NORMAL STRUCTURE OF THE ONE-POINT STABILIZER OF A DOUBLY-TRANSITIVE PERMUTATION GROUP. II

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ABSTRACT. The main result is that the socle of the point stabilizer of a doubly-transitive permutation group is abelian or the direct product of an abelian group and a simple group. Under certain circumstances, it is proved that the lengths of the orbits of a normal subgroup of the one point stabilizer bound the degree of the group. As a corollary, a fixed nonabelian simple group occurs as a factor of the socle of the one point stabilizer of at most finitely many doubly-transitive groups.

Introduction. This paper is a continuation of our study of the normal structure of the one-point stabilizer of a doubly-transitive permutation group, a study which we began in part I [9]. Here we prove the following theorems:

Theorem A. Let G be a doubly-transitive permutation group on a finite set X. Suppose that x is an element of X. Then either

(i) $G_x$ has an abelian normal subgroup $\neq 1$, or
(ii) $G_x$ has a unique minimal normal subgroup, and this minimal normal subgroup is simple.

Actually, we prove slightly more than this. In fact, it is shown that the socle of $G_x$ is either an abelian group or the direct product of an abelian group (possibly 1) and a simple group. Using this and a recent theorem of Aschbacher [11], it follows that in an unknown doubly-transitive group of minimal degree the socle of $G_x$ is either abelian or simple.

Theorem B. Let G be a doubly-transitive permutation group on a finite set X and x an element of X. Suppose $N^X$ is a normal subgroup of $G_x$. Let $|X| = n$ and suppose that on $X - x$ all orbits of $N^X$ have length s. Then either

(i) $N^X$ is semiregular on $X - x$, or
(ii) G is a normal extension of $L_n(q)$ (in its natural doubly-transitive representation), or
(iii) $n < 2(s - 1)^2$.

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In both cases (i) and (ii), it is not possible to bound \( n \) in terms of \( s \). Indeed, even if \( s = 2 \), \( n \) can be arbitrarily large. The bound of (iii), however, does not seem to be the best possible. There is a condition on \( N^x \) under which it can be improved. If \( N^x \lhd G_x \), for other \( y \in X \), we define \( N^y \) to be the unique conjugate of \( N^x \) in \( G_y \). Then, we say \( N^x \) is balanced if \( N^x_y = N^y_x = N^x \cap N^y \).

Equivalently, \( N^x \) is a strongly closed subgroup of \( G_x \) in \( G \).

**Theorem C.** Let \( G \) be a doubly-transitive group on a finite set \( X \), and \( N^x \), a normal subgroup of \( G_x \). Suppose \( |X| = n \) and the orbits of \( N^x \) on \( X - x \) are of length \( s \). Also, suppose that \( N^x \) is a permutation group of rank \( r \) on each of its orbits.

Then if \( N^x \) is balanced and \( N^x_y \neq 1 \) if \( y \in X - x \), it follows that

(i) \( n \leq (s - 1)^2 \), and

(ii) the number of orbits of \( N^x \) on \( X - x \) is no more than \( r(s - 1)(s - 2)/s(s - r) \).

The bounds given here are much sharper. They hold exactly for an infinite number of groups \( G \). (See the remark following the proof of Theorem C in §8.)

In the case \( r = 2 \), we are able to determine the group \( G \) without assuming that \( N^x \) is balanced. Indeed, we have:

**Theorem D.** Let \( G \) be a doubly-transitive group on a finite set \( X \). Suppose \( N^x \) is a normal subgroup of \( G_x \) and \( N^x \) is doubly-transitive on each of its orbits on \( X - x \). Then either

(i) \( N^x \) is transitive on \( X - x \) and \( G \) is triply-transitive, or

(ii) \( G \) is a normal extension of \( L_n(q) \), or

(iii) \( |N^x| = 2 \) and \( G \) has a regular normal subgroup.

By combining Theorem C of Part I with Theorem A and Theorem B above, we obtain:

**Corollary.** Let \( M \) be a fixed nonabelian finite simple group. Then there are only finitely many doubly-transitive groups \( G \) such that \( M \) is a factor of the socle of \( G_x \).

The following result is a consequence of Theorem A above and Theorem A of [8]. However, it is obtained preliminary to the proof of Theorem A:

**Proposition 4.** Let \( G \) be a doubly-transitive permutation group on a finite set \( X \) and suppose \( N^x \) is a normal subgroup of \( G_x \). Then, either

(i) \( N^x \) restricts faithfully to its orbits on \( X - x \), or

(ii) \( G \) is a normal extension of \( L_n(q) \).

The next result may also be of some independent interest:
Proposition 8. Let $G$ be a doubly-transitive group on a finite set $X$ and suppose that $G_x$ admits a system of imprimitivity on $X - x$ having imprimitive block $\Delta$. Suppose $|X| = n$ and $|\Delta| = s$. Suppose that no nonidentity element of $G$ fixes all points of $x \cup \Delta$.

Then, if $g \in G$, $g \neq 1$, $g$ fixes at most $s\sqrt{n/2}$ points of $X$.

The bulk of the paper (the first seven sections) is connected with the proof of Theorem A.

In §1 we prove several counting lemmas. As a consequence we obtain the following result which is often of use:

Lemma 1.3. Let $G$ be a doubly-transitive group on a set $X$, and $N^X$ a normal subgroup of $G_x$ having orbits of length $s$ on $X - x$.

Let $f \in G$, $f \neq 1$.

Set $S(f) = \{y | f \in N^y\}$.

Then, $|S(f)| \leq s - 1$.

The results of this section are crucial to the proofs of all our theorems.

In §2 we begin the proof of Theorem A. As a first step, we show

Lemma 2.3. If $G$ is a doubly-transitive group on $X$, $x \in X$, and $G_x$ has a nonsolvable minimal normal subgroup, this nonsolvable minimal normal subgroup is unique.

The proof of this result is, in fact, quite elementary and depends only on the use of Corollary B of [8].

In §3, we state several results about direct products of simple groups. These results are more or less standard.

In §4, we prove Proposition 4, stated above. Here, again the proof is not too complicated. At one point, we use a result of Glauberman which asserts that if $M$ is a simple group and $P$ a Sylow 2-subgroup of $M$ the subgroup of $\text{Aut}(M)$ which centralizes $P$ is solvable.

In §§5, 6, and 7, $N^X$ is a minimal normal subgroup of the point stabilizer $G_x$ of a doubly-transitive group $G$. We assume $N^X = M_1^X \times \ldots \times M_k^X$, where the $M_i^X$'s are isomorphic nonabelian simple groups. Our goal is to show that $k = 1$. This is done in three cases, according as:

(i) For all $x, y \in X$, $x \neq y$, and for all $i$, $1 \leq i \leq k$, $|(M_i^X)_y|$ is odd, i.e., all involutions of $M_i^X$ fix only $x$.

(ii) For all $x, y \in X$, $x \neq y$, and for all $i, j$, $1 \leq i \leq k$, $1 \leq j \leq k$, we have $M_i^X \cap M_j^y = 1$, i.e., the subgroups $M_i^X$ are T.I. sets.

(iii) $M_i^X \cap M_j^y \neq 1$ for some $x, y \in X$, $x \neq y$, and some $i, j$.

In §5, we treat case (i), in §6 case (ii), and so forth.

In the first case, our first step is to prove that $N^X_y$ is of even order. Then,
if \( j \) is an involution in \( N_x^x \), and \( S(j) \) is defined as in Lemma 1.3 above, we show that \( C_G(j) \) is transitive on \( S(j) \). The rank of \( j \) is defined to be the number of factors of \( N_x^x \) upon which \( j \) projects nontrivially. We then show that if \( j \) is an involution in \( N_x^x \) of minimal rank that \( C_G(j) \mid S(j) \) is a permutation group whose point stabilizer has a semiregular normal 2-subgroup. By invoking a theorem of Shult \([10]\), the precise structure of \( C_G(j) \mid S(j) \) is obtained. The latter yields enough information to contradict the simplicity of the groups \( M_f^x \).

We also note that the special case of Shult's result which we use is implied by some recent theorems of Aschbacher and also some of Goldschmidt.

The second case is more difficult. The methods used resemble closely those used in the proof Theorem A of Part I. Our first goal is to prove that those groups \((M_f^x)_y \) with \((M_f^x)_y \neq 1 \) normalize the groups \((M_f^y)_x \) with \((M_f^y)_x \neq 1 \). In proving this we utilize many of the theorems on \((H, K, L)-configurations of Part I. When the latter result is obtained, it is not too difficult to complete the second case. In the latter part of the proof of the second case, we use the fact that, by the first case, \(|(M_f^x)_y| \) is even.

In the third case, the counting methods of §1 come to the fore. First we obtain an inductive reduction to the case \( k = 2 \). The analysis for \( k = 2 \) is divided into two cases according as \( G_{xy} \) does or does not normalize \( M_1^x \).

When \( G_{xy} \) normalizes \( M_1^x \), we construct a block design and obtain a contradiction by counting. When \( G_{xy} \) does not normalize \( M_1^x \), after some analysis, we force the group \((M_1^x)_y \) to be semiregular on the complement of its fixed point set and \(|M_1^x : (M_1^x)_y| \) to be an odd number. Then, Bender's theorem is applicable to give the precise structure of \( M_1^x \). At this point, formulas for \(|X| \) and the lengths of the orbits of the groups \( M_1^x \) are obtained. Calculating explicitly with these formulas yields the desired contradiction.

In §8, we prove Theorems B and C, and in §9, we prove Theorem D. The proofs involve only the counting arguments of §1, Theorem A of [9], and Proposition 8, stated above.

1. The notations we use are the same as those of part I.

In this section we prove a counting lemma. In subsequent sections we apply this lemma in the proofs of Theorems A, B, and C.

We begin by studying the following general situation: \( G \) is a doubly-transitive group on a set \( X \), and \( G_x \) is a graph on \( X - x \) preserved by \( G_x \).

Then, for other points \( y \in X \), we may define a graph \( G_y \) on \( X - y \), preserved by \( G_y \), in such a way that if \( f \in G \), \( f(G_y) = G_{f(y)} \).

We suppose \(|X| = n \) and the valence of \( G_x \) is \( k \). If \( a, b \in X \), \( a \neq b \), we take \( \tau(a, b) = \{ x \mid \{a, b\} \in G_x \} \). By the double-transitivity of \( G \), \(|\tau(a, b)| \) is independent of the pair \( \{a, b\} \).
Lemma 1.1. \(|\tau(a, b)| = k.\)

Proof. Let \(Y = \{(\{a, b\}, c) \mid \{a, b\} \in G_x\}.\) We determine in two ways the size of \(Y.\)

First, there are \(n\) possible choices for \(c.\) Since \(G_x\) is a graph of valence \(k\) on a set of \(n - 1\) elements, \(|G_x| = (n - 1)k/2.\) Thus, for each choice of \(c,\) there are \((n - 1)k/2\) choices for \(\{a, b\}.\) Thus, \(|Y| = n(n - 1)k/2.\)

Secondly, there are \(n(n - 1)/2\) choices for \(\{a, b\},\) and for each choice of \(\{a, b\},\) there are \(|\tau(a, b)|\) choices for \(c.\) Thus, \(|Y| = n(n - 1)|\tau(a, b)|/2.\)

Next we determine the number of edges in \(G_x \cap G_y,\) i.e., \(|G_x \cap G_y|\) (with \(x \neq y).\) Again, by the double-transitivity of \(G,\) this number does not depend on the pair \(\{x, y\}.\)

Lemma 1.2. If \(x, y \in X, x \neq y,\) \(|G_x \cap G_y| = k(k - 1)/2.\)

Proof. Let \(P = \{(\{a, A\}, \{x, y\}) \mid \{a, b\} \in G_x \text{ and } \{a, b\} \in G_y\}.\) We determine in two ways the size of \(P.\)

There are \(n(n - 1)/2\) choices for \(\{x, y\},\) and for each \(\{x, y\},\) there are \(|G_x \cap G_y|\) choices for \(\{a, b\}.\) So \(|P| = n(n - 1)|G_x \cap G_y|/2.\)

There are \(n(n - 1)/2\) choices for \(\{a, b\},\) and for a fixed \(\{a, b\},\) \(\{x, y\}\) can be any two element subsets of \(\tau(a, b).\) Thus, for each \(\{a, b\},\) there are \(k(k - 1)/2\) choices for \(\{x, y\}.\) Thus, \(|P| = (n(n - 1)/2)(k(k - 1)/2).\)

Most frequently we shall use these lemmas in the following situation. Suppose \(G_x\) is imprimitive on \(X - x\) and \(\Delta(x, y)\) is the predesign function associated with the system of imprimitivity of \(G_x\) or \(X - x.)\) Thus, the set \(\Delta(x, y) - x\) is an imprimitive block of \(G_x.\)

We shall define a graph \(G_x\) on \(X - x\) by \(\{a, b\} \in G_x\) if \(\{a, b\}\) belongs to some \(\Delta(x, y) - x.\) Then, if \(|\Delta(x, y) - x| = s, G_x\) is a graph of valence \(s - 1.\)

In particular we note the following:

Lemma 1.3. Let \(G\) be a doubly-transitive group on \(X,\) and \(N^x\) a normal subgroup of \(G_x\) having orbits of length \(s\) on \(X - x.\)

Let \(f \in G, f \neq 1.\)

Set \(S(f) = \{y \mid f \in N^y\},\) when \(N^y\) is the unique conjugate of \(N^x\) in \(G_y.\)

Then \(|S(f)| \leq s - 1.\)

Proof. We let \(\Delta_N(x, y)\) be the predesign function associated with the orbits of \(N^x\) on \(X - x.\) We let \(G_x\) be the graph associated as above. Then, \(\{a, b\} \in G_x\) if and only if there is a \(g \in N^x\) such that \(g(a) = b.\)

Then, with \(\tau(a, b)\) as above, \(\tau(a, b) = \{x \mid \text{there is a } g \in N^x \text{ such that } g(a) = b.\}

Since \(f \neq 1, f(a) = b,\) with \(a \neq b,\) for some \(a, b \in X.\) Then, \(S(f) \subseteq \tau(a, b).\)

By Lemma 1.1, \(|\tau(a, b)| = s - 1.\)
2. In this section we prove that if the point stabilizer of a doubly-transitive group has a nonsolvable minimal normal subgroup, this minimal normal subgroup is unique.

Then, let \( G \) be a doubly-transitive group on \( X \). Let \( M^x \) and \( N^x \) be minimal normal subgroups of \( G_x \) with \( M^x \neq N^x \). We shall show that at least one of these groups is solvable. Assuming that this is not the case, \( M^x \) is a direct product of isomorphic nonabelian simple groups, as is \( N^x \). Moreover, \( M^x \cap N^x = 1 \) and \([M^x, N^x] = 1\).

As usual, we define \( M^y \) so that \( fM^y f^{-1} = M^f(y) \). Likewise, for \( N^y \). Also, \( M^y_x \) is the subgroup of \( M^x \) which fixes \( y \in X \).

By the Feit-Thompson theorem \([3]\), \( M^x \) and \( N^x \) are both of even order. By Theorem C of part I, if \( M^x \) is semiregular on \( X - x \), \( O_2(M^x) \neq 1 \). Since we are assuming \( M^x \) is nonsolvable, it follows that \( M^x \neq 1 \) if \( y \in X - x \). Likewise, \( N^y \neq 1 \) if \( y \in X - x \).

We take \( B \) and \( C \) to be the families of fixed point sets \( B = \{F_{M^x} | x, y \in X, x \neq y\} \) and \( C = \{F_{N^y} | x, y \in X, x \neq y\} \). By Corollary B1 of \([8]\), \( B \) and \( C \) are block designs with \( |B| = 1 \). Also, as \( M^x \neq 1 \) and \( N^y \neq 1 \), if \( B \in B, |B| < |X| \), and if \( C \in C, |C| < |X| \).

**LEMMA 2.1.** (i) \( M^x \) fixes all blocks of \( C \) which contain \( x \).
(ii) \( N^x \) fixes all blocks of \( B \) which contain \( x \).

**PROOF.** Let \( C \) be a block of \( C \) which contains \( x \). If \( y \in C - x \), then \( C \) is the fixed point set of \( N^y \). Now, as \([M^x, N^x] = 1, [M^x, N^y] = 1\). Thus, \( M^x \) centralizes \( N^y \), and so \( M^x \) fixes \( C \), the fixed point set of \( N^y \). Since \( C \) was any block of \( C \) containing \( x \), (i) follows, and (ii) is the same.

**LEMMA 2.2.** (i) \( M^x \cap N^y = 1 \) if \( x \neq y \);
(ii) \( N^x \cap N^y = 1 \) if \( x \neq y \).

This is immediate from Lemma 2.1 and Lemma 2.8 of \([8]\).

At this point, it is possible to quote Theorem A of part I to conclude that \( G \) is a normal extension of \( L_{n}(q) \), in contradiction to the nonsolvability of \( M^x \). Instead, we draw a direct contradiction.

Let \( C \) be the fixed point set of \( N^y \), \( x, y \in X, x \neq y \), (which is also the fixed point set of \( N^y \), as \( C \) is a block design). By Lemma 2.1, \( M^x \) fixes \( C \). Also, \( N^y \) fixes \( x \) and normalizes \( M^x \). Thus, \( M^x \cdot N^y \) is a subgroup of \( G \) which fixes the set \( C \).

Let \( L \) be the subgroup of \( M^x \) which fixes all points of \( C \). Then, \( L \triangleleft M^x \), and as \( M^x \) is a direct product of isomorphic simple groups, \( M^x = L \times K \). Moreover, as \( M^x \triangleleft G_x \), and \( G_x \) is transitive on \( X - x \), \( M^x \) has no orbit of length 1 on \( X - x \); and so \( K \neq 1 \).
Now as $N^y_x$ normalizes $M^x$ and fixes $C$, $N^y_x$ normalizes $L$. Since $M^x$ is a direct product of simple groups, $N^y_x$ normalizes $K$. Thus, $K \cdot N^y_x$ is a subgroup of $G$.

Moreover, $K \cdot N^y_x$ fixes the set $C$ and $N^y_x$ fixes all points of $C$. Since $K \cap L = 1$, it follows that $N^y_x$ is the subgroup of $K \cdot N^y_x$ which fixes all points of $C$. Therefore, $K$ normalizes $N^y_x$.

Now $K$ does not fix $y$ (or if so $M^x = K \times L$ would fix $y$), so there is an $f \in K$ with $f(y) \neq y$. Since $f$ normalizes $N^y_x$, $N^y_x \leq N^y \cap N^f(y)$, but $N^y \cap N^f(y) = 1$, by Lemma 2.2. Hence, $N^y_x = 1$, contrary to hypothesis.

We have proved:

**Lemma 2.3.** If $G$ is a doubly-transitive group and $G_x$ has a nonsolvable minimal normal subgroup, then this nonsolvable minimal normal subgroup is unique.

It follows from this that the point stabilizer of a doubly-transitive group is either local or has a unique minimal normal subgroup. In the next several sections, our goal is to prove the simplicity of this minimal normal subgroup.

3. In this section we note a few lemmas on direct products which we use in later sections. If $G = G_1 \times \ldots \times G_k$ is a direct product of groups, we let $\pi_i$ be the projection of $G$ on $G_i$. If $S$ is a subset of $\{1, \ldots, k\}$, we let $G_S = \prod_{i \in S} G_i$ and we take $\pi_S$ to be the projection of $G$ onto $G_S$. We say that a subgroup $D$ of $G$ is a diagonal if $\pi_i|D: D \rightarrow G_i$ is an isomorphism for all $i$.

The following is easily proved:

**Lemma 3.1.** Let $D$ be a diagonal of $G = G_1 \times \ldots \times G_k$. Then $N_G(D) = D(Z(G_1) \times \ldots \times Z(G_k))$.

From this it follows quickly:

**Lemma 3.2.** If $G = G_1 \times \ldots \times G_k$ has a normal diagonal, $G_1, \ldots, G_k$ are abelian.

**Lemma 3.3.** If $Z(G_1) = \ldots = Z(G_k) = 1$ and $D$ is a diagonal of $G = G_1 \times \ldots \times G_k$, then $N_G(D) = D$.

If $f \in G = G_1 \times \ldots \times G_k$, we take the support of $f$ to be $\{i | \pi_i(f) \neq 1\}$.

**Lemma 3.4.** Suppose $G_1, \ldots, G_k$ are simple groups and $H$ is a subgroup of $G = G_1 \times \ldots \times G_k$ such that $\pi_i(H) = G_i$ for all $i$. Then:

(i) There is a partition $P_1, \ldots, P_i$ of $\{1, \ldots, k\}$ such that $H$ contains a diagonal $H_i$ of $\Pi_{i \in P_i} G_i = G_{P_i}$.

(ii) $H = H_1 \times \ldots \times H_i$.

(iii) $N_G(H) = H$.

(iv) If $f \in H$, the support of $f$ is a union of $P_i$'s.
PROOF. We say $S$ is a set of support of $H$ if $S$ is the set of support of some $f \in H$. If $S$ is a set of support of $H$, we let $H_S = H \cap G_S$.

First we claim: If $S$ is a set of support of $H$, then $H_S = \pi_S(H)$.

Indeed, if $S = \{1, \ldots, k\}$, this is clear. So we may suppose $S \subset \{1, \ldots, k\}$. Now $\pi_i(H_S) < \pi_i(H)$, as $H_S < H$. But as $S$ is the set of support of some $f \in H$, $\pi_i(H_S) \neq 1$ if $i \in S$. By hypothesis, $\pi_i(H) = G_i$. Since $G_i$ is simple, $\pi_i(H_S) = G_i$. By induction in $G_S$, $N_{G_S}(H_S) = H_S$. Since $H_S < \pi_S(H)$, it follows that $H_S = \pi_S(H)$.

Now if $\{1, \ldots, k\}$ is the only set of support of $H$, $H$ is a diagonal of $G$, and the lemma follows. Otherwise, choose $S$ a maximal set of support of $H$, properly contained in $\{1, \ldots, k\}$.

We claim: $H = H_S \times H_{\{1, \ldots, k\} \setminus S}$.

For take $f \in H$ with $\pi_i(f) \neq 1$ for some $i \not\in S$. Then $f = \pi_S(f)\pi_{\{1\} \setminus S}(f)$. Since $\pi_S(f) \in H_S$, $g = \pi_{\{1\} \setminus S}(f) \in H$. If the support of $g$ is properly contained in $\{i\} \setminus S$, $S$ is not a maximal set of support. Thus, $H \subseteq \pi_S(H) \times \pi_{\{1\} \setminus S}(H) = H_S \times H_{\{1\} \setminus S}$.

Now (i) and (ii) follows by induction, (iii) follows by Lemma 3.3 and (iv) is clear.

It is well known that if $G = G_1 \times \ldots \times G_k$ is a direct product of simple groups and $N < G$, then $N = G_S$ for some subset $S$ of $\{1, \ldots, k\}$.

**Lemma 3.5.** Suppose $G_1, \ldots, G_k$ are simple groups and $H$ is a subgroup of $G = G_1 \times \ldots \times G_k$. Suppose that $H$ contains no normal subgroup $\neq 1$ of $G$. Then $|G : H| \geq 5^k$.

**Proof.** It suffices to show that if $G$ admits a faithful transitive permutation representation on $X$, then $|X| \geq 5^k$.

Let $G_1$ be the factor of $G$ of least order. Let $\Delta_1, \ldots, \Delta_l$ be the orbits of $G_1$ on $X$. Since $G$ is transitive on $X$ and $G_1 \triangleleft G$, there is an $m$ such that $|\Delta_i| = m$ for all $i$.

Now if $G_1$ is the largest subgroup of $G$ which fixes all of the sets $\Delta_1, \ldots, \Delta_l$, then $G/G_1$ acts faithfully and transitively on the $l$ sets $\Delta_1, \ldots, \Delta_l$. By induction, $l \geq 5^{k-1}$. Since $G$ is simple, $|\Delta_i| \geq 5$, and so $|X| \geq 5^k$.

If some other factor, say $G_2$, of $G$, fixes $\Delta_1, \ldots, \Delta_l$, since $G_1$ is transitive on $\Delta_i$ and $[G_2, G_1] = 1$, $G_2$ is semiregular on $\Delta_l$. Since $|\Delta_l| < |G_1| < |G_2|$, $|G_2| = |\Delta_l|$. Thus, also $G_1$ is regular on $\Delta_l$: We may identify the action of $G_1$ as the right regular representation of $G_1$ and that of $G_2$ as the left regular representation. Thus, $G_1 \times G_2$ is the largest subgroup of $G$ fixing $\Delta_1, \ldots, \Delta_l$.

By induction, $l \geq 5^{k-2}$ and $|\Delta_l| = |G_1| \geq 5^2$.

We also note:
Lemma 3.6. Let $G = G_1 \times \ldots \times G_k$ and $\phi$ be an automorphism of $G$ of prime order $p$.

If $G_1, \ldots, G_p$ is an orbit of $\phi$, then the centralizer of $\phi$ on $G_1 \times \ldots \times G_p$ is a diagonal of $G_1 \times \ldots \times G_p$.

4. In this section we prove

Proposition 4. Let $G$ be a doubly-transitive group on $X$ and suppose $N^x \triangleleft G_x$. Then either

(i) $N^x$ restricts faithfully to each of its orbits on $X - x$, or
(ii) $G$ is a normal extension of $L_n(q)$ (in its usual doubly-transitive representation).

We take $G$ a doubly-transitive group on $X$, $x \in X$, $N^x \triangleleft G_x$, such that $N^x$ does not restrict faithfully to one of its orbits on $X - x$. Then, there is a minimal normal subgroup $L$ of $N^x$ which fixes $y \in X - x$. The characteristic closure $K^x$ of $L$ in $N^x$ is either elementary abelian or semisimple. Moreover, $K^x$ does not restrict faithfully to its orbits on $X - x$. If $K^x$ is abelian, by Theorem A of [8], $G$ is a normal extension of $L_n(q)$. Thus, it suffices to obtain a contradiction when $K^x$ is the direct product of isomorphic simple groups. Clearly, we may take $K^x = N^x$.

We let $\Delta(x, y) = x \cup \{f(y) \mid f \in N^x\}$ be the predesign function associated with the orbits of $N^x$. We let $|\Delta(x, y)| = 1 + s$. We let $K(x, y)$ be the kernel of the homomorphism obtained by restricting $N^x$ to $\Delta(x, y)$. By hypothesis, $K(x, y) \neq 1$. By double-transitivity, all the groups $K(x, y)$, with $x, y \in X$, $x \neq y$, are conjugate. $K(x, y)$ may be described as the largest normal subgroup of $N^x$ which fixes the point $y$. From this definition, it is clear that if $K(x, y)$ and $K(x', y')$ fix a common point $y''$, then $K(x, y) = K(x', y')$. Thus, if we define $\Gamma(x, y) = F_{K(x,y)} \Gamma(x, y)$ is a predesign function. It is clear from the definition that $\Delta(x, y) \subseteq \Gamma(x, y)$.

We note first a general lemma:

Lemma 4.1. Let $G$ be a doubly-transitive group on a set $X$ and suppose $N^x$ is a normal subgroup of $G_x$ and that $N^x$ is a direct product of simple groups.

Then, if $x, y \in X$, $x \neq y$, $(N^x, N^y)$ is transitive on $X$.

Proof. Suppose $(N^x, N^y)$ is intransitive on $X$, for some $x, y \in X$, $x \neq y$. By the double-transitivity of $G$, this is true for all pairs $x, y \in X, x \neq y$.

Let $B(x, y) = \{z \mid N^x \subseteq (N^x, N^y)\}$. Clearly, $\{x, y\} \subseteq B(x, y)$. We claim that the family of sets $B = \{B(x, y) \mid x, y \in X, x \neq y\}$ form a block design on $X$ (with $\lambda = 1$). Since $g(B(x, y)) = B(g(x), g(y))$, all the sets $B(x, y)$ have the same size.

To prove that every two element subset of $X$ belongs to exactly one $B(x, y)$, it suffices to show that if $\{a, b\} \subseteq B(x, y)$, $B(a, b) = B(x, y)$.
But if \{a, b\} \subseteq B(x, y), \langle N^a, N^b \rangle \subseteq \langle N^x, N^y \rangle, and by double-transitivity, 
\langle N^a, N^b \rangle = \langle N^x, N^y \rangle, and the claim follows.

It is clear that if \( y \in X - x \), \( N^x \) fixes the set \( B(x, y) \). Thus, \( N^x \) fixes all blocks of \( B \) which contain \( x \). By Theorem B of part I, \( G \) is a normal extension of \( L_n(q) \). Since \( N^x \) is a direct product of simple groups, the latter is a contradiction to the structure of the one-point stabilizer in \( L_n(q) \).

By hypothesis, \( N^x = M_1^x \times \ldots \times M_k^x \), when the \( M_i^x \)'s are isomorphic simple groups.

**Lemma 4.2.** (i) \( N_G(K(x, y)) \subseteq G_x \); (ii) \( N_G(M_i^x) \subseteq G_x \).

**Proof.** It suffices to show that if \( L \lhd N^x, L \neq 1 \), then \( N_G(L) \subseteq G_x \). If \( N_G(L) \not\subseteq G_x \), then also \( L \lhd N^y \) for some \( y \neq x \). Then, \( \langle N^x, N^y \rangle \) fixes \( F_L \).

Since \( F_L \subseteq X \), as \( L \neq 1 \), we have a contradiction by Lemma 4.1.

**Lemma 4.3.** If, \( x, y \in X, x \neq y \), (i) \( \Delta(x, y) \neq \Delta(y, x) \); (ii) \( \Gamma(x, y) \neq \Gamma(y, x) \).

**Proof.** If \( \Delta(x, y) = \Delta(y, x) \), \( \langle N^x, N^y \rangle \) fixes the set \( \Delta(x, y) \). \( \Delta(x, y) \subseteq X \), as \( K(x, y) \lhd 1 \) fixes all points of \( \Delta(x, y) \). By Lemma 4.1, this is a contradiction. A similar contradiction results if \( \Gamma(x, y) = \Gamma(y, x) \).

**Lemma 4.4.** \( K(y, x) \cap N^x = 1 \) if \( x \neq y \).

**Proof.** Let \( D = K(y, x) \cap N^x \) and suppose \( D \neq 1 \). Since \( K(y, x) \) normalizes \( N^x \), \( D \lhd K(y, x) \). Since \( N^y \) is a direct product of simple groups and \( K(y, x) \) normalizes \( N^y \), \( D \lhd N^y \).

Let \( T = \{ t | D \subseteq N^f \} \). Then, \( \{ x, y \} \subseteq T \). Clearly, \( N_G(D) \) fixes \( T \). Thus, \( N^y \) fixes \( T \). Since \( \{ x, y \} \subseteq T \), \( \Delta(y, x) \subseteq T \). Therefore, \( 1 + s \leq |T| \).

On the other hand, by Lemma 1.3, \( |T| \leq s - 1 \). This contradiction shows \( D = 1 \).

**Lemma 4.5.** \[ |K(y, x), N_y^x| = 1 \].

**Proof.** \( K(y, x) \lhd G_{xy} \) and \( N_y^x \lhd G_{xy} \). By Lemma 4.4, \( K(y, x) \cap N_y^x = 1 \).

**Lemma 4.6.** Let \( M_i^x \) be a simple factor of \( N^x \). Then all orbits of \( M_i^x \) are of length 1 or of even length.

**Proof.** Let \( y \in X - x \) and let \( Y \) be the orbit of \( y \) under \( M_i^x \). Suppose \( |Y| \) be odd. We show \( |Y| = 1 \).

Let \( L \) be the stabilizer of \( y \) in \( M_i^x \). Then, as \( |M_i^x| \) is even and \( |Y| \) is odd, \( L \neq 1 \). Also, \( L \subseteq N_y^x \). By Lemma 4.5, \( K(y, x) \) centralizes \( L \). Since \( K(y, x) \) normalizes \( N^x \) and \( M_i^x \) is the unique simple factor of \( N^x \) which contains \( L \), \( K(y, x) \) normalizes \( M_i^x \).

Since \( |Y| = |M_i^x : L| \) is odd, \( K(y, x) \) acts on \( M_i^x \) and centralizes a Sylow
2-subgroup of $M^x_f$. But by a theorem of Glauberman [5], the subgroup of Aut($M^x_f$) which centralizes a Sylow 2-subgroup of $M^x_f$ is solvable. Consequently, $K(y, x)$ centralizes $M^x_f$. Therefore, by Lemma 4.2(i), $M^x_i$ fixes $y$. Hence, $|Y| = 1$.

We now complete the proof of the proposition.

Since $|\Gamma(x, y)| = |\Gamma(y, x)|$ and $\Gamma(x, y) \neq \Gamma(y, x)$ by Lemma 4.3, we may choose $z \in \Gamma(y, x)$, $z \notin \Gamma(x, y)$. Thus, $K(y, x)$ fixes $z$, but $K(x, y)$ does not fix $z$. Therefore, $K(y, x)$ normalizes $K(x, z)$.

Now $K(y, x)$ and $K(x, z)$ are both products of $r$ isomorphic simple groups. Since $K(y, x)$ acts as a permutation group on the $r$ simple factors of $K(x, z)$, it follows by Lemma 3.5, that some simple factor, say $M^x_1$, of $K(y, x)$ fixes all simple factors of $K(x, z)$.

Now if $M^x_1$ centralizes all simple factors of $K(x, z)$, $M^x_1$ centralizes $K(x, z)$, and by Lemma 4.2, $K(x, z)$ fixes $y$, contrary to hypothesis.

Thus, there is some simple factor, $M^x_1$, of $K(x, z)$ such that $M^x_1$ normalizes but does not centralize $M^x_1$.

Let $Y$ be the orbit of $y$ under $M^x_1$. If $Y = \{y\}$, $M^x_1 \leq N^x_1$. Since $K(y, x)$ centralizes $N^x_1$ by Lemma 4.5, $M^x_1$ centralizes $M^x_1$, contrary to choice. Therefore, $|Y| > 1$. By Lemma 4.6, $|Y|$ is even.

Since $M^x_1$ normalizes $M^x_1$ and $M^x_1$ fixes $y$, $M^x_1$ fixes $Y$. We consider the action of $M^x_1 \cdot M^x_1$ on $Y$. If $L$ is the stabilizer of $y$ in $M^x_1$, by Lemma 4.5, $[L, M^x_1] = 1$. Moreover, in the group $M^x_1 \cdot M^x_1$ the stabilizer of $y$ is $L \times M^x_1$.

Since $|Y| > 1$, $L \subset M^x_1$. Therefore, $M^x_1$ is the unique subgroup of its isomorphism type in $L \times M^x_1$. Therefore, $M^x_1$ is a weakly closed subgroup of $L \times M^x_1$ in $M^x_1 \cdot M^x_1$.

By Witt’s theorem, the normalizer of $M^x_1$ in $M^x_1 \cdot M^x_1$ is transitive on the fixed point set of $M^x_1$ in $Y$. But by Lemma 4.2, the normalizer of $M^x_1$ fixes $y$. Thus, the fixed point set of $M^x_1$ in $Y$ is precisely $\{y\}$.

Thus, all orbits of $M^x_1$ on $Y - y$ are of length greater than 1. By Lemma 4.6, $|Y - y|$ is even. Hence, $|Y|$ is odd, a contradiction, in so far as we have already proved that $|Y|$ is even.

This completes the proof of the proposition.

5. In the next three sections we complete the proof of Theorem A. We assume $G$ is a doubly-transitive group on a set $X$ and $N^x$ a normal subgroup of $G_x$. We suppose $N^x = M^x_1 \times \ldots \times M^x_k$, with the $M^x_i$ isomorphic nonabelian simple groups. We shall prove that $k = 1$.

By the proposition of §4, $N^x$ is faithful on its orbits on $X - x$. Hence no $M^x_i$ fixes points in $X - x$.

Our analysis will be in three cases according as:

1. For all $i$, no involutions of $M^x_i$ fix points in $X - x$, i.e., $|(M^x_i)_y|$ is odd for all $i$, $x, y \in X, x \neq y$. 

2. $M_i^x \cap M_j^y = 1$ for all $x, y \in X, x \neq y,$ and all $i, j,$ but $\langle M_i^x \rangle_y$ is even for some $i, x, y \in X, x \neq y.$

3. $M_i^x \cap M_j^y \neq 1$ for some $x, y \in X,$ and some $i, j.$

In this section we assume $\langle M_i^x \rangle_y$ is odd for all $i$ and all $x, y \in X, x \neq y.$

We assume $k \geq 2$ and obtain a contradiction. We take $\pi_i$ to be the projection of $N^x$ on $M_i^x.$

By the Feit-Thompson Theorem [3] and the Brauer-Suzuki Theorem [2] the 2-rank of $M_i^x$ is at least two.

**Lemma 5.1.** \$|N^x| \$ is even.

**Proof.** If \$|N^y| \$ is odd, by Lemma 2.2 of part I, all involutions of $N^x$ are conjugate in $G_x.$ But this is not possible if $N^x \vartriangleleft G_x$ and $N^x$ has at least two simple factors.

**Lemma 5.2.** If $j$ is an involution, let $S(j) = \{ z | j \in N^z \}.$ Then $C_G(j)$ is transitive on $S(j).$

**Proof.** Clearly $C_G(j)$ fixes the set $S(j).$ To prove the lemma, it suffices to show that if $z \in S(j),$ the orbits of $C_G(z)$ on $S(j) - z$ are of even length.

Now if $z \in S(j),$ $j \in N^z,$ and there is an $i$ such that $\pi_i(j) = j_i \neq 1.$ Then \$[j, j_i] = 1$ and $j_i$ is semiregular on $X - x,$ as $j_i \in M_i^x.$

Now if $j$ is an involution of $N^x,$ we define the rank of $j$ to be the size of the support of $j$ in the direct product $M_1^x \times \ldots \times M_k^x.$ By Lemma 5.2, if $j \in N^x$ and $j \in N^y,$ the rank of $j$ in $N^x$ and the rank of $j$ in $N^y$ are the same. Thus, the rank of $j$ does not depend upon $x,$ so long as $x \in S(j).$

**Lemma 5.3.** Let $r$ be the minimal rank of the involutions in $N^x.$ Then, $N^x \cap N^y$ contains an involution of rank $r.$

**Proof.** We suppose that $N^x \cap N^y$ contains no involution of rank $r.$ Then, by double-transitivity no $N^a \cap N^b$ contains an involution of rank $r,$ if $a, b \in X,$ $a \neq b.$ Take $j \in N^x,$ an involution of rank $r.$ Since $j$ does not belong to $N^z$ for $z \neq y,$ $C_G(j) \subseteq G_y.$

Now $j$ permutes the factors $M_i^x$ of $N^x$ by conjugation. If $j$ fixes some factor $M_i^x$ of $N^x,$ then $C_{G_i}(j) \subseteq \langle M_i^x \rangle_y$ is of even order contrary to hypothesis. Therefore, $j$ fixes no factor of $N^x.$

Then we may suppose $j$ interchanges $M_{2i-1}^x$ and $M_{2i}^x$ for $i = 1, \ldots, k/2.$ By Lemma 3.6, $j$ centralizes a diagonal $D_i$ of $M_{2i-1}^x \times M_{2i}^x.$ Hence, $D_i \subseteq N_y^x.$

Thus, $H = D_1 \times \ldots \times D_{k/2} \subseteq N_y^x.$

Now $\pi_i(H) = M_i^x$ for all $i.$ By Lemma 3.4, $N_y^x = D_1 \times \ldots \times D_{k/2},$ as if $N_y^x \vartriangleleft D_1 \times \ldots \times D_{k/2},$ $N_y^x$ contains a factor of $N^x,$ a contradiction.

Therefore, $N_y^x$ is a direct product of isomorphic simple groups, as is $N^x.$
By Theorem A of part I $N^x \cap N^y \neq 1$. Since $N^x \cap N^y < N^y$ and $N^x \cap N^y$ contains some $D_i$.

Therefore, $N^x \cap N^y$ contains an involution of rank 2. As $N^y$ contains no involution of rank 1, $r = 2$, in contradiction to our original assumption, that $N^x \cap N^y$ contains no involution of minimal rank $r$.

We now take $f$ to be an involution in $N^x \cap N^y$ of rank $r$. Then we choose $M_i^x$, $1 \leq i \leq r$, so that $\pi_i(f) \neq 1$.

**Lemma 5.4.** (i) $C_{G_x}(f)$ in its action by conjugation permutes $\{j_1, \ldots, j_r\}$.
(ii) $\langle j_1, \ldots, j_r \rangle < C_{G_x}(f)$.
(iii) $\langle j_1, \ldots, j_r \rangle | S(f) - x$ is semiregular.

**Proof.** (i) $C_{G_x}(f)$ permutes $M_i^x$, $1 \leq i \leq r$, and centralizes $f$, so $C_G(f)$ permutes $j_1, \ldots, j_r$. (ii) is clear. (iii) If $t \in \langle j_1, \ldots, j_r \rangle$ fixes $y \in S(f) - x$, then $t \in N^y$ and $t$ is of rank $< r$. The rank of $t$ is $< r$, $t$ cannot fix $y$, as $r$ is the smallest rank of an involution of $N^y$. If the rank of $t$ is $r$, then $t = f$. Since $\langle f \rangle$ fixes all points of $S(f)$, the result follows.

It follows from Lemma 5.4 that $C_{G_x}(f) | S(f)$ is a permutation group whose point stabilizer has a semiregular normal 2-subgroup. By a theorem of Shult [10], either $C_G(f) | S(f)$ has a regular normal subgroup, or is a normal extension of $S_L(2, 2^a)$, $U_3(2^a)$, or $S_2(2^a)$. In particular, all involutions in $\langle j_1, \ldots, j_r \rangle | \langle f \rangle$ are conjugate under the action of $C_{G_x}(f)$.

**Lemma 5.5.** $r = 3$.

**Proof.** All involutions of $\langle j_1, \ldots, j_r \rangle | \langle f \rangle$ are conjugate under $C_{G_x}(f)$ by inspection. On the other hand, by Lemma 5.4, $\langle j_1, \ldots, j_r \rangle$ is an orbit of $C_{G_x}(f)$. Hence, $2^{r-1} - 1 \leq r$. Thus, $r \leq 3$.

If $r = 2$, $C_G(f) | S(f)$ has a regular normal subgroup. Moreover, the Sylow 2-subgroup of $C_{M_1^x}(f) | S(f)$ is semiregular on $S(f) - x$, as all involutions of $C_{M_1^x}(f)$ are of rank 1. Hence, $C_{M_1^x}(f)$ has cyclic or generalized quaternion Sylow 2-subgroup. Hence, $M_1^x$ has cyclic or generalized quaternion Sylow 2-subgroup, a contradiction, by the Brauer-Suzuki Theorem.

Now we obtain a final contradiction. Since $\langle j_1, j_2, j_3 \rangle | \langle f \rangle$ is elementary of order 4, $C_G(f) | S(f)$ is a normal extension of $A_5$ or $U_3(4)$. Also, if $P_1$ is a Sylow 2-subgroup of $C_{M_1^x}(f_1)$ and $P_2$ that of $C_{M_2^x}(f_2)$, $P_1 \times P_2$ is semiregular on $S(f) - x$, as no involution of $N^x$ of rank $< 2$ fixes points on $X - x$.

But then by the structure of the Sylow 2-subgroups of $A_5$ and $U_3(4)$, $P_1$ is cyclic or generalized quaternion. Again, a contradiction results, as $M_1^x$ has cyclic or generalized quaternion Sylow 2-subgroup.

6. As in the last section, $G$ is a doubly-transitive group on a set $X$, $N^x < G_x$, and $N^x = M_1^x \times \ldots \times M_k^x$ with the $M_i^x$ isomorphic simple groups.
In this section we assume $M_i^x \cap M_j^y = 1$ for all $x, y \in X, x \neq y$, and all $i, j$. By the last section $|\langle M_i^x \rangle_y| \leq 1$ is even for some $x, y \in X, x \neq y$, and some $i$. Again, we derive a contradiction if $k \geq 2$.

**Lemma 6.1.** If $f \in M_i^x$, $f \neq 1$, then $C_G(f) \subset G_x$.

**Proof.** If $g \in C_G(f)$, $f \in M_i^x \cap M_j^y(x)$. As $f \neq 1$, it follows that $g(x) = x$.

We choose $M_1^x, \ldots, M_r^x$ so that $(M_1^x)_y \neq 1, \ldots, (M_k^x)_y \neq 1$, and $(M_{r+1}^x)_y = 1, \ldots, (M_k^x)_y = 1$. Likewise, we choose $M_1^y, \ldots, M_r^y$ so that $(M_1^y)_x \neq 1, \ldots, (M_k^y)_x \neq 1$, and $(M_{r+1}^y)_x = 1, \ldots, (M_k^y)_x = 1$. We let $H^i = (M_i^x)_x$ and $L^j = (M_j^y)_y$.

Since $H^i$ normalizes $M_1^x \ldots M_r^x$ and permutes the factors $M_1^x, \ldots, M_r^x$, $H^i$ normalizes $L^1 \ldots L^r$ and permutes the factors $L^1, \ldots, L^r$.

Likewise, $L^j$ normalizes $H^1 \ldots H^r$ and permutes the factors $H^1, \ldots, H^r$.

We set
\[ H^i_j = N_{H^i}(M_j^y) = N_{L^j}(M_i^x) = N_{L^j}(H^i). \]

Thus, $H^i_j$ is the subgroup of $H^i$ fixing $L^j$, and vice versa.

Our first goal is to show that $H^i = H^i_j$ and $L^j = L^j_i$, i.e., that $H^i$'s and $L^j$'s normalize each other.

**Lemma 6.2.** If $f \in H^i$ and $|f|$ is prime, then for any $j$, $f \in H^i_j$.

**Proof.** We may take $j = 1$. Then if $f$ does not fix $M_1^x$, $M_1^x$ has an orbit $M_1^x, \ldots, M_p^x$ under $f$, where $p = |f|$. Then, $f$ centralizes a diagonal $D$ of $M_1^x \ldots M_r^x$. By Lemma 6.1, $D \subset N_x^y$. Therefore, $\pi_1(N_x^y) = M_1^x$. But $(M_1^x)_y = (M_1^x)^N_x$, contradicting Proposition 4.

Thus, $f$ fixes $M_1^x$ and $f \in H^i_j$.

It follows also from Lemma 6.2 that $H^i_j \neq 1$ and $L^j_i \neq 1$.

**Lemma 6.3.** $[H^i_j, L^j_i] = 1$.

**Proof.** $[H^i_j, L^j_i] \subseteq L^j_i$ and $[H^i, L^j_i] \subseteq H^i$. As $H^i \cap L^j_i = 1$, $[H^i_j, L^j_i] = 1$.

**Lemma 6.4.** If $f \in H^i_j$, $f \neq 1$, then $C_L(f) = L^j_i$.

**Proof.** By Lemma 6.3, $L^j_i \subseteq C_L(f)$. Suppose $g \in L^j_i$ and $g$ centralizes $f$. By Lemma 6.1, $g$ fixes $y$. Since $g$ centralizes $f$ and normalizes $N^y$, $g$ fixes that factor of $N^y$ which contains $f$, namely $M^y_i$. So $g \in L^j_i$.

Of course, a similar result holds with $H$ and $L$ interchanged.

**Lemma 6.5.** If $f \in H^i_j$, $f \neq 1$, then $C_{M^y_i}(f) = L^j_i$. 

Proof. As before, \( L_i \subseteq C_{M_i^f}(f) \). If \( g \in C_{M_j^f}(f) \), then by Lemma 6.1, \( g \in (M_j^f)_s \), and the rest follows by Lemma 6.4.

**Lemma 6.6.** (i) \( |H_i^f| = |L_i^f| \).
(ii) For a given prime \( p \), all Sylow \( p \)-subgroups of \( H_i^f \) and \( L_i^f \) are isomorphic.

Proof. In the language of §4 of part I, using Lemma 6.5, we see that we have a \((H_i^f, M_i^f, L_i^f)\) configuration and a \((L_i^f, M_i^f, H_i^f)\) configuration. By applying Lemma 4.2 of part I to each of the prime divisors of \( |H_i^f| \) and \( |L_i^f| \), (i) and (ii) follow.

**Lemma 6.7.** Suppose \( H^f \) does not fix any \( L_i^f \), then
(i) \( |H^f : H_i^f| = 2 \) and \( H_i^f \) is abelian,
(ii) \( H^f \) is fixed by all \( L_k^f, 1 \leq k \leq r \).

Proof. Suppose some element of \( H^f \) moves \( L_i^k \) to \( L_i^k \) with \( k \neq j \). Then some element of \( H^f \) moves \( L_i^j \) to \( L_i^k \). Since \( L_i^j \) normalizes \( H^f, L_i^k \subseteq H^f L_i^f \).

Now \( H^f \cap L_i^j = 1 \) and, by Lemma 6.4, \( N_{H_i^f L_i^f}(L_i^j) = H_i^f L_i^f \). Thus, \( L_i^k \cdot L_i^f \subseteq H_i^f \cdot L_i^f \). Since \( |L_i^k| = |L_i^f| = |H_i^f| \) (by Lemma 6.6), \( L_i^k \times L_i^f = H_i^f L_i^f \). Since \( H^f \cap L_i^k \times L_i^f = H_i^f \cap L_i^f = 1, H_i^f, L_i^k, L_i^f \) are isomorphic abelian groups.

Now if \( |H^f : H_i^f| > 2 \), there is some \( L_i^k \neq L_i^j, L_i^k \), such that \( L_i^k \times L_i^j \times L_i^k \subseteq H^f L_i^f \), a contradiction to the order of the normalizer of \( L_i^k \) in \( H^f L_i^f \). Thus, (i) follows.

Next we prove (ii).

Take \( L_i^k \neq L_i^j, L_i^k \). Then, \([H^f, L_i^j] \subseteq L_i^j \times L_i^k \) and so \([H^f, L_i^j], L_i^j] = 1\). Now \([H^f, L_i^j]\) \neq 1, as if \([H^f, L_i^j] = 1, H^f \) fixes \( L_i^j \) and so \( H^f \) fixes \( L_i^j, \) the unique factor of \( L_i^j \times \ldots \times L_i^r \) containing \( L_i^j \). This is contrary to the fact that \( H^f \) moves \( L_i^j \). Hence, \([H^f, L_i^j] \neq 1 \). Since \([H^f, L_i^j] \subseteq H^f \) and \( L_i^j \) centralizes \([H^f, L_i^j], L_i^j \) fixes \( H^f \).

Thus, to prove (ii), it suffices to show that \( L_i^j \) and \( L_i^k \) fix \( H^f \). So, we suppose \( L_i^j \) moves \( H^f \). By (i) of this lemma, \( |L_i^j : L_i^j| = 2 \) and \( |L_i^k : L_i^k| = 2 \).

Let \( P \) be the Sylow 2-subgroup of \( H_i^f, Q \) a Sylow 2-subgroup of \( L_i^j \) normalized by \( P \) and \( R \), the Sylow 2-subgroup of \( L_i^j \), with similar definitions for \( Q^k \), and \( R^k \). Then \( |Q : R_i^j| = |Q^k : R_i^k| = 2 \). Also, \( L_i^j \) contains all involutions of \( L_i^j \) by Lemma 6.2, and as \( |L_i^j| \) is even, \( |R_i^j| \neq 1 \). Since \( |L_i^j| = |H_i^f|, P \neq 1 \).

By Lemma 6.4, we have, in the terminology of part I, a constrained \((P, Q^k, R^k)\) configuration and a constrained \((P, Q^k, R^k)\) configuration. By Lemma 4.6 of part I, \( \Omega_1(R^k) \subseteq Z(Q^k) \) and \( \Omega_1(R^k) \subseteq Z(Q^k) \). But \( H_i^f \subseteq L_i^k \times L_i^j \).

So \( P \subseteq R^k \times R^k \), and \( \Omega_1(P) \subseteq Z(Q^k \times Q^k) \). Thus, if \( x \in \Omega_1(P), x \neq 1, x \) centralizes \( Q^k \), a contradiction, by Lemma 6.4.

Therefore, \( L_i^j \) does not move \( H^f \).

We shall show next that all \( H^f \)'s fix all the \( L_i^j \)'s, and vice versa. Now if \( j \) is
an involution interchanging \( x, y \in X \), \( j \) normalizes \( (H^1 \times \ldots \times H') \cdot (L^1 \times \ldots \times L'^j) \) and interchanges \( H^i \)'s and \( L^i \)'s. Thus, if all groups \( H^i \) fix all the groups \( L^i \), all the groups \( L^j \) fix all the groups \( H^i \), and we obtained the desired conclusion. Thus, we may suppose, say \( H^1 \), does not fix all the \( L^i \)'s, and \( L^1 \cong H^1 \), does not fix all the \( H^i \)'s.

By Lemma 6.7, all the \( L^i \)'s fix \( H^1 \) and all the \( H^i \)'s fix \( L^1 \).

**Lemma 6.8.** \( H^1 \) is an abelian group.

**Proof.** Since \( H^1 \) fixes \( L^1 \), \( H^1 \) normalizes \( M_1^x \), and, by Lemma 6.5, we have a constrained \((H^1, M_1^x, L^1)\) configuration. By Proposition 4.9 and Proposition 4.15 of part I, as \( M_1^x \) is a simple group, \( H^1 \) is abelian.

**Lemma 6.9.** \( H^1 = Z_4 \times A \), where \( A \) is an elementary abelian 2-group.

**Proof.** Suppose \( H^1 \) moves \( L^2 \) to \( L^3 \). By Lemma 6.7, \(|H^1 : H_2^1| = 2\).

Also, by the proof of Lemma 6.7, \( L_1^2 \cong H_1^2 \). As \( L^2 \) fixes \( H^1 \), \( L^2 \cong H_2^1 \). Thus, by Lemma 6.4, we have a constrained \((L^2, H^1, H_1^2)\) configuration. Since \(|H^1 : H_2^1| = 2\), \([L^2, H^1] \subseteq H_1^2 \). By Lemma 4.17 of part I, and as \( H^1 \) is abelian, it follows that \( H^1 \) is a 2-group. By Lemma 4.6(i) of part I, \( H_1^2 \) is elementary abelian. By Lemma 6.2, \( H_1^2 \) contains all involutions of \( H^1 \), and so \( H^1 = Z_4 \times A \), with \( A \) an elementary abelian 2-group.

We now obtain a final contradiction to the assumption that not all \( H^i \)'s fix the \( L^i \)'s by studying the constrained \((H^1, M_1^x, L^1)\) configuration.

The following lemma, which we use again shortly, handles the case in which \( A \) of Lemma 6.9 is the identity.

**Lemma 6.10.** There is no \((Z_n^2, G, Z_n^2)\) configuration in which \( G \) is a simple group, and \( n \geq 1 \).

**Proof.** Assume such a configuration exists and let \( Q \) be a Sylow 2-subgroup of \( G \). Since \( G \) is simple \( Q \supseteq Z_n^2 \). By Lemma 4.6 of part I, \( Q \) is dihedral, cyclic, or generalized quaternion. By the Brauer-Suzuki Theorem, \( Q \) is dihedral. By Gorenstein and Walter [6], \( G = PSL(2, q) \), \( q \) odd, or \( A_7 \).

We let \( P \) be the group of automorphisms of \( G \) such that the centralizer of each nonidentity element of \( P \) is \( R \), with \( P = R = Z_n^2 \). We let \( j \) be the involution of \( P \).

If \( G = PSL(2, q) \), then \( j \) induces a field automorphism or a graph automorphism of \( G \). In the first case, \( C_G(j) = PSL(2, \sqrt{q}) \), which is not a 2-group. In the second case, \( C_G(j) \) is dihedral of order \( q + 1 \) or \( q - 1 \), and never cyclic if \( q > 3 \).

If \( G = A_7 \), an outer automorphism of \( A_7 \) centralizes an element of order 3.

We now begin the proof that the hypothesis of this section leads to a contradiction.
It follows from the last lemma, as \( M_1^x \) is simple, that \( A \) has order greater than 1.

Now let \( T = C_{M_1^x}(L^1) \). By Lemma 4.6(i) of part I, \( L^1 \) is a Sylow 2-subgroup of its centralizer. Hence, \( T = L^1 \times O(T) \). Since \( H^1 \) is noncyclic, \( O(T) = \langle C_{O(T)}(j) | j \in H^1 \rangle \), \( j \neq 1 \). Since \( C_{M_1^x}(j) \subseteq L^1 \) if \( j \in H^1 \), \( j \neq 1 \), \( O(T) \subseteq L^1 \) and \( O(T) = 1 \). Thus \( T = L^1 \).

By Lemma 4.1 of part I, \([N_{M_1^x}(L^1), H^1] \subseteq L^1 \). Thus, in the terminology of part I, we have a constrained \((H, K, L)\)-configuration of type \( A \). By Proposition 4.26 of part I, \( L^1 \not< M_1^x \), a contradiction, or \( L^1 \) is elementary abelian, again a contradiction.

Thus, we may assume that all \( H^i \)'s normalize \( L^i \)'s, and vice versa.

**Lemma 6.11.** \( H^1, \ldots, H^r, L^1, \ldots, L^r \) are isomorphic abelian groups.

**Proof.** By choosing an involution \( t \) which interchanges \( x, y \in X \), we may number the groups so that \( H^1 \cong L^1, \ldots, H^r \cong L^r \). By Lemma 6.4 we then have a constrained \((H^i, M^i, L^i)\) configuration. By part I, Propositions 4.9 and 4.15, \( H^i \) and \( L^i \) are abelian. Since \( H^i = H_j \) and \( L^i = L_i \), and by Lemma 6.6, all Sylow subgroups of \( H^i \) and \( L^i \) are isomorphic, \( H^i \) and \( L^i \) are isomorphic, for all \( i \) and \( j \).

**Lemma 6.12.** For all \( i, 1 < i < k, (M_i^x)_y = 1 \). (In other words, \( r = k \).)

**Proof.** Assume the lemma is false, and \( (M_{r+1}^x)_y = \ldots = (M_k^x)_y = 1 \). We study the action of \( H^1 = (M_1^x)_x \) on \( K = M_{r+1}^x \times \ldots \times M_k^x \). We take \( \Lambda = \{r + 1, \ldots, k\} \). If \( f \in H^1 \), we speak of the action of \( f \) on \( \Lambda \) to mean the action of \( f \) on the corresponding factors of \( K \).

First we claim that \( H^1 \) acts semiregularly on \( \Lambda \). It suffices to verify this if \( f \in H^1 \) is of prime order \( p \). Indeed, if \( f \) fixes \( M^x \), \( t \in \Lambda \), as \( p \mid |M^x| \), \( C_{M^x}(t) \neq 1 \). But then, by Lemma 6.1, \( (M^x)_y = 1 \), a contradiction. Thus, the claim follows.

By the hypothesis of this section, some \( (M_i^x)_y \) is of even order. So by Lemma 6.11, \( H^1 \) is of even order. We claim next that \( H^1 \) is a cyclic 2-subgroup.

Take \( f \in H^1 \) of prime order and let \( \Gamma_1, \ldots, \Gamma_s \) be the orbits of \( f \) on \( \Lambda \). By Lemma 3.6, \( f \) centralizes a diagonal \( D_\lambda \) of \( \Pi_{\lambda \in \Gamma_i} M^x \). By Lemma 6.1, \( D_\lambda \subseteq K_y \). Thus, if \( i \in \Lambda, \pi_i(K_y) = M^x \). Then, by Lemma 3.4, there is a partition \( \Delta_1, \ldots, \Delta_u \) of \( \Lambda \) such that \( K_y \) is a product of diagonals of \( \Pi_{\lambda \in \Delta_i} M^x \).

Moreover, each orbit of \( f \) on \( \Lambda \) is a union of sets \( \Delta_i \), where \( f \) may be any element of \( H^1 \) of prime order.

Now if it is possible, choose two elements \( f, g \in H^1, f, g \) of prime order, and \( \langle f \rangle \neq \langle g \rangle \). Then, \( \langle f \rangle \) has orbits \( \Gamma_1, \ldots, \Gamma_s \) on \( \Lambda \) and \( \langle g \rangle \) has orbits \( \Gamma_1', \ldots, \Gamma_s' \). Since \( \langle f, g \rangle \) is semiregular on \( \Lambda \), \( |\Gamma_i \cap \Gamma_i'| \leq 1 \), for all choices of \( \Gamma_i \) and \( \Gamma_i' \). Since both \( \Gamma_i \) and \( \Gamma_i' \) are the union of sets \( \Delta_i \), it follows that \( |\Delta_1| = \ldots = |\Delta_u| = 1 \).
But then if \( l \in \Lambda \), \( M^x_l \) fixes \( y \), a contradiction. Thus, the choice of the previous paragraph is not possible. Therefore, \( H^1 \) is a cyclic 2-group.

Next we consider the constrained \((H^1, M^x_1, L^1)\) configuration. Since \( H^1 = L^1 = Z_{2^n} \), by Lemma 6.10, we obtain a final contradiction proving Lemma 6.12.

The final contradiction of \( \S 6 \) will follow immediately from the next lemma.

**Lemma 6.13.** For all \( i \) and \( j \), \( F_{H^1} = F_{L^1} \).

**Proof.** Take \( z \in F_{H^1} \). We show that \( z \in F_{L^1} \). Now \( H^1 = (M^x_1)_z \) and \( L^1 = (M^x_1)_y \). Since all \( H^1 \)'s normalize \( L^1 \)'s, \( H^1 \) normalizes \( L^1 \) and fixes \( M^x_1 \). By Lemma 6.12 and double-transitivity, \( (M^x_1)_z \neq 1 \). By Lemma 6.11 and double-transitivity, \( |(M^x_1)_z| = |(M^x_1)_y| \). Since \( H^1 \) fixes \( z \) and normalizes \( M^x_1 \), \( H^1 \) normalizes \( (M^x_1)_z \). By Lemma 6.5, if \( f \in H^1 \), and \( f \neq 1 \), the centralizer of \( f \) on \( (M^x_1)_z \) is precisely \( (M^x_1)_z \). Therefore, we have a \((H^1, (M^x_1)_z, (M^x_1)_y)\) configuration with \( |H^1| = |(M^x_1)_z| \). Applying Lemma 4.2 of part I to each Sylow subgroup of \( H^1 \) and \( (M^x_1)_z \), it follows that \( (M^x_1)_z = (M^x_1)_y \). Since \( |(M^x_1)_z| = |(M^x_1)_y| \), \( (M^x_1)_z = (M^x_1)_y \). Therefore, \( z \in F_{L^1} \).

Thus, \( F_{H^1} \subset F_{L^1} \). Similarly, \( F_{L^1} \subset F_{H^1} \).

We can now bring \( \S 6 \) to a close. Set \( Y = F_{H^1} \). By Lemma 6.13, \( Y = F_{L^1} = F_{L^2} \). Now \( L^2 \) centralizes \( M^x_1 \) and therefore \( M^x_1 \) fixes the set \( Y \). But the kernel of the restriction map \( M^x_1 \rightarrow M^x_1|Y \) is not 1 as \( L^1 \neq 1 \) and \( L^1 \) fixes all points of \( Y \). Since \( M^x_1 \) is simple, \( M^x_1 \) fixes \( y \), a contradiction.

7. In this section we complete the proof of Theorem A. Here, \( G \) is a doubly-transitive group on \( X \), \( N^x < G_x \), and \( N^x = M^x_1 \times M^x_2 \times \ldots \times M^x_k \), with \( M^x_i \) isomorphic simple groups. By the last section we may assume \( M^x_i \cap M^x_j \neq 1 \) for some \( x, y \in X, x \neq y \). For definiteness, we assume \( M^x_1 \cap M^x_2 \neq 1 \).

We derive a contradiction if \( k \geq 2 \).

Crucial to many of our arguments will be the counting methods of \( \S 1 \).

We take \( s_1, \ldots, s_k \) to be the lengths of the orbits of \( y \) under \( M^x_1, \ldots, M^x_k \) and \( t_1, \ldots, t_k \) to be the lengths of the orbits of \( x \) under \( M^x_1, \ldots, M^x_k \), respectively.

We define a graph \( G_x \) on \( X - x \) preserved by \( G_x \) as follows: \( G_x = \{ (a, b) | f(a) = b \text{ for some } f \in M^x_i \text{ and some } i \} \). Thus, \( \{a, b\} \in G_x \) if \( \{a, b\} \) lies in some orbit of some \( M^x_i \), \( 1 \leq i \leq k \). We let \( v \) be the valence of \( G_x \).

**Lemma 7.1.** \( v \leq (s_1 - 1) + (s_2 - 1) + \ldots + (s_k - 1) \).

**Proof.** Fix \( a \in X - x \) and let \( \Delta_1, \ldots, \Delta_k \) be the orbits of \( a \) under \( M^x_1, \ldots, M^x_k \) respectively. Then \( \{a, b\} \in G_x \) if and only if \( b \in \bigcup_{i=1}^k (\Delta_i - a) \).

Since the latter set has size at most \( (s_1 - 1) + \ldots + (s_k - 1) \), the lemma follows.

**Lemma 7.2.** If \( f \in G, f \neq 1 \), set
Then \( |T(f)| \leq v \).

**Proof.** Take \( a, b \in X, a \neq b \), such that \( f(a) = b \). Set \( \tau(a, b) = \{z \mid \{a, b\} \in G_z \} \). By Lemma 1.1 of §1, \( |\tau(a, b)| = v \). Now if \( z \in T(f), f \in M^*_i, \) and \( f(a) = b \). Thus, \( \{a, b\} \in G_z \). So \( T(f) \subseteq \tau(a, b) \), and \( |T(f)| \leq v \).

Our first step will be to show that \( k = 2 \). So, we first assume \( k \geq 3 \).

**Lemma 7.3.** Either (i) \( s_1 > s_2, \ldots, s_1 > s_k \), or (ii) \( k = 3 \) and \((M_2^x)_y = (M_3^x)_y = 1\).

**Proof.** If (i) fails, we may take \( s_2 > s_1 \) and \( s_2 > s_3, \ldots, s_2 > s_k \). Set \( Y = \{z \mid M_1^x \cap M_2^x \subseteq M_i^x \text{ for some } i \} \). By Lemmas 7.1 and 7.2, \( |Y| \leq v \leq (s_1 - 1) + \ldots + (s_k - 1) \). Clearly, \( \{x, y\} \subseteq Y \). Moreover, \( C_G(M_1^x \cap M_2^x) \) fixes \( Y \).

Now take \( \bar{M}_1 = \bar{M}_2 \times \ldots \times \bar{M}_k \). Since \( [\bar{M}_1, \bar{M}_2] = 1 \), \( \bar{M}_1 \) fixes \( Y \). Thus, \( Y \) contains the orbit of \( y \) under \( \bar{M}_1^x \). The length of this orbit is \( |\bar{M}_1^x : (\bar{M}_1^x)_y| \).

Thus, \( |\bar{M}_1^x : (\bar{M}_1^x)_y| < s_1 + \ldots + s_k \).

Let \( \pi \) be the projection of \( \bar{M}_1^x \) onto \( M_2^x \times \ldots \times M_k^x \). Let \( T = \pi((\bar{M}_1^x)_y) \). Then, \( |(\bar{M}_1^x)_y| = |T| \cdot |(M_2^x)_y| \). So \( |\bar{M}_1^x : (\bar{M}_1^x)_y| = |M_2^x : (M_2^x)_y| \cdot |M_3^x \times \ldots \times M_k^x : T| \). Since \( s_2 > s_1 \) for all \( i, k > |M_3^x \times \ldots \times M_k^x : T| \).

If \( T \) contains no normal subgroup \( \neq 1 \) of \( M_3^x \times \ldots \times M_k^x \), by Lemma 3.5, \( k > 5^{k-2} \). Hence, \( k \leq 2 \).

So we may assume \( T \) contains \( M_3^x \). Let \( L \) be the largest normal subgroup of \( M_3^x \times \ldots \times M_k^x \) contained in \( T \). Then, there is a subgroup \( H \) of \( \bar{M}_1^x \), such that \( (M_2^x)_y \subseteq H \) and \( \pi(H) = L \). Let \( \bar{H} \) be the projection of \( \bar{M}_1^x \) onto \( M_2^x \). Let \( \bar{H} = \ker(\pi_2|H) \). Then, \( \bar{H} \subseteq M_3^x \times \ldots \times M_k^x \) and \( \pi(\bar{H}) = \bar{H} \). Since \( \bar{H} \) fixes \( y \), if \( \bar{H} \neq 1 \), we have a contradiction by the proposition of §4. Hence, \( \bar{H} = 1 \). Therefore, \( H \) is isomorphic to a subgroup of \( M_2^x \). Since also \( H \) has homomorphic image \( L \) and \( L \) contains \( M_3^x \), \( |H| = |M_3^x| \), and \( H \) is a diagonal of \( M_2^x \) and \( M_3^x \). Since \( H \) fixes \( y \), it follows that \( (M_2^x)_y = (M_3^x)_y = 1 \).

Since \( T/M_3^x \) contains no normal subgroup of \( M_3^x \times \ldots \times M_k^x/M_3^x \), by Lemma 3.5, \( k > 5^{k-3} \). Hence, \( k \leq 3 \). Therefore, \( k = 3 \) and \((M_2^x)_y = (M_3^x)_y = 1 \).

**Lemma 7.4.** Either (i) \( s_1 > s_2, \ldots, s_1 > s_k \) and \( t_1 > t_2, \ldots, t_1 > t_k \), or

\[
\begin{align*}
(\text{ii}) \quad & k = 3 \text{ and } (M_2^x)_y = (M_3^x)_y = 1, \\
& (M_2^x)_x = (M_3^x)_x = 1.
\end{align*}
\]

**Proof.** Since Lemma 7.3 also applies to \( N^y \), it suffices to show that if \( k = 3 \) and \((M_2^x)_y = (M_3^x)_y = 1 \), then \((M_2^x)_x = (M_3^x)_x = 1 \). But if \( k = 3 \) and
(M_x^2)_y = (M_3^3)_y = 1, M_2^x and M_3^x are both regular on their orbits which contain \( y \). By double-transitivity, two of the factors of \( N^y \) are regular on their orbits which contain \( x \). Since \( M_1^y \) is not regular on its \( x \)-orbit, as \( 1 \neq M_1^x \cap M_1^y \subseteq (M_1^y)_x \), \( M_2^x \) and \( M_3^x \) are regular on their \( x \)-orbits. Hence, \( (M_2^y)_x = (M_3^y)_x = 1 \).

**Lemma 7.5.** \( M_i^x \cap M_j^y = 1 \) if \( (i, j) \neq (1, 1) \).

**Proof.** We use Lemma 7.4. In case (ii) of Lemma 7.4, Lemma 7.5 is clear, as \( (M_i^x)_y = 1 \) if \( i \neq 1 \) and \( (M_j^y)_x = 1 \) if \( j \neq 1 \). Hence, \( s_i > s_j \) if \( i \neq 1 \) and \( t_i > t_j \) if \( j \neq 1 \).

Now if \( M_i^x \cap M_j^y \neq 1 \) for \( (i, j) \neq (1, 1) \), we may suppose, say \( i \neq 1 \). But the proof of Lemma 7.4 is based only on the fact that \( M_i^x \cap M_j^y \neq 1 \). Hence, if \( M_i^x \cap M_j^y \neq 1 \) for \( i \neq 1 \), by Lemma 7.4, \( s_i > s_1 \), a contradiction.

It follows that for each \( a, b \in X \), \( a \neq b \), there is a unique \( (i, j) \) such that \( M_i^a \cap M_j^b \neq 1 \). We set

\[
B(a, b) = \{ c | 1 \neq M_i^a \cap M_j^b \subseteq M_k^c \text{ for some } k \}.
\]

**Lemma 7.6.** \( \{ B(a, b) | a, b \in X, a \neq b \} \) form a block design (with \( \lambda = 1 \)) on \( X \) preserved by \( G \).

**Proof.** By double-transitivity, if \( M_i^a \cap M_j^b \neq 1 \), then \( |M_i^a \cap M_j^b| = |M_i^x \cap M_j^y| \).

To prove Lemma 7.6, it suffices to show that if \( \{c, d\} \subseteq B(a, b) \), then \( B(a, b) = B(c, d) \). But if \( \{c, d\} \subseteq B(a, b) \), there are subscripts \( k \) and \( l \) such that \( 1 \neq M_i^a \cap M_j^b \subseteq M_k^c \) and \( 1 \neq M_i^a \cap M_j^b \subseteq M_l^d \). Hence, \( 1 \neq M_i^a \cap M_j^b \subseteq M_k^c \cap M_l^d \). By the previous paragraph, \( M_i^a \cap M_j^b = M_k^c \cap M_l^d \). Thus, \( B(a, b) = B(c, d) \).

**Lemma 7.7.** \( k = 2 \).

**Proof.** Suppose \( k \geq 3 \) and let \( B \) be the block design of Lemma 7.7. Let \( B = B(x, y) \). Then \( G_x^B \mid B \) is doubly-transitive. Let \( \bar{M}_1^x = M_2^x \times \ldots \times M_k^x \). Since \( \bar{M}_1^x \) centralizes \( M_1^x \cap M_1^y \), \( \bar{M}_1^x \) fixes the set \( B \). Since \( M_1^x \cap M_1^y \) fixes all points of \( B \) and \( M_1^x \) is simple, \( M_1^x \) does not fix the set \( B \). Therefore, \( \bar{M}_1^x \) is the largest normal subgroup of \( N^x \) which fixes \( B \). Therefore, \( \bar{M}_1^x \leq (G_B^* \mid B)_x \).

Since \( k \geq 3 \), \( \bar{M}_1^x \) has at least two simple factors. By induction, we obtain a contradiction.

Before proceeding to the case \( k = 2 \), we prove some general lemmas which we shall use later on.

**Lemma 7.8.** Let \( G \) be a doubly-transitive group on a set \( X \) and \( K^x \) a normal subgroup of \( G_x \) of prime index \( p \) in \( G_x \). Suppose also

- (i) \( |X| \equiv 1 \pmod{p} \), and
- (ii) if \( x, y \in X, x \neq y, K_x^x = K_y^y = K_x^x \cap K_y^y \). Then, there is a normal subgroup \( \bar{G} \) of \( G \) such that \( \bar{G}_x = K_x^x \).
Proof. Take $Q$ a Sylow $p$-subgroup of $G_x$ and set $P = Q \cap K^x$. By (i), $Q$ is a Sylow $p$-subgroup of $G$. We use Grun's theorem [7], to show that $G$ has a normal subgroup $\overline{G}$ of index $p$ such that $\overline{G} \cap Q = P$.

Thus, it suffices to show that $[N_G(Q), Q] \subseteq P$ and if $Q'$ is another Sylow $p$-subgroup of $G$, $[Q', Q'] \cap Q \subseteq P$.

Since $Q$ fixes only the point $x$, $N_G(Q) = N_{G_x}(Q)$. Since $K^x \triangleleft G_x$ and $K^x \cap Q = P$, $[N_{G_x}(Q), Q] \subseteq P$. Likewise, if $Q'$ is a Sylow $p$-subgroup of $G$ and $Q' \subseteq G_x$, $[Q', Q'] \cap Q \subseteq P$.

So we may suppose $Q'$ is a Sylow $p$-subgroup of $G$ and $Q' \subseteq G_y$ for some $y \neq x$. Then, $[Q', Q'] \subseteq K^y$. Thus, $[Q', Q'] \cap Q \subseteq K^x = K^y$. Since $K^x \cap Q = P$, $[Q', Q'] \cap Q \subseteq P$.

From this it follows:

Lemma 7.9. Let $G$ be a doubly-transitive group on $X$, $x \in X$, $K^x$ a normal subgroup of $G_x$ of prime index $p$ in $G_x$ such that $K^x$ is intransitive on $X - x$.

Then there is a normal subgroup $\overline{G}$ of $G$ such that $\overline{G}_x = K^x$.

Proof. Since $|G_x : K^x| = p$ and $K^x$ is intransitive on $X - x$, if $f \in G_x$ fixes $y \in X - x$, $f \in K^x$. Thus, $K^x_x \subseteq K^x$ and so $K^x = K^y$. Since $K^x$ has $p$ orbits of equal length of $X - x$, $|X - x| \equiv 0 \pmod{p}$.

Now we begin the analysis with $k = 2$. We proceed in two cases according as $G_{xy}$ normalizes $M_1^x$ or $G_{xy}$ does not normalize $M_1^x$. First we treat the case in which $G_{xy}$ normalizes $M_1^x$.

Lemma 7.10. $G_{xy}$ normalizes $M_1^x, M_2^x, M_1^y, M_2^y$.

Now set $L^x = N_{G_x}(M_1^x) = N_{G_x}(M_2^x)$. By Lemma 2.3, $L^x \subseteq G_x$. Thus, $L^x$ is of index 2 in $G_x$. Since $G_{xy}$ is contained in $L^x$, $L^x$ has two orbits of length, say $l$, on $X - x$. By Lemma 7.9, there is a normal subgroup $\overline{G}$ of $G$ such that $\overline{G}_x = L^x$.

Then, $\overline{G}$ is a rank 3 permutation group having subdegrees 1, 1, 1. Since $M_1^x \subseteq \overline{G}$, $\overline{G}$ is even. Therefore, $l$ is even and both orbits of $\overline{G}_x$ are self-paired. Since $G_{xy} \subseteq \overline{G}$, for all $x, y \in X$, $x \neq y$, and since $\overline{G}$ contains elements interchanging $x$ and $y$, $G_{xy} \subseteq \overline{G}$.

Now $M_1^x$ and $M_2^x$ are not conjugate in $\overline{G}$. Thus, we may index $M_1^x$ and $M_2^x$ for other $y \in X - x$ so that $M_1^x$ and $M_1^y$ are conjugate in $\overline{G}$. If $j$ is an element of $\overline{G}$ interchanging $x$ and $y$, then $j(M_1^x)j^{-1} = M_1^y$. It follows that the $y$-orbit of $M_1^x$ has the same length as the $x$-orbit of $M_1^y$, which we denote by $s_1$. $s_2$ is defined similarly.

Because of our reindexing we need no longer have $M_1^x \cap M_1^y \neq 1$. However, still for some $i, j$, $M_1^x \cap M_1^y \neq 1$.

Lemma 7.11. $M_1^x \cap M_1^y = M_2^x \cap M_1^y = 1$. 
Proof. Suppose $M_1^x \cap M_2^x \neq 1$ and set $Y = \{ z | M_1^x \cap M_2^x \subseteq M_1^z \}$. By Lemma 7.2, $|Y| \leq (s_1 - 1) + (s_2 - 1)$. Also, \{x, y\} \subseteq Y.

Suppose for definiteness that $s_2 \geq s_1$. Since $[M_1^x, M_1^x \cap M_2^x] = 1$, $M_2^x$ fixes the set $Y$. We study the orbits of $M_2^x$ on $Y - x$. The orbit of $y \in Y - x$ under $M_2^x$ is of length $s_2$. Any other orbit $M_2^x$ on $Y - x$ is of length $\geq s_1$.

But as $s_1 + s_2 > (s_1 - 1) + (s_2 - 1)$, $M_1^x$ has only one orbit on $Y - x$.

Since $[M_1^x, M_1^x \cap M_2^x] = 1$, $M_1^x$ fixes $Y$. Since $M_1^x$ does not fix $x$, $M_1^y$ moves $x$ into $Y - x$. Thus, $(M_2^x, M_1^y)$ is doubly-transitive on $Y$. Also, $(M_2^x, M_1^y) \subseteq G$. Thus, there is a $j \in \langle M_2^x, M_1^y \rangle$ such that $j$ interchanges $x$ and $y$, and it follows that $j(M_1^x)^{-1} = M_1^x$. Therefore, $M_1^x$ fixes $Y$, a contradiction, as $M_1^x \cap M_2^x$ fixes all points of $Y$ and $M_1^x$ is simple.

Now by hypothesis $M_1^x \cap M_1^y \neq 1$ for some $i$, $j$. If both $M_1^x \cap M_1^y \neq 1$ and $M_2^x \cap M_2^y \neq 1$, if necessary, renumber $M_1^x$ and $M_2^x$ so that $s_2 > s_1$. Otherwise, choose the numbering so that $M_1^x \cap M_1^y = 1$ and $M_2^x \cap M_2^y = 1$.

Now set $B = \{ z | M_1^x \cap M_2^y \subseteq M_1^z \}$. Lemma 7.12. \{g(E) | g \in G\} form a block design on $X$ (with $\lambda = 1$) preserved by $G$. We call this block design $B$.

Proof. If $M_2^x \cap M_2^y = 1$, the proof of Lemma 7.6 is valid. So we may suppose that $M_2^x \cap M_2^y \neq 1$. By our choice of numbering, $s_2 > s_1$.

Since $[M_2^x, M_1^x \cap M_2^y] = 1$, $M_2^x$ fixes $B$. By Lemmas 7.1 and 7.2, $|B| \leq (s_1 - 1) + (s_2 - 1)$. Also, \{x, y\} \subseteq B. Studying the orbits of $M_2^x$ on $B - x$, it follows, as in Lemma 7.11, that $M_2^x$ is transitive on $B - x$, and that $\langle M_2^x, M_2^y \rangle$ is doubly-transitive on $B$.

Since $G_{xy}$ normalizes $M_1^x$ and $M_1^y$, $M_2^x \cap M_2^y \subseteq G_{xy}$. Therefore, $G_{xy}$ fixes $B$.

Now since $G_B^x |B$ is doubly-transitive and $G_{xy}$ fixes $B$, it is readily verified that the translates of $B$ under the action of $G$ form a block design preserved by $G$.

Take $\Delta(x, y)$ the orbit of $y$ under $L^x$. If $z \in x - \Delta(x, y)$, $\Delta(x, z)$ is the orbit of $z$ under $L^x$.

Lemma 7.13. If $C \in B$ and $x \in C$, either (i) $C \subseteq \Delta(x, y)$ and $M_2^x$ fixes $C$, or (ii) $C \subseteq \Delta(x, z)$ and $M_1^x$ fixes $C$.

Proof. First take $C = B$, the block containing $x$ and $y$. It is clear from the definition of $B$ that $M_2^x$ fixes $B$. Moreover, $M_1^x$ does not fix $B$ as $M_1^x$ is simple and $M_1^x \cap M_1^y$ fixes all points of $B$. Therefore, $M_2^x$ is the largest normal subgroup of $N^x$ which fixes $B$. Therefore, $M_2^x \subseteq (G_B^x)^{x}$. Therefore, $(G_B^x)^{x} \subseteq L^x$.

Since also $(G_B^x)^{x}$ is transitive on $B - x$, $B \subseteq \Delta(x, y)$.

Now if $C$ is any block of $B$ containing $x$, $f(B) = C$ for some $f \in G_x$. If
Let $f \in L^x$, $C \subseteq \Delta(x, y)$ and $M_2^x$ fixes $C$. If $f \in G_x - L^x$, $C \subseteq \Delta(x, z)$ and $M_1^z$ fixes $C$.

We are now in a position to obtain a final contradiction to our assumption that $G_{xy}$ normalizes $M_1^x$. Recall $|\Delta(x, y)| = 1 + l$ and $|X| = 1 + 2l$. Set $|B| = 1 + r$.

Take $B = \{x_1, \ldots, x_{r+1}\}$. Then there is a $y_i \in X - x_i$ such that $B \subseteq \Delta(x_i, y_i)$, by Lemma 7.13. Take $z_i \in X - \Delta(x_i, y_i)$.

Now for each $x_i \in B$, there is an index $t$ such that $M_1^x \cap M_2^z \subseteq M_1^x_t$. By Lemma 7.13, $M_1^x_t$ fixes either all the blocks of $B$ containing $x_i$ in $\Delta(x_i, y_i)$ or those in $\Delta(x_i, z_i)$. Since $B \subseteq \Delta(x_i, y_i)$ and $M_1^x \cap M_2^z$ fixes all points of $B$, $M_1^x_t$ does not fix $B$. Therefore, $M_1^x_t$ fixes all blocks contained in $\Delta(x_i, z_i)$. Therefore, $M_1^x \cap M_2^z$ fixes all the blocks of $B$ which contain $x_i$ and lie in $\Delta(x_i, z_i)$.

Moreover, if $C$ is such a block, as $M_1^x_t$ fixes $C$, $M_1^x \cap M_2^z$ has a nontrivial orbit on $C$.

Since each $\Delta(x_i, z_i)$ contain $l/r$ blocks which contain $x_i$ and since there are $r + 1$ choices for $x_i \in B$, we obtain a family $C$ of $(1 + r)/r$ distinct blocks such that $M_1^x \cap M_2^z$ fixes each $C \in C$ and $M_1^x \cap M_2^z$ has a nontrivial orbit on $C$.

Since no two blocks of $B$ have in common a pair of points, no two blocks of $C$ have in common a nontrivial orbit of $M_1^x \cap M_2^z$.

Then, if $Y$ is the union of the nontrivial orbits of $M_1^x \cap M_2^z$, $|Y| \geq 2(1 + r)/l_r$.

Since $M_1^x \cap M_2^z$ also fixes the $1 + r$ points of $B$, it follows that $(1 + r) + 2(1 + r)/l_r \leq |X| = 1 + 2l$. Therefore, $1 + r + 2l \leq 1 + 2l$, a contradiction.

Thus, in the remainder of this section we may assume that $G_{xy}$ does not normalize $M_1^x$.

Again, we set $L^x = N_{G_x}(M_1^x) = N_{G_x}(M_2^x)$. Since $G_x = L^x \cdot G_{xy}$, it follows that $L^x$ is transitive on $X - x$.

We suppose that $M_1^x \cap M_2^z \neq 1$. We take $\Delta_4(x, y)$ to be the union of $x$ and the orbit of $y$ under $M_1^x$ and $\Delta_4(y, x)$ to be the union of $y$ and the orbit of $x$ under $M_2^z$. We define $\Delta_2(x, y)$ and $\Delta_2(y, x)$ analogously.

Since $L^x$ is transitive on $X - x$, all orbits of $M_1^x$ on $X - x$ are of the same length $s$. Since $M_1^x$ and $M_2^z$ are conjugate in $G_x$, the same applies to $M_2^z$. Thus, $|\Delta_1(x, y)| = |\Delta_2(x, y)| = |\Delta_1(y, x)| = |\Delta_2(y, x)| = s + 1$.

**Lemma 7.14.** Let $f \in M_1^x \cap M_2^z$, $f \neq 1$. Set $T(f) = \{z \mid f \in M_1^z \text{ for some } k\}$. Writing $N^x = M_1^x \times M_2^x$ and $N^y = M_1^y \times M_2^y$, we have $T(f) = \Delta_4(x, y) = \Delta_4(y, x)$.

**Proof.** By Lemmas 7.1 and 7.2, $|T(f)| \leq 2s - 2$. Also, $\{x, y\} \subseteq T(f)$, and $M_1^x$ fixes $T(f)$, as $M_1^x$ centralizes $M_1^x \cap M_2^y$. Thus, $\Delta_4(x, y) \subseteq T(f)$. By the foregoing inequality on $|T(f)|$ and since each orbit of $M_1^x$ on $T(f) - x$ is of length $s$, it follows that $\Delta_4(x, y) = T(f)$. Similarly, $\Delta_4(y, x) = T(f)$. 


Lemma 7.15. (i) $|M_1^x \cap M_1^y| = |M_2^x \cap M_2^y|$; (ii) $M_1^x \cap M_2^y = M_2^x \cap M_1^y = 1$.

Proof. Since $M_1^x \cap M_1^y \neq 1$, by Lemma 7.14, $\Delta_2(x, y) = \Delta_2(y, x)$. If also $M_1^x \cap M_2^y \neq 1$, then $\Delta_2(x, y) = \Delta_1(x, y)$, again by Lemma 7.14. Thus, $\Delta_1(x, y) = \Delta_1(y, x)$. Since $M_2^y$ is transitive on $\Delta_2(y, x)$ and $M_2^y$ fixes $\Delta_2(y, x)$, while centralizing $M_2^y$, $M_2^y$ is semiregular on $\Delta_2(y, x)$, a contradiction as $1 \neq M_2^y \cap M_1^x \subseteq (M_1^y)_x$. 

Thus $M_1^x \cap M_2^y = M_2^x \cap M_1^y = 1$.

Now take $t \in G_{xy}$ such that $t(M_1^x)t^{-1} = Aff$. Then, $t(M_1^x \cap M_1^y)t^{-1} = M_2^x \cap M_1^y$. Since $M_2^x \cap M_1^y = 1$, $t(M_1^x \cap M_1^y)t^{-1} = M_2^x$. Thus, $|M_1^x \cap M_1^y| = |M_2^x \cap M_1^y|$

Lemma 7.16. (i) $L_1^x = L_1^y = L^x \cap L^y$; (ii) $|X|$ is even and $s$ is odd.

Proof. (i) Since $I^*$ normalizes $M_1^x$, $M_1^x \cap M_1^y \neq 1$, and $M_1^x \cap M_2^y = 1$, $L_1^x$ normalizes $M_1^y$. Thus, $L_1^x \subseteq L_1^y$. So $L_1^x = L_1^y = L^x \cap L^y$.

(ii) Now if $|X|$ is odd, by Lemma 7.8, there is a normal subgroup $\overline{G}$ of $G$ such that $\overline{G}_x = L_1^x$. Since $L_1^x$ is transitive on $X - x$, $\overline{G}$ is doubly-transitive on $X$. On the other hand, $Aff \leq G_x$, a contradiction by Lemma 2.3.

Thus, $|X|$ is even. As $s | |X - x|$, $s$ is odd.

Lemma 7.17. $(M_1^1)_y = M_1^x \cap M_1^y = (M_1^y)_x$.

Proof. Set $B = A_2(x, y) = A_2(y, x)$ (using Lemma 7.14). Thus, $(M_2^x, M_2^y)$ is doubly-transitive on $B$. Now $B = \{z \mid M_1^x \cap M_1^y \subseteq M_1^z\}$, by Lemma 7.14. By Lemma 7.16(i), $(M_1^x)_y$ normalizes $M_1^y$. Therefore, $M_1^x \cap M_1^y \subseteq (M_1^y)_y$. Thus, also $(M_1^y)_y$ fixes $B$.

Since $M_2^x$ is transitive on $B - x$, and $(M_1^y)_y$ fixes $y \in B - x$ and centralizes $M_2^x$, $(M_1^y)_y$ fixes all points of $B$.

Now $(M_1^y)_y$ normalizes $M_2^y$ (as it normalizes $(M_1^y)_y$). Therefore, $(M_2^y) \cdot (M_1^y)_y$ is a group which fixes the set $B$. The subgroup of $M_2^y \cdot (M_1^y)_y$ fixing all points of $B$ is precisely $(M_1^y)_y$. Therefore, $M_2^y$ normalizes $(M_1^y)_y$. Since $M_2^y$ does not fix $x$, $(M_1^y)_y \subseteq M_1^x \cap M_1^y$ for some $x' \neq x$ and some $i$. But by Lemma 7.15, $|M_1^x \cap M_1^y| = |M_1^x \cap M_1^y|$. Therefore, $|M_1^y| \leq |M_1^x \cap M_1^y|$. Since also $M_1^x \cap M_1^y \subseteq (M_1^y)_y$, $M_1^x \cap M_1^y = (M_1^y)_y$.

Lemma 7.18. Let $B$ be the fixed point set of $(M_1^y)_y$. Then (i) $B = \Delta_2(x, y) = \Delta_2(y, x)$, and (ii) $(M_1^y)_y$ is semiregular on $X - B$.

Proof. Suppose $f \in (M_1^y)_y$, $f \neq 1$, and suppose $f$ fixes $z$. We show $z \in \Delta_2(x, y)$, proving both (i) and (ii).

By Lemma 7.15 and double-transitivity, either $M_1^x \cap M_1^y \neq 1$ and $M_1^x \cap M_2^y = 1$ or $M_1^x \cap M_1^y = 1$ and $M_1^x \cap M_2^y \neq 1$. So, for definiteness, say $M_1^x \cap M_1^y \neq 1$.

By Lemma 7.17 and double-transitivity, $(M_1^1)_y = (M_1^y)_x$. Therefore, $f \in M_1^y$. Since
Lemma 7.19. (i) $M_1^x$ is $SL(2, q)$, $Sz(q)$, $U_3(q)$, with $q = 2^a$.
(ii) $(M_1^x)_y$ is a Sylow 2-subgroup of $M_1^x$.

Proof. Now $|M_1^x : (M_1^x)_y| = s$ is odd, by Lemma 7.16 and $(M_1^x)_y$ is semi-
regular on the complement of its fixed point set in $A_2(x, y)$.

Set $H = N_{M_1^x}((M_1^x)_y)$. If $T$ is a subgroup of $(M_1^x)_y$ and $T 
eq 1$, then $(M_1^x)_y$
and $T$ have the same fixed point set. Therefore, $N_{M_1^x}(T) \subseteq H$.

In particular, as $s$ is odd, $(M_1^x)_y$ contains a Sylow 2-subgroup $P$ of $M_1^x$, and
$N_{M_1^x}(P) \subseteq H$. Also, if $j$ is an involution of $H$, $j \in (M_1^x)_y$ (as $(M_1^x)_y < H$ and
$|H : (M_1^x)_y|$ is odd), and $C_{M_1^x}(j) \subseteq H$. Thus, $H$ is a strongly embedded subgroup
of $M_1^x$.

By Bender's Theorem [1], and since $M_1^x$ is simple, $M_1^x$ is $SL(2, q)$, $Sz(q)$,
$U_3(q)$, with $q = 2^a > 2$. Moreover, $H$ is a Sylow 2-normalizer in $M_1^x$.

Next we claim: $P = (M_1^x)_y$.

By the structure of $SL(2, q)$, $Sz(q)$, and $U_3(q)$, it follows that if $a \in N_{M_1^x}(P)$,
a $\neq 1$, and $|a|$ is odd, then $M_1^x = \langle P, N_{M_1^x}(a) \rangle$. Thus, if $(M_1^x)_y$ contains an
element $a \neq 1$ of odd order, $P$ and $N_{M_1^x}(a)$ are contained in $H$ and $H = M_1^x$, a
contradiction.

Lemma 7.20. (i) $\Delta_1(x, y) = \Delta_2(x, y) = \Delta_2(y, x)$.

Proof. Now all orbits of $M_1^x$ on $X - x$ are of length $s$. Thus, each point
$y \in X - x$ is fixed by a unique Sylow 2-subgroup of $M_1^x$.

By Lemma 7.18, $\Delta_2(x, y) - x$ is the fixed point set of a Sylow 2-subgroup
of $M_1^x$. By Lemma 7.19, the size of $\Delta(x, y) - x$ is as given. Thus, $|X - x|$ is
$|\Delta_1(x, y) - x|$ times the number of Sylow 2-subgroups of $M_1^x$.

Lemma 7.21. The family of sets $\{g(\Delta_1(x, y)) | g \in G\}$ form a $(b, v, r, k, \lambda)$-
design on $X$ with $\lambda = 2$.

Proof. We must show that each 2 element subset of $X$ belongs to exactly
two of the sets $g(\Delta_1(x, y))$. It suffices to show that if $\{x, y\} \subseteq g(\Delta_1(x, y))$, then
g(\Delta_1(x, y)) = \Delta_1(x, y)$ or $\Delta_2(x, y)$.

Then, $g(a) = x$ and $g(b) = y$ for $\{a, b\} \subseteq \Delta_1(x, y)$. Since $(M_2^x, M_2^x)$ is
doubly-transitive on $\Delta_1(x, y)$, there is an $h$ which fixes $\Delta_1(x, y)$ such that
$h(x) = a$ and $h(y) = b$.
Then \((g \cdot h)(\Delta_1(x, y)) = g(\Delta_1(x, y))\) and \(g \cdot h \in G_{xy}\).

Now \(|G_{xy} : L^x_y| = 2\) and \(L^x_y\) fixes \(\Delta_1(x, y)\). Thus, if \(gh \in L^x_y\), \(g(\Delta_1(x, y)) = \Delta_1(x, y)\) and if \(g \cdot h \in G_{xy} - L^x_y\), \(g(\Delta_1(x, y)) = \Delta_2(x, y)\).

We now obtain a final contradiction, and complete the proof of Theorem A.

\(v = |X|\) and \(k = |\Delta(x, y)|\). Then \(b\) is the number of sets \(\{g(\Delta_1(x, y))\}\) and satisfies \(b = 2v(v - 1)/k(k - 1)\).

Therefore, \(2v(v - 1) \equiv 0\pmod{k(k - 1)}\).

Since \((v - 1)(k - 1)\) is the number of Sylow 2-subgroups of \(M^x_1\) and \(k - 1\) is prime to \(k\), so \((v - 1)(k - 1)\) is relatively prime to \(k\), and so \(2v \equiv 0\pmod{k}\).

In case (i), \(v = 1 + (1 + q)^2(q - 1)\) and \(k = 1 + (1 + q)(q - 1)\). Thus, \(2(1 + (1 + q)^2(q - 1)) \equiv 0\pmod{q^2}\). Thus, \(2(q^3 + q^2 - q) \equiv 0\pmod{q^2}\). So \(q/2\), in contradiction to the simplicity of \(M^x_1\).

In case (ii), \(2(1 + (1 + q^2)^2(q - 1)) \equiv 0\pmod{(1 + (1 + q^2)(q - 1) + 1)}\), and as \((1 + q^2)(q - 1) + 1 = q^3 - q^2 + q\), \(2(1 + (1 + q^2)^2(q - 1)) \equiv 0\pmod{q^2 - q + 1}\). Since the modulus is odd and \(1 + q^2 \equiv q\pmod{q^2 - q + 1}\), \(1 + q^2(q - 1) \equiv 0\pmod{q^2 - q + 1}\). Then, \(q^2 - 2q + 2 \equiv 0\pmod{q^2 - q + 1}\). So \(q - 1 \equiv 0\pmod{q^2 - q + 1}\). It follows that \(q^2 - q + 1 \leq q - 1\), or \(q^2 \leq 2(q - 1)\), a contradiction.

In case (iii), a similar procedure works, completing the proof of Theorem A.

8. In this section we prove Theorem B. First, however, we must bound the size of the fixed point sets of elements of a doubly-transitive group under certain conditions.

**Proposition 8.** Let \(G\) be a doubly-transitive group on \(X\) and suppose \(G_x\) admits a system of imprimitivity on \(X - x\) having imprimitive block \(\Delta\). Suppose \(|X| = n\) and \(|\Delta| = s\). Suppose no nonidentity element of \(G\) fixes all points of \(x \cup \Delta\).

Then, if \(g \in G\), \(g \neq 1\), \(g\) fixes at most \(s\sqrt{n/2}\) points of \(X\).

**Proof.** Let \(Y\) be the fixed point set of \(g\), and suppose \(t = |Y|\). We take \(\Delta(x, y) = x \cup \Delta\) and obtain a predesign function for \(G\). As usual, we take \(G_x\) to be a graph on \(X - x\) with \(\{a, b\} \in G_x\) if \(\Delta(x, a) = \Delta(x, b)\). Also, \(\tau(a, b) = \{x \in (a, b) \subseteq \Delta(x, y) - x\}\) for some \(y \in X - x\). Then, \(G_x\) is of valence \(s - 1\) and \(|\tau(a, b)| = s - 1\) by Lemma 1.1.

We take \(\Omega\) to be the set of ordered pairs \((x, \{a, b\})\) where

(i) \(g\) fixes \(x\),

(ii) there is some \(y \in X - x\) such that \(\{a, b\} \subseteq \Delta(x, y) - x\),

(iii) \(a \neq b\) and \(a\) and \(b\) lie in the same orbit of \(g\).
Note that (ii) is equivalent to the statement that \( x \in \tau(a, b) \). Also, it follows from these hypotheses that \( g(\Delta(x, y)) = \Delta(x, y) \).

We bound \(|Y|\) by estimating in two different ways the size of \( \Omega \). Clearly, it suffices to do this when \( g \) is of prime order \( p \).

First we obtain a lower bound for \(|\Omega|\). Since no element of \( G \) fixes all points of \( \Delta(x, y) \), the number of fixed points of \( g \) on \( \Delta(x, y) - x \) is at most \( s - p \). Thus, for a fixed \( x \in Y \), the number of sets \( \Delta(x, z) \) on which \( g \) fixes a point of \( \Delta(x, z) - x \) is greater than or equal to \((t - 1)/(s - p)\). Thus, the number of possibilities for \( \Delta(x, y) \) in the definition of \( \Omega \) is at least \( t(t - 1)/(s - p) \).

On each such set \( \Delta(x, y) \), \( g \) has at least one orbit of length \( p \), and so for this \( \Delta(x, y) \), the number of possibilities for \( \{a, b\} \) is at least \( p(p - 1)/2 \). Therefore,

\[
(t(t - 1)/(s - p))(p(p - 1)/2) \leq |\Omega|.
\]

Next we obtain an upper bound for \(|\Omega|\). On \( X - Y, g \) has \((n - t)/p \) orbits of length \( p \). Thus, there are at most \((n - t)/p \cdot p(p - 1)/2 \) possibilities for \( \{a, b\} \) in the definition of \( \Omega \). As \( x \in \tau(a, b) \) if \( (x, \{a, b\}) \in \Omega \), for a given \( \{a, b\} \), there are at most \( s - 1 \) choices for \( x \). Therefore, \(|\Omega| \leq ((n - t)/p)(p(p - 1)/2)(s - 1) \).

It follows that \( t(t - 1) \leq (s - 1)(s - p)(n - t)/p \). Since \( p \geq 2 \), we obtain \( t(t - 1) \leq (s - 1)(s - 2)(n - t)/2 \).

It follows from this that \( t^2 \leq s^2n/2 \). For if not, \( s^2n < 2t + (s - 1)(s - 2) \cdot (n - t) \). Then, \((3s - 2)n < (2 - (s - 1)(s - 2))t \). If \( s \geq 3 \), we have the contradiction, \((3s - 2)n \leq 0 \). If \( s = 1 \) or \( 2 \), it follows from the fact that no element \( \neq 1 \) of \( G \) fixes all points of \( x \in \Delta \) that \( G_{xy} = 1 \) and so \( t \leq 1 \). Thus, \( t \leq s\sqrt{n/2} \).

Remark. Later we shall use the inequality \( t(t - 1) \leq (s - 1)(s - 2)(n - t)/2 \).

Next we prove:

**Theorem B.** Let \( G \) be a doubly-transitive group on \( X, x \in X, and suppose \( N^x \) is a normal subgroup of \( G^x \). Suppose \( |X| = n \) and that the orbits of \( N^x \) on \( X - x \) are of length \( s \). Then, one of the following holds:

(i) \( N^x \) is semiregular on \( X - x \), or

(ii) \( G \) is a normal extension of \( L_n(q) \), or

(iii) \( n < 2(s - 1)^2 \).

**Proof.** We take \( \Delta(x, y) = x \cup \{f(y) \mid f \in N^x\} \). \( \Delta(x, y) \) is the predesign function associated with the orbits of \( N^x \) on \( X - x \). By hypothesis, \(|\Delta(x, y)| = 1 + s \).

We define a graph \( G_x \) on \( X - x \) by connecting \( a, b \in X - x \) if \( a \) and \( b \) belong to the same orbit of \( N^x \). Equivalently, \( a \) and \( b \) are connected in \( \{a, b\} \subseteq \Delta(x, y) - x \) for some \( y \in X - x \). \( G_x \) is a graph of valence \( s - 1 \). By Lemma 1.2, if \( x \neq y \), \(|G_x \cap G_y| = (s - 1)(s - 2)/2 \).

Now if \( N^x \) is semiregular on \( X - x \), (i) holds. If \( N^x \) is not semiregular on
$X - x$, but $N^x \cap N^y = 1$ if $x \neq y$, by Theorem A of part I, (ii) holds. Thus we may take $f \in N^x \cap N^y$, $f$ of prime order $p$.

If $N^x$ does not restrict faithfully to $\Delta(x, y)$, by Theorem A, (ii) holds. Thus, we may suppose $N^x$ restricts faithfully to $\Delta(x, y)$.

We claim that no element of $G$ fixes all points of $\Delta(x, y)$. Indeed, let $W = G_{\Delta(x, y)} \neq 1$. Then, as $N^x$ fixes $\Delta(x, y)$, $[N^x, W] \subseteq W$. As $W$ fixes $x$, $[N^x, W] = N^x$. By the last paragraph, $N^x \cap W = 1$. Thus, $[N^x, W] = 1$. Therefore, $C_{G_x}(N^x)$ is not semiregular on $X - x$. By Corollary B.3 of [8], there is a block design $B$, such that $N^x$ fixes all blocks of $B$ which contain $x$. By Lemma 2.8 of [8], it follows that $N^x \cap N^y = 1$ if $x \neq y$, contrary to hypothesis. Thus, the claim follows.

Let $t$ be the number of fixed points of $f$. By Proposition 8 of this section, $t(t - 1) \leq (s - 1)(s - 2)(n - t)/2$.

Since $f \in N^x \cap N^y$, all pairs of points in orbits of $f$ lie in $G_x \cap G_y$. Since $f$ has $(n - t)/p$ orbits in the complement of its fixed point set and since each orbit contains $p(p - 1)/2$ pairs of points, it follows that

$$\frac{(n - t) \cdot p(p - 1)}{2} \leq \frac{(s - 1)(s - 2)}{2}.$$ 

Thus, $n - t \leq (s - 1)(s - 2)/(p - 1)$. Since $p \geq 2$, $n - t \leq (s - 1)(s - 2)$.

By the previous paragraph, $t(t - 1) \leq (s - 1)^2(s - 2)^2/2$. Thus, $t \leq (s - 1)^2/\sqrt{2}$. Thus, $n \leq (s - 1)^2/\sqrt{2} + (s - 1)(s - 2) < 2(s - 1)^2$.

**Remark.** In case (i) and (ii) of Theorem C, no bound of $n$ in terms of $s$ is possible. Indeed, if $s = 2$, $n$ can be arbitrarily large.

There are certain circumstances under which the bound of (iii) can be sharpened. We say the normal subgroup $N^x$ of $G_x$ is balanced if $N^x \neq N^y = N^x \cap N^y$. In other words, $N^x$ is balanced if $N^x$ is a strongly closed subgroup of $G_x$ in $G$.

In all known doubly-transitive groups, if $N^x$ is intransitive on $X - x$, either $N^x$ is balanced or $N^x \cap N^y = 1$ if $x \neq y$.

Under the condition of balance, we can obtain stronger bounds for the degree of $G$.

**Theorem C.** Let $G$ be a doubly-transitive group on $X$, $x \in X$, and $N^x$, a normal subgroup of $G_x$.

Suppose that the orbits of $N^x$ on $X - x$ are of length $s$ and that $N^x$ is a permutation group of rank $r$ on each of these orbits.

Suppose that $N^x$ is balanced and $N^y \neq 1$ if $y \in X - x$. Then (i) $n \leq (s - 1)^2$, and (ii) the number of orbits of $N^x$ on $X - x$ is less than or equal to $r(s - 1)(s - 2)/s(s - r)$.
Before proceeding with the proof of this result, we remark that if \( r < s/2 \) (which is certainly true if \( N^x \) is primitive on its orbits in \( X - x \)), then the number of orbits of \( N^x \) on \( X - x \) is less than \( 2r \).

If \( r = 2 \), then \( N^x \) is transitive on \( X - x \). If \( r = 3 \), \( N^x \) has at most 3 orbits on \( X - x \).

The proof of (i) follows the lines of Theorem B.

**Lemma 8.1.** Suppose \( G \) is doubly-transitive on \( X, x \in X \), and \( N^x \) is a balanced normal subgroup of \( G_\times \). Suppose the orbits of \( N^x \) on \( X - x \) are of length \( s \).

Let \( f \in N^x, f \neq 1 \). Then \( f \) fixes at most \( s - 1 \) points of \( X \).

**Proof.** Let \( Y \) be the fixed point set of \( f \). Then if \( y \in Y, f \in N^y_\times = N^x_\times \). Thus, \( f \in N^y \). In the notation of Lemma 1.3, it follows that \( Y \leq S(f) \). By Lemma 1.3, \( |S(f)| \leq s - 1 \), and \( |Y| \leq s - 1 \).

To complete the proof of (i) of Theorem C, let \( t \) be the number of fixed points \( N^x \cap N^y \), if \( x, y \in X, x \neq y \). By the proof of Theorem B, it follows that \( n - t \leq (s - 1)(s - 2) \). Since also by the previous lemma, \( t \leq s - 1 \), it follows that \( n \leq (s - 1)^2 \).

Next we prove (ii) of Theorem C with \( \Delta(x, y) \) and \( G_\times \) defined as in the proof of Theorem B.

**Lemma 8.2.** Let \( a_1, \ldots, a_r \) be positive real numbers and suppose \( a_1 + \ldots + a_r = s \). Then \( a_1^2 + \ldots + a_r^2 \geq s^2/r \).

**Proof.** Apply the Cauchy-Schwarz inequality to the vectors \( a = (a_1, a_2, \ldots, a_r) \) and \( b = (1, 1, \ldots, 1) \). Then \( |\Sigma a_i| = s \leq (a_1^2 + \ldots + a_r^2)^{1/2}(1/2) \). Thus, \( a_1^2 + \ldots + a_r^2 \geq s^2/r \).

**Lemma 8.3.** Let \( G \) be a group having two permutation representations of rank \( r \) with respective point stabilizers \( H_1 \) and \( H_2 \).

Then \( H_2 \) has no more than \( r \) orbits on cosets of \( H_1 \).

**Proof.** Let \( \chi_1 \) and \( \chi_2 \) be the respective permutation characters. Since \( G \) is of rank \( r \), \( (\chi_1, \chi_1) = (\chi_2, \chi_2) = r \). Then the number of orbits of \( H_2 \) on cosets of \( H_1 \) is

\[
(\chi_1, \chi_2, 1) = (\chi_1, \chi_1) \leq (\chi_1, \chi_1)^{1/2}(\chi_2, \chi_2)^{1/2} = r.
\]

We now prove (ii). Since \( N^x_\times = N^y_\times \), all orbits of \( N^x_\times \) lie in the sets \( \Delta(y, z) \), for some \( z \in X - y \). Thus, any two element subset of an orbit of \( N^x_\times \) lies in \( G_\times \cap G_\times \).

We estimate the number of two element subsets of \( X \) contained in the orbits of \( N^x_\times \) contained in some \( \Delta(x, z), z \in X - x \). Let the lengths of the orbits of \( N^x_\times \) on \( \Delta(x, z) - x \) be \( a_1, \ldots, a_r \). Then, \( a_1 + \ldots + a_r = s \). By
Lemma 4.2, \(a_1^2 + \ldots + a_t^2 \geq s^2/r\). By Lemma 4.3, \(t \leq r\). Thus, \(a_1^2 + \ldots + a_t^2 \geq s^2/r\). Thus, \(a_1(a_1 - 1)/2 + \ldots + a_t(a_t - 1)/2 \geq 1/2(s^2/r - s)\). Thus, the number of two element subsets of \(\Delta(x, z)\) contained in some orbit of \(N^x_y\) is greater than or equal to \(1/2(s^2/r - s)\).

Since the number of choices for \(\Delta(x, z)\) is \(m\), with \(m\) the number of orbits of \(N^x\) on \(X - x\), it follows that \(|G_x \cap G_y| \geq m/2(s^2/r - s)\).

By Lemma 3.2, \(|G_x \cap G_y| = (s - 1)(s - 2)/2\). Thus, \(m \leq r(s - 1)(s - 2)/s(s - r)\), proving (ii).

Remark. The bounds of Theorem C do not seem to be susceptible to much improvement. Indeed, if \(q = 2m\), there is a doubly-transitive group of degree \(q^2\) and order \(q^2(q^2 - 1) \cdot 2\), such that \(G_x\) has a normal subgroup \(N^x\) of order \((q + 1) \cdot 2\). The nontrivial orbits of \(N^x\) are of length \((q + 1)\) and of rank \(q/2 + 1\). Thus, in this case, \(n = (s - 1)^2\) and the number of orbits of \(N^x\) is the greatest integer less than \(r(s - 1)(s - 2)/s(s - r)\).

The crucial condition in the proof of Theorem C is that \(N^x\) be balanced. We note some conditions under which this holds.

If \(N^x_y\) is simple, then, as \(N^x \cap N^y < N^x_y\), either \(N^x \cap N^y = 1\) or \(N^x = N^y = N^x \cap N^y\). By Theorem A of part I, we cannot have \(N^x \cap N^y = 1\) and \(N^x\) simple. Thus, \(N^x_y = N^x\), and \(N^y\) is balanced.

If we assume \(N^x_y\) is the largest normal subgroup of \(G_x\) which fixes the orbits of \(N^x\), there are other conditions under which \(N^x\) is balanced. If, for example, \(N^x_y\) is an intravariant subgroup of \(N^x\), then \(N^x_y\) fixes some point in each orbit of \(N^x\) on \(X - x\). By Theorem B of [8], \(N^x_y\) fixes some point on each orbit of \(N^x\) on \(X - x\). Thus, \(N^x_y\) fixes all orbits of \(N^x\) on \(X - x\). Thus, \(N^x_y \subseteq N^x\). It follows that \(N^x\) is balanced.

Also, if \(N^x_y\) is a Hall subgroup of \(N^x\), \(N^x\) is balanced. Indeed, under these conditions, if \(p\) is a prime dividing \(N^x_y\), a Sylow \(p\)-subgroup \(p\) of \(N^x_y\) fixes some point in each orbit of \(N^x\). By Lemma 2.6 of [8], some Sylow \(p\)-subgroup of \(N^x_y\) fixes a point in each orbit of \(N^x\). Since \(p\) was any prime divisor of \(|N^x_y|\), it follows that \(N^x_y\) fixes all orbits of \(N^x\). Thus, \(N^x\) is balanced.

We have proved

**Lemma 8.4.** Let \(G\) be a doubly-transitive group on \(X, x, y \in X, x \neq y\). Suppose \(N^x\) is a normal subgroup of \(G_x\). Suppose also that \(N^x\) is the largest normal subgroup of \(G_x\) fixing all orbits of \(N^x\). Then \(N^x\) is balanced if

(i) \(N^x_y\) is an intravariant subgroup of \(N^x\), or
(ii) \(N^x_y\) is a Hall subgroup of \(N^x\).

9. In this section we prove:

**Theorem D.** Let \(G\) be a doubly-transitive group on a set \(X\). Suppose \(x \in X\) and \(N^x < G_x\). Suppose \(N^x \neq 1\) is doubly-transitive on each of its orbits on
Then either
(i) \( G \) is triply-transitive on \( X \), or
(ii) \( G \) is a normal extension of \( L_n(q) \), or
(iii) \(|N^X| = 2\) and \( G \) has a regular normal subgroup.

As usual, we let \( \Delta(x, y) \) be the predesign function associated with the orbits of \( N^X \). We suppose \(|\Delta(x, y)| = 1 + s\).

**Lemma 9.1.** Let \( r^k \) be the highest power of the prime \( r \) dividing \( s - 1 \).
Let \( R \) be a Sylow \( r \)-subgroup of \( N^x_y \). Then all orbits of \( R \) are of length 1 or of length divisible by \( r^k \).

**Proof.** Since \( N^X \) is doubly-transitive of degree \( s \), \( R \) is a Sylow \( r \)-subgroup of \( N^X \). Thus, on any orbit \( \Delta(x, z) - x \) of \( N^X \) on \( X - x \), \( R \) fixes some point \( z' \) of \( \Delta(x, z) - x \). So \( R \) is a Sylow \( r \)-subgroup of \( N^x_z \). Since \( N^x_z \) is transitive on \( \Delta(x, z) - \{x, z'\} \), all other orbits of \( R \) or \( \Delta(x, z) - x \) are of length divisible by \( r^k \).

**Lemma 9.2.** \( N^y_x \) is transitive on \( \Delta(x, y) - \{x, y\} \).

**Proof.** Take \( R' \) a Sylow \( r \)-subgroup of \( N^y_x \) for some prime \( r \) dividing \( s - 1 \). Let \( r^k \) be the highest power of \( r \) dividing \( s - 1 \). We claim: all orbits of \( R' \) on \( \Delta(x, y) - \{x, y\} \) are of length divisible by \( r^k \). If this is false, by Lemma 9.1, \( R' \) fixes some point \( z \in \Delta(x, y) - \{x, y\} \). By Lemma 2.6 of [8], some Sylow \( r \)-subgroup \( R \) of \( N^x_z \) also fixes \( z \). But all orbits of \( R \) on \( \Delta(x, y) - \{x, y\} \) are of length divisible by \( r^k \), as \( N^y_x \) is transitive on \( \Delta(x, y) - \{x, y\} \). This contradiction proves the claim.

We now prove Theorem D.

Since all orbits of \( N^y_x \) on \( X - y \) are contained in \( \Delta(y, z) \) for some \( z \in X - y \), it follows that \( \Delta(x, y) - \{x, y\} \subseteq \Delta(y, z) \). Thus, \(|(\Delta(x, y) - x) \cap (\Delta(y, z) - y)| = s - 1|\\text{.}

Applying Lemma 9.2 to \( N^x_y \) instead of \( N^y_x \), it follows that \( N^y_x \) is transitive on \( \Delta(y, x) - \{x, y\} \). Thus there is a point \( z' \in X - x \) such that

\(|(\Delta(y, x) - y) \cap (\Delta(x, z') - x)| = s - 1|

As in §1, we define \( G_x \) so that \( \{a, b\} \in G_x \) if \( a \) and \( b \) belong to the same orbit of \( N^X \). Then \( G_x \) is of valence \( s - 1 \), and by Lemma 1.2, if \( x \neq y \),

\(|G_x \cap G_y| = (s - 1)(s - 2)/2|

Now all pairs of points in \( (\Delta(x, y) - x) \cap (\Delta(y, z) - y) \) and \( (\Delta(y, x) - y) \cap (\Delta(x, z') - x) \) belong to \( G_x \cap G_y \). Thus, if either \( \Delta(x, y) \neq \Delta(x, z') \) or \( \Delta(y, x) \neq \Delta(y, z) \), we have

\((\Delta(x, y) - x) \cap (\Delta(y, z) - y) \cap (\Delta(y, x) - y) \cap (\Delta(x, z') - x) = \emptyset)\)
and so \(|G_x \cap G_y| \geq (s - 1)(s - 2)|\), a contradiction if \(s > 2\).

Thus, if \(s \neq 2\),

\[
(\Delta(x, y) - x) \cap (\Delta(y, z) - y) = (\Delta(y, x) - y) \cap (\Delta(x, z') - x).
\]

Therefore, \(\Delta(x, y) = \Delta(y, x)\). It follows (by Lemma 1.4 of \([8]\)) that \(\{\Delta(x, y) | x, y \in X, x \neq y\}\) form a block design preserved by \(G\) and that \(N^x\) fixes all blocks which contain \(x\). Moreover, as \(s > 2\), \(N^y_x \neq 1\). By Theorem B of part I, if \(\Delta(x, y) \subset X\), \(G\) is a normal extension of \(L^\alpha_n(q)\). If \(\Delta(x, y) = X\), of course, \(G\) is triply-transitive.

Thus, we need only consider the case \(s = 2\). Then, \(N^x\) is an elementary abelian \(2\)-group. If \(|N^x| > 2\), by Theorem A of \([8]\), \(G\) is a normal extension of \(L^\alpha_n(2)\). If \(|N^x| = 2\), by Glauberman’s \(Z^*-\)Theorem \([4]\), \(O(G) \neq 1\), and \(G\) has a regular normal subgroup.

**BIBLIOGRAPHY**