BV-FUNCTIONS, POSITIVE-DEFINITE FUNCTIONS
AND MOMENT PROBLEMS

BY

P. H. MASERICK

ABSTRACT. Let $S$ be a commutative semigroup with identity 1 and involution. A complex valued function $f$ on $S$ is defined to be positive definite if $\prod f(\Delta_j) \geq 0$ where the $\Delta_j$'s belong to a certain class of linear sums of shift operators. For discrete groups the positive definite functions defined herein are shown to be the classically defined positive definite functions. An integral representation theorem is proved and necessary and sufficient conditions for a function to be the difference of two positive-definite functions, i.e. a BV-function, are given. Moreover the BV-function defined herein agrees with those previously defined for semilattices, with respect to the identity involution. Connections between the positive-definite functions and completely monotonic functions are discussed along with applications to moment problems.

1. Introduction. Let $S$ be an involution semigroup, $\Gamma$ a subset of the complex-valued semicharacters on $S$ equipped with a locally compact topology and $M$ a subcollection of real regular Borel measures on $\Gamma$. Solvability of the moment problem

$$f(x) = \int_{\Gamma} x(x) \, d\mu(x) \quad (\mu \in M)$$

is considered. It should be noted that all of the classically studied moment problems mentioned in [12] admit appropriate abstractions when put in this setting. A well-known instance of this is the abstraction of the trigonometric moment problem and subsequent solution by Raikov, cf. [9, p. 410]. More recent theorems of this type appear in [1], [5], [6], [7], [8] and [10].

A new notion of finite difference is introduced herein and those functions which admit an integral representation of the form (1.0.1) are characterized in terms of this difference operator, much in the flavor of Hausdorff's solution to the "little moment problem", cf. [8]. The main results of the work are Theorems 2.3 and 3.7. The first considers integral equations of the type (1.0.1) with respect to nonnegative $\mu$ and the second considers the same equation for signed $\mu$. The

Received by the editors September 6, 1973.


Key words and phrases. BV-functions, positive-definite function, completely monotonic function, semigroup, moment problem, semicharacter, integral representation, finite difference, vector lattice, Banach algebra, convolution of measures.
first mentioned result implies a new characterization of both positive-definite functions on groups and *-definite functions, cf. [7]. Applications of the theory to inverse and Hermitian semigroups are given in §4 and §5.

2. Positive-definite functions. Let \( S \) be a commutative semigroup with identity 1 and involution \(*\), cf. [7, p. 771], and \( C \) be the class of all complex valued functions \( f \) on \( S \). The displacement operator \( E_x \colon C \to C \) is defined by \( (E_x f)(y) = f(xy) \) (\( x, y \in S \)) and frequently \( f_x \) will be written instead of \( E_x f \), and \( J \) instead of \( E_1 \). Let \( \{ \eta_j \} \), \( \{ \alpha_j \} \) and \( \{ x_j \} \) (\( j = 1, \ldots, k \)) be finite sequences of nonnegative integers, fourth roots of unity and members of \( S \) respectively. A difference operator \( \nabla \) on \( C \) is defined by

\[
\nabla f(\cdot; \{ \alpha_j \eta_j x_j \}) = \prod_{j=1}^{k} \left( I + \frac{\alpha_j}{2} E_x + \frac{\alpha_j^*}{2} E_x^* \right)^{n_j} f.
\]

It follows that

\[
\nabla_{k+1} f(x; \alpha_1 x_1, \ldots, \alpha_{k+1} x_{k+1}) = \nabla_k f(x; \alpha_1 x_1, \ldots, \alpha_k x_k) + (\alpha_{k+1}/2) \nabla_k f(x x_{k+1}; \alpha_1 x_1, \ldots, \alpha_k x_k) + (\alpha_{k+1}^*/2) \nabla_k f(x x_{k+1}; \alpha_1 x_1, \ldots, \alpha_k x_k).
\]

Here and throughout this paper \(*\) will denote both the semigroup involution and complex conjugation. Since \( E_x E_y = E_{xy} \), the commutivity of \( S \) implies that \( \nabla \) is independent of the order in which the increments \( x_j \) are chosen so that the operator \( \nabla \) is well defined. A subset \( X \) of \( S \) such that every \( y \in S \) except possibly the identity admits a finite factorization of the form \( \Pi_j x_j \) where \( x_j \in X \cup X^* \) (\( X^* = \{ x \in S | x^* \in X \} \)) will be called a generator set for \( S \). Henceforth \( X \) will denote an arbitrary but fixed generator set for \( S \).

If \( f \in C \) and both \( f(1) \geq 0 \) and \( \nabla f(1; \{ \alpha_j x_j \}) \geq 0 \) for all choices of \( x_j \in X \) and for all choices of fourth roots of unity \( \alpha_j \), then \( f \) will be called positive \( X \)-definite. Later it will be shown that this notion of positive definite is equivalent to the classical concept when \( S \) is a discrete group and agrees with that introduced in [7] when \( X = S \).

The difference operator \( \Delta \) used in [8] can be extended by defining

\[
\Delta_{n+1} f(\cdot; \alpha_1 x_1, \ldots, \alpha_{n+1} x_{n+1}) = \Delta_n f(\cdot; \alpha_1 x_1, \ldots, \alpha_n x_n) - \alpha_{n+1} \Delta_n f x_{n+1} (\cdot; \alpha_1 x_1, \ldots, \alpha_n x_n).
\]
If $S$ is Hermitian \cite[p. 772]{7} i.e. $x = x^*$ then

\begin{equation}
\nabla f(\cdot; \{\sigma_j \rangle_j}) = \prod_j \left[ I + \left( \frac{\sigma_j + \sigma_j^*}{2} \right) \mathcal{A} \right] f = \Delta f(\cdot; \left\{ - \frac{\sigma_j + \sigma_j^*}{2} \right\} \langle_j).
\end{equation}

Thus $f \in K_X$ if and only if $\Delta f(1; \{\sigma_j \langle_j \}) \geq 0$ for all choices of $\langle_j \in X$ and square roots of unity $\sigma_j$.

If $S$ is a group, $\{\langle_j \}$ $\subset S$ and $\{\sigma_j \}$ is a set of fourth roots of unity for $j = 1, 2, \ldots, k$ then $\prod_j (2I + \alpha_j E_{\langle_j} + \alpha_j^* E_{\langle_j}^*) = \prod_j (I + \alpha_j E_{\langle_j} + \alpha_j^* E_{\langle_j} + \alpha_j \alpha_j^* E_{\langle_j \langle_j})$ so that the difference operator $\nabla$ defined herein is related to the classical difference operator $\Delta$ used in \cite{8} by

\begin{equation}
2^k \nabla f(\cdot; \{\sigma_j \langle_j \}) = \Delta f(\cdot; \{- \sigma_j \langle_j, - \sigma_j^* \langle_j \} \}
\end{equation}

Thus in terms of differences $f$ is positive definite if and only if all differences of the form $\Delta f(1; \{\sigma_j \langle_j, \alpha_j^* \langle_j \})$ are nonnegative where $\{\langle_j \}$ is an arbitrary finite subset of a generator set $X$.

Let $K_X$ denote the cone of positive $X$-definite functions. If $X$ is a $*$-semi-character i.e. if $x$ is a not necessarily bounded\(^{(2)}\) complex-valued function on $S$ such that $x \neq 0$, $x(xy) = x(x)x(y)$ and $x^*(x) = x(x^*)$ then

\begin{equation}
\nabla x(\langle ; \{\sigma_j \langle_j \}) = x(\langle) \prod_j \left( 1 + \frac{\sigma_j}{2} x(\langle_j) + \frac{\sigma_j^*}{2} x^*(\langle_j) \right)
\end{equation}

thus $x$ is positive $X$-definite if and only if both

\begin{equation}
|\text{Re} \; x(\langle)| \leq 1 \quad \text{and} \quad |\text{Im} \; x(\langle)| \leq 1
\end{equation}

for all $\langle \in X$. The class of all such $*$-semicharacters is denoted by $\Gamma_X$. The cone $K_X$ has a natural description in terms of $\Gamma_X$. The following proposition is essential to the development.

**Proposition 2.1.** Let $f \in K_X$, $\{\langle_j \}_{j=1,2,\ldots,k} \subset X \cup X^*$ and $\langle \in S$. Then

\begin{equation}
\left| f\left( \prod_j \langle_j \right) \right| \leq 2^k f(1),
\end{equation}

\begin{equation}
f(\langle^*) = f^*(\langle).
\end{equation}

**Proof.** Let $A_k$ denote the set of all functions $\sigma \equiv \sigma(\langle)$ on the first $k$ natural numbers whose range is contained in the three element set $\{0, 1, *\}$. If $\beta^4 = 1$ then

\[^{(2)}\] In \cite{7}, all $*$-semicharacters were, by definition assumed bounded. Here they are not.
\[ \sum_{\prod_j \alpha_j = \beta} \nabla f(\cdot ; \{ \alpha_j x_j \}) = \sum_{\prod_j \alpha_j = \beta} \Pi_j \left( I + \frac{\alpha_j}{2} E_{x_j} + \frac{\alpha_j^*}{2} E_{x_j^*} \right) f \]

\[ = \sum_{\prod_j \alpha_j = \beta} \Pi_j \left( \frac{\alpha_j}{2} \right) f_{x_1 \cdots x_k} \]

\[ = 4^{k-1} f + 2^{k-2} \beta^* f_{x_1 \cdots x_k} + 2^{k-2} \beta f_{x_1^* \cdots x_k^*} \]

\[ + \sum_{(\sigma \neq 1, *, 0)} \sum_{\prod_j \alpha_j = \beta} \Pi_j \left( \frac{\alpha_j}{2} \right) f_{x_1^* \cdots x_k^*}. \]

But \( \Sigma_{\prod_j \alpha_j = \beta} \Pi_j (\alpha/2)^j = 0 \) so that if \( x = \Pi_j x_j \) then

\[ (2.1.3) \sum_{\prod_j \alpha_j = \beta} \nabla f(\cdot , \{ \alpha_j x_j \}) = [2^{k-1} f + \frac{1}{2} \beta f_{x} + \frac{1}{2} \beta^* f_{x^*}] \]

for all functions \( f \).

For \( f \in K \) and \( x_j \in X \), evaluation of (2.1.3) at 1 shows

\[ 2^{k-1} f(1) + \frac{1}{2} \beta f(x) + \frac{1}{2} \beta^* f(x^*) \geq 0. \]

Setting \( \beta = 1 \) and \( i \) shows

\[ \text{Re } f(x) = \frac{1}{2} \{ f(x) + f(x^*) \}, \quad \text{Im } f(x) = (i/2) \{ f(x) - f(x^*) \} \]

from which (2.1.1) and (2.1.2) follow.

**PROPOSITION 2.2.** Every extreme point of \( B_X \) is a \( * \)-semicharacter in \( \Gamma_X \).

**PROOF.** First observe that \( \nabla f(\cdot ; \alpha x) \) (\( \alpha^4 = 1 \)) is in \( K_X \) whenever \( f \in K_X \) and \( x \in X \). Indeed if \( x = x_{k+1} \) then

\[ \nabla_k [\nabla f(\cdot ; \alpha x)] (1; \alpha_1 x_1, \ldots, \alpha_k x_k) = \left[ \prod_j (I + \alpha_j E_{x_j} + \alpha_j^* E_{x_j^*}) f \right](1) \]

\[ = \nabla_{k+1} f(1; \alpha_1 x_1, \ldots, \alpha_{k+1} x_{k+1}). \]

Now suppose \( f \) is an extreme point of \( B_X \). Direct computations shows \( f = \frac{1}{4} \sum_{\alpha^4 = 1} \nabla f(\cdot ; \alpha x) \) so that \( f - \frac{1}{4} \nabla f(\cdot ; \alpha x) \in K_X \). Since \( f \) is on an extreme ray of \( K_X \) there exists \( \lambda_\alpha > 0 \) such that \( \lambda_\alpha f = \nabla f(\cdot ; \alpha x) \). Evaluation at \( y = 1 \) shows \( \lambda_1 = 1 + \frac{1}{2} f(x) + \frac{1}{2} f(x^*) \) and \( \lambda_i = 1 + (i/2) f(x) - (i/2) f(x^*) \). Evaluation at arbitrary \( y \) gives

\[ f(x)f(y) + f(x^*)f(y) = f(xy) + f(x^*y), \]

\[ f(x)f(y) - f(x^*)f(y) = f(xy) - f(x^*y). \]
Thus \( f(x)f(y) = f(xy) \) for all \( x \in X \) and \( y \in S \) and since \( X \) is a generator set this same multiplicative property holds for arbitrary \( x \in S \). The assertion follows from (2.0.7) and (2.1.2).

For each \( x \in S \) let \( \hat{x} \) denote the evaluation function on \( \Gamma_X \), i.e. \( \hat{x}(\chi) = \chi(x) \) for all \( \chi \in \Gamma_X \).

**Theorem 2.3.** A real-valued function \( f \) is positive \( X \)-definite if and only if there exists a necessarily unique nonnegative regular Borel measure \( \mu_f \) on \( \Gamma_X \) such that \( f(x) = \int_{\Gamma_X} \hat{x} \ d\mu_f \).

**Proof.** The existence of the representing measure follows from the Krein-Milman Theorem and Proposition 2.2. Since \( x \) is a continuous linear functional on \( E_X \), uniqueness is a consequence of the Stone-Weierstrass theorem and follows along the lines of [5, Theorem 1.2]. If \( f \) admits such an integral representation then we must have

\[
\nabla f(1, \{a_j x_j\}) = \int_{\Gamma_X} \prod_j \left( 1 + \frac{a_j}{2} \hat{x}_j + \frac{a_j^*}{2} \hat{x}_j^* \right) d\mu_f
\]

(2.3.1)

\[
\int \prod_{\sigma_f \text{ real}} \left( 1 \pm \text{Re} \chi(x_j) \right) \prod_{\sigma_f \text{ imag}} \left( 1 \pm \text{Im} \chi(x_j) \right) d\mu_f > 0
\]

and the assertion follows.

**Remark.** Theorem 2.3 may be used to argue as in [5, Corollary 1.3] that \( \Gamma_X \) is precisely the set of extreme points of \( B_X \).

It follows from (2.1.1) that the semicharacters in \( \Gamma_S \) are bounded so that \( |\chi(x)| \leq 1 \) for all \( \chi \in \Gamma_S \) and all \( x \in S \). If the integral representation theorem above is compared with [7, Theorem 2.1] then the formally different notion of \( * \)-definite defined therein is seen to agree with the concept of positive \( S \)-definite here. Moreover if \( S \) is a Hermitian semigroup then \( 0 \leq \nabla f(1; \pm x) \) implies

(2.3.2)

\[
|f(x)| = |\text{Re} f(x)| \leq f(1) = 1
\]

for all \( f \in K_X \) and all \( x \in X \). But if \( f \) is a semicharacter then (2.3.2) holds for all \( x \in S \) so that again \( \Gamma_S = \Gamma_X \) and hence the notions of positive \( X \)-definite and Hermitian definite introduced in [7] coincide for all choices of \( X \). Finally if \( S \) is an inverse semigroup, cf. [2, §1.9], then \( |\chi| = 1 \) for all \( \chi \in \Gamma_X \) so that \( \Gamma_S = \Gamma_X \) also. In particular for \( S \) a group, the classical notion of positive definite is synonymous with that of positive \( X \)-definite.

**Corollary 2.4.** A complex valued function \( f \) on an involution semigroup \( S \) is \( * \)-definite if and only if \( f \) is positive \( S \)-definite. If \( f \) is bounded in absolute value by \( f(1) \) or if \( S \) is either an inverse semigroup or a Hermitian semigroup then \( f \) is positive \( X \)-definite if and only if \( f \) is positive \( S \)-definite.
Proof. The only unproved assertion is the case where $f$ is bounded in absolute value by $f(1)$. If $f$ is positive $S$-definite then the integral representation theorem implies $|f| \leq f(1)$. To prove the converse let $K$ denote the closed cone of all $f \in K_X$ such that $|f| \leq f(1)$. Then $B = B_X \cap K$ is a compact base for $K$ and the argument that the extreme points of $B$ are contained in $\Gamma_X$ follows as in Proposition 2.2. But since each extreme point is a bounded function the set of all extreme points of $B$ is contained in $\Gamma_S$ so that the integral representation of Theorem 2.3 holds. The converse assertion follows.

Example 2.5. Consider the product $S = N \times N$ of the additive semigroup $N$ of nonnegative integers with itself, and with involution $(m, n)^* = (n, m)$. If $X = \{(1, 0), (0, 1)\}$ then $\Gamma_X$ is the set of all maps $\chi_x$ defined by $\chi_x(m, n) = z^m z^*(z^*)^n$ where $|\text{Re} z| \leq 1$ and $|\text{Im} z| \leq 1$ $(0^0 = 1)$ while $\Gamma_S = \{\chi_x \in \Gamma_X \mid |z| \leq 1\}$. Thus $\Gamma_S$ and $K_S$ are properly contained in $\Gamma_X$ and $K_X$ respectively. The cones $K_X$ and $K_S$ provide all solutions with nonnegative measures to the following respective moment problems

\begin{align*}
(2.5.1) & \quad a_{m,n} = \int_{|\text{Re} z| \leq 1; |\text{Im} z| \leq 1} z^m z^*(z^*)^n \, d\mu(z) \\
(2.5.2) & \quad a_{m,n} = \int_{|z| \leq 1} z^m z^*(z^*)^n \, d\mu(z), \quad m, n = 0, 1, 2, \ldots
\end{align*}

3. BV-functions. Theorem 2.3 implies that the map $f \mapsto \mu_f$ of the real linear span of $E_X$ onto its representing measure is an isomorphism. Let $|\mu|$ denote the variation of a measure $\mu$; then the lattice properties of the regular Borel measures can be imposed on $E_X$ by defining

\begin{equation}
(3.0.1) \quad f \vee (-f) \equiv |f|(x) = \int_{\Gamma_X} \hat{x} \, d|\mu_f|(x).
\end{equation}

Also define $\|f\| = |f|(1) = \|\mu_f\|$.

An explicit description of those functions in $E_X$ as well as a characterization of their variation will be given. First the total variation of regular Borel measures on $\Gamma_X$ will be described in terms of certain partitions of unity on $\Gamma_X$.

For this let $P$ denote the collection of all partitions obtained upon expansion of products of the form

$$\prod_i \frac{1}{2^{n_j}} \left[ \left( 1 + \frac{\alpha_j}{2} \hat{x}_j + \frac{\alpha_j^*}{2} \hat{x}_j^* \right) + \left( 1 - \frac{\alpha_j}{2} \hat{x}_j - \frac{\alpha_j^*}{2} \hat{x}_j^* \right) \right]^{n_j}$$

where $\sigma = 1, i$ and $x \in X$. A typical member of $P$ is then a set

$$\mathcal{P} = \{P_{i_1, \ldots, i_k} \mid i_j = 0, \ldots, n_j, j = 1, \ldots, k\}$$

of functions.

$$P_{i_1, \ldots, i_k} = \frac{1}{2^n} \prod_{j} \binom{n_j}{i_j} \left( 1 + \frac{\alpha_j}{2} \hat{x}_j + \frac{\alpha_j^*}{2} \hat{x}_j^* \right)^{i_j} \left( 1 - \frac{\alpha_j}{2} \hat{x}_j - \frac{\alpha_j^*}{2} \hat{x}_j^* \right)^{n_j-i_j}$$
where $n = \Sigma \eta_j$, $\sigma_j = 1$ or $i$ and $x_j \in X$. If $\mu_\nu$ represents $f$ then (2.3.1) implies

$$
(3.0.2) \quad \int_{\Gamma_X} \sum_{i_1 \cdots i_k} d\mu_\nu = \frac{1}{2^n} \prod_{j} \left( ^{n_j} \binom{\eta_j}{i_j} \right) \nabla f(1; \{\sigma_j x_j, \sigma_j(i_j - \eta_j) x_j\}).
$$

Thus if

$$
(3.0.3) \quad \|f\|_{\{x_j\}, \{\eta_j\}, \kappa} = \frac{1}{2^n} \sum_{\sigma} \prod_{j} \left( ^{n_j} \binom{\eta_j}{i_j} \right) \|\nabla f(1; \{\sigma_j x_j, \sigma_j(i_j - \eta_j) x_j\})|,
$$

then

$$
(3.0.4) \quad \|f\| \geq \sup \|f\|_{\{x_j\}, \{\eta_j\}, \kappa}
$$

where the $x_j$'s form a finite and possible repetitious sequence of elements in $X$, the $n_j$'s are nonnegative integers and $j = 1, 2, \ldots, k$ where $k$ is allowed to vary.

**Lemma 3.1.** If $f \in E_X$ then $\|f\| = \sup \|f\|_{\{x_j\}, \{\eta_j\}, \kappa}$.

**Proof.** The inequality (3.0.4) will be reversed by showing that $\hat{P}$ is rich enough to describe the total variation of the measures $\mu$ on $\Gamma_X$. That is

$$
(3.1.1) \quad \|\mu\| = \sup_{P \in \mathcal{P}} \sum_{P \in \mathcal{P}} \left| \int P \, d\mu \right|.
$$

Let $x_1$ and $x_2$ be two distinct members of $\Gamma_X$. There exists $x \in X$ such that $x_1(x) \neq x_2(x)$. Suppose $\text{Re } x_1(x) \neq \text{Re } x_2(x)$. Let $G_1$ and $G_2$ be disjoint open sets about $x_1(x)$ and $x_2(x)$ respectively. Since $\text{Re } \hat{x}$ is continuous on $\Gamma_X$, their inverse images $(\text{Re } \hat{x})^{-1}[G_i]$ ($i = 1, 2$) are open and disjoint subsets of $\Gamma_X$.

By an appropriate change of variables, Bernstein polynomials on the interval $[-1, 1]$ take the form

$$
(B_{n\delta})(t) = \frac{1}{2^n} \sum_{i_1 = 0}^{n} \left( ^{n} \binom{n}{i_1} \right) g \left( 1 - 2 \frac{i_1}{n} \right) (1 + t)^{i_1} (1 - t)^{n-i_1}.
$$

From [4, p. 6] there exists a sufficiently large integer $N$ such that if $n > N$ then the partition of unity

$$
\left\{ \frac{1}{2^n} \left( ^{n} \binom{n}{i_1} \right) (1 + t)^{i_1} (1 - t)^{n-i_1} \right\}_{i_1}
$$

contains a subpartition which is arbitrarily close to unity on $G_1$ and arbitrarily close to zero on $G_2$. Thus

$$
\left\{ \frac{1}{2^n} \left( ^{n} \binom{n}{i_1} \right) \left( 1 + \frac{\hat{x}}{2} + \frac{\hat{x}^*}{2} \right)^{i_1} \left( 1 - \frac{\hat{x}}{2} - \frac{\hat{x}^*}{2} \right)^{n-i_1} \right\}_{i_1}
$$

has the same separation property with respect to the open $\text{Re } \hat{x}^+(G_1)$ and $\text{Re } \hat{x}^-(G_2)$. If $\text{Im } x_1(x) \neq \text{Im } x_2(x)$ then a similar argument yields a separating.
partition of the form
\[
\left\{ \frac{1}{2^n} \binom{n}{i_1} \left(1 + \frac{i}{2} x - \frac{i}{2} x^*\right)^{i_1} \left(1 - \frac{i}{2} x + \frac{i}{2} x^*\right)^{n-i_1} \right\}.
\]

The arguments of Lemmas 3.1 and 3.2 of [8] can now be used to prove the lemma.

**Lemma 3.2.** Let \( j = 2, 3, \ldots, k \) and \( \sigma, \sigma_2, \ldots, \sigma_k \) be fourth roots of unity. Then

\[
\sum_{i_1=0}^{n} \binom{n}{i_1} \left| \nabla f(1; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right| \leq \frac{1}{2} \sum_{i_1=0}^{n+1} \binom{n+1}{i_1} \left| \nabla f(1; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n-1)x) \right|.
\]

**Proof.**

\[
\left| \nabla f(1; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
\leq \frac{1}{2} \left| \nabla f(1; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
+ \frac{\sigma}{2} \left| \nabla f(x; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
+ \frac{\sigma^*}{2} \left| \nabla f(x^*; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
+ \frac{1}{2} \left| \nabla f(1; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
- \frac{\sigma}{2} \left| \nabla f(x; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
- \frac{\sigma^*}{2} \left| \nabla f(x^*; \{\sigma_j x_j\}, \sigma_1 x, \sigma(i_1 - n)x) \right|
\]

The assertion follows upon applying (2.0.2), regrouping and observing that

\[
\binom{n}{i_1} + \binom{n}{i_1 - 1} = \binom{n+1}{i_1} \quad \text{if} \quad n \gg i_1 > 0.
\]

In particular the above lemma implies that \( \|f\|_{\{x_j\}, \{n_j\}, k} \) is an increasing function of \( n_j \) and \( k \). Moreover duplicate entries among \( \{x_j\} \) may be eliminated as follows. Suppose \( x_1 = x_2 \) then

\[
\|f\|_{\{x_j\}, \{n_j\}, k} = \|f\|_{\{x_2, \ldots, x_k\}, \{n_1 + n_2, n_3, \ldots, n_k\}, k}
\]

since
When the $x_j$'s are distinct with $X' = \{x_j\}$ and $n_j = n$ for all $j$ the notation $(\{x_j\}, \{n_j\}, k)$ will be replaced by $(X', n)$. Thus

\begin{equation}
\sup \|f\|_{(x_j), \{n_j\}} = \lim \|f\|_{(x', n, k)}
\end{equation}

where the set $(X', n)$ is directed by $(X', n) \supseteq (X'', n')$ whenever $X' \supseteq X''$, $n \geq n'$. For arbitrary $f \in C$ the total variation $\|f\|$ is defined by (3.2.1). If $f(x) = f^*(x^*)$ and if $\|f\| < \infty$ then $f$ will be called a BVX-function or equivalently $f \in \text{BVX}(S)$. Lemma 3.1 implies $\text{BVX}(S) \supseteq E_X$ and the main theorem, Theorem 3.7 of this section will show that equality holds. Lemmas 3.3–3.6 pave the way for this result.

**Lemma 3.3.** Let $f \in \text{BVX}$, $x \in X$ and $\alpha$ be a fourth root of unity. Then

$$\|f\| = \frac{1}{2} \|\nabla f(\cdot; \alpha x)\| + \frac{1}{2} \|\nabla f(\cdot; -\alpha^* x^*)\|.$$ 

**Proof.** From (3.2.1),

$$\|f\| \leftarrow \frac{1}{2^{kn+1}} \sum_i \left( \begin{array}{c} n \\ i_0 \end{array} \right) \left( \begin{array}{c} n \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n \\ i_k \end{array} \right)$$

$$\cdot |\nabla f(1; \{\sigma_j x_j, \sigma_j(i_j-n)x_j^*\}, \alpha_i x, \alpha^*(i_i-1)x^*)|$$

$$= \frac{1}{2} \cdot \frac{1}{2^{kn}} \sum_i \left( \begin{array}{c} n \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n \\ i_k \end{array} \right) |\nabla \left( f + \frac{\alpha}{2} f_x + \frac{\alpha^*}{2} f_x^* \right) (1; \{\sigma_j x_j, \sigma_j(i_j-n)x_j^*\})|$$

$$+ \frac{1}{2} \cdot \frac{1}{2^{kn}} \sum_i \left( \begin{array}{c} n \\ i_1 \end{array} \right) \cdots \left( \begin{array}{c} n \\ i_k \end{array} \right) |\nabla \left( f - \frac{\alpha}{2} f_x - \frac{\alpha^*}{2} f_x^* \right) (\sigma_j x_j, \sigma_j(i_j-n)x_j^*)|$$

$$\rightarrow \frac{1}{2} \|\nabla f(\cdot; \alpha x)\| + \frac{1}{2} \|\nabla f(\cdot; \alpha^* x^*)\|.$$ 

In particular $\nabla f(\cdot; \{\alpha_j x_j\}) \in \text{BVX}$ whenever $f \in \text{BVX}$.

**Corollary 3.4.** Let $f \in \text{BVX}$, $T_k$ be a set of distinct 4th root of unity-valued functions $\alpha(\cdot)$ on $\{j|j = 1, \ldots, k\}$ and $n_\alpha$ a nonnegative integer for each $\alpha$. If $x_1, \ldots, x_k \in X$ then

$$\sum_{\alpha \in T_k} n_\alpha \|\nabla f(\cdot; \{\alpha_j x_j\})\| = \left\| \sum_{\alpha \in T_k} n_\alpha \nabla f(\cdot; \{\alpha_j x_j\}) \right\|.$$ 

**Proof.** Let $T_k'$ denote the set of all 4th root of unity-valued functions on $\{1, 2, \ldots, k\}$ which are not in $T_k$ and $n = \max_\alpha n_\alpha$. Then
A second application of the triangle inequality verifies the assertion.

**Lemma 3.5.** Let \( f \) be a complex valued function on \( S \), \( T_k \) be all fourth root of unity valued functions \( \alpha \) defined on \( \{1, \ldots, k\} \) and \( x_1, \ldots, x_k \in S \). Then

\[
\begin{align*}
4^k n \| f \| = & \left\| \sum_{\alpha_1} \sum_{\alpha_2} \cdots \sum_{\alpha_k} n \|
\right. \\
& \left. \sum f(\cdot; \{\alpha_j x_j\}) \| (\alpha \in T_k \cup T_k') \right. \\
\leq & \left\| \sum_{\alpha \in T_k'} n \|
\right. \\
& \left. \sum f(\cdot; \{\alpha_j x_j\}) \| + \left\| \sum_{\alpha \in T_k} (n - n_{\alpha}) \|
\right. \\
& \left. \sum f(\cdot; \{\alpha_j x_j\}) \| \right. \\
\leq & \sum_{\alpha \in T_k} n_{\alpha} \||
\left. \sum f(\cdot; \{\alpha_j x_j\}) \| + \left\| \sum_{\alpha \in T_k} (n - n_{\alpha}) \|
\right. \\
& \left. \sum f(\cdot; \{\alpha_j x_j\}) \| \right. \\
\leq & n \sum_{\alpha \in T_k \cup T_k'} \||
\left. \sum f(\cdot; \{\alpha_j x_j\}) \| \right. \\
= & 4^k n \sum_{\alpha_1} \cdots \sum_{\alpha_{k-1}} 4 \||
\left. \sum f(\cdot; \{\alpha_j x_j\}) \| \right. \\
& \{\alpha_j x_j \mid j = 1, \ldots, k - 1\}\| = 4^k n \| f \|.
\end{align*}
\]

Proof. Let \( A_k \) be as defined in the proof of Proposition 2.1. Then the above summation can be rewritten as

\[
\begin{align*}
\frac{1}{2^k} \sum_{\alpha} \sum_{\alpha_k} \sum_{\sigma \in A_k} \Pi_j \left( \frac{\alpha_j}{2} \right)^{\sigma} f(\Pi_j x_j) \\
= & f(\Pi_j x_j) + \frac{1}{2^k} \sum_{\sigma \neq 1} \left( \sum_{\alpha} \Pi_j \alpha_j^{\sigma} \left( \frac{\alpha_j}{2} \right)^{\sigma} \right) f(\Pi_j x_j) \\
= & f(\Pi_j x_j) + \frac{1}{2^k} \sum_{\sigma \neq 1} \left( \sum_{\alpha \in T_k} \prod_{i=1}^{k-1} \left( \alpha_k^{\sigma} \alpha_i^{\sigma} \right) \sum_{\alpha_k} \alpha_k^{\sigma} f(\Pi x_j) \right).
\end{align*}
\]

But a trivial computation shows that \( \Sigma_{\alpha_k} \alpha_k^{\sigma} \alpha_k^{\sigma} = 0 \) if \( \sigma_k \neq 1 \) and the assertion is proved.

Let \( f \in E_X \) with representing measure \( \mu_f \) and \( x_1, \ldots, x_k \in X \cup X^* \). Then if \( x = \Pi_j x_j \), Lemma 3.5, and formulas (2.0.6) and (3.0.1) imply
The variation $|f|$ of $f$ can be characterized by (3.5.1) below:

$$
|f|(x) = \frac{1}{2^k} \sum_{\alpha \in \mathbb{T}_k} \left( \prod_{j} \alpha_j^* \right) \nabla |f|(1; \{\alpha_j x_j\})
$$

and hence the variation $|f|$ of $f$ can be characterized by (3.5.1) below:

$$
|f|(x) = \frac{1}{2^k} \sum_{\alpha} \left( \prod_{j} \alpha_j^* \right) \| \nabla f(\cdot; \{\alpha_j x_j\}) \|.
$$

Since Lemma 3.3 implies $\| \nabla f(\cdot; \{\alpha_j x_j\}) \| < \infty$ whenever $f \in BVX$, (3.5.1) will be taken as a definition for $|f|$ for all $f \in BVX$.

It remains to be seen that $|f|$ is well defined. For this let $x_1, \ldots, x_k \in X$ and $x$ denote their product. Then (2.1.3) implies

$$
(3.5.2) \quad \text{Re} |f|(x) = \frac{1}{2} \| 2^{k-1} f + \frac{1}{2} f_x + \frac{1}{2} f_x^* \| - \frac{1}{2} \| 2^{k-1} f - \frac{1}{2} f_x - \frac{1}{2} f_x^* \|.
$$

Thus $|f|(x)$ is at worst dependent on the number of factors of $x$ rather than the factors themselves. To see that $f$ is also independent of this number observe that if $x$ admits a factorization by $k + m$ elements from $X$ and if $|f|_0(x)$ denotes the variation of $f$ at $x$ computed with respect to this latter factorization then

$$
\text{Re} |f|_0(x) = \frac{1}{2} \left\| 2^{m-1} (2m - 1) f + (2^{k-1} f + \frac{1}{2} f_x + \frac{1}{2} f_x^* ) \right\|
$$

and

$$
\text{Im} |f|_0(x) = \frac{1}{2} \| 2^{m-1} (2m - 1) f + (2^{k-1} f - \frac{1}{2} f_x - \frac{1}{2} f_x^* ) \|.
$$

where $|f|(x)$ is the variation defined by the factors $x_j$. 

Lemma 3.6. If $\beta$ is a fourth root of unity and $f \in BVX$ then

$$
(3.6.1) \quad \frac{1}{2} |\nabla f(\cdot; \beta x)| + \frac{1}{2} |\nabla f(\cdot; -\beta^* x^*)| = |f|,
$$
(3.6.2) \[ |\nabla f(\cdot; \beta x)| - |\nabla f(\cdot; -\beta^* x^*)| = \beta|f|_x + \beta^*|f|_{x^*}, \]
for all \( x \in X \).

**Proof.** Evaluation of (3.6.1) at 1 is just Lemma 3.3. If \( x_1, \ldots, x_k \in X \cup X^* \) and \( y = \Pi_j x_j \) then
\[ |\nabla f(\cdot; \beta x)|(y) = \frac{1}{2^k} \sum_{\alpha} \prod_j \alpha_j\|\nabla f(\cdot; \{\alpha_j x_j\}, \beta x)\|, \]
so that equality holds in (3.6.1) for evaluation at \( y \) from Lemma 3.3. To verify (3.6.2) observe that both

(3.6.3) \[ |\nabla f(\cdot; \beta x)|(1) = \frac{1}{4} \sum_{\alpha^4 = 1} \|\nabla f(\cdot; \beta x, \alpha 1)\|, \]

(3.6.4) \[ |f|_1(x) = |f|(x) = \frac{1}{4} \sum_{\alpha^4 = 1} \alpha^*\|\nabla f(\cdot; \alpha x)\|. \]

To prove that (3.6.2) holds when evaluated at 1, substitute the above two equations into (3.6.2) and show that positive and negative real as well as positive and negative imaginary parts of each side of the equation agree respectively. This results in verifying sixteen equations which can be written succinctly as

(3.6.5) \[ \|f(\cdot; \alpha x)\| + \|\nabla f(\cdot; \alpha \beta^2 x)\| = \|\nabla f(\cdot; \alpha \beta^* x, \beta 1)\| + \|\nabla f(1; -\alpha \beta^* x, -\beta 1)\|. \]

That (3.6.5) holds for all fourth roots of unity \( \alpha \) and \( \beta \) follows from Lemma 3.3 and (3.6.6) below.

(3.6.6) \[ \nabla g(\cdot; \beta 1) = \begin{cases} 0 & \text{if } \beta = -1, \\ g & \text{if } \beta^2 = -1, \\ 2g & \text{if } \beta = +1. \end{cases} \]

The right side of (3.6.2), when evaluated at \( y \), expands to
\[ \frac{1}{2^k} \sum_{\alpha} \prod_j \alpha^*_j \left[ \frac{\beta}{2} \sum_{\alpha_{k+1} x_{k+1}} \alpha^*_{k+1} \|\nabla f(\cdot; \{\alpha_j x_j\}, \alpha_{k+1} x_{k+1})\| + \frac{\beta^*}{2} \sum_{\alpha_{k+1} x_{k+1}} \alpha^*_{k+1} \|\nabla f(\cdot; \{\alpha_j x_j\}, \alpha_{k+1} x_{k+1}^*)\| \right] \]
\[ = \frac{1}{2^k} \sum_{\alpha} \prod_j \alpha^*_j \left[ \beta|\nabla f(\cdot; \{\alpha_j x_j\})|_x(1) + \beta^*|\nabla f(\cdot; \{\alpha_j x_j\})|_{x^*}(1) \right] \]
\[ = \frac{1}{2^k} \sum_{\alpha} \prod_j \alpha^*_j \left[ |\nabla f(\cdot; \{\alpha_j x_j\}, \beta x)|(1) - |\nabla f(\cdot; \{\alpha_j x_j\}, -\beta^* x^*)|(1) \right] \]
\[ = |\nabla f(\cdot; \beta x)|(y) - |\nabla f(\cdot; -\beta^* x^*)|(y). \]

**Theorem 3.7.** \( B V X = E_X \).
Proof. From the remarks following Lemma 3.1, the only part of the theorem yet to be proved is BVX \subset E_X. This will be done by showing that each BVX-function \( f \) is the difference of two positive \( X \)-definite functions. Adding half of (3.6.2) to (3.6.1) gives \( \nabla |f| (\cdot; \beta x) = |\nabla f (\cdot; \beta x)| \) for all \( x \in X \). Thus if \( \{x_j\} \subset X \),

\[
\nabla |f| (\cdot; \{a_j x_j\}) = \left[ \prod_j \left( I + \frac{a_j}{2} E_{x_j} + \frac{a^*_j}{2} E_{x^*_j} \right) \right] |f| \\
= \left| \prod_j \left( I + \frac{a_j}{2} E_{x_j} + \frac{a^*_j}{2} E_{x^*_j} \right) f \right| = |\nabla f (\cdot; \{a_j x_j\})|.
\]

In particular, \( |f| \) is positive \( X \)-definite. Let \( f^\pm = \frac{1}{2} (|f| \pm f) \). Then

\[
\nabla f^\pm (1; \{a_j x_j\}) = \frac{1}{2} \nabla |f| (1; \{a_j x_j\}) \pm \frac{1}{2} \nabla f (1; \{a_j x_j\}) \\
= \frac{1}{2} \| \nabla f (1; \{a_j x_j\}) \| \pm \frac{1}{2} \| \nabla f (1; \{a_j x_j\}) \|.
\]

But if \( g \) is any complex valued function on \( S \) such that \( \#(1) \) is real then

\[
\| g \| > \frac{1}{2} | g(1) + \frac{1}{2} g(x) + \frac{1}{2} g(x^*) | + \frac{1}{2} | g(1) - \frac{1}{2} g(x) - \frac{1}{2} g(x^*) | \\
> | g(1) | > | g(1) |.
\]

Since \( f^*(x) = f(x^*) \) for all \( x \in S \) and \( 1^* = 1 \) it follows that \( \nabla f (1; \{a_j x_j\}) \) is real so that \( f^\pm \) is positive \( X \)-definite. The assertion follows since \( f = f^+ - f^- \).

Remark. The BVX-functions are precisely those functions which can be expressed as the difference of two positive \( X \)-definite functions. As in Corollary 2.4 if \( S \) is either a Hermitian semigroup or an inverse semigroup then \( BVS = BVX \) for all choices of generator sets \( X \). If \( S \) is both square root closed and Hermitian then as remarked in [7, Proposition 4.2] the positive definite functions are just the completely monotonic (CM) function and in particular if \( S \) is a linearly ordered semilattice the CM-functions are just the nonincreasing nonnegative functions. For this classical case the usual notion of Bounded Variation agrees with that introduced here and the reader will recall that in the classical setting the BV-functions are those which can be expressed as the difference of two nonnegative, nonincreasing functions. Finally it should be noted that for arbitrary \( S \), [7, Corollary 2.2] implies \( BVS \) is a Banach algebra under pointwise multiplications since the convolution of the representing measure of two BVS-functions is the representing measure of their product.

4. Moment problem with respect to characters. A semicharacter \( \chi \in \Gamma_S \) will be called a character if \( |\chi|^2 = |\chi| \). Let \( \Gamma_0 \) denote the set of all characters equipped with the topology of pointwise convergence and consider the moment problem.
(4.0.1) \[ f(x) = \int_{\Gamma_0} \chi(x) \, d\mu_f(x), \]
where \( \mu_f \) is nonnegative (or signed).

**Theorem 4.1.** A complex valued function \( f \) on \( S \) admits an integral representation of the form (4.0.1) if and only if \( f \) is positive \( S \)-definite (or \( f \in BV(S) \)) and

(4.1.1) \[ f[(xx^*)^2] = f(xx^*). \]

**Proof.** If \( f \) admits such an integral representation then

\[ f(xx^*) = \int_{\Gamma_0} |\chi(x)|^2 \, d\mu_f(x) = \int_{\Gamma_0} |\chi(x)|^4 \, d\mu_f(x) = f[(xx^*)^2] \]

and the work of §2 and 3 imply \( f \) is positive \( S \)-definite (or \( f \in BV(S) \)).

As in §2 let \( B_S = \{ f \in K_S | f(1) = 1 \} \) and set \( B_0 = \{ f \in B_S | f[(xx^*)^2] = f(xx^*) \} \). Since \( B_0 \) is convex and closed relative to the topology of simple convergence the converse assertion can be established by showing that \( B_0 \) is an extremal subset of \( B_S \). For then it will follow from the remark following Theorem 2.3 that \( \Gamma_0 \) is the set of extreme points of \( B_0 \). Since \( \Gamma_0 \) is closed the results of [11] apply. To see that \( B_0 \) is extremal let \( f = \frac{1}{2} f_1 + \frac{1}{2} f_2 \) where \( f_j \in B(S) \) \( (j = 1, 2) \) and \( f \in B_0 \). Then

\[ f_1(xx^*) + f_2(xx^*) = 2f[(xx^*)^2] = f_1[(xx^*)^2] + f_2[(xx^*)^2] \]

for all \( x \in S \). That \( f = f_j \) follows since \( f_j(xx^*) = f_j[(xx^*)^2] \geq 0 \).

In particular if \( S \) is a group (with \( xx^* = x^{-1} \)) or a semilattice (with \( xx^* = x \)) then

(4.1.2) \[ (xx^*) = (xx^*)^2 \quad \text{for all} \quad x \in S. \]

In general (4.1.2) holds whenever \( S \) is an inverse semigroup. Motivated by [2, §1.9] \( S \) will be called *-regular if \( xx^*x = x \) for all \( x \in S \). Note that every *-regular semigroup is invertible and as is well known admits enough bounded semicharacters to separate points. Clearly every bounded semicharacter is a character.

For arbitrary \( S \) consider the evaluation map \( x \rightarrow \hat{x} \) where \( \hat{x}(\chi) = \chi(x) \) for all \( \chi \in \Gamma_0 \) and \( x \in S \). Since \( \hat{x}(\hat{x})^* \hat{x} = \hat{x} \), \( S \) is *-regular and *-isomorphism to \( S \) if and only if \( S \) admits enough characters to separate points. It can be shown that \( \hat{S} \) is the maximal *-regular, *-homeomorphic image of \( S \), cf. [2, §1.5]. But \( \Gamma(S) = \Gamma_0(S) \) so that the positive \( S \)-definite (BV\( S \)) functions which satisfy (4.1.1) can be uniquely identified with the positive \( S \)-definite (BV\( S \)) functions by the lifting map \( f \rightarrow \hat{f} \) where \( \hat{f}(x) = f(\hat{x}(x)) \, d\mu_f \).

5. BV-functions on Hermitian semigroups. Let \( S \) be Hermitian. It follows from (2.0.4) that
(5.0.1) \[ \|f\| = \lim_{n \to \infty} \frac{1}{2^n} \sum_{i_1, \ldots, i_k} \|f(1; \{-i_jx_j, (n - i_j)x_j\})\| \]

where the limit is taken with respect to \( n \) and all finite subsets \( \{x_1, \ldots, x_k\} \) of a fixed generator set \( X \) for \( S \). Moreover (3.5.1) implies

(5.0.2) \[ |f(x)| = \frac{1}{2^k} \sum_{\sigma} \left( \prod_{j} \sigma_j \right) \|\Delta f(\sigma; \{-\sigma_jx_j\})\| \]

where the summation is taken over all square roots of unity \( \sigma \). In particular if \( S = X \) then \( |f(x)| = \frac{1}{2} \|f + f_x\| - \frac{1}{2} \|f - f_x\| \). For convenience the BV-functions defined in [8] will be denoted by BV(CM). Then BV(CM) \( \subset \) BV with \( \|f\|_{CM} = \|f\| \) and \( |f|_{CM} = |f| \) whenever \( \|f\|_{CM} < \infty \).

If \( S \) is a Hermitian group (i.e. a group such that \( x^2 = 1 \) for all \( x \in S \)) then the only CM-functions on \( S \) are constant functions thus the BV(CM) functions form a one-dimensional space while the BV-functions contain all of the characters. Also the equalities \((I - E_x)(I + E_x) = 0\) and \((I \pm E_x)^p = 2^{p-1}(I \pm E_x)\) reduce (5.0.1) quite simply to

\[ \|f\| = \lim_{n} \frac{1}{2^k} \sum_{i_1, \ldots, i_k} \|\Delta f(1; \{\sigma_jx_j\})\|. \]

In this case it is easy to see that \( S \) admits a base, i.e. a subset \( X \) which is maximal with respect to the property that \( \Pi x_0^\beta = 1 \), where \( \beta_j \) is a 0-1 valued function on the first \( k \) natural numbers. It follows that \( X \) is a minimal generator set.

Finally the moment problem \( f(n) = \int_{0}^{1} t^n \, d\mu(t) \) can be put into the setting of this work by selecting \( S \) to be the Hermitian semigroups of nonnegative additive integers. The \(*\)-semicharacters are then just the maps \( n \mapsto t^n \ (-1 < t < 1) \). Since the integer 1 is a generator set for \( S \), (5.0.1) reduces to

\[ \|f\| = \lim_{n} \frac{1}{2^n} \sum_{i_1, \ldots, i_k} \|\Delta f(0; \{-1, \ldots, -1, 1, \ldots, 1\})\|. \]

Recall that Hausdorff's solution to the little moment problem \( f(n) = \int_{0}^{1} t^n \, d\mu(t) \) is given by

\[ \|f\| = \lim_{n} \sum_{i_1} \left( \frac{n}{i_1} \right) \|\Delta_{n-i} f(0; -1, \ldots, -1)\|, \]

cf. [8, Corollary 6.1].

It is interesting to note that the semicharacters \( f: n \mapsto t^n \ (-1 \leq t < 0) \) while positive-definite are not BV(CM). Indeed, if \( f \) is such a semicharacter then
\[ \|f\|_{(CM)} = \lim_{n} \sum_{i_1} \left(1 + i_1 \right) \left(1 - i_1 \right)^n = \lim_{n} \sum_{i_1} (-1)^{i_1} (1 - i_1)^n \]
\[ = [(1 - t) - t]^n = (1 - 2t)^n \to \infty. \]

REFERENCES

1. H. Bauer, *Konvexität in topologischen Vektorräumen*, Lecture Notes, University of Hamburg, West Germany.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802