A DECOMPOSITION FOR CERTAIN
REAL SEMISIMPLE LIE GROUPS

BY

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ABSTRACT. For a class of real semisimple Lie groups, including those for which $G$ and $K$ have the same rank, Kostant introduced the decomposition $G = KN_0K$, where $N_0$ is a certain abelian subgroup of $N$, and conjectured that the Jacobian of the decomposition with respect to Haar measure, as well as the spherical functions, would be polynomial in the canonical coordinates of $N_0$. We compute here the Jacobian, which turns out to be polynomial precisely when the equality of ranks is satisfied. We also compute those spherical functions which restrict to polynomials on $N_0$.

1. Some preliminaries concerning root systems. Let $V$ be a Euclidean space with inner product $(\cdot, \cdot)$. Let $\Delta$ be a root system in $V$. For $\alpha \in \Delta$ we indicate by $s_\alpha$ the Weyl reflection with respect to $\alpha$

\[(s_\alpha(v) = v - 2(\alpha, v)\alpha/(\alpha, \alpha)).\]

The group generated by $\{s_\alpha | \alpha \in \Delta\}$ is called the Weyl group and will be designated by $W$.

1.1. Proposition. Let $s$ be an involutive element of $W$ with $\pm 1$-eigenspaces $V_\pm$, respectively. Then $s$ can be written in the form $s = s_{\gamma_1} \cdots s_{\gamma_n}$, where $\{\gamma_1, \ldots, \gamma_n\}$ is an orthogonal basis of $V_-$ and $\gamma_i \pm \gamma_j \notin \Delta$ for $i, j = 1, \ldots, n$.

Proof. Let $v$ be a relatively regular element of $V_+$. Now introduce any ordering in $V$, and let $\gamma_1$ be the largest element of $\Delta \cap V_-$ with respect to that ordering. Because of (1), $\gamma_1$ exists. Now, having chosen $\gamma_1, \ldots, \gamma_k$, if $s \neq s_{\gamma_1} \cdots s_{\gamma_k}$, let $\gamma_{k+1}$ be the largest element of $\Delta \cap V_-$ orthogonal to $\gamma_1, \ldots, \gamma_k$. $\gamma_{k+1}$ exists

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by (1), applied to the involutive element $s s_{\gamma_1}, \ldots, s s_{\gamma_k} \in W$. Since $V$, hence $V_-$, is finite-dimensional, the process terminates, and we have $s = s s_{\gamma_1} \cdots s s_{\gamma_n}$. $\gamma_1, \ldots, \gamma_n$ is an orthogonal basis of $V_-$. If $\gamma_i + \gamma_j \in \Delta$, $i < j$, then $\gamma_i + \gamma_j > \gamma_j$ and would have had to be chosen in preference to $\gamma_i$. Thus $\gamma_i + \gamma_j \notin \Delta$, and for $i \neq j$, $\gamma_i - \gamma_j = s s_{\gamma_j}(\gamma_i + \gamma_j) \notin \Delta$. $\gamma_i - \gamma_j = 0 \notin \Delta$.

A subset of $\Delta$ having the property that no sum or difference of two of its elements belongs to $\Delta$ and no sum of two of its elements is zero is called a set of strongly orthogonal roots. Two distinct strongly orthogonal roots are always orthogonal. For if $\alpha$ and $\beta$ are nonproportional elements of $\Delta$, then if $\langle \alpha, \beta \rangle < 0$, $\alpha + \beta \in \Delta$, and if $\langle \alpha, \beta \rangle > 0$, $\alpha - \beta \in \Delta$; whereas if $\alpha$ and $\beta$ are proportional then either $\alpha = \pm \beta$, a case we have excluded, or $\alpha = \pm 2\beta$ (or $\beta = \pm 2\alpha$), whence $\alpha + \alpha$ (resp., $\beta + \beta$) is an element of $\Delta$. We have show that the $-1$-eigenspace of an involutive element of $W$ has a basis of strongly orthogonal roots. (The most interesting case is, of course, the case $s = -1$, $V_- = V$, in case $-1 \in W$.)

Now let $\sigma$ be a linear involution of $V$ (not necessarily an element of $W$) with the property that $\sigma(\alpha) - \alpha \in \Delta$ for $\alpha \in \Delta$. Let $P = \{\frac{1}{2}(\alpha + \sigma(\alpha)) | \alpha \in \Delta\}$. Then $P$ is a root system [1, Proposition 2.1] in the $+1$-eigenspace of $\sigma$. The elements of $P$ will be called "restricted roots", while those of $\Delta$ will be called simply "roots". For $\alpha \in P$ the multiplicity of $\alpha$ (denoted $m_\alpha$) is defined as the number of roots $\beta$ satisfying $\frac{1}{2}(\beta + \sigma(\beta)) = \alpha$. The following lemma is obvious, since $\sigma$ acts as an involution on $\{\beta | \frac{1}{2}(\beta + \sigma(\beta)) = \alpha\}$ and fixes $\beta$ iff $\beta = \alpha$.

1.2. Lemma. For $\alpha \in P, \alpha \in \Delta$ iff $m_\alpha$ is odd [1, Proposition 2.2].

We now relate sets of strongly orthogonal restricted roots to sets of strongly orthogonal roots.

1.3. Proposition. Let $\gamma_1, \ldots, \gamma_n$ be restricted roots. Then $\{\gamma_1, \ldots, \gamma_n\}$ is a set of strongly orthogonal roots iff it is a set of strongly orthogonal restricted roots of odd multiplicities.

Proof. If the multiplicities of the $\gamma_i$ are odd, then the $\gamma_i$ are roots, and $\gamma_i \pm \gamma_j \in \Delta$ if $\frac{1}{2}[(\gamma_i \pm \gamma_j) + \sigma(\gamma_i \pm \gamma_j)] = \gamma_i \pm \gamma_j \in P$. Conversely, if $\{\gamma_1, \ldots, \gamma_n\}$ is a set of strongly orthogonal roots contained in $P$, then each $\alpha_i$ is a restricted root of odd multiplicity, and if $\gamma_i \pm \gamma_j \in P$, then $\gamma_i \pm \gamma_j \neq 0$ and $\gamma_i \pm \gamma_j + v \in \Delta$ for some $v$ with $\sigma(v) = v$. But then

$$0 < \frac{2\langle \gamma_i, \gamma_j \rangle + \gamma_j + v \gamma_i \pm \gamma_j + v \rangle}{\gamma_i \pm \gamma_j + v \gamma_i \pm \gamma_j + v} < 1,$$

an impossibility because $\Delta$ is a root system.

Now assume that $-1$ belongs to the Weyl group of $P$, so that the $+1$-eigenspace of $\sigma$ has a basis $\{\gamma_1, \ldots, \gamma_q\}$ of strongly orthogonal restricted roots.
1.4. Proposition. Every \( \alpha \in P \) is of the form \( \alpha = \frac{1}{2} \sum_{i=1}^{a} n_i \gamma_i \), where
the \( n_i \) are integers. If \( \alpha \) is proportional to \( \gamma_i \), then \( \alpha = \pm \gamma_i \) or \( \alpha = \pm \frac{1}{2} \gamma_i \).

Proof.

\[
\alpha = \frac{1}{2} \sum_{i=1}^{a} 2(\alpha, \gamma_i) \gamma_i,
\]

and \( 2(\alpha, \gamma_i/\gamma_i, \gamma_i) \) is an integer. If \( \alpha \) is proportional to \( \gamma_i \), then \( \alpha = \pm \gamma_i \), \( \alpha = \pm \frac{1}{2} \gamma_i \), or \( \alpha = \pm 2 \gamma_i \). But \( 2 \gamma_i = \gamma_i + \gamma_i \) and \( -2 \gamma_i = s_i (\gamma_i + \gamma_i) \) are not restricted roots.

Let \( H \) be any linear functional on the \( +1 \)-eigenspace of \( \sigma \). Let \( t_i = 2 \sinh \frac{1}{2} H(\gamma_i) \), and let

\[
J(t_1, \ldots, t_q) = \prod_{\alpha \in P} \frac{|\sinh H(\alpha)|^m}{\prod_{i=1}^{q} \cosh \frac{1}{2} H(\gamma_i)}.
\]

The following lemma will be useful in the study of the function \( J \).

1.5. Lemma. The function from \( R^r \) to \( R \) defined by

\[
f(h_1, \ldots, h_r) = (-1)^{2^{r-1}} \prod \sinh(\pm n_1 h_1 \pm \cdots \pm n_r h_r),
\]

where \( n_1, \ldots, n_r \) are fixed integers and the product is extended over all combinations of signs, is a polynomial in \( \sinh h_1, \ldots, \sinh h_r \). It is the square of a polynomial in all cases except the case \( r = 1, n_1 \) even.

Proof. \( f = g^2 \), where

\[
g(h_1, \ldots, h_r) = \prod \sinh(n_1 h_1 \pm \cdots \pm n_r h_r),
\]

where the product is extended over all combinations of signs. For \( r = 1 \), it is well known that \( g^2 \) is a polynomial in \( \sinh h \) and \( g \) is a polynomial in \( \sinh h_1 \) iff \( n_1 \) is odd. For \( r \geq 2 \),

\[
\sinh(a_1 + \cdots + a_r) = \sum f_i(a_1) \cdots f_r(a_r),
\]

\( f_i \in \{ \sinh, \cosh \} \), \( i \mid f_i = \sinh \) of odd cardinality.

Thus \( \Pi \sinh(a_1 \pm \cdots \pm a_r) \) is a sum of terms of the form

\[
\prod_{i=1}^{r} (\sinh a_i)^{k_i}(\cosh a_i)^{2^{r-1}-k_i}.
\]

Because of the evenness in each \( a_i, k_i \) is even for each \( i \) in each term (3). But then \( 2^{r-1} - k_i \) is even, and \( \Pi \sinh(a_1 \pm \cdots \pm a_r) \) is a polynomial in
\[
\sinh^2 a_1, \ldots, \sinh^2 a_r. \text{ Thus } g \text{ is a polynomial in } \sinh^2 n_1 h_1, \ldots, \sinh^2 n_r h_r \\
\text{and hence in } \sinh h_1, \ldots, \sinh h_r.
\]

1.6. Proposition. \( J \) is the absolute value of a polynomial iff \( m_{\gamma_i} \) is odd for \( i = 1, \ldots, q \).

Proof. We may partition \( P \) into orbits of the subgroup of its Weyl group generated by the \( s_{\gamma_i} \). The orbit of \( \alpha = \sum_{i=1}^q n_i \gamma_i \) has the form \( \{ \pm \sum_{i=1}^q n_i \gamma_i \} \) for all combinations of signs. By Lemma 1.5, if \( \{ i_1, \ldots, i_r \} \) is the subset of \( \{ 1, \ldots, q \} \) on which \( n_i \neq 0 \),

\[
\prod |\sinh(\pm n_{i_1} \gamma_{i_1} \pm \cdots \pm n_{i_r} \gamma_{i_r})|^{\frac{1}{2}}
\]

is the absolute value of a polynomial in the \( t_i \) except in the case \( r = 1, n_i \) even. In that case \( n_i = \pm 2, \alpha = \pm \gamma_{i_1} \), the orbit of \( \alpha \) has the form \( \{ \pm \gamma_{i_1} \} \), and we consider the factor of \( J \):

\[
\frac{|\sinh H(\gamma_{i_1})|^{\frac{1}{2}m_{\gamma_{i_1}}}}{\cosh \frac{1}{2}H(\gamma_{i_1})} \cdot \frac{|\sinh H(-\gamma_{i_1})|^{\frac{1}{2}m_{\gamma_{i_1}}}}{\cosh \frac{1}{2}H(\gamma_{i_1})} = \frac{|\sinh H(\gamma_{i_1})|^{m_{\gamma_{i_1}}}}{\cosh \frac{1}{2}H(\gamma_{i_1})} = \left| t_{i_1} \right|^{m_{\gamma_{i_1}} \left[ \frac{1}{2}(t_{i_1}^2 + 4) \right]} \frac{1}{2}(m_{\gamma_{i_1}} - 1)
\]

which is the absolute value of a polynomial iff \( m_{\gamma_{i_1}} \) is odd. Thus \( J \) is the absolute value of a polynomial iff all the \( m_{\gamma_i} \) are odd.

1.7. Corollary. \( J \) is the absolute value of a polynomial iff the +1-eigenspace of \( \sigma \) has a basis of strongly orthogonal roots.

We give in the table below the explicit formula for \( J \), for each restricted root system with \(-1\) in its Weyl group, in terms of the multiplicities. We shall use for convenience the following abbreviated notations.

\[
P(w, x, y, z) = (w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^4
\]
\[
+ w^4x^4(w^2 + 4)^2(x^2 + 4)^2 + y^4z^4(y^2 + 4)^2(z^2 + 4)^2
\]
\[
- 2(w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^2w^2x^2(w^2 + 4)(x^2 + 4)
\]
\[
- 2(w^2x^2 + 2w^2 + 2x^2 - y^2z^2 - 2y^2 - 2z^2)^2y^2z^2(y^2 + 4)(z^2 + 4)
\]
\[
- 2w^2x^2y^2z^2(w^2 + 4)(x^2 + 4)(y^2 + 4)(z^2 + 4).
\]

\[
Q(t, u, v) = [2t^2(t^2 + 4) - u^2v^2 - 2u^2 - 2v^2]^2 - u^2v^2(u^2 + 4)(v^2 + 4).
\]

We have listed in the table only irreducible types. Clearly, for a reducible root system, \( J \) is the product over the irreducible direct factors.
The degree $d_i$ in $t_i$ of the polynomial whose absolute value is $J$ (or, more generally, half the degree of $J^2$) can be read off in each case from Table 1. We prefer, however, to relate $d_i$ to the structures of the root systems $P$ and $\Delta$.

1.8. **Proposition.** $J^2$ is a polynomial in $t_1, \ldots, t_q$ whose degree in $t_i$ is

$$2d_i = -2 + 2 \sum_{\alpha \in P} \frac{m_\alpha |\gamma_i, \alpha|}{\langle \gamma_i, \gamma_i \rangle} = -2 + 2 \sum_{\beta \in \Delta} \frac{|\gamma_i, \beta|}{\langle \gamma_i, \gamma_i \rangle}.$$ 

**Proof.**

$$J^2(t_1, \ldots, t_q) = \frac{\prod_{\alpha \in P} |\sinh H(\alpha)|^{m_\alpha}}{\prod_{i=1}^q \cosh^2 \frac{1}{2}H(\gamma_i)}$$

where the numerator is a product of factors of the form

$$\prod \left| \sinh \left( \pm \frac{\langle \gamma_i, \alpha \rangle}{\langle \gamma_i, \gamma_i \rangle} H(\gamma_i) \right) \right|^{m_\alpha},$$

such a factor occurring for each orbit in $P$ of the subgroup of the Weyl group of $P$ generated by the $s_{\gamma_i}$. Such a factor is of degree $2^{r+1}m_\alpha \langle \gamma_i, \alpha \rangle / \langle \gamma_i, \gamma_i \rangle$ in $t_i = 2 \sinh \gamma_i$ and is counted $2r$ times in the product over all $\alpha \in P$. The denominator is of degree $2$ in $t_i$. The first equality of the proposition is now proven. To prove the second equality we note that, for $\beta \in \Delta$, $\langle \gamma_i, \beta \rangle = \langle \gamma_i, \frac{1}{2} (\beta + \sigma(\beta)) \rangle$.

Note that the formula for $d_i$ depends only on $\Delta$ and $\gamma_i$, not on $\sigma$.

Now assume that the $m_{\gamma_i}$ are odd, so that $J$ is the absolute value of a polynomial. Its degree in $t_i$ is

$$d_i = -1 + \sum_{\beta \in \Delta} \frac{|\gamma_i, \beta|}{\langle \gamma_i, \gamma_i \rangle}.$$ 

Assume further that $\Delta$ is irreducible and reduced. We can then express $d_i$ in terms of the coefficients of the highest root of $\Delta$ in terms of a simple system (with respect to some ordering).

1.9. **Proposition.** *If the highest root of $\Delta$ is expressed in terms of the simple system $\{\alpha_1, \ldots, \alpha_n\}$ as $\Sigma_{j=1}^n k_j \alpha_j$, then*

$$d_i = -1 + 2 \sum_{j=1}^n k_j \frac{\langle \alpha_j, \gamma_i \rangle}{\langle \gamma_i, \gamma_i \rangle}.$$ 

**Proof.** [2, proof of Proposition 31, Chapter VI, §1, 1.11].
<table>
<thead>
<tr>
<th>Type of $P$</th>
<th>Notation for $\gamma_i$</th>
<th>Notation for $t_i$</th>
<th>Weyl group orbits in $P$</th>
<th>Notation for $m\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$BC_q$</td>
<td>$\gamma_1, \ldots, \gamma_q$</td>
<td>$t_1, \ldots, t_q$</td>
<td>${\pm \gamma_i}$, ${\pm \gamma_i + \gamma_j \mid i &lt; j}$, ${\pm \gamma_i}$</td>
<td>$s$</td>
</tr>
<tr>
<td>(including $C_q$ for $s = 0$)</td>
<td></td>
<td></td>
<td></td>
<td>$m$ $l$</td>
</tr>
<tr>
<td>$B_{2r}$</td>
<td>$\delta_1, \ldots, \delta_r$</td>
<td>$e_1, \ldots, e_r$</td>
<td>$u_1, \ldots, u_r$</td>
<td>${\pm \delta_i \pm \gamma_j \mid i \neq j}$</td>
</tr>
<tr>
<td>(including $D_{2r}$ for $s = 0$)</td>
<td></td>
<td></td>
<td></td>
<td>$e_1$ $r$</td>
</tr>
<tr>
<td>$B_{2r+1}$</td>
<td>$\gamma$</td>
<td>$\delta_1, \ldots, \delta_r$</td>
<td>$e_1, \ldots, e_r$</td>
<td>$u_1, \ldots, u_r$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$\gamma$</td>
<td>$\delta_1, \delta_2, \delta_3$</td>
<td>$e_1 = e_4$, $e_2 = e_5$, $e_3$</td>
<td>$u_1, u_2, u_3$, $v_1 = v_4$, $v_2 = v_5$, $v_3$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$\delta_1 = \delta_5$, $\delta_2 = \delta_6$, $\delta_3 = \delta_7$, $\delta_4$, $e_1 = e_5$, $e_2 = e_6$, $e_3 = e_7$, $e_4$</td>
<td>$u_1 = u_5$, $u_2 = u_6$, $u_3 = u_7$, $u_4$</td>
<td>$v_1 = v_5$, $v_2 = v_6$, $v_3 = v_7$, $v_4$</td>
<td>${\delta_i \pm \epsilon_j \mid i \neq j}$, ${\delta_i \pm \delta_i \pm \epsilon_j \mid i \neq j}$, ${\delta_i \pm \epsilon_j \pm \delta_i \pm \epsilon_j \mid i \neq j}$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$\gamma_1, \ldots, \gamma_4$</td>
<td>$t_1, \ldots, t_4$</td>
<td>${\pm \gamma_i \pm \gamma_j \mid i &lt; j}$, ${\pm \gamma_i \pm \gamma_j \pm \gamma_k \mid i &lt; j, k}$</td>
<td>$s$ $l$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$\gamma$</td>
<td>$\delta$</td>
<td>$t$</td>
<td>$u$</td>
</tr>
</tbody>
</table>
TABLE 1 (Continued)

\[ \mathcal{X}(t_1, \ldots, t_q) \]

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\mathfrak{k})^{q+q(\ell-1)/2+q(q-1)m} \prod_{j=1}^{q} t_j^{((\ell_j^2 + 4)(\ell_j-1)/2) \prod_{1&lt;i&lt;j&lt;q} (t_i^2 - t_j^2)^m} \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{r(6q-3)+r(r-1)+s} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{r(6q+1)+r(r-1)+s} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{1951/2-7/2} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{1721-4} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{1412-12} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
<tr>
<td>((\mathfrak{k})^{91/2+92/2-1} \prod_{1&lt;i&lt;j&lt;r} P(u_i, v_i, u_j, v_j) \right</td>
<td></td>
</tr>
</tbody>
</table>
1.10. Corollary. If $h$ is the Coxeter number of $\Delta$ and $\gamma_i$ is a root of minimal length, then $d_i = 2h - 3$.

Proof. [2, loc. cit.].

2. Application to Lie algebras. Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commutative subspace of $\mathfrak{p}$. $\mathfrak{a}$ can be extended to a maximal commutative subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and such an $\mathfrak{h}$ has the form $\mathfrak{h} = \mathfrak{h}_+ + \mathfrak{a}$, where $\mathfrak{h}_+ \subset \mathfrak{h}$ [4, p. 221]. The non-zero eigenvalues of the adjoint representation of $\mathfrak{h}$ on the complexification $\mathfrak{g}C$ of $\mathfrak{g}$ form a reduced root system $\Delta$ in $i\mathfrak{h}_+^* + \mathfrak{a}^*$, with inner product $(.,.)$ dual to the killing form $B$ of $\mathfrak{g}$. (The stars denote real dual vector spaces, and $C\mathfrak{h}_+^* + C\mathfrak{a}^*$ is naturally identified with $C\mathfrak{h}_+^*$.) We let $\sigma$ be the linear involution of $i\mathfrak{h}_+^* + \mathfrak{a}^*$ which is $-1$ on $i\mathfrak{h}_+^*$ and $+1$ on $\mathfrak{a}^*$. Then the restricted root system $P$ defined by $\sigma$ is the set of nonzero eigenvalues of the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$, and the multiplicity $m_\alpha$ of $\alpha \in P$ is equal to the dimension of its eigenspace in $\mathfrak{g}$. (For details of the above, see e.g. [1].)

$B|_{l \times l}$ is negative definite, while $B|_{p \times p}$ is positive definite. Let $\theta$ be the symmetry; i.e., the linear involution of $\mathfrak{g}$ equal to $+1$ on $\mathfrak{l}$, to $-1$ on $\mathfrak{p}$. $\theta$ is an algebra automorphism of $\mathfrak{g}$. For $\alpha \in P$ let $H_\alpha \in \mathfrak{a}$ be the unique element such that $\alpha(H) = B(H, H_\alpha)$ for all $H \in \mathfrak{a}$.

Now let $\{\gamma_1, \ldots, \gamma_r\}$ be a set of strongly orthogonal restricted roots. Let $X_i$ be an element of the eigenspace of $\gamma_i$ in $\mathfrak{g}$ such that $-B(X_i, \theta X_i) = 2/\gamma_i(B(H_{\gamma_i}))$. Let $Y_i = -\theta X_i, Z_i = 2H_{\gamma_i}/\gamma_i(B(H_{\gamma_i})).$

2.1. Proposition. For the $X_i, Y_i, Z_i$ we have the following multiplication table:

\[ [X_i, Y_j] = [Y_i, X_j] = [Z_i, Z_j] = 0, \quad [Z_i, X_j] = 2\delta_{ij}X_j, \]
\[ [X_i, Y_j] = \delta_{ij}Z_j, \quad [Z_i, Y_j] = -2\delta_{ij}Y_j. \]

Furthermore, $X_i - Y_i \in \mathfrak{l}, X_i + Y_i \in \mathfrak{p}, Z_i \in \mathfrak{p}$.

Proof (as in [4, Chapter VI, Lemma 3.1]). $Z_i \in \mathfrak{a}$, which is commutative; for $i \neq j$, $[X_p, Y_p], [X_p, X_j],$ and $[Y_p, Y_j]$ belong to $(\pm \gamma_i \pm \gamma_j)$-eigenspaces of $ad_\mathfrak{a}(\mathfrak{a})$, which are all $\{0\}$, and $[Z_p, X_j] = \gamma_j(Z_i)X_j = 0 = -\gamma_j(Z_i)Y_j = [Z_p, Y_j]$. For $i = j$,

\[ [Z_p, X_j] = \gamma_j(Z_i)X_j = 2X_j, \quad [Z_p, Y_j] = -\gamma_j(Z_i)Y_j = -2Y_j, \]

and $[X_p, Y_j]$ belongs to the 0-eigenspace of $ad_\mathfrak{a}(\mathfrak{a})$. Also

$\theta([X_p, Y_j]) = [\theta X_p, \theta Y_j] = [-Y_p, X_j] = [Y_p, X_j] = [-X_p, Y_j]$. Therefore $[X_p, Y_j] \in \mathfrak{p}$, and so $[X_p, Y_j] \in \mathfrak{a}$, by maximality of $\mathfrak{a}$ in $\mathfrak{p}$. Now, for $H \in \mathfrak{a}$,

\[ B(H, [X_p, Y_j]) = B([H, X_p], Y_j) = \gamma_i(H)B(X_p, Y_p) = 2\gamma_i(H)/\gamma_i(H_\delta). \]
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Therefore \([X_\mu, Y_\nu] = Z_\tau\). \(\theta(X_\mu - Y_\nu) = -Y_\mu + X_\mu = X_\mu - Y_\mu\). Therefore \(X_\mu - Y_\mu \in \mathfrak{t}\). \(\theta(X_\mu + Y_\nu) = -Y_\mu - X_\mu = -(X_\mu + Y_\nu)\). Therefore \(X_\mu + Y_\nu \in \mathfrak{p}\). Finally, \(Z_\tau \in \mathfrak{a} \subset \mathfrak{p}\).

2.2. Corollary. \(X_1, \ldots, X_\mu, Y_1, \ldots, Y_\nu\), and \(Z_1, \ldots, Z_\tau\) generate (as a vector space) a subalgebra of \(\mathfrak{g}\) isomorphic to the Lie algebra direct sum of \(r\) copies of \(\mathfrak{sl}(2, \mathbb{R})\) and having a Cartan decomposition compatible with that of \(\mathfrak{g}\). Specifically, the Lie algebra generated by \(X_\mu, Y_\nu\), and \(Z_\tau\) is mapped isomorphically onto \(\mathfrak{sl}(2, \mathbb{R})\) by the linear mapping defined on the given basis by

\[
X_\mu \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_\nu \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z_\tau \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and the Cartan decomposition

\(\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{so}(2) + \{t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}\)

is compatible with the Cartan decomposition of \(\mathfrak{g}\).

Proof. Direct computation.

We now determine a necessary and sufficient condition on \(\mathfrak{g}\) for \(\mathfrak{a}^*\) to have a basis of strongly orthogonal roots. (If we require only a basis of strongly orthogonal restricted roots, a necessary and sufficient condition is simply that \(-1\) belong to the Weyl group of \(P\).)

2.3. Proposition. \(\mathfrak{a}^*\) has a basis of strongly orthogonal roots if and only if \(\mathfrak{t}\) contains a maximal commutative subalgebra of \(\mathfrak{g}\).

Proof. Let \(\{\gamma_1, \ldots, \gamma_\alpha\}\) be a basis of strongly orthogonal roots for \(\mathfrak{a}^*\), and let \(X_\mu, Y_\nu, \mathfrak{h}_-,\) and \(\mathfrak{h}_+\) be as above.

Claim. A maximal commutative subalgebra of \(\mathfrak{g}\) contained in \(\mathfrak{t}\) is given by

\[\exp \left( \text{ad}_{\mathfrak{g}^*} \left( \frac{\pi}{4} \sum_{i=1}^{\alpha} (X_i + Y_i) \right) \right) (\mathfrak{h}_+ + i\mathfrak{a}).\]

Proof of claim. \(X_\mu\) and \(Y_\nu\) commute with \(\mathfrak{h}_+\) because the \(\gamma_\iota\) vanish on \(\mathfrak{h}_+\). For \(H = \sum_{i=1}^\alpha h_i Z_i\), a typical element of \(\mathfrak{a}\),

\[
\exp \left( \text{ad}_{\mathfrak{g}^*} \left( \frac{\pi}{4} \sum_{i=1}^{\alpha} (X_i + Y_i) \right) \right) (H)
\]

\[= \sum_{i=1}^{\alpha} \left[ \sum_{k=0}^\infty \frac{(\pi/2)^{2k}}{(2k)!} (X_i + Y_i) - \sum_{k=0}^\infty \frac{(\pi/2)^{2k+1}}{(2k + 1)!} (X_i - Y_i) \right]
\]

\[= \sum_{i=1}^{\alpha} \left[ \left( \cos \frac{\pi}{2} \right) (X_i + Y_i) - i \left( \sin \frac{\pi}{2} \right) (X_i - Y_i) \right] = -i \sum_{i=1}^{\alpha} (X_i - Y_i) \in \mathfrak{t}.\]
The commutativity and maximality follow from the same properties for \( \mathfrak{h} \).

To prove the converse (along the lines of [4, Chapter VIII, Proposition 7.4]), we assume that \( \widetilde{\mathfrak{h}} \subset \mathfrak{f} \) is a maximal commutative subalgebra of \( \mathfrak{g} \). We let \( \widetilde{\Delta} \) be the root system of nonzero eigenvalues of \( \text{ad}_{\mathfrak{C}}(C \widetilde{\mathfrak{h}}) \). \( \widetilde{\Delta} = \widetilde{\Delta}_c \cup \widetilde{\Delta}_n \), where the eigenspaces of \( \widetilde{\Delta}_c \) are contained in \( C \mathfrak{f} \), while those of \( \widetilde{\Delta}_n \) are contained in \( C \mathfrak{p} \).

Introduce an ordering in the span of \( \widetilde{\Delta}_n \), and choose \( \widetilde{X}_\beta \) in the \( \beta \)-eigenspace for each \( \beta \). Let \( \widetilde{Y}_\beta = \sigma \widetilde{X}_\beta \), where \( \sigma \) is the linear involution of \( \mathfrak{g}_C \) which is +1 on \( \mathfrak{g} \) and -1 on \( \mathfrak{p} \). Since \( \widetilde{\Delta} \subset \mathfrak{i} \), \( \widetilde{Y}_\beta \) belongs to the \( -\beta \)-eigenspace. Clearly \( \widetilde{X}_\beta + \widetilde{Y}_\beta \in \mathfrak{g} \). Since \( 0 \notin [\widetilde{X}_\beta, \widetilde{Y}_\beta] \in C \mathfrak{h} \subset C \mathfrak{f} \), \( \widetilde{Y}_\beta \notin C \mathfrak{f} \). Therefore \( \widetilde{Y}_\beta \in C \mathfrak{p} \); \( \widetilde{X}_\beta + \widetilde{Y}_\beta \in C \mathfrak{p} \cap \mathfrak{g} = \mathfrak{p} \). In fact \( \mathfrak{p} = \sum_{\beta \in \mathfrak{X}_n} R(\widetilde{X}_\beta + \widetilde{Y}_\beta) \).

Now let \( \gamma_1 \) be the highest root in \( \widetilde{\Delta}_n \), and, given \( \gamma_1, \ldots, \gamma_k \), let \( \gamma_{k+1} \) be the highest root in \( \widetilde{\Delta}_n \) such that \( \{\gamma_1, \ldots, \gamma_{k+1}\} \) is a strongly orthogonal set (if such a root exists; if not, the process terminates). Let \( \{\gamma_1, \ldots, \gamma_q\} \) be the full sequence of strongly orthogonal roots obtained in this manner. Let \( \tilde{a} = \sum_{\gamma \in \mathfrak{X}_n} R(\tilde{X}_{\gamma_1} + \tilde{Y}_{\gamma_1}) \). Clearly \( \tilde{a} \) is commutative. To show that \( \tilde{a} \) is maximal commutative in \( \mathfrak{p} \), consider any element \( X \) of \( \mathfrak{p} \).

\[
X = \sum_{\beta \in \widetilde{\Delta}_n} t_\beta (\tilde{X}_\beta + \tilde{Y}_\beta),
\]

and assume that \( X \) commutes with \( \tilde{a} \) but \( X \notin \tilde{a} \). Let \( r \) be the smallest index such that \( t_\beta \neq 0 \) for some \( \beta \) with \( \{\gamma_1, \ldots, \gamma_r, \beta\} \) not strongly orthogonal. Then in \( [X, \tilde{X}_{\gamma_r} + \tilde{Y}_{\gamma_r}] = 0 \) we must have

\[
t_\beta [\tilde{X}_\beta, \tilde{X}_{\gamma_r}] = t_{2\gamma_r - \beta} [\tilde{X}_{2\gamma_r - \beta}, \tilde{Y}_{\gamma_r}] \neq 0.
\]

But then either \( \gamma_r < \beta \in \widetilde{\Delta}_n \) or \( \gamma_r < 2\gamma_r - \beta \in \widetilde{\Delta}_n \). Thus either \( \{\gamma_1, \ldots, \gamma_{r-1}, \beta\} \) or \( \{\gamma_1, \ldots, \gamma_{r-1}, 2\gamma_r - \beta\} \) is a set of roots which is not strongly orthogonal. But we assumed that \( r \) was the minimal index for which such a set could be constructed.

Now we can show by a computation similar to (4) that

\[
\exp\left( \text{ad}_{\mathfrak{g}}\left( \sum_{i=1}^{q} B(\tilde{X}_{\gamma_i}, \tilde{Y}_{\gamma_i}) \right) \right)(\tilde{a}) \subset \mathfrak{i}\tilde{h}.
\]

We can therefore view the \( \gamma_i \) as roots of the conjugate of \( \mathfrak{i}\tilde{h} \),

\[
\exp\left( \text{ad}_{\mathfrak{g}}\left( -\sum_{i=1}^{q} B(\tilde{X}_{\gamma_i}, \tilde{Y}_{\gamma_i}) \right) \right)(\mathfrak{i}\tilde{h}),
\]

which is of the form \( \tilde{a} + \mathfrak{i}\tilde{h}_+ \), \( \tilde{h}_+ \subset \mathfrak{p} \). The \( \gamma_i \) vanish on \( \mathfrak{h}_+ \) and can therefore be regarded as forming a basis of \( \tilde{a}^* \). Any given maximal commutative subspace \( a \) of \( \mathfrak{p} \) is \( \text{Int}(\mathfrak{f}) \)-conjugate to \( \tilde{a} \).
3. The Horn-Thompson-Kostant decomposition. Let $g$, $f$, $p$, $a$, and $a^*$ be as in §2, and assume that $a^*$ has a basis of strongly orthogonal restricted roots (not necessarily roots). Let $X_i$ and $Z_i$ be as in Proposition 2.1, and let $n_0 = \Sigma_{i=1}^q X_i$. Then $n_0$ is a commutative subalgebra of $g$.

Now let $G$ be any analytic group having Lie algebra $g$. Let $K$, $A$, and $N_0$ be the analytic subgroups of $G$ corresponding to $f$, $a$, and $n_0$, respectively.

3.1. Proposition. The element $\exp 2 \Sigma_{i=1}^q h_i Z_i$ of $A$ belongs to the same coset in $K \backslash G/K$ as the element $\exp 2 \Sigma_{i=1}^q \sinh h_i X_i$ of $N_0$.

Proof. Because of Corollary 2.2, it is enough to prove the proposition for $g = \mathfrak{sl}(2, \mathbb{R})$. Because the center of $G$ is contained in $K$, it is enough to prove the proposition for one analytic group having Lie algebra $\mathfrak{sl}(2, \mathbb{R})$; say, for $G = SL(2, \mathbb{R})$.

In $\mathfrak{sl}(2, \mathbb{R})$, since

\[
\left( \begin{array}{cc}
    e^h & 0 \\
    0 & e^{-h}
\end{array} \right) \left( \begin{array}{cc}
    e^h & 0 \\
    0 & e^{-h}
\end{array} \right) = \left( \begin{array}{cc}
    e^{2h} & 0 \\
    0 & e^{-2h}
\end{array} \right)
\]

is similar to

\[
\left( \begin{array}{cc}
    1 & 2 \sinh h \\
    0 & 1
\end{array} \right) \left( \begin{array}{cc}
    1 & 0 \\
    2 \sinh h & 1
\end{array} \right) = \left( \begin{array}{cc}
    1 + 2 \sinh^2 h & 2 \sinh h \\
    2 \sinh h & 1
\end{array} \right).
\]

$\exp hZ$ and $\exp 2 \sinh X$ belong to the same double coset of $K = SO(2)$.

3.2. Corollary. We have the decomposition (announced in [8])

$$G = KN_0K.$$  

Proof. The corollary follows from Proposition 3.1 and the well-known decomposition of Cartan $G = KAK$ [8, (4.2.8)].

The decomposition (5) was called by Barker the Thompson-Kostant decomposition. Kostant later added the name Horn upon discovering that Thompson’s result for $SL(2, \mathbb{R})$, later generalized by Kostant, had previously been discovered by Horn.

3.3. Corollary. The Haar integral on $G$ is given (up to normalization by a constant factor) by the formula

$$
\int_G f(g) \, dg = \int_K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_K f \left( \prod_{i=1}^q t_i X_i k_2 \right) \cdot J(t_1, \ldots, t_q) \, dk_2 \, dt_1 \cdots dt_q \, dk_2,
$$
where $dk_1 = dk_2$ is the Haar measure on $K$ and $J$ is defined by (2) (and given, for $G$ simple, by Table 1).

**Proof.** The corollary follows from Proposition 3.1 and the well-known formula [4, Chapter X, Proposition 1.17]

$$\int_G f(g) \, dg = \int_K \int_a \left( \int_K f(k_1 \exp H k_2) \prod_{\alpha \in P} |\sinh \alpha(H)|^{\frac{1}{2}m} \, dk_1 \, dH \, dk_2, $$

where $dH$ is Lebesgue measure on the Euclidean space $a$.

Kostant conjectured that the Jacobian appearing in (6) would be a polynomial. We see from Corollary 1.7 and Proposition 2.3 that Kostant's conjecture is true precisely when $\mathfrak{f}$ contains a maximal commutative subalgebra of $\mathfrak{g}$, the case for which Kostant stated in [8,(5.1.1)] the decomposition (5).

We conclude this section by computing the radial part on $N_0$ of the Casimir operator $\Omega$ of $G$, which will be useful in the next section.

3.4. **Corollary.** If $f$ is any smooth $K$-bi-invariant function on $G$, then

$$\Omega f \left( \exp \sum_{i=1}^q t_i X_i \right) = \left[ \sum_{i=1}^q \frac{\gamma_i \gamma_i}{4} \left( t_i^2 + 4 \right) \frac{\partial^2}{\partial t_i^2} \right.
\left. + \left[ 2t_i + (t_i^2 + 4) \frac{\partial \log J}{\partial t_i} \right] \exp \sum_{i=1}^q t_i X_i \right] \frac{\partial}{\partial t_i} \left( f \left( \exp \sum_{i=1}^q t_i X_i \right) \right),$$

wherever $J(t_1, \ldots, t_q) \neq 0$.

**Proof.** The corollary follows from Proposition 3.1 and Helgason's formula for the radial part of $\Omega$ on $A$ (as in [5, Theorem 3.3]); namely, for $H \in a$,

$$\Omega f(\exp H) = D(\exp H)^{-\frac{1}{2}} \Delta_a \left[ D(\exp H)^{\frac{1}{2}} f(\exp H) \right]$$

(7)

$$- D(\exp H)^{-\frac{1}{2}} \Delta_a \left[ D(\exp H)^{\frac{1}{2}} f(\exp H) \right],$$

where $D(\exp H) = \prod_{\alpha \in P} |\sinh \alpha(H)|^{\frac{1}{2}m} \alpha$ and $\Delta_a$ is the Laplacian of the Euclidean space $a$. Formula (7) is valid wherever $D(\exp H) \neq 0$.

4. **Spherical polynomials.** Assume that $G$ has finite center, so that $K$ is compact.

Kostant conjectured in [8, Remark 5.1.1], that the $(G, K)$-spherical functions, which, due to Corollary 3.2, are determined by their values on $N_0$, might have a polynomial nature there. In case $P$ is of type $C_\ell$ or $B_{2\ell}$, we do indeed find a sequence of spherical functions whose restrictions to $N_0$ are polynomials in the canonical coordinates $t_1, \ldots, t_q$. These polynomials can all be expressed in
terms of the hypergeometric function $F$. For other simple types we find that the only spherical polynomial is the constant 1.

4.1. Lemma. If $f$ is a $K$-bi-invariant eigenfunction of $\Omega$ whose restriction to $N_0$ is a polynomial in the canonical coordinates $t_1, \ldots, t_q$, and if $f|_{N_0}$ has an extremal term of the form $a_1 t_1^{n_1} \cdots t_q^{n_q}$, then

$$\Omega f = \sum_{i=1}^{q} [n_i^2 + \frac{1}{2}(d_i + 1)n_i] \langle \gamma_i, \gamma_i \rangle f,$$

where $d_i$ is as in Proposition 1.8.

Proof. Apply Corollary 3.4 and equate coefficients of $t_1^{n_1} \cdots t_q^{n_q}$.

We now introduce in $a^*$ the lexicographic ordering with respect to the ordered basis $(\gamma_1, \ldots, \gamma_q)$. With respect to that ordering we let $G = KAN$ be the Iwasawa decomposition; $a_+$ and $a_+^*$ be the positive Weyl chambers in $a$ and $a^*$, respectively; and $\rho$ be the half-sum of the positive restricted roots with multiplicities.

4.2. Lemma.

$$d_i + 1 \geq \langle 4\rho, \gamma_i \rangle/\langle \gamma_i, \gamma_i \rangle.$$

Equality holds for $i = 1$. If $G$ is simple, equality holds only for $i = 1$.

Proof. The inequality, as well as the equality for $i = 1$, follows from Proposition 1.8. If $G$ is simple, then for $i \in \{1, \ldots, q\}$ there exists a finite sequence $(\delta_1, \ldots, \delta_r)$ from $\Delta$ such that $\delta_1 = \gamma_1, \delta_r = \gamma_i$, and $\langle \delta_1, \delta_{j+1} \rangle \neq 0$. Now let $(\delta_1, \ldots, \delta_r)$ be such a sequence of minimal length. $\langle \delta_1, \delta_{j+1} \rangle = 0$; otherwise we could obtain a shorter sequence by omitting $\delta_{j+1}$. But now there exists a root of the form $\delta_{j+1} \pm \delta_{j+2}$, and $\langle \delta_1, \delta_{j+1} \pm \delta_{j+2} \rangle = \langle \delta_1, \delta_{j+1} \rangle \neq 0$, $\langle \delta_{j+1} \pm \delta_{j+2}, \delta_{j+3} \rangle = \pm \langle \delta_{j+2}, \delta_{j+3} \rangle \neq 0$; so we may obtain a shorter sequence by substituting $\delta_{j+1} \pm \delta_{j+2}$ for $\delta_{j+1}$ and $\delta_{j+2}$ whenever $2 < j + 1 < j + 2 \leq r - 1$. Therefore $r = 3$, and $\delta_2$ is not orthogonal to either $\gamma_1$ or $\gamma_i$. By applying Weyl reflections with respect to $\gamma_1$ and $\gamma_i$, we may assume that $\langle \gamma_1, \delta_2 \rangle > 0 > \langle \gamma_i, \delta_2 \rangle$. Then $\delta_2 > 0$, and

$$\langle \gamma_i, \rho \rangle = \frac{1}{2} \sum_{\beta > 0} \langle \gamma_i, \beta \rangle < \frac{1}{2} \sum_{\beta < 0} \langle \gamma_i, \beta \rangle = \frac{1}{4}(d_i + 1)\langle \gamma_i, \gamma_i \rangle.$$

4.3. Corollary. If $f$ is a $K$-bi-invariant function whose restriction to $N_0$ is a polynomial in $t_1, \ldots, t_q$, and if $\Omega f = cf$, then

$$c = \sum_{i=1}^{q} \left[ n_i^2 + \frac{1}{2}(d_i + 1)n_i \right] \langle \gamma_i, \gamma_i \rangle \geq \left\langle \rho + \sum_{i=1}^{q} n_i \gamma_i, \rho + \sum_{i=1}^{q} n_i \gamma_i \right\rangle - \langle \rho, \rho \rangle,$$
where \( f|_{N_0} \) has an extremal term of the form 
\[ a_1 t_1^{2n_1} \cdots t_q^{2n_q} \]
as in Lemma 4.1. Equality holds if \( n_i = 0 \) for \( i \geq 2 \), and for \( G \) simple only in that case.

**Proof.** The corollary follows from Lemmas 4.1 and 4.2.

**4.4. Lemma.** If \( f \) is a \( K \)-bi-invariant function on \( G \) such that \( f|_{N_0} \) is a polynomial in \( t_1, \ldots, t_q \) and \( e^{-\mu(H)}f(\exp H) \) is bounded away from 0 and \( \infty \) for \( H \) in the closure of \( a_+ \), where \( \mu \) is some element in the closure of \( a_+^* \); then \( \mu = 2\Sigma_{i=1}^q n_i \gamma_i \) for some nonnegative integers \( n_1, \ldots, n_q \), and

\[
\begin{align*}
&\text{\( f(\exp \sum_{i=1}^q t_i X_i) = a_{n_1} \cdots a_{n_q} t_1^{2n_1} \cdots t_q^{2n_q} + \sum_{m_1, \ldots, m_q} a_{m_1, \ldots, m_q} t_1^{2m_1} \cdots t_q^{2m_q} \)} & \\
&\text{\( \left\{ m_1, \ldots, m_q \right\} \left( \sum_{i=1}^q (n_i - m_i) \gamma_i, a_+ \right) \geq 0, \)} & \\
&\text{\( (m_1, \ldots, m_q) \neq (n_1, \ldots, n_q) \)} & 
\end{align*}
\]

for some coefficients \( a_{m_1, \ldots, m_q} \).

**Proof.** Since \( f \) is invariant under the Weyl reflection with respect to each \( \gamma_i \), \( f(\exp \sum_{i=1}^q t_i X_i) \) is even in each \( t_i \). The degree follows from Proposition 3.1.

We now apply Corollary 4.3 and Lemma 4.4 to the problem of determining which spherical functions have polynomial restrictions to \( N_0 \). The spherical functions on \( G \) are indexed by \( a_+^* \) (modulo the Weyl group of \( P \)) and given by the formula

\[
\phi_{\lambda}(g) = \int_K e^{i(\lambda - \rho)(H(g))} \, dk,
\]

for \( \lambda \in a_+^* \), where \( H(g) \) is the element of \( a_+ \) such that \( g \in K \exp(H(g))N \). If \( \lambda \in a_+^* + i a_+^* \) we can transform the integral formula for \( \phi_{\lambda}(a) \) (for \( a \in \exp a_+ \)) to an integral over \( \tilde{N} \), the analytic subgroup of \( G \) corresponding to the sum of the negative restricted root spaces. We have, as in [6, Lemma 2.3],

\[
\phi_{\lambda}(a) = \exp [(\lambda - \rho)(\log a)] \int_{\tilde{N}} \exp [(\lambda - \rho)(H(\tilde{a}^{-1}))] \exp [(-\lambda - \rho)(H(\tilde{n}))] \, d\tilde{n},
\]

where \( d\tilde{n} \) is the Haar measure on \( \tilde{N} \) such that \( \int_{\tilde{N}} \exp [-2\rho(H(\tilde{n}))] \, d\tilde{n} = 1 \). We see that for \( e^{-\mu(\log a)}\phi_{\lambda}(a) \) to be bounded away from 0 and \( \infty \) on the closure of \( a_+ \), we must have \( \mu \leq i \lambda - \rho + i a_+^* \). In case \( i \lambda - \rho \) is in the closure of \( a_+^* \), we have indeed

\[
0 < c(\lambda) = \int_{\tilde{N}} \exp [(-i \lambda - \rho)(H(\tilde{n}))] \, d\tilde{n} \leq \exp [(-i \lambda + \rho)(\log a)] \phi_{\lambda}(a)
\]

\[
\leq \int_{\tilde{N}} \exp [-2\rho(H(\tilde{n}))] \, d\tilde{n} = 1.
\]
Furthermore, $\Omega \phi_\lambda = (-\lambda, \lambda - \langle \rho, \rho \rangle) \phi_\lambda$.

Now assume that $\phi_{\lambda,|_{V_0}}$ is a polynomial in $t_1, \ldots, t_q$. By Lemma 4.4,

$$\phi_{\lambda} \left( \exp \sum_{i=1}^{q} t_i X_i \right) = a_{n_1 \cdots n_q} t_1^{2n_1} \cdots t_q^{2n_q} + \text{"lower order" terms},$$

where $\lambda - \rho = \sum_{i=1}^{q} n_i \gamma_i$ is in the closure of $a_\lambda^+$. (We may have $\lambda - \rho = 0$, $\phi_{\lambda} = \phi_{-\rho} \equiv 1$.) Furthermore,

$$-\langle \lambda, \lambda - \langle \rho, \rho \rangle \rangle = \left( \rho + \sum_{i=1}^{q} n_i \gamma_i, \rho + \sum_{i=1}^{q} n_i \gamma_i \right) - \langle \rho, \rho \rangle$$

$$= \sum_{i=1}^{q} \left[ n_i^2 + \frac{1}{2}(d_i + 1)n_i \right] \langle \gamma_i, \gamma_i \rangle.$$

By Corollary 4.3 we must have, for $G$ simple, $n_i = 0$ for $i \gg 2$.

Now by considering the asymptotic behavior at $\infty$ of $\phi_{\lambda}$ in all Weyl chambers, we conclude that $\phi_{\lambda,|_{V_0}}$ must have an extremal term of the form $a_\beta\prod_{i=1}^{r} \gamma_i$ whenever $\beta = \frac{1}{2}\sum_{i=1}^{r} \gamma_i$ belongs to the Weyl group orbit of $\gamma_1$. But if $P$ is of a simple type other than $C_q$ or $BC_q$, then we may set $\beta = \frac{1}{2} \gamma_i + \frac{1}{2} \gamma_j + \frac{1}{2} \gamma_k + \frac{1}{2} \gamma_l$ for some choice of $i, j, k, l$. (We have assumed for convenience that $\gamma_1$ is of maximal length.) The number of the indices $i, j, k, l$ equal to $r \in \{1, \ldots, q\}$ is either 0 or $\langle \gamma_1, \gamma_1 \rangle / \langle \gamma_r, \gamma_r \rangle$. Then we must have, by comparison of eigenvalues of $\Omega$, that

$$\left[ n_1^2 + \frac{1}{2}(d_1 + 1)n_1 \right] \langle \gamma_1, \gamma_1 \rangle$$

$$= n_1^2 + \frac{1}{2}n_1 \left( 4 + d_i \langle \gamma_1, \gamma_1 \rangle + d_j \langle \gamma_2, \gamma_2 \rangle + d_k \langle \gamma_3, \gamma_3 \rangle + d_l \langle \gamma_4, \gamma_4 \rangle \right)$$

$$\geq \left[ n_1^2 + (\frac{1}{2}d_1 + 1)n_1 \right] \langle \gamma_1, \gamma_1 \rangle,$$

whence $n_1 = 0$. (We have used that $d_1 \leq \min\{d_i, d_j, d_k, d_l\}$ and that $i, j, k, l$ are not all equal.) We have proven the following

4.5. Theorem. If $P$ is of a simple type other than $C_q$ or $BC_q$, then the only spherical function on $G$ restricting on $N_0$ to a polynomial in $t_1, \ldots, t_q$ is $\phi_{-\rho} \equiv 1$.

In case $P$ is of type $C_q$ or $BC_q$, we find the polynomial solution

$$p_n \left( \exp \sum_{i=1}^{q} t_i X_i \right)$$

$$= \frac{-2m(q - 1)^2}{q(s + l + 1) + 2(q - 1)m + 1} + \frac{s + l + 2(q - 1)m + 1}{q(s + l + 1) + 2(q - 1)m}$$

$$\cdot \sum_{i=1}^{q} F(-n, \frac{1}{2}s + l + (q - 1)m + \frac{1}{2}; \frac{1}{2}s + \frac{1}{2}l + (q - 1)m + \frac{1}{2}; -\lambda_i^2)$$

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to the differential equation on $N_0$ for a $K$-bi-invariant eigenfunction of $\Omega$ with eigenvalue $[n^2 + \frac{1}{2}(d_1 + 1)l] (\gamma_1, \gamma_1)$. (Here $s$, $m$, and $l$ are as in Table 1.) Since $p_n$ is even in each $t_i$ and symmetric in the $t_i$, it extends to a $K$-bi-invariant function on $G$. Now I claim that $p_n(n_0) = \phi_{-i(n_1 + 1)}(n_0)$ for $n_0 \in N_0$. For $p_n$ is a $K$-bi-invariant function satisfying

$$\frac{\Omega p_n}{p_n} = \frac{\Omega \phi_{-i(n_1 + 1)}}{\phi_{-i(n_1 + 1)}} \quad \text{and} \quad 0 \leq p_n \leq \phi_{-i(n_1 + 1)}.$$

Since $\phi_{-i(n_1 + 1)}$ is a minimal $K$-bi-invariant eigenfunction of $\Omega$ (see [7]), $p_n = k\phi_{-i(n_1 + 1)}$ for some $k \in [0, 1]$. But $p_n(e) = \phi_{-i(n_1 + 1)}(e) = 1$. Therefore $k = 1$. We have proven the following theorem.

4.6. **Theorem.** If $P$ is of type $C_q$ or $BC_q$, then the spherical functions on $G$ restricting on $N_0$ to polynomials in $t_1, \ldots, t_q$ are precisely

$$\phi_{-i(n_1 + 1)} \left( \exp \sum_{i=1}^q t_i X_i \right)$$

$$= \frac{-2m(q-1)^2}{qs + l + 1 + 2(q-1)m + 1} + \frac{s + l + 2(q-1)m + 1}{qs + l + 1 + 2(q-1)m} \cdot \sum_{i=1}^q F(-n, \frac{1}{2}s + l + (q-1)n + n; \frac{1}{2}s + \frac{1}{2}l + (q-1)m + \frac{1}{2}; -\frac{1}{4}t_i^2).$$

The formula of the theorem is valid (by the same proof) for all $n \geq 0$ and by analytic continuation for all $n \in \mathbb{C}$, although $\phi_{-i(n_1 + 1)}$ is polynomial in $t_1, \ldots, t_q$ only for $n$ a nonnegative integer. Our result includes in particular Harish-Chandra's formula for all spherical functions on a rank-one symmetric space [3].

**REFERENCES**

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