nonsmooth point of \( \Sigma_d(\pi) \) and (ii) \( x_j \in S_{i_j}(\pi) - S_{i_j+1}(\pi) \). As in the proof of Proposition 7.4, we observe that we may fix \((i_1, \ldots, i_d)\) [with \( d \leq d_* \)] and prove that there is a dense open subset of \( G(n, n - m - 1) \) whose points correspond to \((n - m - 1)\)-subspaces \( L \subseteq \mathbb{P}^n \) such that \( \pi = \pi_L | V \) has the desired properties relative to the given \( d \)-tuple \((i_1, \ldots, i_d)\).

We define \( G_0 \subseteq G(n, n - m - 1) \) and \( U_d \subseteq V \times \cdots \times V \) as before. We consider a point \((z_1, \ldots, z_d) \in U_d \) and \( L \in G_0 \) such that \( \pi(z_1) = \cdots = \pi(z_d) \) and \( z_j \in S_{i_j}(\pi) - S_{i_j+1}(\pi) \) for \( j = 1, \ldots, d \). By Propositions 7.1 and 7.2, \( \Sigma_d(\pi) \) fails to be smooth at \((z_1, \ldots, z_d)\) if and only if there exists a linear subspace \( \Lambda' \subseteq \cap_{j=1}^{d} \pi_L(t_{V,z_j}) \) such that
\[ \dim(\Lambda') = m - d(m - r) - \sum_{j=1}^{d} i_j + 1. \]

Setting \( \Lambda = \pi^{-1}_L(\Lambda') \), we have

\[ \dim(\Lambda) = \dim(\Lambda') + n - m = n - d(m - r) - \sum_{j=1}^{d} i_j + 1, \]

\[ \dim(\Lambda \cap t_{V,x_j}) \geq \dim(\Lambda') + i_j \text{ for } j = 1, \ldots, d. \]

For any integer \( \sigma \geq 0 \), let \( E_{\sigma} \subset U_d \times G(n, q) \), where \( q = n - m - 1 \), be the subset consisting of all closed points \( (x_1, \ldots, x_d, L) \) such that \( L \cap \bar{V} = \emptyset \) and

(a) \( \dim(L \cap L) \geq d - 2 \), where \( L \) is spanned by \( x_1, \ldots, x_d \);
(b) \( \dim(L \cap t_{V,x_j}) \geq i_j - 1 \) for \( j = 1, \ldots, d \);
(c) there is a subspace \( L \subset P^d \) with \( \dim(\Lambda) = q + \sigma + 1 \), \( L \subset \Lambda, x_j \in \Lambda \), and \( \dim(t_{V,x_j} \cap L) \geq \sigma + i_j \) for \( j = 1, \ldots, d \).

Since we want to show that \( \Sigma(\alpha) \) is smooth if \( L \) is chosen to lie in some dense open subset of \( G(n, q) \), the proof will be finished if we can show that \( E_{\sigma} \) is a constructible subset of \( U_d \times G(n, q) \) with \( \dim(E_{\sigma}) < \dim(G(n, q)) \) in the case that \( \sigma = m - d(m - r) - \sum_{j=1}^{d} i_j + 1 \).

We will estimate \( \dim(E_{\sigma}) \) by studying \( F_{\sigma} \), which is defined to be the subset of \( U_d \times G(n, \nu) \times G(n, q) \), where \( \nu = q + \sigma + 1 \), consisting of all \( (x_1, \ldots, x_d, \Lambda, L) \) such that \( L \cap \bar{V} = \emptyset \), (a) and (b) above are satisfied, and

(c') \( L \subset \Lambda, x_j \in \Lambda \) and \( \dim(\Lambda \cap t_{V,x_j}) \geq \sigma + i_j \) for \( j = 1, \ldots, d \).

By Lemma 8 of [11], \( F_{\sigma} \) is locally closed in \( U_d \times G(n, \nu) \times G(n, q) \). Let \( p_{13}: U_d \times G(n, \nu) \times G(n, q) \to U_d \times G(n, q) \) be the projection; then \( E_{\sigma} = p_{13}(F_{\sigma}) \). We conclude that \( E_{\sigma} \) is constructible, and \( \dim(E_{\sigma}) \leq \dim(F_{\sigma}) \).

Let \( G_{\sigma} \subset U_d \times G(n, \nu) \) be the closed subset consisting of all \( (x_1, \ldots, x_d, \Lambda) \) with \( x_j \in \Lambda \) and \( \dim(\Lambda \cap t_{V,x_j}) \geq \sigma + i_j \) for \( j = 1, \ldots, d \). Using Lemma 7.11 below, we see that the dimension of \( G_{\sigma} \) is

\[ dr + (\nu + 1)(n - \nu) - d\sigma((n - \nu) - (r - \sigma)) - d(n - \nu) \]
\[ - ((n - \nu) - (r - 2\sigma)) \sum_{j=1}^{d} i_j - \sum_{j=1}^{d} i_j^2. \]

Recalling that \( \nu = n - m + \sigma \), so that \( n - \nu = m - \sigma \), we find

\[ \dim(G_{\sigma}) = dr + (n - m + \sigma + 1)(m - \sigma) - d\sigma(m - r) - d(m - \sigma) \]
\[ - (m - r + \sigma) \sum_{j=1}^{d} i_j - \sum_{j=1}^{d} i_j^2. \]

The projection \( p_{12}: U_d \times G(n, \nu) \times G(n, q) \to U_d \times G(n, \nu) \) induces a surjection \( \phi: F_{\sigma} \to G_{\sigma} \). If \( (x_1, \ldots, x_d, \Lambda) \) is a point of \( G_{\sigma} \), then the fibre over this point
is isomorphic to the locally closed subvariety of $G(\nu, q)$ which parameterizes all $L \subset \Lambda$ which satisfy conditions (a) and (b) and also $L \cap \overline{V} = \emptyset$. By Corollary 7.10 below [applied with $\nu$ replacing $\nu$], the dimension of any fibre of $\phi$ is $(q + 1)(\nu - q) - (d - 1)(\nu - q - 1)$. Recalling that $\nu = q + \sigma + 1$ and $q = n - \sigma - m$, we obtain

$$\dim(F_\sigma) = \dim(G_\sigma) + (n - m)(\sigma + 1) - (d - 1)\sigma - (m - r) \sum_{j=1}^{d} i_j$$

$$- \sum_{j=1}^{d} i_j^2 + \sigma \left( m - d(m - r) - \sum_{j=1}^{d} i_j \right) - \sigma^2.$$ 

In the case $\sigma = m - d(m - r) - \sum_{j=1}^{d} i_j + 1$, this becomes

$$\dim(F_\sigma) = \dim(G(n, n - m - 1)) - (m - r) \sum_{j=1}^{d} i_j + \sum_{j=1}^{d} (i_j - i_j^2) - 1.$$ 

Therefore, $\dim(F_\sigma) < \dim(G(n, n - m - 1)) - 1$; in particular, $\dim(F_\sigma) < \dim(G(n, n - m - 1))$. Q.E.D.

**Lemma 7.9.** Let $L_1, \ldots, L_d$ be subspaces of $\mathbb{P}^n$, and let $x_j \in L_j$ for $j = 1, \ldots, d$. Let $r_j = \dim(L_j)$, and assume that $L_1, \ldots, L_d$ span a subspace of $\mathbb{P}^n$ of the largest possible dimension, viz. $\Sigma_{j=1}^{d} (r_j + 1) - 1$. Let $M \subset \mathbb{P}^n$ be the $(d - 1)$-subspace spanned by $x_1, \ldots, x_d$, let $i_1, \ldots, i_d$ be nonnegative integers with $i_j < r_j$, and let $q$ satisfy $n > q > \Sigma_{j=1}^{d} i_j + d - 2$. Finally, let $Z \subset G(n, q)$ be the locally closed subset consisting of all $q$-subspaces $L \subset \mathbb{P}^n$ such that

1. $x_j \notin L$ for $j = 1, \ldots, d$,
2. $\dim(L \cap L_j) \geq i_j + 1$ for $j = 1, \ldots, d$,
3. $\dim(L \cap M) \geq d - 2$.

Then, $\dim(Z) \leq (q + 1)(n - q) - (d - 1)(n - q - 1) - \Sigma_{j=1}^{d} i_j(n - q - 1 - r_j + i_j)$, and equality holds if and only if $q < n - r_j + i_j - 1$ for all $j$.

We note that condition (1) is vacuous for those values of $j$ with $i_j = 0$.

**Corollary 7.10.** Let the notation be as in Lemma 7.9. Assume that there is an integer $\sigma \geq 0$ such that $n = q + \sigma + 1$ and $r_j = \sigma + i_j$ for $j = 1, \ldots, d$. Then $\dim(Z) = (q + 1)(n - q) - (d - 1)(n - q - 1)$.

**Proof of Lemma 7.9.** Let $G_0 \subset G(d - 1, d - 2)$ parameterize $(d - 2)$-subspaces $M' \subset M$ with $x_j \notin M'$ for $j = 1, \ldots, d$. When $1 \leq j \leq d$, let $G_j \subset G(r_j, i_j - 1)$ be the dense open subset which parameterizes $(i_j - 1)$-subspaces $M_j \subset L_j$ with $x_j \notin M_j$. (If $i_j = 0$, we take $G_j = \text{Spec}(k)$.) Let $Y = G_0 \times \cdots \times G_d$, and let $T \subset G(n, q) \times Y$ consist of all $(L, M_0, \ldots, M_d)$ with $M_j \subset L$ for $j = 0, \ldots, d$. Let $p_1$ and $p_2$ be the projections of $G(n, q) \times Y$ to the first and
second factors, and let $q_2 : T \to Y$ be the restriction of $p_2$. Then $Z = p_1(T)$, so that $\dim(Z) \leq \dim(T)$; equality holds if and only if there exists a $q$-subspace $L \subset \mathbb{P}^n$ such that equality holds in (1) and (2). On the other hand, let $y = (M_0, \ldots, M_d) \in Y$. Then $(L, M_0, \ldots, M_d) \in T$ if and only if $M_\ast \subset L$, where $M_\ast$ is spanned by $M_0, \ldots, M_d$. Hence, $q_2^{-1}(y)$ is irreducible and of the same dimension as $G(n - e - 1, q - e - 1)$, where $e = \dim(M_\ast) = \sum_{j=1}^d i_j + d - 2$.

Therefore

$$\dim(T) = \dim(Y) + \dim(G(n - e - 1, q - e - 1))$$

$$= d - 1 + \sum_{j=1}^d i_j(r - i_j + 1) + (q - e)(n - q)$$

$$= (q + 1)(n - q) - (d - 1)(n - q - 1) - \sum_{j=1}^d i_j(n - q - 1 - r + i_j). \quad Q.E.D.$$

**Lemma 7.11.** Let $L_1, \ldots, L_d$ be $r$-subspaces of $\mathbb{P}^n$, and let $x_j \in L_j$ be a closed point, for $j = 1, \ldots, d$. Assume that $\dim(L) = (d + 1)(r + 1) - 1$, where $L$ is spanned by $L_1, \ldots, L_d$. Let $\nu, \sigma, i_1, \ldots, i_d$ be nonnegative integers such that $\sigma + i_j \leq r$ for $j = 1, \ldots, d$, and

$$d(\sigma + 1) + \sum_{j=1}^d i_j - 1 \leq \nu \leq n - 1.$$

Let $H \subset G(n, \nu)$ consist of all $\nu$-subspaces $\Lambda \subset \mathbb{P}^n$ such that $x_j \in \Lambda$ and $\dim(\Lambda \cap L_j) \geq \sigma + i_j$ for $j = 1, \ldots, d$. Then $H$ is irreducible, and

$$\dim(H) \leq (\nu + 1)(n - \nu) - d(\nu\nu) - (r - \sigma)(n - \nu) - d(n - \nu)$$

$$- ((n - \nu) - (r - 2\sigma)) \sum_{j=1}^d i_j - \sum_{j=1}^d i_j^2.$$

Equality holds if and only if $\nu \leq n - r + \sigma + i_j$ for all $j$.

The proof of this lemma is similar to the proof of Lemma 9 of [11]. We omit the details.

8. The morphism $\Phi$. Let $V$ be a nonsingular closed subvariety of $\mathbb{P}^n = \text{Proj } k[X_0, \ldots, X_n]$ and let $m$ be an integer such that $n > m \geq \dim(V)$. We construct a morphism which parameterizes projections of $V$ to $\mathbb{P}^m$ (see Definition 8.1) and study its behavior under certain coordinate changes.

Let $N = (m + 1)(n + 1) - 1$ and $\mathbb{P}^N = \text{Proj } k[\tau_{ij}]_{0 \leq i \leq m}; 0 \leq j \leq n}$. The homogeneous coordinates of a closed point can be regarded as entries of an $(m + 1) \times (n + 1)$ matrix. Let $U \subset \mathbb{P}^N$ consist of all points where this matrix has rank $(m + 1)$. If $z = (b_{ij}) \in U$, let $L_z \subset \mathbb{P}^n$ be the $(n - m - 1)$-subspace
given by the vanishing of the linear forms $\sum_{i=0}^{n} \beta_{ij} T_j$, $i = 0, \ldots, m$. Since $m \geq \dim(V)$, there is a dense open subset $Z \subset U$ such that $L_z \cap V = \emptyset$ for all $z \in Z$.

**Definition 8.1.** Let $z = (\beta_{ij}) \in Z \subset P^N$. We define $\Phi_z : V \rightarrow P^m$ to be the unique morphism such that if $x = (a_0, \ldots, a_n) \in V \subset P^n$, then $\Phi_z(x) = (c_0, \ldots, c_m) \in P^m = \Proj k[Y_0, \ldots, Y_m]$, where $c_i = \sum_{j=0}^{n} \beta_{ij} a_j$. [Equivalently, we can specify that $\Phi_z^* : \Gamma(P^m, O(1)) \rightarrow \Gamma(P^n, O(1))$ satisfies $\Phi_z^*(Y_i) = \sum_{j=0}^{n} \beta_{ij} X_j$, $i = 0, \ldots, m$.] Finally, we define $\Phi : V \times Z \rightarrow P^m \times Z$ to be the unique morphism which satisfies $\Phi(x, z) = (\Phi_z(x), z)$. (The product sign "\times" without a subscript will henceforth denote the product $\times_k$ (taken over Spec($k$)).)

**Lemma 8.2.** With the above notation, $\Phi_z : V \rightarrow P^m$ is induced by projection from $L_z$. If $z$ and $z'$ are points of $Z$ such that $L_z = L_{z'}$, then $\Phi_z$ and $\Phi_{z'}$ differ by an automorphism of $P^m$.

**Lemma 8.3.** Let $\{S_0, \ldots, S_m\}$ and $\{T_0, \ldots, T_n\}$ be bases of $\Gamma(P^m, O(1))$ and $\Gamma(P^n, O(1))$ respectively. Let $z_0 \in Z$. Assume that $L_{z_0} \subset P^n$ is given by the vanishing of $T_0, \ldots, T_n$ and that $\Phi_{z_0}^*(S_i) = T_i$, $i = 0, \ldots, m$. Then there is a basis $\{a_{ij}\}$ $0 \leq i \leq m$ and $0 \leq j \leq n$ of $\Gamma(P^n, O(1))$ such that

(i) if $z \in Z \subset P^N$ has homogeneous coordinates $(b_{ij})$ relative to $\{a_{ij}\}$, then $\Phi_z^*(S_i) = \sum_{j=0}^{m} b_{ij} T_j$, for $i = 0, \ldots, m$;

(ii) $z_0$ has homogeneous coordinates $(\delta_{ij})$ relative to $\{a_{ij}\}$, where $\delta_{ij} = 1$ (resp. 0) if $i = j$ (resp. $i \neq j$).

**Proof.** $X_\nu = \sum_{j=0}^{n} \alpha_{\nu j} T_j$, $\nu = 0, \ldots, n$, and $S_i = \sum_{\lambda=0}^{m} \gamma_{i\lambda} Y_\lambda$, $i = 0, \ldots, m$, for suitable $\alpha_{\nu j}$ and $\gamma_{i\lambda} \in k$. We set

$$
\sigma_{ij} = \sum_{\lambda=0}^{m} \sum_{\nu=0}^{n} \gamma_{i\lambda} \alpha_{\nu j} \tau_{\lambda\nu}.
$$

The matrix of coefficients which expresses $\{\sigma_{ij}\}$ in terms of $\{r_{\lambda\nu}\}$ is invertible because it is the Kronecker product of two invertible matrices. If $z \in Z$ has homogeneous coordinates $(b_{ij})$ relative to $\{a_{ij}\}$, then

$$
b_{ij} = \sum_{\lambda=0}^{m} \sum_{\nu=0}^{n} \gamma_{i\lambda} \alpha_{\nu j} \beta_{\lambda\nu},
$$

where $\{\beta_{\lambda\nu}\}$ is a set of homogeneous coordinates of $z$ relative to $\{r_{\lambda\nu}\}$. Therefore

$$
\Phi_z^*(S_i) = \sum_{\lambda=0}^{m} \gamma_{i\lambda} \Phi_z^*(Y_\lambda)
$$

$$
= \sum_{\lambda=0}^{m} \sum_{\nu=0}^{n} \sum_{j=0}^{n} \gamma_{i\lambda} \beta_{\lambda\nu} \alpha_{\nu j} T_j = \sum_{j=0}^{n} b_{ij} T_j.
$$

This proves (i), and (ii) is clear. Q.E.D.
9. The singularity subschemes of $\Phi$. We again fix a nonsingular projective variety $V$ and an integer $m \geq r = \dim(V)$. Let $d_\ast = \lceil m/(m-r) \rceil + 1$ (respectively $d_\ast = r+1$) if $m \geq r+1$ (respectively $m = r$). We set $q_\ast = \lceil r/(m-r+1) \rceil + 1$ in either case. (The brackets denote integer part.) We also assume that the embedding $V \subset \mathbb{P}^n$ satisfies:

\[ d = d_\ast \text{ and } b_1 = \cdots = b_d = q_\ast + 1. \]

(See also Lemma 6.4.) We set $N = (m+1)(n+1) - 1$ and consider the dense open subset $Z \subset \mathbb{P}^N$ and the morphism $\Phi$: $V \times Z \to \mathbb{P}^n \times Z$ as in Definition 8.1.

**Theorem 9.1.** Let $V \subset \mathbb{P}^n$ be as above. Assume that $d \leq d_\ast$ and $0 \leq q_j \leq q_\ast$, $j = 1, \ldots, d$. Then $\Sigma_d(\Phi; q_1, \ldots, q_d)$ is smooth over $k$ and has pure dimension $d r - (d-1)m - \sum_{j=1}^d q_j(m-r+1) + \dim(Z)$. In particular, $S_1(\Phi)$ has pure codimension $q(m-r+1)$ in $V \times Z$ whenever $0 \leq q \leq q_\ast$.

**Proof.** Corollary 5.2 implies that every irreducible component of $\Sigma_d(\Phi; q_1, \ldots, q_d)$ has dimension $\geq d r - (d-1)m - \sum_{j=1}^d q_j(m-r+1) + \dim(Z)$. In proving the opposite inequality we will regard $\Sigma_d(\Phi; q_1, \ldots, q_d)$ as a subscheme of $V \times \cdots \times V \times Z$. Therefore,

\[ (x_1, \ldots, x_d, z) \in \Sigma_d(\Phi; q_1, \ldots, q_d) \]

if and only if (a) $\Phi_2(x_1) = \cdots = \Phi_2(x_d)$ and (b) $(x_j, z) \in S_1^{(q_j)}(\Phi), j = 1, \ldots, d$. If $\text{Spec}(k) \to Z$ corresponds to the closed point $z \in Z$, then $S_1^{(q_j)}(\Phi_2) \cong \text{Spec}(k) \times Z S_1^{(q_j)}(\Phi)$. Thus (b) holds if and only if $x_j \in S_1^{(q_j)}(\Phi_2), j = 1, \ldots, d$.

Let $(x_1, \ldots, x_d, z_0)$ be a closed point of $W = \Sigma_d(\Phi; q_1, \ldots, q_d)$. We will show that there is a subspace $\Lambda \subset \mathbb{P}^N$ such that

(i) $z_0 \in \Lambda$;

(ii) $\dim(\Lambda) = (d-1)m + \sum_{j=1}^d q_j(m-r+1)$;

(iii) if $S = (x_1, \ldots, x_d) \times \Lambda$, then $S \cap W$ is the point $(x_1, \ldots, x_d, z_0)$ with reduced structure.

In other words, $S \cap W$ is nonsingular and of pure codimension $(d-1)m + \sum_{j=1}^d q_j(m-r+1)$ in $S$. Since $(x_1, \ldots, x_d, z_0)$ is an arbitrary closed point of $W$, we will obtain the desired conclusion.

In constructing $\Lambda$, we fix homogeneous coordinate systems $\{T_0, \ldots, T_n\}$ on $\mathbb{P}^n$ and $\{S_0, \ldots, S_m\}$ on $\mathbb{P}^m$ which satisfy (1) through (6) below.

(1) $x_i = (1, 0, \ldots, 0)$, and $x_j = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$, $j = 2, \ldots, d$. (The 0th and $(n-j+2)$th coordinates of $x_j$ are nonzero.)

(2) $L_{z_0} \subset \mathbb{P}^n$ is the subspace $T_0 = \cdots = T_m = 0$, and $\Phi_2^{z_0}: \Gamma(\mathbb{P}^m, \mathcal{O}(1)) \to \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ satisfies $\Phi_2^{z_0}(S_i) = T_i, i = 0, \ldots, m$. (In particular, $\Phi_2^{z_0}(x_1) = \cdots = \Phi_2^{z_0}(x_d) = y = (1, 0, \ldots, 0) \in \mathbb{P}^n$.)
(3) For \( j = 1, \ldots, d \), the \( (n-r) \)-subspace \( T_0 = \cdots = T_{r-1} = 0 \) meets the tangent space \( t_{v,x_j} \subset \mathbb{P}^n \) in exactly one point.

This is possible by Proposition 7.1. (Note that \( (x_j, z_0) \notin S_2(\Phi) \), so that \( x_j \notin S_2(\Phi_{z_0}) \).)

We now work with affine coordinate functions \( t_1, \ldots, t_n \) on \( V \) (we set \( t_0 = 1 \)). By (*) we can modify \( t_{n-d+1}, \ldots, t_n \) to obtain the following stronger version of (1), where \( (0_{x_j}, m_{x_j}) = 0_{v,x_j} \).

(4) If \( 2 \leq v \leq d \) and \( 1 \leq j \leq d \), then
\[
\begin{align*}
  t_{n-v+2} &\equiv 1 \pmod{m_{x_j}^{q_j+1}} & \text{if } v = j, \\
  t_{n-v+2} &\equiv 0 \pmod{m_{x_j}^{q_j+1}} & \text{if } v \neq j.
\end{align*}
\]

By (3), the elements \( t_i = \Phi_{s_i}(s_i), \ i = 1, \ldots, r-1 \), are linearly independent modulo \( m_{x_j}^2 \) for all \( j \), where \( s_1, \ldots, s_m \) are affine coordinates on \( \mathbb{P}^m \) in a neighborhood of \( y \). By (*) we may also assume

(5) If \( q_i > 0 \), then \( t_1, \ldots, t_{r-1}, t_{m+j} \) generate \( m_{x_j} \) while \( t_{m+j'} \in m_{x_j}^{q_{j'}+2} \) if \( q_{j'} > 0 \) and \( j' \neq j \).

(6) If \( q_i > 0 \) and \( 1 \leq l \leq q_\ast \), then \( t_{v'(j,l)} \equiv t_{m+j} \pmod{m_{x_j}^{q_j+1}} \), where \( v'(j,l) = m + (l-1)d + j \), while if \( q_{j'} > 0 \) and \( j' \neq j \), then \( t_{v'(j',l)} \equiv 0 \pmod{m_{x_j}^{q_j+1}} \).

Having good coordinate systems on \( \mathbb{P}^m \) and \( \mathbb{P}^n \), we now fix a homogeneous coordinate system \( \{ \sigma_{iv}, 0 \leq i \leq m \text{ and } 0 \leq v \leq n \} \) on \( \mathbb{P}^V \) which satisfies the conclusions of Lemma 8.3. We take \( \Lambda \subset \mathbb{P}^V \) to be the linear subspace given by the vanishing of all \( \sigma_{iv} - \delta_{iv}\sigma_{00} \) except those which satisfy either

(7a) \( n - d + 2 \leq v \leq n \) and \( 1 \leq i \leq m \), or

(7b) \( r < i \leq m \) and \( v = v(j,l) \) for some \( j, l \) such that \( q_j > 0 \) and \( 1 \leq l \leq q_j \).

I claim that (i), (ii) and (iii) hold for this \( \Lambda \). Of course, (i) and (ii) are clear. To establish (iii), we get \( Z' = Z \cap \Lambda \) and consider the morphism \( \Phi' = \Phi|_{V \times Z'}: V \times Z' \rightarrow \mathbb{P}^m \times Z' \). Thus if \( \Phi \) is regarded as a morphism of schemes over \( Z \), then \( \Phi' \) is obtained by the base extension \( Z' \rightarrow Z \). Therefore \( S'_1(\Phi') = S_1(\Phi) \times_Z Z' \) for all \( q \geq 0 \), by Proposition 3.3. Using elementary properties of fibre products one shows \( S_d(\Phi') = S_d(\Phi) \times_Z Z' \). In other words \( S'_1(\Phi') = S_1(\Phi) \cap (V \times Z') \) in \( V \times Z \), and \( S_d(\Phi') = S_d(\Phi) \cap (V \times \cdots \times V \times Z') \) in \( V \times \cdots \times V \times Z \). Moreover,
\[
\Sigma_d(\Phi'; q_1, \ldots, q_d) = \Sigma_d(\Phi') \cap p^{-1}_1(\Sigma_1(\Phi')) \cap \cdots \cap p^{-1}_d(\Sigma_d(\Phi'))
\]
where \( p_j: V \times \cdots \times V \times Z \rightarrow V \times Z \) sends \( (x_1, \ldots, x_d, z) \rightarrow (x_j, z) \). Therefore, (iii) is a consequence of the following two lemmas.

**Lemma 9.2.** If \( q_j > 0 \), then \( S'_{d,q_j}(\Phi') \cap \{ \{ x_j \} \times Z' \} = \{ x_j \} \times (\Lambda_j \cap Z') \), where
$\Lambda$ is the subspace of $\Lambda$ given by the vanishing of $\{a_{iv} | r \leq i \leq m$ and $v = v(j, l)\}$ for some $l \in \{1, \ldots, q_j\}$.

**Lemma 9.3.** $\Sigma_{d}(\Phi') \cap ((x_1, \ldots, x_d) \times Z') = (x_1, \ldots, x_d) \times (\Lambda_0 \cap Z')$, where $\Lambda_0 \subset \Lambda$ is the subspace of $\Lambda$ given by the vanishing of $\{a_{iv} | 1 \leq i \leq m$ and $n - d + 2 \leq v \leq n\}$.

**Proof of Lemma 9.2.** Consider the statement about $\Sigma_{d}(\Phi') \cap ((x_1, \ldots, x_d) \times Z')$. To simplify notation we set $x = x_1$ and $q = q_1$. Let $(A, m_A)$ and $(B, m_B)$ be the local rings on $\mathbb{P}^m \times Z'$ at $(y, z_0)$ and on $V \times Z'$ at $(x, z_0)$ respectively, and let $\Phi': A \to B$ be induced by $\Phi$. In particular, $A$ and $B$ are localizations of $\mathcal{O}_y \otimes_k R$ and $\mathcal{O}_x \otimes_k R$, where $(\mathcal{O}_y, m_y) = \mathcal{O}_{V \times Z'}(y, z_0)$, and $(R, m_R) = \mathcal{O}_{Z'}(z_0)$. Moreover, $\Phi'$ is an $R$-algebra homomorphism. As minimal generating set of $m_R$ we take $\{a_{iv}\}$ where $(i, v)$ runs through all pairs such that either (7a) or (7b) holds. Then

1. $\Phi'(a_{iv}) = a_{iv}$ for all $(i, v)$;
2. $\Phi'(s_i) = u_i$ for $i = 1, \ldots, r - 1$;
3. $\Phi'(s_i) = u_i + \sum_{(j, l)} a_{il} v_{(j, l)} t_{j' l}$ for $i = r, \ldots, m$,

where $u_i = t_i + \sum_{v=1}^{n-d+2} a_{iv} v_T, i = 1, \ldots, m$. $(j, l)$ runs through all pairs such that $q_l > 0$ and $1 \leq l \leq q$. (We have written $a_{iv}, s_i$ and $t_{j' l}$ to denote the elements $1 \otimes a_{iv}, s_i \otimes 1$ and $t_{j' l} \otimes 1$, respectively.)

Since $\{u_1, \ldots, u_{r-1}, t_{m+1}\}$ is a minimal generating set of $m_B$, Proposition 3.10 implies that there are $R$-linear differential operators $D^{(\mu)}: B \to B$, $\mu = 1, \ldots, q$, such that

1. $D^{(\mu)} b \equiv (\prod_{v=1}^{r-1} u_v^{l}) t_{m+1} \pmod{m_x^{q+1} B}$, then
2. $D^{(\mu)} b \equiv \prod_{v=1}^{r-1} u_v^{l} \pmod{m_x^{q+1} B}$;
3. $\Delta_q(P_{B/A}^B)$ is generated by the elements $D^{(\mu)} \Phi'(s_i)$, where $1 \leq \mu \leq q$ and $1 \leq i \leq m$.

By (5), (6), and (9), we see that if $1 \leq l \leq q$, then

1. $D^{(\mu)} t_{(i, j)} \equiv 1 \pmod{m_x B}$ if $\mu = l$,
2. $0 \pmod{m_x B}$ if $\mu \neq l$.

Using this, together with (8c) and the $R$-linearity of $D^{(\mu)}$, and observing that (4) implies $D^{(\mu)} u_i \in m_x B, i = r, \ldots, m$, we conclude that $D^{(\mu)} \Phi'(s_i) \equiv a_{iv} t_{(i, l)} \pmod{m_x B}$ whenever $1 \leq \mu \leq q$ and $r \leq i \leq m$. By (10), this implies that $m_x B + \Delta_q(P_{B/A}^B) = m_x B + J$, where $J \subset B$ is generated by $\{a_{iv} t_{(i, l)} | r \leq i \leq m$ and $1 \leq l \leq q\}$.

Since $\Delta_q(P_{B/A}^B) \subset B$ corresponds to the subscheme $S_{d}(\Phi')$, the proof of the case $j = 1$ is complete.

The proof is entirely similar in the case $2 \leq j \leq d$. The idea is that equations like (8b) and (8c) hold with $s_i$ and $u_i$ replaced by $s_i - \sigma_{i, n-j+2}$ and $u_i - \sigma_{i, n-j+2}$.
respectively. With this change, Proposition 3.10 can be applied as in the case considered above. Q.E.D.

The proof of Lemma 9.3 is straightforward and will be omitted. Let us simply observe that if \( z = (b_{iv}) \) is a point of \( Z' \), then \( \Phi_z(x_1) = \cdots = \Phi_z(x_d) \) if and only if \( b_{iv} = 0 \) for all \( i \in \{1, \ldots, m\} \) and \( v \in \{n - d + 2, \ldots, n\} \). Thus, it is clear that our description of \( \Sigma_d(\Phi') \cap ((x_1, \ldots, x_d) \times Z') \) is set-theoretically correct.

10. Proof of Theorem A. We fix an integer \( m \geq r = \dim(V) \) and an embedding \( V \subset \mathbb{P}^n \) which satisfies assumption (\(*\)) of §9. Let \( Z \subset \mathbb{P}^n \) and \( \Phi: V \times Z \to \mathbb{P}^m \times Z \) be as in Definition 8.1. Using Plücker coordinates, we obtain a dominant morphism \( g: Z \to G(n, n - m - 1) \). Therefore, Theorem A will follow if we can show that \( \pi = \Phi_z \) satisfies the conclusions of Theorem A when \( z \) lies in a suitable dense open subset of \( Z \).

Let \( \text{Spec}(k) \to Z \) correspond to the closed point \( z \in Z \). Then \( \Sigma_d(\Phi_z; q_1, \ldots, q_d) \cong \Sigma_d(\Phi; q_1, \ldots, q_d) \times Z \text{Spec}(k) \). If we regard \( \Sigma_d(\Phi; q_1, \ldots, q_d) \) as a closed subscheme of \( V \times \cdots \times V \times Z \) (product taken over \( k \), as usual), this is equivalent to

\[
\Sigma_d(\Phi_z; q_1, \ldots, q_d) \times \{z\} = \Sigma_d(\Phi; q_1, \ldots, q_d) \cap (V \times \cdots \times V \times \{z\}).
\]

By Theorem 9.1, \( \dim(S_1^{(q^*)}(\Phi)) = r - q_d(m - r + 1) + \dim(Z) < \dim(Z) \). Therefore \( (V \times \{z\}) \cap S_1^{(q^*)}(\Phi) = \emptyset \) if \( z \) lies in a suitable dense open subset of \( Z \). Equivalently, we may replace \( Z \) by a dense open subset and assume

\( S_1^{(q)}(\Phi) = \emptyset \) when \( q \geq q_\ast \).

A similar argument shows that if \( Z \) is replaced by a still smaller dense open subset, we may assume

(IIa) if \( m > r \), then \( \Sigma_d(\Phi) = \emptyset \) when \( d \geq d_\ast \);

(IIb) if \( m = r \), then \( \Sigma_d(\Phi; 1, \ldots, 1) = \emptyset \) when \( d \geq d_\ast \).

In the case \( m \geq r + 1 \) it follows that only finitely many \( \Sigma_d(\Phi; q_1, \ldots, q_d) \) are nonempty; if \( m = r \) we conclude that only finitely many of the \( \Sigma_d(\Phi; q_1, \ldots, q_d) \) with \( q_j > 0 \) for all \( j \) are nonempty. Since the intersection of finitely many dense open subsets of \( Z \) is dense, we may therefore fix \( q_1, \ldots, q_d \) and consider the morphism \( p: \Sigma_d(\Phi; q_1, \ldots, q_d) \to Z \) induced by \( V \times \cdots \times V \times Z \to Z \).

Specifically, we must show that if \( z \) lies in some dense open subset of \( Z \), then

(i') \( p^{-1}(z) \) has pure dimension \( dr - (d - 1)m - \sum_{j=1}^{d} q_j(m - r + 1) \);

(ii') if \( \text{char}(k) \nmid (q_j + 1) \) for all \( j \), then \( p^{-1}(z) \) is smooth over \( k \);

(iii') in general, the dimension of the nonsmooth locus of \( p^{-1}(z) \) is \( \leq \text{card}(Q) - 1 \), where \( Q = \{j | \text{char}(k) \text{ divides } q_j + 1\} \).

Since \( \dim(\Sigma_d(\Phi; q_1, \ldots, q_d)) - \dim(Z) = dr - (d - 1)m - \sum_{j=1}^{d} q_j(m - r + 1) \), (i') follows from well-known properties of the fibres of a morphism, regardless of the characteristic. If \( \text{char}(k) = 0 \), then (ii') follows.
from Theorem 2.9, and (iii') is meaningless. If char(k) > 0, then (ii') and (iii') follow from Lemma 10.3 below. Q.E.D. for Theorem A.

Remark 10.1. Let m = r, suppose that d > d*, and (x_1, ..., x_d) \in \Sigma_d(\Phi, q_1, ..., q_d). By (IIb), we may assume that q_{d_*} = \cdots = q_d = 0 and x_j \notin \Sigma_1(\Phi), j = d_*, ..., d. By Proposition 5.4, smoothness of \Sigma_d(\Phi, q_1, ..., q_d) at (x_1, ..., x_d) follows from smoothness of \Sigma_{d_*-1}(\Phi, q_1, ..., q_{d_*-1}) at (x_1, ..., x_{d_*-1}). [With W_j = \Sigma^{(q_j)}_j(\Phi), we have Ker(\rho_j) = 0, j = d_*, ..., d.] This explains why it suffices to consider the values d = 1, ..., d_* - 1 when proving the case m = r of Theorem A.

Definition 10.2. Let T be a subscheme of V \times \cdots \times V \times Z which is smooth over k and of pure dimension, with dim(T) \geq dim(Z). Let \rho: T \to Z be induced by the projection V \times \cdots \times V \times Z \to Z. We define the nonsmooth locus of T over Z, denoted NS(T/Z), to be \rho^{-1}(p), where \nu = dim(T) - dim(Z) + 1.

As in the proof of Theorem 2.9, we observe that s \in T is a closed point of NS(T/Z) if and only if either
(a) \rho^{-1}(\rho(s)) is not smooth (over k) at s, or
(b) s is contained in an irreducible component of \rho^{-1}(\rho(s)) which has dimension > dim(T) - dim(Z).

Lemma 10.3. Let 0 \leq d \leq d_* and 0 \leq q_j \leq q_* - 1, j = 1, ..., d. Let W = \Sigma_d(\Phi; q_1, ..., q_d) be identified with a subscheme of V \times \cdots \times V \times Z. Then dim(NS(W/Z)) \leq dim(Z) + card(Q) - 1, where Q = \{j|\forall k divides q_j + 1\}.

The proof of Lemma 10.3 is given in the next section. Similar methods can also be used to prove the following more detailed result, of which Theorem 10.5 is an immediate consequence.

Lemma 10.4. Let 1 \leq e \leq d \leq d_* and let q_1, ..., q_d, q'_1, ..., q'_e be integers such that 0 \leq q_j \leq q_* - 1, j = 1, ..., d, and 0 \leq q'_j \leq q_j, j = 1, ..., e. Let W = \Sigma_d(\Phi; q_1, ..., q_d), W' = \Sigma_e(\Phi; q'_1, ..., q'_e), and Y = W \cap (NS(W'/Z) \times V). Then dim(Y) \leq dim(W) - \delta - 1, where

\delta = r - (e - 1)(m - r) - \sum_{j=1}^e q'_j(m - r + 1) - card(Q) - card(Q'),

Q = \{j|1 \leq j \leq e, q'_j = q_j, and char(k) divides q_j + 1\}, and
Q' = \{j|1 \leq j \leq e and q'_j < q_j\}.

In other words, (x_1, ..., x_d, z) \in Y if and only if (x_1, ..., x_d, z) \in W and (x_1, ..., x_e, z) \in NS(W'/Z). We obtain Lemma 10.3 by taking e = d and q'_j = q_j, j = 1, ..., d.
Theorem 10.5. Let \( d, e, q_1, \ldots, q_d, \) and \( q'_1, \ldots, q'_e \) be as above. Let \( \pi: V \to \mathbb{P}^m \) be a generic projection, and let \( \Theta \subset V \times \cdots \times V \) (d-fold product) consist of all \((x_1, \ldots, x_d)\) such that
(a) \((x_1, \ldots, x_d) \in \Sigma_d(\pi; q_1, \ldots, q_d)\),
(b) \((x_1, \ldots, x_d)\) is a nonsmooth point of \( \Sigma_e(q_1, \ldots, q_e) \).
Then \( \dim(\Theta) \leq \dim(\Sigma_d(\pi; q_1, \ldots, q_d)) - \delta - 1 \), where \( \delta \) is defined as above.

Corollary 10.6. Assume, as always, that the projective embedding satisfies \((*)\). If \( \pi: V \to \mathbb{P}^m \) is a generic projection and \( x \) is a nonsmooth point of \( S_{d}(\pi) \), then
(i) \( x \notin S_{d}(\pi; q_1, \ldots, q_d) \),
(ii) \( \pi^{-1}(\pi(x)) = \{x\} \) if \( m > r \),
(iii) \( \pi^{-1}(\pi(x)) \cap S_{d}(\pi) = \{x\} \) if \( m = r \).

Remark 10.7. With the hypotheses of Theorem 10.5, let \((x_1, \ldots, x_d) \in \Sigma_d(\pi; q_1, \ldots, q_d)\) and set \( y = \pi(x_1) = \cdots = \pi(x_d) \in \mathbb{P}^m\). Set \( (0_y, m_y) = \mathcal{O}_{\mathbb{P}^m,y}, (0_{x_j}, m_{x_j}) = \mathcal{O}_{V,x_j} \), and let \( I_j \subset 0_{x_j} \) be the ideal defining the subscheme \( S_{d+1}^{(q_j)}(\pi) \subset V \) locally, \( j = 1, \ldots, e \). Let \( N_j = \text{Ker}(\rho_j) \subset m_y/m_y^2 \), where \( \rho_j: m_y/m_y^2 \to m_{x_j}/(I_{x_j} + m_{x_j}^2) \) is induced by \( \pi_j: 0_y \to 0_{x_j} \). For simplicity, assume that \( x_j \not\in S_{d+1}^{(q_j+1)}(\pi) \) if \( 1 \leq j \leq e \) and \( q_j = q'_j \). Lemma 11.2 below implies that \( \delta = m - \sum_{j=1}^{e} \dim_k(N_j) \). By Proposition 5.4 we conclude that if \( \delta < 0 \), then \((x_1, \ldots, x_e)\) cannot be a smooth point of \( \Sigma_e(\pi; q_1', \ldots, q'_e) \). Of course, this does not imply that \( \Sigma_e(\pi; q_1', \ldots, q'_e) \) is empty.

11. Dimension of the nonsmooth locus. We will keep the assumptions of \( \S 9 \). In particular \( Z \subset \mathbb{P}^N \) and \( \Phi: V \times Z \to \mathbb{P}^m \times Z \) are as in Definition 8.1. As we have noted before, \( \Sigma_d(\Phi_2 \times \{z\}) = \Sigma_d(\Phi) \cap (V \times \cdots \times V \times \{z\}) \). Since \((*)\) implies condition \((\alpha)\) of \( \S 7 \), Theorem 7.6 implies that we may replace \( Z \) with a dense open subset and assume
\( (A) \) If \( 0 \leq d \leq d_1 \), then \( \Sigma_d(\Phi) \) is smooth over \( Z \) and of pure relative dimension \( dr - (d-1)m \). (We regard \( \Sigma_d(\Phi) \) as a scheme over \( Z \), with \( p: \Sigma_d(\Phi) \to Z \) induced by the projection \( V \times \cdots \times V \times Z \to Z \).)

Proof of Lemma 10.3. We fix \( d \leq d_1 \) and \( q_j, \ j = 1, \ldots, d, \) such that \( 0 \leq q_j \leq q_e - 1 \). Let \( e = \text{card}(\{j|q_j > 0\}) \). If \( e = 0 \), then \( q_1 = \cdots = q_d = 0 \), and smoothness follows directly from \( (A) \). Thus, we may assume that \( q_1 > 0 \). If \( d \geq 2 \), we set \( W_0 = \Sigma_d(\Phi; 0, q_2, \ldots, q_d) \) and \( Q_0 = Q \setminus \{2, \ldots, d\} \).

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We now fix a closed point \((x_1, \ldots, x_d, z_0) \in W\) such that

1. \(W_0\) is smooth over \(Z\) at \((x_1, \ldots, x_d, z_0)\). (Thus, \((x_1, \ldots, x_d)\) is a smooth point of \(\Sigma_d(\Phi_{z_0}; 0, q_2, \ldots, q_d)\).)

(If \(d = 1\), we simply fix a point \((x, z_0) \in S[q](\Phi)\)) Let \(Q' = \{i| (x_i, z_0) \in S'_1(q+1)(\Phi)\}\) and \(Q'' = Q - Q \cap Q'\). Let \(W_1 = \Sigma_d(\Phi; q'_1, \ldots, q'_d)\), where \(q'_j = q_j + 1\) (resp. \(q''_j = q_j\)) if \(j \in Q'\) (resp. \(j \notin Q'\)), and let \(Y = W_1 \cap NS(W/Z)\).

I claim that \(\dim(Y) < \dim(Z) + \text{card}(\beta'') - (m - r)\text{card}(\beta') - 1\), or equivalently that \(\dim(Y) < \dim(W_1) - \delta - 1\), where

\[
\delta = r - (d - 1)(m - r) - \sum_{j=1}^{d} q_j(m - r + 1) - \text{card}(\beta') - \text{card}(\beta'').
\]

This will imply the desired conclusion because \(Q'' \subseteq Q\). We may assume that \(\delta > 0\); there is nothing to prove otherwise. Finally, a descending induction on \(\text{card}(Q')\) shows that we only need to study irreducible components of \(Y\) which contain \((x_1, \ldots, x_d, z_0)\).

We will establish the claim by finding a linear subspace \(\Lambda \subseteq P^N\) such that the following statements hold with \(Z' = Z \cap \Lambda\).

1. \(z_0 \in Z'\), and \((x_1, \ldots, x_d) \times Z' \subseteq W_1\).
2. \(\dim((x_1, \ldots, x_d) \times Z') \cap NS(W/Z)) < \dim(\Lambda) - \delta - 1\).

[Since \(W_1\) is smooth over \(k\), (P1) and (P2) will in fact imply that \(Y\) has codimension \(\geq \delta + 1\) in \(W_1\)].

We will use homogeneous coordinate systems \(\{T_0, \ldots, T_n\}\) on \(P^n\) and \(\{S_0, \ldots, S_m\}\) on \(P^m\) which satisfy (2) through (6) below.

1. \(x_1 = (1, 0, \ldots, 0) \in P^n\), and \(T_0(x_j) \neq 0, j = 1, \ldots, d\).
2. \(L_{x_0} \subseteq P^n\) is the subspace \(T_0 = \cdots = T_m = 0, \) and \(\Phi_{z_0} : \Gamma(P^n, 0(1)) \rightarrow \Gamma(P^n, 0(1))\) satisfies \(\Phi_{x_0}(S_i) = T_i, i = 0, \ldots, m\).

Thus, \(y = (1, \ldots, 0) \in P^m\). Henceforth, we use affine coordinates \(t_1, \ldots, t_n\) (resp. \(s_1, \ldots, s_m\)) on an affine open subset of \(P^n\) (resp. \(P^m\)). Since \(x_1 \in S_1(\Phi_{z_0}) - S_2(\Phi_{z_0})\), we may assume

1. \(t_1, \ldots, t_{r-1}, t_{m+1}\) generate the maximal ideal \(m_{x_1} \subset \mathcal{O}_{x_1} = \mathcal{O}_{X,x_1}\), and \(t_v \in m_{x_1}^2, v = r, \ldots, m\).

If \(x \in X\) with \((\mathcal{O}_x, m_x) = \mathcal{O}_{X,x}\), we set \(T(X)x_0 = m_x/m_x^2\), i.e., the cotangent space. \(\Phi_{x_0} : V \rightarrow P^n\) induces linear maps \(\lambda_{x_0} : T(P^n)x_0 \rightarrow T(V)x_0\) and (if \(d > 2\)) \(\rho_j, z_0 : T(P^m)x_0 \rightarrow T(S(q)(\Phi_{z_0}))x_0, j = 2, \ldots, d\). Let \(N^0_1 = \text{Ker}(\lambda_{x_0})\) and \(N^0_j = \text{Ker}(\rho_j, z_0), j = 2, \ldots, d\). By (4), \(\{s_1, \ldots, s_m\}\) is a basis of \(N^0_1\), where \(s_i \in m_y/m_y^2 = T(P^m)x_0\) is the residue class of \(s_i\). In the case \(d > 2\), Proposition 5.4 implies that we may assume

2. There are disjoint subsets \(E_2, \ldots, E_d \subseteq \{1, \ldots, r - 1\}\) such that \(\{\tilde{s}_j, l \in E_j\}\) is a basis of \(N^0_j, j = 2, \ldots, d\).

Finally, by assumption (*) of §9 we may assume
(6) If $1 \leq l \leq q_1$, and $1 \leq \mu \leq r - 1$, then there is an integer $\gamma(l, \mu)$ such that (a) $t_{\gamma(l, \mu)} \equiv t_{\mu l + 1} (\text{mod } m_1^{q_1 + 2})$, and (b) $t_{\gamma(l, \mu)} \in m_1^{q_1 + 2}$ if $2 \leq j \leq d$.

Having obtained good coordinate systems on $\mathbb{P}^d$ and $\mathbb{P}^n$, we now fix a homogeneous coordinate system $\{\sigma_{iv}, 0 \leq i \leq m$ and $0 \leq v \leq n\}$ on $\mathbb{P}^N$ which satisfies the conclusions of Lemma 8.3. If $d = 1$, we set $N = (0) \in T(\mathbb{P}^n)_\Phi$ and $c = 0$. If $d > 2$, we set $N = N_2 + \cdots + N_d$ and $c = \dim(N_2) + \cdots + \dim(N_d)$; in this case we may assume that $E_2 \cup \cdots \cup E_d = \{r - c, \ldots, r - 1\}$. We take $\Lambda \subset \mathbb{P}^N$ to be the linear subspace given by the vanishing of all $\sigma_{iv} - \delta_{iv}\sigma_{00}$ such that $(i, v) \notin \Delta$, where

$$\Delta = \{(i, v), r \leq i \leq m \text{ and } v = \gamma(l, \mu) \text{ for some } (l, \mu) \}$$

such that $1 \leq l \leq q_1$ and $1 \leq \mu \leq r - c - 1$.

An argument similar to the proof of Lemma 9.2 shows that $Z' = Z \cap \Lambda$ satisfies (P1); moreover

(7) If $z \in Z'$, then $(x_j, z) \in S_1^{(q_1 + 1)}(\Phi)$ if and only if $j \in Q'$. For $z \in Z'$, $\Phi_\alpha : V \to \mathbb{P}^n$ induces linear maps $\lambda_2 : T(\mathbb{P}^n)_\Phi \to T(V)_\Phi$ and $\rho_{j,z} : T(\mathbb{P}^n)_\Phi \to T(S_1^{(q_1)}(\Phi_\alpha))_\Phi, j = 2, \ldots, d$. Lemma 11.1 below implies that $\{s_i | i \in E_j\}$ is a basis of Ker(\rho_{j,z}). Thus, Corollary 5.6 implies

(8) $(x_1, \ldots, x_d)$ is a smooth point of $\Sigma_d(\Phi_\alpha; q_1, \ldots, q_d)$ if and only if there exist elements $b_i \in I_1, i = 1, \ldots, q_1(m - r + 1)$, where $I_1 \subset O_{x_1} = O_{V, x_1}$ is the ideal defining $S_1^{(q_1)}(\Phi_\alpha) \subset V$, such that the elements $b_i \in m_{x_1}/m_{x_1}^2, i = 1, \ldots, q_1(m - r + 1)$, are linearly independent modulo the subspace of $m_{x_1}/m_{x_1}^2$ spanned by $\{y_1 | r - c \leq i \leq r - 1\}$.

Specifically, Lemma 3.8 implies that $I_1$ is generated by the elements $D^{(\alpha)}(\Phi_\alpha(s_i))$, $r \leq i \leq m$ and $1 \leq \lambda \leq q_1$, where the $D^{(\alpha)} : O_{x_1} \to O_{x_1}$ are $k$-linear differential operators such that

(9) If $b \in O_{x_1}$ and $b \equiv ur_l^{m+1} (\text{mod } m_1^{q_1 + 2})$ where $u$ is a monomial in $t_1, \ldots, t_{r-1}$, then $D^{(\alpha)}b \equiv \delta_{ur_m^{l+1}} (\text{mod } m_1^{q_1 + 2})$.

Let $\Lambda_0 \subset \Lambda$ be the complement of the hyperplane $\sigma_{00} = 0$. We set $\sigma_{00} = 1$ to obtain a set of linear coordinate functions on $\Lambda_0$, which we denote $\{\sigma_{iv}(l, \nu) \in \Delta\}$. If $z \in Z'$ and $\Phi_\alpha : O_{Y} \to O_{x_1}$ is induced by $\Phi_\alpha$, then

$$\Phi_\alpha(s_i) = t_p, \quad i = 1, \ldots, r - 1,$$

$$\Phi_\alpha(s_i) = t_i + \sum_v \sigma_{iv}(z)t_v, \quad i = r, \ldots, m,$$

where $\nu$ takes the values $\nu = \gamma(l, \mu)$, with $1 \leq \mu \leq r - c - 1$ and $1 \leq l \leq q_1$. By (6), we may therefore assume that the following congruences hold modulo $m_1^{q_1 + 2}$:

$$\Phi_\alpha(s_i) \equiv t^{q_1 + 1}_{m+1} + \sum_{\mu=1}^{r-c-1} \sum_{l=1}^{q_1} \sigma_{\mu l r}^l(z)t_{\mu l + 1}^{m+1} + \xi_r \text{ if } i \notin Q';$$
\[
\Phi^*_z(s_{i}) = \sum_{\mu=1}^{r-c-1} \sum_{l=1}^{q_1} a'_{\mu r}(z) t_{\mu} t_{m+1}^l + \xi_i \quad \text{if } 1 \in Q',
\]

\[
\Phi^*_v(s_{i}) = \sum_{\mu=1}^{r-c-1} \sum_{l=1}^{q_1} a''_{\mu i}(z) t_{\mu} t_{m+1}^l + \xi_i, \quad i = r + 1, \ldots, m,
\]

where \(a'_{\mu i} = a_{\mu i} + a_{\mu i}^\prime\) for some \(a_{\mu i}^\prime \in k\) [similarly for \(a''_{\mu i}\)], and \(\xi_1, \ldots, \xi_m\) are polynomials in \(t_1, \ldots, t_{r-1}, t_{m+1}\) which do not involve the monomials \(t_{\mu} t_{m+1}^l, 1 \leq \mu \leq r-c-1,\) or powers of \(t_{m+1}\). These congruences, together with (9), imply that the following hold modulo \(m_2^1\):

\[
(10a) \quad D(q_1) \Phi^*_z(s_{i}) \equiv (q_1 + 1) t_{m+1} + \sum_{\mu=1}^{r-c-1} a'_{\mu r q_1}(z) t_{\mu} + D(q_1) \xi _r \quad \text{if } 1 \notin Q';
\]

\[
(10b) \quad D(q_1) \Phi^*_v(s_{i}) \equiv \sum_{\mu=1}^{r-c-1} a''_{\mu i q_1}(z) t_{\mu} + D(q_1) \xi_i \quad \text{if } 1 \in Q';
\]

\[
(11) \quad D^{(\lambda)} \Phi^*_v(s_{i}) \equiv \sum_{\mu=1}^{r-c-1} a''_{\mu i \lambda}(z) t_{\mu} + D^{(\lambda)} \xi_i \quad \text{if } (i, \lambda) \neq (r, q_1).
\]

The linear term of \(D^{(\lambda)} \xi_i\) involves \(t_{r-c-1}, \ldots, t_{r-1}\). By (8) it does not matter.

We now consider a matrix \(M\), whose entries are to be the \(a'_{\mu i}\) (distinct indeterminates, lying in the coordinate ring of \(\Lambda_0\)). \(M\) will have \(r-c-1\) columns, indexed by the integers \(\mu = 1, \ldots, r-c-1\). If \(1 \notin Q'\) or \(Q''\) (i.e., if \(x_1 \notin S(q_1 + 1)\) and \(\text{char}(k) \nmid (q_1 + 1)\)), we take the rows of \(M\) to be indexed by \(\{(i, \lambda)| r \leq i \leq m, 1 \leq \lambda \leq q_1, \text{and } (i, \lambda) \neq (r, q_1)\}\). [Since \(t_{m+1}\) occurs nontrivially in the linear form of \(D^{(q_1)} \Phi^*_v(s_{i})\), we do not need to consider the coefficients of \(t_1, \ldots, t_{r-c-1}\) in (10a).] If \(1 \in Q'\) or if \(\text{char}(k) \nmid (q_1 + 1)\), \(M\) will have one more row—we do not exclude \((r, q_1)\) from the indexing set. It follows from (8) that \((x_1, \ldots, x_d, z) \in NS(W/Z)\) if and only if \(M\) has rank \(< \#(\text{rows})\). If \(d = 1\), then \(M\) has \(r-1\) columns. If \(d \geq 2\), then Lemma 11.2, below, and the fact that \(c = \dim(N_2) + \cdots + \dim(N_d)\) imply

\[
\#(\text{columns}) + 1 = r - (d - 1)(m - r) - \sum_{j=2}^{d} q_j (m - r + 1) - \text{card}(Q'_1) - \text{card}(Q''_1),
\]

where \(Q'_1 = Q' \cap \{2, \ldots, d\}\) and \(Q''_1 = Q'' \cap \{2, \ldots, d\}\). Therefore

\[
\#(\text{columns}) - \#(\text{rows}) + 1 = \delta + 1.
\]

It follows that the subvariety of \(\Lambda_0\) determined by the condition \(\text{rank}(M) < \#(\text{rows})\) has codimension \(\delta + 1\). (See [10, Propositions 1 and 2] and [2, Theorem 3].) This implies (P2). Q.E.D.

**Lemma 11.1.** With the notation of the proof of Lemma 10.3, let \(z\) be an
arbitrary point of $Z'$. Then $T(S^{(q)}(\Phi_z))_{x_j}$ and $T(S^{(q)}(\Phi_{z_0}))_{x_j}$ are identical subspace of $T(V)_{x_j}$ for $j = 2, \ldots, d$.

**Proof.** We set $x = x_j$ to simplify notation. $\Phi_z^*$ and $\Phi_{z_0}^*$ induce $\mathcal{O}_y$-algebra structures on $\mathcal{O}_x$. In each case, we get a surjective $\mathcal{O}_x$-algebra homomorphism $u: P^q_{0,x/k} \rightarrow P_{0,y/\mathbb{O}_y}$, where $q = q_j$. The two kernels are generated by $\{d^q\Phi_z^*(a) | a \in \mathcal{O}_y\}$ and $\{d^q\Phi_{z_0}^*(a) | a \in \mathcal{O}_y\}$ respectively. Since $\Phi_z^*(a) \equiv \Phi_{z_0}^*(a) \pmod{m^{q+2}}$ and $d^q$ is a differential operator of order $q$, we conclude that $d^q\Phi_z^*(a) \equiv d^q\Phi_{z_0}^*(a) \pmod{m^{2}}$. Using a free basis of $P^q_{0,x/k}$ to compute the two Fitting ideals, we see that corresponding generators of the Fitting ideals are congruent modulo $m^2$. Q.E.D.

**Lemma 11.2.** Let $f: X^r \rightarrow Y^m$ be a morphism of nonsingular varieties, where $r \leq m$. Let $q \geq 0$, let $x$ be a closed point of $S = S^{(q)}(f)$, and let $g = f|_{S}$. Assume that $S$ is smooth at $x$ and locally of codimension $q(m - r + 1)$ in $X$. Then the linear map $(dg)_x: T(S)_x \rightarrow T(Y)_{f(x)}$ satisfies
\[
\text{rank}((dg)_x) = r - q(m - r + 1) \quad \text{if} \quad x \notin S^{(q+1)}(f) \quad \text{and} \quad \text{char}(k)^q \overset{+}{=} (q + 1),
\]
\[
\text{rank}((dg)_x) = r - q(m - r + 1) - 1 \quad \text{otherwise}.
\]

**Proof.** Let $(A, m_A) = \mathcal{O}_Y, f(x)$ and $(B, m_B) = \mathcal{O}_X, x$ and let $\rho = f^*: A \rightarrow B$ be induced by $f$. Let $\{s_1, \ldots, s_m\} \subseteq m_A$, $\{t_1, \ldots, t_r\} \subseteq m_B$ and the differential operators $D^{(1)}, \ldots, D^{(q)}$ be as in Lemma 3.8. The cokernel of the linear map $(df)_x^*: T(Y)_{f(x)}^* \rightarrow T(X)_x^*$ is just $m_B/(m_AB + m_B^2)$. Therefore, we must show that it is possible to find a pair $(i, I)$ such that $D^{(i)}\rho(u_i) \notin m_AB + m_B^2$ if and only if $(i) \ x \notin S^{(q+1)}(f)$ and $(ii) \ \text{char}(k)^{q+1}$. Let $\rho(u_r), \ldots, \rho(u_m)$ be expanded (modulo $m_B^{q+2}$) as power series in $t_1, \ldots, t_r$. Then $x \notin S^{(q+1)}(f)$ if and only if the monomial $t_r^{q+1}$ occurs in one of these expansions. By Lemma 3.7, $D^{(i)}(t_r^{q+1}) \equiv (q + 1)t_r \pmod{m_B^2}$, while $D^{(i)}t_r \in m_B^2$ if $i < q$ and $\nu > q + 1$. Therefore, $D^{(i)}(\rho(u_i)) \notin m_AB + m_B^2$ for some $(i, I)$ if and only if $(i)$ and $(ii)$ hold. [Observe that $m_B/(m_AB + m_B^2)$ is spanned by the residue class of $t_r$.]

12. Further consequences of smoothness. The main result of this section is Theorem 12.1. We will prove that Theorem B is a direct consequence of Theorem 12.1.

Consider a finite morphism $f: X^r \rightarrow Y^m$ of nonsingular varieties over $k (m \geq r)$ and distinct closed points $x_1, \ldots, x_d \in X$ such that $f(x_1) = \cdots = f(x_d) = y \in Y$. We fix nonnegative integers $q_1, \ldots, q_d$ and assume
\[
\begin{align*}
(1) \quad & x_j \in S^{(q)}(f) - S^{(q+1)}(f), \quad j = 1, \ldots, d; \\
(2) \quad & S^{(q)}(f) \text{ is smooth and of dimension } r - q(m - r + 1) \text{ near } x_j; \\
(3) \quad & S^{(1)}(f) \times Y \cdots \times Y S^{(d)}(f) \text{ is smooth and of dimension } dr - (d - 1)m - \sum_{j=1}^{d} q_j(m - r + 1) \text{ near } (x_1, \ldots, x_d).
\end{align*}
\]
We have homomorphisms $f^*_i: \hat{\mathcal{O}}_{Y,y} \rightarrow \hat{\mathcal{O}}_{X,x_j}, j = 1, \ldots, d$, induced by $f$. Let $f^*_i: \hat{\mathcal{O}}_{Y,y} \rightarrow \prod_{j=1}^d \hat{\mathcal{O}}_{X,x_j}$, where $f^*_i(\mathcal{O}) = (f^*_i(\mathcal{O}), \ldots, f^*_d(\mathcal{O}))$. We want to identify the various complete local rings with formal power series rings in a way that will give a canonical form for $f^*_i$.

In this paragraph, we fix a value of $j$ and set $q = q_j$. Let $(A, m_A) = k[[u_1, \ldots, u_m]]$, and let $(B_j, m_j)$ be a copy of $k[[t_1, \ldots, t_r]]$. By Lemma 4.5, there are isomorphisms $\phi: \hat{\mathcal{O}}_{Y,y} \cong A$ and $\psi_j: \hat{\mathcal{O}}_{X,x_j} \cong B_j, j = 1, \ldots, d$, such that $\psi_j = \psi_j \circ f^*_i \circ \phi^{-1}: A \rightarrow B_j$ satisfies

$$(4) \ g_j(u_1) = t_1, \ i = 1, \ldots, r - 1;$$

$$(5) \ g_j(u_r) = t_r^2 + 1 + \sum_{\nu=1}^{q-1} t_r^{(m-r)+\nu} t_r^\nu \quad \text{if char}(k) \nmid (q + 1),$$

$$(6) \ g_j(u_{r+i}) = \sum_{\nu=1}^{q} t_r^{(m-r)+\nu} t_r^\nu \quad \text{if char}(k) | (q + 1);$$

(If $q = 0$, we are to take $g_j(u_r) = t_r$ and $g_j(u_{r+i}) = 0, i = 1, \ldots, m - r$.) Let $I_j = \Delta_q (P_{B_j/A})$ [with $q = q_j$] and $N_j = \text{Ker}(m_A/m_A^2 \rightarrow m_j/(m_j^2 + I_j))$.

Since each $I_j$ is the “extension to $B$” of the ideal defining $S^f_{j+1}(f)$ near $x_j$, assumption (3) implies $\dim(N_1 + \cdots + N_d) = \dim(N_1) + \cdots + \dim(N_d)$.

(See Proposition 5.4.) For $j = 1, \ldots, d$, there are generating sets $\{t_{j_1}, \ldots, t_{j_r}\} \subset m_j$ and $\{s_{j_1}, \ldots, s_{j_m}\} \subset m_A$ such that (4), (5), and (6) hold with $u_i = s_{j_i}$, $t_r = t_r$ and $q = q_j$. By Proposition 4.10, there is a subset $E_j \subset \{s_{j_1}, \ldots, s_{j_m}\}$ such that $\{s_{j_i}, t_{j_i} \in E_j\}$ is a basis of $N_j \subset m_A/m_A^2$. Specifically

$$(7) \ \dim(N_1 + \cdots + N_d) = \dim(N_1) + \cdots + \dim(N_d).$$

By (7), $E^*_j = E_1 \cup \cdots \cup E_d$ is a subset of a minimal generating set $E = \{s_1, \ldots, s_m\}$ of $m_A$.

It is easily seen that $g_j(E - E_j) \subset m_j$ is a set of elements whose residues are linearly independent modulo $(m_j^2 + I_j)$. For each value of $j$, we can change $\{t_{j_1}, \ldots, t_{j_m}\}$ by replacing those $t_r = t_{j_r}$ not appearing on the right-hand sides of (5) and (6) with suitable elements of $g_j(E - E_j)$. If this is done, then (4), (5), and (6) will hold for all $j$ if $u_1, \ldots, u_m$ is taken to be a rearrangement of $s_1, \ldots, s_m$. The following theorem summarizes our conclusions.

**Theorem 12.1.** Let $f: X^r \rightarrow Y^m$ be as above, and assume that (1), (2), and (3) hold. Then there exist

(a) isomorphisms $\psi: \hat{\mathcal{O}}_{Y,y} \cong A = k[[s_1, \ldots, s_m]]$ and $\phi_j: \hat{\mathcal{O}}_{X,x_j} \cong B = k[[t_1, \ldots, t_r]], j = 1, \ldots, d,$
(b) permutations \( \sigma_1, \ldots, \sigma_d \) of \( \{1, \ldots, m\} \), and
(c) disjoint subsets \( E_1, \ldots, E_d \) of \( \{s_1, \ldots, s_m\} \), having the properties:

(i) The homomorphisms \( g_j = \phi_j \circ f^*_j \circ \psi^{-1} : A \to B, \ j = 1, \ldots, d, \) satisfy (4), (5), and (6) with \( g = g_j, q = q_j, \) and \( u_i = s_{q_j}(i), \ i = 1, \ldots, m; \)
(ii) the subsets \( E_j, j = 1, \ldots, d, \) satisfy (8) with \( s_{ij} = s_{q_j}(i). \)

Proof of Theorem B. By Theorem 12.1 there are isomorphisms \( \psi' : \)

\[ \begin{array}{ccc}
\delta_{Y', y'} & \rightarrow & A, \ 
\psi'' : \delta_{Y, y} & \rightarrow & A, \ 
\phi' : \delta_{X', x_j} & \rightarrow & B, \ 
\phi'' : \delta_{X, x_j} & \rightarrow & B \end{array} \]

(\( 1 \leq j \leq d \)) such that the homomorphisms \( g_j = \phi''_j \circ f^*_j \circ (\psi'')^{-1} \) and \( g'_j = \phi''_j \circ f^*_j \circ (\psi')^{-1} \) both satisfy (4), (5), and (6). Changing \( \psi' \) by an automorphism

which permutes the variables, we may assume that \( g_j = g'_j \) for all \( j. \) Let \( \psi = (\psi')^{-1} \circ \psi'' \) and \( \phi_j = (\phi''_j)^{-1} \circ \phi'' \). Then \( f^*_j \circ \psi'' \circ \psi^{-1}, j = 1, \ldots, d. \)

Q.E.D.

13. Application to generic projections. Let \( m \geq r + 1, \) and let \( \pi : V \to \mathbb{P}^m \) be a generic projection, starting from a projective embedding of \( V \) which satisfies assumption (\( * \)) of \( \S 9. \) (We will regard as generic those projections which satisfy the conclusions of Theorem A and Proposition 7.4 and have the property that \( \pi : V \to V' = \pi(V) \subset \mathbb{P}^m \) is finite and birational.)

Let \( y \in V' \) and \( \pi^{-1}(y) = \{x_1, \ldots, x_d\} \subset V. \) Then

\[ \delta_{V', y} \cong \delta_{\mathbb{P}^m, y} / \text{Ker}(\pi^*) \cong \text{Im}(\pi^*) \subset \prod_{j=1}^d \delta_{V, x_j}, \]

where \( \pi^* : \delta_{\mathbb{P}^m, y} \to \prod_{j=1}^d \delta_{V, x_j} \) is induced by \( \pi. \) Assume now that

(a) \( S_2(\pi) \cap \pi^{-1}(y) = \emptyset; \)
(b) \( x_j \in S_{(q_j)}(\pi) - S_{(q_j+1)}(\pi) \) for \( j = 1, \ldots, d; \)
(c) \( (x_1, \ldots, x_d) \) is a smooth point of \( \Sigma_d(\pi; q_1, \ldots, q_d). \)

[If \( \text{char}(k) \leq q_j + 1, \) \( j = 1, \ldots, d, \) then smoothness is automatic in (c) since \( \pi \)

is generic.] The hypotheses of Theorem 12.1 are thus satisfied with \( X = V \) and \( Y = \mathbb{P}^m. \) It follows that \( \text{Im}(\pi^*) \) can be explicitly described as being isomorphic to a certain subring of the direct product of \( d \) copies of \( k[[t_1, \ldots, t_d]]. \)

(13.1) The case \( m = r + 1. \) Instead of using the general approach outlined above, we can consider the ring homomorphisms \( \pi^*_j : \delta_{\mathbb{P}^m, y} \to \delta_{V, x_j} \) induced by \( \pi. \) Let \( P_j = \text{Ker}(\pi^*_j) \subset \delta_{\mathbb{P}^m, y} \cong k[[X_1, \ldots, X_{r+1}]]. \) Then \( P_1, \ldots, P_d \)

are principal prime ideals, and \( \text{Ker}(\pi^*) = P_1 \cap \cdots \cap P_d. \) Since \( \delta_{V', y} = \delta_{\mathbb{P}^m, y} / \text{Ker}(\pi^*), \) we conclude that \( \delta_{V', y} \cong k[[X_1, \ldots, X_{r+1}] / (f_1 \cdots f_d)], \)

where \( f_1, \ldots, f_d \) are (distinct) irreducible power series.

(13.2) The case \( m = r + 1, r \leq 5. \) If \( m = r + 1, \) then \( S_2(\pi) \)

has pure dimension \( r - 6. \) Hence \( S_2(\pi) = \emptyset \) when \( r \leq 5, \) so that (a) holds automatically. We will now list the various possibilities for \( \pi^{-1}(y) \) and the corresponding structure of \( \delta_{V', y}. \)

(0) \( \pi^{-1}(y) \) consists of \( d \) points of \( V - S_1, \) where \( d \leq r + 1. \) This can occur
along a subset of dimension $r - d + 1$, and $\delta_{V',y} \cong k[[X_1, \ldots, X_{r+1}]]/(X_1 \cdots X_d)$.

(1a) $\pi^{-1}(y)$ consists of one point of $S_1 - S_1^{(2)}$. This can occur when $r \geq 2$, along a subset of dimension $r - 2$. If $\text{char}(k) \neq 2$, then

$$\delta_{V',y} \cong k[[t_1, \ldots, t_{r-1}, t_1 t_r, t_r^2]] \subset k[[t_1, \ldots, t_r]]$$

$$\cong k[[X_1, \ldots, X_{r+1}]]/(X_1^2 - X_1^2 X_{r+1}).$$

If $\text{char}(k) = 2$ and $x$ is a smooth point of $S_1$, then

$$\delta_{V',y} \cong k[[t_1, \ldots, t_{r-1}, t_1 t_r, t_2 t_r + t_r^2]]$$

$$\cong k[[X_1, \ldots, X_{r+1}]]/(X_1^2 + X_1^2 X_{r+1} + X_1 X_2 X_r).$$

**Remark.** When $\text{char}(k) = 2$, smoothness of $S_1$ implies $r > 3$. Finitely many nonsmooth points of $S_1$ can, of course, occur. These have been studied only in the case $m = 2r - 1$. (See Theorem 3 of [11].)

(1b) $\pi^{-1}(y)$ consists of one point of $S_1^{(2)} - S_1^{(3)}$. This can occur when $r \geq 4$, along a subset of dimension $r - 4$. If $\text{char}(k) \neq 3$, then

$$\delta_{V',y} \cong k[[t_1, \ldots, t_{r-1}, t_1 t_r + t_2 t_r + t_r^2]]$$

where

$$\Phi_4 = X_1^2 X_3 X_r - X_1^3 X_{r+1} + 2X_2 X_3 X_r^2 - 3X_1 X_2 X_r X_{r+1},$$

$$\Phi_5 = X_2^2 X_3^2 X_r - X_1 X_2^2 X_3 X_{r+1} - X_3^2 X_{r+1}.$$

If $\text{char}(k) = 3$ and $x$ is a smooth point of $S_1^{(2)}$, then

$$\delta_{V',y} \cong k[[t_1, \ldots, t_{r-1}, t_1 t_r + t_2 t_r + t_3 t_r + t_4 t_r + t_r^2]].$$

(Observe that $r > 5$ when $\text{char}(k) = 3$ and $S_1^{(2)}$ is smooth.)

(2a) $\pi^{-1}(y) = \{x_1, x_2\}$, where $x_1 \in S_1 - S_1^{(2)}$ and $x_2 \notin \cal{V} - S_1$. This can occur when $r \geq 3$, along a set of dimension $r - 3$. If $\text{char}(k) \neq 2$, then

$$\delta_{V',y} \cong k[[X_1, \ldots, X_{r+1}]]/(X_1 (X_r^2 - X_2^2 X_{r+1})).$$

If $\text{char}(k) = 2$ and $(x_1, x_2)$ is a smooth point of $\Sigma_2(\pi; 1, 0)$, then

$$\delta_{V',y} \cong k[[X_1, \ldots, X_{r+1}]]/(X_1 (X_1^2 + X_2^2 X_{r+1} + X_2 X_3 X_r)).$$

**Proof.** We consider only the first isomorphism. The homomorphisms

$$\pi_j^*: \delta_{p_{m_j},y} \rightarrow \delta_{V',y}, j = 1, 2,$$ can be identified with homomorphisms $g_j$:

$$k[[X_1, \ldots, X_{r+1}]] \rightarrow k[[t_1, \ldots, t_r]], j = 1, 2,$$ where

$$g_1(X_i) = t_i, \quad i = 1, \ldots, r-1;$$

$$g_1(X_r) = t_2 t_r, \quad g_1(X_{r+1}) = t_r^2;$$
\[ g_2(X_i) = t_{i-1}, \quad i = 2, \ldots, r + 1; \quad g_2(X_1) = 0. \]

(In the notation of §12, \( E_1 = \{X_2, X_r, X_{r+1}\} \) and \( E_2 = \{X_1\} \); notice that these sets are disjoint.) Then \( \text{Ker}(g_1) = (X_2^2 - X_2^2 X_{r+1}) \) and \( \text{Ker}(g_2) = (X_1) \). Q.E.D.

(2b) \( \pi^{-1}(y) = \{x_1, x_2\} \), where \( x_1 \in S_1^{(2)} - S_1^{(3)} \) and \( x_2 \in V - S_1 \). This can occur when \( r \geq 5 \), along a set of dimension \( r - 5 \). If \( \text{char}(k) \neq 3 \), then
\[ \Omega_{\nu - 1} = k\left[\left[X_1, \ldots, X_{r+1}\right]\right]/(X_1(X_2^2 X_{r+1} + \ldots)). \]

(2c) \( \pi^{-1}(y) \) consists of two points of \( S_1 - S_1^{(2)} \). This can occur when \( r \geq 5 \), along a set of dimension \( r - 5 \). If \( \text{char}(k) \neq 2 \), then
\[ \Omega_{\nu - 1} = k\left[\left[X_1, \ldots, X_{r+1}\right]\right]/(X_1 X_2 X_3 (X_2 - X_2 X_{r+1})). \]

(3) \( \pi^{-1}(y) = \{x_1, x_2, x_3\} \), where \( x_1 \in S_1 - S_1^{(2)} \) and \( x_i \in V - S_1, i = 2, 3 \). This can occur when \( r \geq 4 \) along a set of dimension \( r - 4 \). If \( \text{char}(k) \neq 2 \), then
\[ \Omega_{\nu - 1} = k\left[\left[X_1, \ldots, X_{r+1}\right]\right]/(X_1 X_2 X_3 (X_2^2 - X_2 X_{r+1})). \]

(4) \( \pi^{-1}(y) = \{x_1, x_2, x_3, x_4\} \), where \( x_1 \in S_1 - S_1^{(2)} \) and \( x_j \in V - S_1, j = 2, 3, 4 \). This can occur when \( j \geq 5 \), along a set of dimension \( r - 5 \). If \( \text{char}(k) \neq 2 \), then
\[ \Omega_{\nu - 1} = k\left[\left[X_1, \ldots, X_{r+1}\right]\right]/(X_1 X_2 X_3 (X_2^2 - X_2 X_{r+1})). \]

Proposition 13.3. The above list of fibres \( \pi^{-1}(y) \) are the only types which occur when \( m = r + 1 \) and \( r \leq 5 \).

Proof. Since \( \Sigma_d(\pi) \) has pure dimension \( r - d + 1 \), we have \( \Sigma_d(\pi) = \emptyset \) when \( d > r + 2 \). Thus we must have \( d \leq r + 1 \) in (0). More generally, \( \Sigma_d(\pi; q_1, \ldots, q_d) \) has pure dimension \( r - d + 1 - 2(q_1 + \cdots + q_d) \). Therefore, \( \Sigma_d(\pi; q_1, \ldots, q_d) \) can be nonempty only when \( 2(q_1 + \cdots + q_d) \leq r - d + 1 \). The following table specifies which nontrivial fibres can occur.

<table>
<thead>
<tr>
<th>( r )</th>
<th>( d )</th>
<th>( r - d + 1 )</th>
<th>Values which can occur</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>( q = 1 )</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>( q = 1 )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td>( (q_1, q_2) = (1, 0) )</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>( q = 1, 2 )</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
<td>( (q_1, q_2) = (1, 0) )</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td></td>
<td>( (q_1, q_2, q_3) = (1, 0, 0) )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>( q = 1, 2 )</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td></td>
<td>( (q_1, q_2) = (1, 0), (2, 0) ) or ( (1, 1) )</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td></td>
<td>( (q_1, q_2, q_3) = (1, 0, 0) )</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td></td>
<td>( (q_1, q_2, q_3, q_4) = (1, 0, 0, 0) )</td>
</tr>
</tbody>
</table>
REFERENCES

15. S. Suzuki, Differentials of commutative rings, Queen’s Papers in Pure and Appl. Math., no. 29, Queen’s University, Kingston, Ont., 1971.

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