THE PREDOUAL THEOREM TO THE JACOBSON-BOURBAKI THEOREM

BY

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ABSTRACT. Suppose \( R \xrightarrow{\varphi} S \) is a map of rings. \( S \) need not be an \( R \) algebra since \( R \) may not be commutative. Even if \( R \) is commutative it may not have central image in \( S \). Nevertheless the ring structure on \( S \) can be expressed in terms of two maps

\[
S \otimes_R S \xrightarrow{(s_1 \otimes s_2 \mapsto s_1 s_2)} S, \quad R \xrightarrow{\varphi} S,
\]

which satisfy certain commutative diagrams. Reversing all the arrows leads to the notion of an \( R \)-coring.

Suppose \( R \) is an overring of \( B \). Let \( C_B = R \otimes_B R \). There are maps

\[
C_B = R \otimes_B R \xrightarrow{(r_1 \otimes r_2 \mapsto r_1 \otimes 1 \otimes r_2)} R \otimes_B R \otimes_B R = (C_B) \otimes_R (C_B),
\]

\[
C_B = R \otimes_B R \xrightarrow{(r_1 \otimes r_2 \mapsto r_1 r_2)} R.
\]

These maps give \( C_B \) an \( R \)-coring structure. The dual \( *C_B \) is naturally isomorphic to the ring \( \text{End}_{B-R} \) of \( B \)-linear endomorphisms of \( R \) considered as a left \( B \)-module. In case \( B \) happens to be the subring of \( R \) generated by \( 1 \), we write \( C_Z \). Then \( *C_Z \) is \( \text{End}_{Z-R} \), the endomorphism ring of \( R \) considered as an additive group. This gives a clue how certain \( R \)-corings correspond to subrings of \( R \) and subrings of \( \text{End}_{Z-R} \), both major ingredients of the Jacobson-Bourbaki theorem.

\( 1 \otimes 1 \) is a "grouplike" element in the \( R \)-coring \( C_Z \) (and should be thought of as a generic automorphism of \( R \)). Suppose \( R \) is a division ring and \( B \) a subring which is a division ring. The natural map \( C_Z \xrightarrow{\pi} C_B \) is a surjective coring map. Conversely if \( C_Z \xrightarrow{\pi} D \) is a (surjective) coring map then \( \pi(1 \otimes 1) \) is a grouplike in \( D \) and \( \{ r \in R \mid \pi(r \otimes 1) = \pi(1 \otimes 1) r \} \) is a subring of \( R \) which is a division ring. This gives a bijective correspondence between the quotient corings of \( C \otimes_Z C \) and the subrings of \( R \) which are division rings.

We show how the Jacobson-Bourbaki correspondence is dual to the above correspondence.

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Another viewpoint. $B \otimes_{Z} B$ is a $B$-bimodule. For a subring $A \subset B$, $K = \ker(B \otimes_{Z} B \rightarrow B \otimes_{A} B)$ is a sub-$B$-bimodule of $B \otimes_{Z} B$.

$K$ is generated by \{a \otimes 1 - 1 \otimes a\}_{a \in A}$ as a $B$-bimodule. There are sub-$B$-bimodules not of this form.

$K$, as above, lies in

\[(*) \text{ kernel}
\begin{array}{c}
B \otimes_{Z} B \\
\text{multiplication}
\end{array} \rightarrow B,
\]

and if $e_{1} : B \otimes_{Z} B \rightarrow B \otimes_{Z} B \otimes_{Z} B$, $\Sigma b_{i} \otimes \beta_{i} \rightarrow \Sigma b_{i} \otimes 1 \otimes \beta_{i}$, then

\[(**) \quad e_{1}(K) \subset B \otimes_{Z} K + K \otimes_{Z} B.
\]

\[(*) \text{ and } (**) \text{ amount to the same thing as saying that } K \text{ is a coideal. Thus the language of corings merely gives us a convenient way of describing the sub-}
\]

$B$-bimodules of $B \otimes_{Z} B$ generated by \{a \otimes 1 - 1 \otimes a\}_{a \in A}$.

Infinite duality. The theory as we have developed it with $B \otimes_{Z} B$ as a coring can be dualized in general. The dual to $B \otimes_{Z} B$ is End $B$. However when there are not finiteness assumptions—in the dual theorems—closedness assumptions and considerations of linear compactness appear. This duality has been worked out by Dieudonné [2].

1. Corings over general rings. If $A$ is a commutative ring, an $A$-coalgebra is an $A$-module $C$ together with maps $\Delta: C \rightarrow C \otimes_{A} C$, $e: C \rightarrow A$ having certain properties [7, p. 4]. Since $A$ is commutative there is no real difference between left and right $A$-modules and "$C \otimes_{A} C$" is well defined. If $A$ were not commutative, then $C \otimes_{A} C$ is only well defined if $C$ has both a left and right $A$-module structure and the tensor product is with respect to the right $A$-module structure of the left $C$ and the left $A$-module structure of the right $C$.

Following Cartan-Eilenberg, we use $AM$ to indicate that $M$ is a left $A$-module and $MA$ indicates that $M$ is a right $A$-module. If $B$ is another ring, we use $AMB$ to indicate that $A$, $M$, $B$ and the following condition is satisfied:

\[(am)b = a(mb), \quad a \in A, m \in M, b \in B.
\]

$M$ is an $A$-bimodule means "$A'M_{A}$" and not simply "$A'M$ and $M_{A}$". If $M$ is an $A$-bimodule, and we write $M \otimes_{A} M$, this is always the tensor product $(M_{A}) \otimes_{A} (A'M)$. The natural $A$-bimodule structure on $M \otimes_{A} M$ is determined by

\[a \cdot (\sum m_{i} \otimes m'_{i}) \cdot b \equiv \sum (am_{i}) \otimes (m'_{i}b), \quad a, b \in A, \sum m_{i} \otimes m'_{i} \in M \otimes_{A} M.
\]

The natural $A$-bimodule structure on $A$ itself is the one induced by left and right multiplication.

1.1. Definition. For a not necessarily commutative ring $A$, an $A$-coring
is an \( A \)-bimodule \( M \) together with \( A \)-bimodule maps \( \Delta: M \to M \otimes_A M \), \( \epsilon: M \to A \) satisfying the following three commutative diagrams:

\[
\begin{align*}
M & \xrightarrow{\Delta} M \otimes_A M \\
& \downarrow{\Delta} \quad \text{coassociativity} \\
M \otimes_A M & \xrightarrow{(\Delta \otimes I)} M \otimes_A M \otimes_A M \\
& \downarrow{(I \otimes \Delta)}
\end{align*}
\]

1.2. Example. This is the main example. Suppose \( A \) and \( B \) are not necessarily commutative rings and \( \varphi: A \to B \) a ring map. The natural \( B \)-bimodule structure on \( B \) gives rise to an \( A \)-bimodule structure on \( B \) via \( \varphi \). Form \( B \otimes_A B \) and call this \( M \). The natural \( B \)-bimodule structure on \( M \) is determined by

\[
b \cdot (c \otimes d) \cdot e = (bc) \otimes (de), \quad b, c, d, e \in B.
\]

Then \( M \otimes_B M = (B \otimes_A B) \otimes_B (B \otimes_A B) \) which is naturally isomorphic to \( B \otimes_A B \otimes_A B \). The (Amitsur complex) map

\[
e_1 : B \otimes_A B \to B \otimes_A B \otimes_A B, \quad b \otimes \beta \to b \otimes 1 \otimes \beta.
\]

determines \( \Delta \) via the commutative diagram

\[
\begin{align*}
B \otimes_A B & \xrightarrow{e_1} B \otimes_A B \otimes_A B \\
M & \xrightarrow{\Delta} M \otimes_B M.
\end{align*}
\]

The multiplication map

\[
B \otimes_A B \to B, \quad b \otimes \beta \to b \beta
\]
determines \( e \) via the commutative diagram

\[
\begin{array}{c}
M \\
\downarrow e \\
B \\
\uparrow
\end{array}
\begin{array}{c}
\downarrow \\
B \otimes_A B
\end{array}
\]

It is left to the reader to verify that \((M, \Delta, e)\) is a \(B\)-coring in the sense of (1.1).

1.3. Definition. If \((M, \Delta, e)\) and \((M', \Delta', e')\) are \(A\)-corings, then a coring map \(g: M \to M'\) is an \(M\)-bimodule map where \(e'g = e\) and \(\Delta'g = (g \otimes g)\Delta\). A coideal (of \(M\)) is a sub-\(A\)-bimodule \(J \subset M\) where \(J \subset \ker e\) and

\[\Delta(J) \subset \ker(M \otimes_A (M/J) \otimes_A (M/J)).\]

Here \(\pi\) is the canonical \(A\)-bimodule map \(M \to M/J\). Of course \(\ker \pi \otimes \pi\) is

\[\text{Im}(J \otimes_A M \otimes \pi \to M \otimes_A M) + \text{Im}(M \otimes_A J \otimes \pi \to M \otimes_A M),\]

where \(\iota\) is the natural inclusion \(J \to M\).

As in classical coalgebra theory, if \(J\) is a coideal of \(M\), then \(M/J\) has a unique \(A\)-coring structure whereby \(\pi: M \to M/J\) is a coring map. Moreover, the kernel of a coring map is a coideal and the expected isomorphism theorem holds. (A source for elementary coalgebra theory is [7, pp. 3-48].)

1.4. Definition. If \(R\) and \(S\) are rings and \(R M_S, R N_S\), then \(\text{Hom}_{R\otimes S}(M, N)\) is used to denote the maps from \(M\) to \(N\) which are simultaneously left \(R\)-module maps and right \(S\)-module maps. Let \(\text{Hom}_{R\otimes S}(M, N)\) denote the left \(R\)-module maps from \(M\) to \(N\). Let \(\text{Hom}_{S}(M, N)\) denote the right \(S\)-module maps from \(M\) to \(N\). If \(L\) is an \(A\)-bimodule let \(*L*\) denote \(\text{Hom}_{A\otimes A}(L, A)\).

1.5. Proposition. Suppose \((M, \Delta, e)\) is an \(A\)-coring. Then \(*M*\) has a ring structure where, for \(f, g \in *M*\), the product \(fg\) is the composite

\[
\begin{array}{c}
M \Delta \\
\downarrow \Delta \\
M \otimes_A M
\end{array}
\begin{array}{c}
\downarrow \otimes \pi \\
\downarrow \\
A \otimes_A A
\end{array}
\]

and the unit of \(*M*\) is \(e\).

Proof. Similar to the construction of the dual algebra in classical coalgebra theory [7, pp. 6-10], and left to the reader. Q.E.D.

It is easily verified that if \(M\) and \(M'\) are \(A\)-corings and \(F: M \to M'\) a coring map, then the induced map \(*F*: *M* \to *M*\) is a ring map.

1.6. Example. Let \(M\) be the \(B\)-coring of Example 1.2 so \(M = B \otimes_A B\). Any \(f \in *M*\) is determined by \(f(1 \otimes 1)\). If \(f(1 \otimes 1) = x \in B\), then \(a \in A\),
\( ax = af(1 \otimes 1) = f(a \otimes 1) = f(1 \otimes a) = f(1 \otimes 1)a = xa. \)

Conversely if \( x \) lies in the centralizer of \( \varphi(A) \), then \( B \otimes_A B \rightarrow B, b \otimes \beta \rightarrow bx\beta \) gives a \( B \)-bimodule map \( g \) with \( g(1 \otimes 1) = x \). Hence \( ^*M^* \) is naturally isomorphic to the centralizer of \( \varphi(A) \) in \( B \).

1.7. Definition. If \( (M, \Delta, \varepsilon) \) is an \( A \)-coring, then \( g \in M \) is called grouplike if \( \varepsilon(g) = 1 \) and \( \Delta(g) = g \otimes g \). A grouplike cannot be zero since \( \varepsilon(g) = 1 \).

1.8. Definition. The trivial \( A \)-coring structure on \( A \) is the \( A \)-coring \((A, \Delta, \varepsilon)\) where \( \Delta \) and \( \varepsilon \) are determined by the commutative diagrams:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes_A A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varepsilon} & A
\end{array}
\]

1.9. Proposition. Suppose \( A \) and \( B \) are rings and \( \varphi: A \rightarrow B \) a ring map. Let \( M = B \otimes_A B \) have the \( B \)-coring structure of 1.2.

(a) Let \( g_{B/A} \) denote \( 1 \otimes 1 \in B \otimes_A B; g_{B/A} \) is a grouplike element of \( M \).

(b) Let \( C \) be a \( B \)-coring and let \( G_{\varphi}(C) \) denote \( \{ g \in C \mid g \text{ is grouplike and } \varphi(a)g = g\varphi(a), a \in A \} \).

For each \( g \in G_{\varphi}(C) \) there is a unique coring map \( \xi: M \rightarrow C \) with \( \xi(g_{B/A}) = g \).

For each coring map \( \xi: M \rightarrow C \), the element \( \xi(g_{B/A}) \in G_{\varphi}(C) \).

Proof. Left to the reader. Q.E.D.

2. Correspondence theorem.

2.1. Fundamental Theorem. Suppose \( A \subset B \) are division rings. Let \( M = B \otimes_A B \) be the \( B \)-coring formed by \( A \hookrightarrow B \) as in 1.2 and let \( C \) denote the set of coideals of \( M \). Let \( \mathcal{D} \) denote the set of (intermediate) division rings \( D \) with \( A \subset D \subset B \).

(a) For \( D \in \mathcal{D} \) the kernel of the natural surjective map \( M = B \otimes_A B \rightarrow B \otimes_D B \) is a coideal. Denote it by \( J_D \).

(b) For a coideal \( J \in C \), let \( \pi: M \rightarrow M/J \) be the natural surjective map. Then \( \{ b \in B \mid b\pi(g_{B/A}) = \pi(g_{B/A})b \} \) is a division ring intermediate between \( A \) and \( B \). Denote it by \( D_J \).

(c) The maps

\[
\begin{array}{ccc}
C & \xrightarrow{\rho} & \mathcal{D}, \\
J & \xrightarrow{D_J} & D
\end{array}
\]

are inverse to each other, thus establishing a bijective correspondence.
Proof. (a) Let $N$ be the $B$-coring formed by $D \hookrightarrow B$ as in 1.2. Then the natural map $M = B \otimes_A B \rightarrow B \otimes_D B = N$ is a coring map and so has a kernel which is a coideal.

(b) Let $L$ be any $B$-bimodule and $l \in L$. If $bl = lb$, then

$$b^{-1}l = b^{-1}ll^{-1} = b^{-1}lb^{-1} = b^{-1}blb^{-1} = ll^{-1} = lb^{-1}.$$  

Hence $\{b \in B | bl = lb\}$ is a subring of $B$ which is a division ring.

For $g_{B/A} = 1 \otimes 1 \in B \otimes_A B$, $ag_{B/A} = g_{B/A}a$ for all $a \in A$. Since $\pi: M \rightarrow M/J$ is a $B$-bimodule map, it follows that $a\pi(g_{B/A}) = \pi(g_{B/A})a$ for all $a \in A$.

Hence $\{b \in B | b\pi(g_{B/A}) = \pi(g_{B/A})b\}$ is a subring of $B$ which is a division ring and contains $A$.

(c) $(\rho A = I)$ Suppose $D \in D$. Then $J_D$ is the kernel of the natural map $\pi: B \otimes_A B \rightarrow B \otimes_D B$ and $\pi(g_{B/A}) = 1 \otimes 1 \in B \otimes_D B$. Hence

$$D_{J_D} = \{b \in B | b(1 \otimes 1) = (1 \otimes 1)b \in B \otimes_D B\} = \{b \in B | (b \otimes 1) = (1 \otimes b) \in B \otimes_D B\}.$$  

For $e \in B$ with $e \notin D$, the set $\{1, e\}$ is $D$-linearly independent when $B$ is considered as a left $D$-module. Hence for $x, y \in B$, if $0 = x \otimes 1 + y \otimes e \in B \otimes_D B$, then both $x$ and $y$ must equal zero. In particular, $0 \neq (-e) \otimes 1 + 1 \otimes e$, which proves that $D_{J_D} \subset D$. The opposite inclusion is obvious and hence $D_{J_D} = D$.

$(\rho D = I)$ Let $J$ be a coideal of $M$, and $\pi: M \rightarrow M/J$ the natural coring map. By 1.9 there is a unique coring map $\xi: B \otimes_D B \rightarrow M/J$ with $\xi(g_{B/DJ}) = \pi(g_{B/A})$. We have the commutative diagram

$$\begin{array}{ccc}
B \otimes_A B & \rightarrow & B \otimes_D B \\
\pi' & \downarrow \pi & \xi \\
B \otimes_D B & \rightarrow & M/J
\end{array}$$  

where $\pi'$ is the natural surjection. (Note, the diagram must commute since $\xi\pi'(g_{B/A}) = \xi(g_{B/DJ}) = \pi(g_{B/A})$ and $B \otimes_A B$ is generated by $g_{B/A}$ as a $B$-bimodule and all the maps are $B$-bimodule maps.)

The coring $B \otimes_A B$ is generated by $g_{B/A}$ as a $B$-bimodule; hence, the quotient $M/J$ is generated by the grouplike $\pi(g_{B/A})$ as a $B$-bimodule. By definition,

$$D_J = \{b \in B | b\pi(g_{B/A}) = \pi(g_{B/A})b\}.$$  

Hence by the following lemma (2.2), the map $\xi$ is a coring isomorphism. From the commutative diagram we deduce $\ker \pi' = \ker \pi$. Hence $J_{D_J} = \ker \pi' = \ker \pi = J$. Q.E.D.
2.2. Fundamental Lemma. Let $B$ be a division ring and $C$ a $B$-coring. Suppose $g$ is a grouplike element of $C$ and $D = \{b \in B \mid bg = gb\}$. Then $D$ is a subring of $B$ which is a division ring, and the unique coring map from 1.11, with $\xi(1 \otimes 1) = C$, $\xi: B \otimes_D B \rightarrow C$, is injective. $\xi$ is an isomorphism if $C$ is generated by $g$ as a $B$-bimodule.

**Proof.** That $D$ is a division ring follows from the first few lines in the proof of 2.1(b).

Explicitly the map $\xi: B \otimes_D B \rightarrow C$ is given by $\Sigma b_i \otimes \beta_i \xrightarrow{\xi} \Sigma b_i g \beta_i$, $\{b_i\} \cup \{\beta_i\} \subset B$. Suppose $\ker \xi \neq 0$. Choose a nonzero element $x = \Sigma^t b_i \otimes \beta_i \in \ker \xi$ with $t$ minimal.

$t \neq 1$: if $x = b_1 \otimes \beta_1 \in \ker \xi$, then $b_1 g \beta_1 = 0$. Since $g \neq 0$ by the remark following 1.9, and since $B$ is a division ring, it follows that $b_1$ or $\beta_1$ must equal zero. This contradicts the fact that $x \neq 0$.

$t > 1$: Since $t$ is minimal it follows that $\beta_t \neq 0$. The element $0 \neq x' = \Sigma b_i \otimes \beta_i \beta_t^{-1}$ still lies in $\ker \xi$ and has the same rank $t$. Let $\beta'_t = \beta_t \beta_t^{-1}$, so that $\beta'_t = 1$. Since $x' = \Sigma b_i \otimes \beta'_i$, and this is a minimal length expression for $x'$, it follows that

(a) $\{b_i\}$ is a $D$-linearly independent set in $B_D$,
(b) $\{\beta'_i\}$ is a $D$-linearly independent set in $D_B$.

Hence since $\beta'_t = 1 \in D$, it follows from (b) that $\beta'_t \notin D$.

Thus $g \beta'_t \neq \beta'_t g$, and there is $h: C \rightarrow B$ a left $B$-module map (i.e. $h(bz) = bh(z)$, $b \in B$, $z \in C$) with $h(g \beta'_t - \beta'_t g) \neq 0$. Such $h$ exists because as a left $B$-module $C$ is free; $B$ is a division ring after all.

Let $x'' = \Sigma b_i \otimes h(g \beta'_t - \beta'_t g)$. Then,

(c) since $\beta'_t = 1$, $g \beta'_t = \beta'_t g$ and $x'' = \Sigma^{t-1} b_i \otimes h(g \beta'_i - \beta'_i g)$, i.e. $x''$ has an expression of length less than $t$.

(d) $0 \neq x''$, since $0 \neq h(g \beta'_t - \beta'_t g)$ and $\{b_i\}$ is a $B_D$, $D$-linearly independent set.

(e) $\xi(x'') = \sum b_i g h(g \beta'_i - \beta'_i g) = \sum b_i g h(g \beta'_i) - \sum b_i g h(\beta'_i g)$.

The term
$$\sum b_i g h(\beta'_i g) = \sum b_i g h(\beta'_i) h(g) = \xi(x') h(g) = 0.$$

The term $\Sigma b_i g h(g \beta'_i)$ is the image of $x'$ under the composite

$$B \otimes_A B \xrightarrow{\xi} C \xrightarrow{\Delta} C \otimes_D C \xrightarrow{I \otimes h} C \otimes_B B = C,$$

but
$$\frac{(I \otimes h) \Delta \xi(x')}{0} = 0.$$

Hence $x'' \in \ker \xi$. 

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But (c), (d) and (e) contradict the minimality of $t$. Hence $\ker t = 0$.

Since $\text{Im} \, t$ is the sub-$B$-bimodule of $C$ generated by $g$, it follows that $t$ is an isomorphism when $C = BgB$. Q.E.D.

2.3. **Corollary.** Suppose $B$ is a division ring and $P$ is the prime field contained in $B$. So that $P \cong \mathbb{Z}/p\mathbb{Z}$ if $B$ has characteristic $p$ and $P \cong \mathbb{Q}$ if $B$ has characteristic zero. Then the correspondence in 2.1 gives a bijective correspondence between the set of all coideals in $B \otimes_P B$ on the one hand and the set of all subdivision rings of $B$ on the other hand.

3. **Semiduality.** This section consists of technical, boring, but important results on duality. The reader is advised to skip to §4 and refer to this section as needed.

3.1. **Definition.** If $M$ is a $B$-bimodule let $M^*$ denote $\text{Hom}_{-B}(M, B)$, the set of right $B$-module maps from $M$ to $B$, and let $M^*$ denote $\text{Hom}_{B-}(M, B)$, the set of left $B$-module maps from $M$ to $B$.

The picture is:

\[
\begin{align*}
\text{Hom}_Z(M, B) & \cup *M \cup *M^* \\
*M \cap M^* &= *M^*
\end{align*}
\]

3.2. **Proposition.** Suppose $(C, \Delta, e)$ is a $B$-coring.

(a) $*C$ has a ring structure where, for $f, g \in *C$, the product $gf$ is the composite

\[
C \xrightarrow{\Delta} C \otimes_B C \xrightarrow{I \otimes f} C \otimes_B B = C \xrightarrow{\varepsilon} B,
\]

and the unit of $*C$ is $e$.

(b) $C^*$ has a ring structure where, for $f, g \in C^*$, the product $gf$ is the composite

\[
C \xrightarrow{\Delta} C \otimes_B C \xrightarrow{g \otimes I} B \otimes_B C = C \xrightarrow{f} B,
\]

and the unit of $C^*$ is $e$.

(c) The map $\lambda: B \rightarrow *C$, determined by $\lambda(b)(c) = e(cb)$, $b \in B, c \in C$, is a ring antimap, i.e. reverses multiplication

(d) The map $\rho: B \rightarrow C^*$, determined by $\rho(b)(c) = e(bc)$, $b \in B, c \in C$, is a ring antimap.

If $C'$ is another $B$-coring and $\Xi: C \rightarrow C'$ a $B$-coring map then:

(e) The natural map $*(C') \rightarrow *C$ is a ring map making the diagram commute:
The natural map \((C')^* \xrightarrow{\cong} C^*\) is a ring map making the diagram commute:

\[ \begin{array}{ccc}
(C')^* & \xrightarrow{\cong} & C^* \\
\downarrow \rho & & \downarrow \rho \\
B & = & B \\
\end{array} \]

**Proof.** (a) For \(f, g, h \in \ast C\) the product \(h(gf)\) or \((hg)f\) both are the composite

\[ C \xrightarrow{\Delta_2} C \otimes_B C \otimes_B C \xrightarrow{I \otimes I \otimes f} C \otimes_B C \otimes_B B = C \otimes_B C \xrightarrow{I \otimes g} C \otimes_B B = C \xrightarrow{h} B, \]

where \(\Delta_2\) is either of the composites

\[ C \xrightarrow{\Delta} C \otimes_B C \xrightarrow{I \otimes \Delta} C \otimes_B C \otimes_B C, \]

\[ C \xrightarrow{\Delta} C \otimes_B C \xrightarrow{\Delta \otimes I} C \otimes_B C \otimes_B C, \]

which are the same by coassociativity.

The rest of the ring axioms are easily verified.

(b) Similar to (a).

The other verifications are left to the reader. Q.E.D.

**3.3. Example.** Suppose \(B\) is a ring with subring \(A\). We have the inclusion \(A \hookrightarrow B\) and form \(M = B \otimes_A B\) as in 1.2. The natural correspondence

\[ \ast M = \text{Hom}_{B-L}(B \otimes_A B, B) = \text{Hom}_{A-L}(B, B) = \text{End}_{A-B} \]

is given as follows: \(f \in \ast M\) corresponds to \(B \twoheadrightarrow B, b \mapsto f(g_{B/A}b)\) where \(g_{B/A} = 1 \otimes 1 \in B \otimes_A B\) the grouplike element of 1.9.

It is left to the reader to show that the above correspondence \(\ast M = \text{End}_{A-B}\) is a ring isomorphism.

Similarly the natural correspondence

\[ M^* = \text{Hom}_{B-R}(B \otimes_A B, B) = \text{Hom}_{A-R}(B, B) = \text{End}_{A-B} \]

is given as follows: \(f \in M^*\) corresponds to \(B \twoheadrightarrow B, b \mapsto f(g_{B/A})\). It is left to
the reader to verify that the correspondence $M^* = \text{End}_A B$ is a ring anti-isomorphism.

3.4. Linear Lemma. Suppose $X$ and $Y$ are rings and there are modules

$Y_R$, $X_S Y$, $T_X$, $U_Y$, $X_Z$.

(a) The right $X$-module structure on $\text{Hom}_Y(S, U)$ arises from $X_S$ and the left $X$-module structure on $\text{Hom}_X(T, X)$ arises from $X_T$.

There is a natural map

$$\eta: \text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T, X) \rightarrow \text{Hom}_Y(T \otimes_X S, U);$$

for $f \in \text{Hom}_Y(S, U)$, $g \in \text{Hom}_X(T, X)$, $t \in T$, $s \in S$,

$$\eta(f \otimes g)(t \otimes s) = f(g(t)s).$$

$\eta$ is bijective if $T$ is a finitely generated projective right $X$-module.

(b) The left $Y$-module structure on $\text{Hom}_X(S, Y)$ arises from $S_Y$ and the right $X$-module structure on $\text{Hom}_Y(R, Y)$ arises from $Y_Y$.

There is a natural map

$$\mu: \text{Hom}_X(R, Y) \otimes_Y \text{Hom}_Y(S, V) \rightarrow \text{Hom}_X(S \otimes_Y R, V);$$

for $f \in \text{Hom}_X(R, Y)$, $g \in \text{Hom}_Y(S, V)$, $s \in S$, $r \in R$,

$$\mu(f \otimes g)(s \otimes r) = g(sf(r)).$$

$\mu$ is bijective if $R$ is a finitely generated projective left $Y$-module.

Suppose $W$ and $Z$ are also rings and we have the additional hypothesis:


(c) If the domain and range of $\eta$ have the left $W$-module structure arising from $W_U$ and the right $Z$-module structure arising from $Z_T$, then $\eta$ is a left $W$-module map and right $Z$-module map.

(d) If the domain and range of $\mu$ have the left $W$-module structure arising from $R_W$ and the right $Z$-module structure arising from $V_Z$, then $\mu$ is a left $W$-module map and right $Z$-module map.

Proof. We leave it to the reader to verify that $\eta$ and $\mu$ are well defined, and that (c) and (d) are true.

(a) Suppose $T$ is the direct sum of right $A$-modules $T_1 \oplus T_2$. Then

$$\text{Hom}_X(T, X) = \text{Hom}_X(T_1, X) \oplus \text{Hom}_X(T_2, X)$$

and so

$$\text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T, X) = [\text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T_1, X)]$$

$$\oplus [\text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T_2, X)].$$
Similarly the range of \( \eta \) decomposes

\[
[\text{Hom}_Y(T \otimes_X S, U)] = \left[\text{Hom}_Y(T_1 \otimes_X S, U)\right] \oplus \left[\text{Hom}_Y(T_2 \otimes_X S, U)\right].
\]

Letting \( \eta \) denote the "\( \eta \)-map" for \( T \) and letting \( \eta_1 \) denote the "\( \eta \)-map" for \( T_1 \), the following diagram commutes (verification left to the reader).

\[
\begin{array}{ccc}
\text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T, X) & \xrightarrow{\eta} & \text{Hom}_Y(T, U) \otimes_X \text{Hom}_X(T, X) \\
\downarrow & & \downarrow \\
\text{Hom}_Y(T \otimes_X S, U) = \left[\text{Hom}_Y(T_1 \otimes_X S, U)\right] \oplus \left[\text{Hom}_Y(T_2 \otimes_X S, U)\right]
\end{array}
\]

\( (*) \) Thus \( \eta \) is bijective if and only if both \( \eta_1 \) and \( \eta_2 \) are bijective.

Now suppose \( T = X \) as a right \( X \)-module. Then \( \text{Hom}_X(T, X) = X \) as a left \( X \)-module, and the domain of \( \eta \) is \( \text{Hom}_Y(S, U) \otimes_X X = \text{Hom}_Y(S, U) \).

The range of \( \eta \) reduces to \( \text{Hom}_Y(X \otimes_X S, U) = \text{Hom}_Y(S, U) \), and it is left to the reader to verify that \( \eta \) reduces to the identity map. Hence \( \eta \) is bijective when \( T = X \).

When \( T \) is a free right \( X \)-module with finite basis, it follows from \( (*) \) and the result for \( T = X \) that \( \eta \) is bijective. Again using \( (*) \), it follows that if \( T \) is a direct summand of a free \( X \)-module with finite basis, then \( \eta \) is bijective.

In other words, \( \eta \) is bijective when \( T \) is a finitely generated projective \( X \)-module. This concludes the proof of (a).

(b) is proved similarly. Q.E.D.

3.5. Duality Lemma. Suppose \( X \) is a ring with modules \( _XR \) and \( _TX \).

(a) \( T^* = \text{Hom}_X(T, X) \) has the left \( X \)-module structure induced by \( _X \) and \( _XR = \text{Hom}_X(R, X) \) has the right \( X \)-module structure induced by \( X_X \). If \( T \) is a finitely generated projective right \( X \)-module, then \( T^* \) is a finitely generated projective left \( X \)-module. If \( R \) is a finitely generated projective left \( X \)-module, then \( _XR \) is a finitely generated projective right \( X \)-module.

(b) \( T^* \) has the left \( X \)-module structure of (a) which permits us to form \( *(T^*) \). There is a natural map \( \beta: T \rightarrow *(T^*) \) where for \( t \in T, f \in T^* \), \( \beta(t)(f) = f(t) \). If \( *(T^*) \) has the right \( X \)-module structure of (a), then \( \beta \) is a right \( X \)-module map.

(c) \( \beta \) is bijective if \( T \) is a finitely generated projective right \( X \)-module.

(d) If \( T \) is a free right \( X \)-module with a basis of finite cardinality \( n \), then \( T^* \) is a free left \( X \)-module with a basis of cardinality \( n \).

(e) \( _XR \) has the right \( X \)-module structure of (a) which permits us to form \( (*R)^* \).

There is a natural map \( \alpha: R \rightarrow (*R)^* \) where, for \( r \in R, g \in R^* \), \( \alpha(r)(g) = \)
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\( g(r) \). If \((^*R)^*\) has the left \(X\)-module structure of (a), then \( \alpha \) is a left \(X\)-module map.

(f) \( \alpha \) is bijective if \( R \) is a finitely generated projective left \(X\)-module.

(g) If \( R \) is a free left \(X\)-module with a basis of finite cardinality \( n \), then \(^*R\) is a free right \(X\)-module with a basis of cardinality \( n \).

**Proof.** (b) and (d) are left to the reader. We sketch a proof of (c).

As in the proof of 3.4(a), if \( T = T_1 \oplus T_2 \), a direct sum of right \(X\)-modules, then \( \beta \) breaks up accordingly into \( \beta_1 \oplus \beta_2 \), and \( \beta \) is an isomorphism if and only if both \( \beta_1 \) and \( \beta_2 \) are isomorphisms.

In case \( T = X \), then \(^*T = X \), and \(^*(^*T) = X \), and \( \beta \) corresponds to the identity map which is bijective. This, together with the behavior of \( \beta \) for direct sums, shows that if \( T \) is isomorphic to a finite direct sum of copies of \( X \), then \( \beta \) is an isomorphism. And so if \( T \) is a direct summand of a finite direct sum of copies of \( X \), then \( \beta \) is an isomorphism. And all finitely generated projective right \(X\)-modules arise as direct summands of a finite direct sum of copies of \( X \).

This proves (c).

(e) is proved similarly.

(a), (d) and (g) have a similar direct sum argument type proof with no maps involved. Simply \( T^* \cong T_1^* \oplus T_2^* \). Then if \( T = B \), it follows \( T^* = B \) etc. Q.E.D.

3.6. **Example.** Suppose \((C, \Delta, \epsilon)\) is a \(B\)-coring. Let the modules \( R, S, T \) of 3.4 all be \( C \). Let the modules \( U, V \) of 3.4 both be \( B \). Let the rings \( W, X, Y, Z \) of 3.4 all be \( B \). The diagonalization \( \Delta: C \rightarrow C \otimes_B C \) is then a map \( \delta: C \rightarrow T \otimes_X S \) and \( \Delta \) is a left \( Z \) and right \( Y \)-module map if \( \epsilon C_Y = \epsilon C_B \). This \( \delta \) induces \( \text{Hom}(\delta, U): \text{Hom}_Y(T \otimes_X S, U) \rightarrow \text{Hom}_Y(C, U) \) which is a left \( W \) and right \( Z \)-module map. Forming the composite \( \text{Hom}(\delta, U) \eta \) gives a map \( \text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T, X) \rightarrow \text{Hom}_Y(C, U) \), which is a map \( C^* \otimes_B C^* \rightarrow C^* \). The reader can check that this map is precisely the opposite multiplication map of 3.2(b). That is \( f \otimes g \rightarrow \text{product } gf \) as described in 3.2(b). Thus the dual algebra structure on \( C^* \) given in 3.2(b) is opposite to the structure induced by 3.4(a).

Similarly the dual algebra structure on \( *C \) given in 3.2(a) is opposite to the structure induced by 3.4(b).

With suitable “finitely-generated-projective-module” type assumptions, the situation \( \varphi: B \rightarrow E \), where \( B \) and \( E \) are rings and \( \varphi \) a ring antimap, can be dualized to obtain a \( B \)-coring. Of course there is left duality and right duality.

Suppose \( B \) and \( E \) are rings and \( \varphi: B \rightarrow E \) a ring antimap. We wish to apply 3.3 and 3.4 with

all the rings \( W, X, Y, Z \) equal to \( B \),

the modules \( R, S, T \) equal to \( E \),

the modules \( U, V \) equal to \( B \).

\( B \) has the usual \( B \)-bimodule structure.
The right and left $B$-module structures on $E$ are induced by $\varphi$ as follows:

For $b \in B$, $e \in E$,

$$b \cdot e \equiv e\varphi(b), \quad e \cdot b \equiv \varphi(b)e,$$

product in $E$.

This gives $E$ left and right $B$-module structures since $\varphi$ is a ring antihomomorphism. And $E$ is a $B$-bimodule.

Note that if $C$ is a $B$-coring and $E = ^*C$ (respectively $C^*$), and $\varphi = \lambda$ (respectively $\rho$) as in 3.2, then the $B$-bimodule structure on $E$ induced by $\varphi$ agrees with the $B$-bimodule structure where $^*_B C$ is induced by $C_B$ and $^*_B C^*$ is induced by $B_B$ (respectively, $B^*_C$ is induced by $B_B$ and $C_B^*$ is induced by $B_C$.) Hence $E$ has the appropriate $B$-module structure to apply 3.5.

3.7. **Dual Coring Theorem.** (a) In the above setting, if $E^* = \text{Hom}_B(F, B)$ has the left $B$-module structure as in 3.5(a), and the right $B$-module structure induced by $B^*E$, and $E$ is finitely generated and projective as a right $B$-module, then there is a unique $B$-coring structure on $E^*$ whereby the bijection $\beta: E \rightarrow ^*(E^*)$ 3.5(b) is a ring isomorphism. Here $^*(E^*)$ has the ring structure of 3.2(a).

Moreover the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\beta} & ^*(E^*) \\
\downarrow{\varphi} & & \downarrow{\lambda} \\
B & & \\
\end{array}
$$

commutes, where $\lambda$ is defined in 3.2(c).

(b) In the above setting, if $E^* = \text{Hom}_{B^*}(E, B)$ has the right $B$-module structure as in 3.5(a), and the left $B$-module structure induced by $E_B$, and $E$ is finitely generated and projective as a left $B$-module, then there is a unique $B$-coring structure on $E^*$ whereby the bijection $\alpha: E \rightarrow (*E)^*$ 3.5(b) is a ring isomorphism. Here $(*E)^*$ has the ring structure of 3.2(b).

Moreover the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\alpha} & (*E)^* \\
\downarrow{\varphi} & & \downarrow{\rho} \\
B & & \\
\end{array}
$$

commutes, where $\rho$ is defined in 3.2(d).

**Proof.** (a) Define $\epsilon: E^* \rightarrow B$ by $\epsilon(f) = f(1), f \in E^*, 1 \in E$. Then it is easily checked that $\epsilon$ is a $B$-bimodule map.
With the choices of \( W, X, Y, Z, S, T, U \) indicated just before this theorem, the map \( \eta \) of 3.4(a) is an isomorphism. Thus \( \eta^{-1} \) is an isomorphism

\[ (*) \quad \eta^{-1} : \text{Hom}_Y(T \otimes_X S, U) \to \text{Hom}_Y(S, U) \otimes_X \text{Hom}_X(T, X), \]

and by 3.4(c), \( \eta^{-1} \) is a left \( W \) and right \( Z \)-module map. Since \( T = E = S \), the map \( \text{tlum} : E \otimes_B E \to E, e \otimes f \to fe \), induces

\[ (**) \quad T \otimes_X S \to E \]

which is a left \( Z \)-module map and right \( Y \)-module map. Hence \( B E_B \) becomes \( Z E_Y \).

Thus \( (**) \) induces

\[ (***) \quad \text{Hom}_Y(E, U) \to \text{Hom}_Y(T \otimes_X S, U) \]

which is a left \( W \)-module map and right \( Z \)-module map.

Composing the maps at \( (*) \) and \( (***) \) gives the map

\[ (****) \quad \text{Hom}_Y(E, U) \to \text{Hom}_Y(T \otimes_X S, U) \]

With the choices of \( W, X, Y, Z, S, T, U \), \( (****) \) is a bimodule map, \( E^* \to E^* \otimes_B E^* \), denoted \( \Delta \).

That \( (E^*, \Delta, e) \) satisfies the necessary commutative diagrams to be a \( B \)-coring follows from commutative diagrams that \( E \) satisfies and the duality established.

That \( \beta : E \to \ast(E^*) \) of 3.5(b) is a ring isomorphism, and \( \beta \beta = X \) is left to the reader to verify. The uniqueness of the coring structure is also left to the reader to verify.

(b) goes similarly. Q.E.D.

4. Jacobson-Bourbaki-Hochschild. Throughout this section \( B \) is a division ring and \( \text{End} B \) denotes \( \text{End}_B B \), the ring of additive group endomorphisms of \( B \). \( l \) and \( r \) denote the maps

\[ B \to \text{End} B, \quad b \to b^l, \quad \beta \to \beta^r \]

where \( b, \beta \in B, b^l(\beta) = b\beta = \beta^r(b) \).

Since \( B \) has a unit, \( l \) and \( r \) are injective. \( \text{Im} \ l \) is denoted \( B^l \), and \( \text{Im} \ r \) is denoted \( B^r \). Since \( l \) is a ring map, \( B \cong B^l \) as rings. Since \( r \) is a ring antihomomorphism, \( B^r \) is isomorphic to the opposite ring of \( B \).

It is amusing to verify that \( B^r \) is the centralizer of \( B^l \) in \( \text{End} B \) and \( B^l \) the centralizer of \( B^r \). This is left to the reader.

Let \( F \) be a subring of \( \text{End} B \) with \( \text{End} B \supset F \supset B^r \). Since \( F \supset B^r \), the centralizer of \( F \) lies in \( B^l \). Let \( A \) denote \( \{ a \in B \mid a(fb) = af(b), b \in B, f \in F \} \).

It is easily verified (and left to the reader) that

(i) \( A \) is a subring of \( B \) and \( A \) is a division ring,

(ii) \( A^l = \{ a^l \}_{a \in A} \) is the centralizer of \( F \) in \( \text{End} B \)
(iii) \( B' \subset C \subset \text{End}_{A-}B \subset \text{End} B, \)

(iv) \( A = \{ b \in B \mid f(b) = b f(1), f \in F \}. \)

4.1. Theorem. If \( F \) has finite dimension as a right \( B \)-module or \( B \) has finite dimension as a left \( A \)-module, i.e., either \( \dim_B F_B < \infty \) or \( \overline{\dim}_A A_B, \) then both are finite and equal and \( F = \text{End}_{A-}B. \)

Proof. The idea behind the proof is to show that \( F^* \) equals or is naturally isomorphic to the \( B \)-coring \( B \otimes_A B \) of 1.2. Then it will follow from 3.3 and 3.7 that \( F = *(F^*) = \text{End}_{A-}B. \)

Now the proof actually begins.

Let \( X \) be an \( A \)-basis for \( A_B \) and let \( \{ \delta_x \}_{x \in X} \subset \text{End}_{A-}B \) be determined by

\[
\delta_x(y) = \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases} \quad x, y \in X.
\]

Then \( \{ \delta_x \} \) is a \( B \)-linearly independent set for \( (\text{End}_{A-}B)_B \) and \( \{ \delta_x \} \) is a \( B \)-basis for \( (\text{End}_{A-}B)_B \) if \( \dim_A A_B < \infty. \) Thus

\[
\dim_B(\text{End}_{A-}B)_B = \begin{cases} \dim_A A_B & \text{if } \dim A_A B < \infty, \\ \infty & \text{if } \dim A_A B = \infty. \end{cases}
\]

Since \( F \subset \text{End}_{A-}B, \) it follows that \( \dim_B F_B < \infty \) if \( \dim A_A B < \infty. \)

Thus assume for the rest of the proof that \( \dim_B F_B < \infty. \)

For \( f \in F, \Sigma b_i \otimes \beta_i \in B \otimes_A B, \) define

\[
\left\langle f, \sum b_i \otimes \beta_i \right\rangle = \sum b_i f(\beta_i) \in B.
\]

For fixed \( \sum b_i \otimes \beta_i \in B \otimes_A B, \) the function \( \langle \cdot, \sum b_i \otimes \beta_i \rangle \) lies in \( F^*, \) and for fixed \( f \in F, \) the function \( (f, \cdot) \) lies in \( *(B \otimes_A B). \)

For \( 0 \neq f \in F \) let \( b \in B \) with \( f(b) \neq 0. \) Then

\[
K: B \otimes_A B \rightarrow F^*, \quad \sum b_i \otimes \beta_i \mapsto \left\langle f, \sum b_i \otimes \beta_i \right\rangle
\]

has dense image in \( F^* \) since \( \langle f, 1 \otimes b \rangle = f(b) \neq 0. \)

Using 3.5 and \( \dim_B F_B < \infty, \) it follows, just like in commutative linear algebra, that \( K \) is surjective.

Dualizing once again yields \( *(B \otimes_A B) \leftarrow *^K *(F^*). \)

Identifying \( *(F^*) \) with \( F \) as a ring by 3.7(a) and

\[
(4.3)
\]

identifying \( *(B \otimes_A B) \) with \( \text{End}_{A-}B \) as a ring by 3.3, then \( *^K \) reduces to the inclusion.

Thus \( *^K \) is a ring map which implies that \( K \) is a \( B \)-coring map. (This type of duality result we are using without including a proof.)
Let \( g = 1 \otimes 1 \in B \otimes_A B \), the distinguished grouplike element of 1.9(a).

Let \( c = K(g) \), a grouplike in \( F^* \), since \( K \) is a coring map. Since \( B \otimes_A B = BgB \), it follows that \( F^* = BcB \) since \( K \) is a surjective \( B \)-bimodule map.

If \( A = \{ a \in B | ac = ca \} \), then, by 2.2, \( K \) is injective and, hence, an isomorphism. This implies that \( *K \) is an isomorphism. By (4.3) it follows that \( F = \text{End}_A B \). Then by (4.2) and the assumption \( \dim_B F_B < \infty \), it follows that

\[
\infty > \dim_A AB = \dim_B (\text{End}_A B)_B = \dim_B F_B,
\]

which would conclude the proof.

\[
c = K(g) = \langle 1 \otimes 1 \rangle.\]

For \( b \in B \), \( cb = \langle 1 \otimes b \rangle \) and \( bc = \langle b \otimes 1 \rangle \).

Thus for \( f \in F \), \( (bc)(f) = b(f(1)) \), \( (cb)(f) = f(b) \).

Hence by (iv)—immediately above 4.1—\( A = \{ b \in B | bc = cb \} \). Q.E.D.

In [3] Hochschild studies an object other than \( \text{End}_A B \). Suppose \( Z \) is a division ring intermediate between \( A \) and \( B \), \( A \subset Z \subset B \). Then one can consider \( \text{Hom}_A-(B, Z) \), which of course is a right ideal in \( \text{End}_A-(B) \) if we take

\[
\text{Hom}(B, \text{inclusion}): \text{Hom}_A-(B, Z) \rightarrow \text{Hom}_A-(B, B)
\]
as an inclusion. As a right ideal, \( \text{Hom}_A-(B, Z) \) has a product structure but not a multiplicative identity (in general). Although \( \text{Hom}_A-(B, Z) \) is a right ideal, it is a left \( Z \)-submodule via \( r \) of \( (\text{End}_A-(B))_{B \supset Z} \). \( \text{Hom}_A-(B, Z) \) is the \( Z \)-dual, \( \text{Hom}_Z-(\cdot, Z) \), of \( Z \otimes_A B \). Such a \( Z \otimes_A B \) is a left \( Z \)-module and right \( B \)-module.

There is a natural diagonal map

\[
Z \otimes_A B \rightarrow Z \otimes_A B \otimes_A B = (Z \otimes_A B) \otimes_Z (Z \otimes_A B), \quad z \otimes b \mapsto z \otimes 1 \otimes b,
\]

but there is no natural counit making \( Z \otimes_A B \) into a \( Z \)-coring. The theory of such coassociative corings without counit could be developed and would yield the Galois theorem to which [3, Theorem 2.1, p. 447] is dual in the finite case.

**BIBLIOGRAPHY**


