TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 215, 1976

AUTOMORPHISMS OF $GL_n(R)$

BY

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ABSTRACT. Let R be a commutative ring and S a multiplicatively closed subset of R having no zero divisors. The pair $\langle R, S\rangle$ is said to be stable if the ring of fractions of R, $S^{-1}R$, defined by S is a ring for which all finitely generated projective modules are free. For a stable pair $\langle R, S\rangle$ assume 2 is a unit in R and V is a free R-module of dimension $\geqslant 3$. This paper examines the action of a group automorphism of $GL(V)$ (the general linear group) on the elementary matrices relative to a basis B of V. In the case that R is a local ring, a Euclidean domain, a connected semilocal ring or a Dedekind domain whose quotient field is a finite extension of the rationals, we obtain a description of the action of the automorphism on all elements of $GL(V)$.

(I). Introduction. Let R be a ring and $GL_n(R)$ the general linear group of n by n invertible matrices over R. If Λ is a group automorphism of $GL_n(R)$ then a basic problem is that of obtaining a description of the action of Λ on elements of $GL_n(R)$.

First, what are the standard automorphisms? If $\sigma\colon R \to R$ is a ring automorphism then clearly σ induces an isomorphism of the n by n matrix ring $(R)_n \to (R)_n$ and since a unit maps to a unit we obtain an automorphism $GL_n(R) \to GL_n(R)$ where $A \to A^\sigma$. For A in $GL_n(R)$ we also have $A \to A^*$ where $A^* = (A^t)^{-1}$ (transpose-inverse). Each Q in $GL_n(R)$ provides an inner-automorphism $A \to QAQ^{-1}$. Finally, for suitable group morphisms $\chi\colon GL_n(R) \to \text{center}(GL_n(R))$ then $A \to \chi(A)A$ is a group automorphism. Precisely, if Λ is an automorphism of $GL_n(R)$ then does

$$\Lambda(A) = \chi(A)QA^\sigma Q^{-1} \quad \text{for all } A \text{ in } GL_n(R)$$

or

$$\Lambda(A) = \chi(A)QA^{*\sigma}Q^{-1} \quad \text{for all } A \text{ in } GL_n(R)$$

for a suitably chosen Q, χ and σ?

The answer is essentially yes if R is a commutative field (Schreier and van der Waerden, 1928 [15]), a division ring (Dieudonné, 1951 [6]), the rational integers (Hua and Reiner, 1951 [7]), a principal ideal domain (Landin and Reiner,

Received by the editors July 24, 1974.
AMS (MOS) subject classifications (1970). Primary 20G99, 20H25; Secondary 15A33.
Key words and phrases. Automorphisms, general linear group, commutative ring.

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1957 [9]), a domain (O'Meara, 1968 [11]) and a local ring (Pomfret and McDonald, 1972 [14] and Keenan, 1965 [8]).

By "essentially" it is often assumed that the ring has sufficient number of elements, 2 is a unit and $n \geqslant 3$. The case $n = 2$ is very difficult. A striking result here by Cohn [4] is that if A is any free associative algebra over a commutative field k on at most countably many free generators then $GL_2(A)$ is isomorphic to $GL_2(k[X])$. A recent paper by Dull summarizes and extends the known results [17] for $n = 2$.

The point of departure for this paper is the observation that each of the above rings or a suitable ring of fractions of the ring has all its projective modules free. We also work only with a commutative ring as the scalar ring. Let R be a commutative ring and S a multiplicative subset of R containing no zero divisors. The pair $\langle R, S\rangle$ is called *stable* if the ring of fractions $S^{-1}R$ has all projective modules free.

§(II) deals with free modules over R and $\bar{R} = S^{-1}R$ for a stable pair $\langle R, S\rangle$ and basic definitions and concepts.

§(IV) concerns the description of the automorphisms of $GL_n(R)$. The approach is essentially that of Yen [16] and Pomfret and McDonald [14] with more care about zero divisiors, that is, we mount an attack on the group $E_B(V)$ of elementary matrices relative to a basis B of V. Precisely, we let Λ be an automorphism of G where G is a group satisfying $E_B(V) \subseteq G \subseteq GL(V)$, e.g. special linear group, group of transvections, etc. Unfortunately, the obstruction to this approach is that $E_B(V)$ is not normal in $GL(V)$ and we fall short in achieving a complete description of the action of the automorphism Λ on an element of G. In our opinion this is as far as the group $E_B(V)$ can be exploited in the automorphism problem. Perhaps an alternate approach which may prove fruitful would be to utilize the stable pair setting in O'Meara's ([11], [12]) beautiful treatment of the domain case. Here the action of Λ is first examined on the group transvections $T(V)$ of V (which is normal). Indeed, we obtain complete results when $T(V) = E_B(V)$.

(II). **Basic concepts.** Let R denote a commutative ring and S a multiplicatively closed subset of R containing no zero divisors. We will say that the pair $\langle R, S\rangle$ is *stable* if the ring of fractions of R defined by S, denoted $S^{-1}R$, is a ring for which all finitely generated projective modules are free. The following are stable pairs

(a) $\langle R, S\rangle$ where R is a domain and $S = R - \{0\}$. Here $S^{-1}R$ is the field of fractions of R.

(b) $\langle R, S\rangle$ where R is a domain and S the complement of a prime ideal.

(c) $\langle R, S\rangle$ where R is a ring having all projective modules free and $S =$ units of R, e.g. R a local ring, a connected semilocal ring or a principal ideal domain. Here $S^{-1}R = R$.

(d) $\langle R, S\rangle$ where S is the complement of the zero divisors provided the zero divisiors Z form an ideal. Recall $S^{-1}R$ is local where $S = R - Z$ if and only if Z is an ideal.

(e) $\langle R, S\rangle$ where R is Noetherian and S is the complement of the zero divisors provided $S^{-1}R$ is connected, i.e. $S^{-1}R$ has only trivial idempotents. In this case, $S^{-1}R$ is always semilocal [3, p. 120], however $S^{-1}R$ must be connected for projective modules to be free [3, p. 113].

Let $\langle R, S\rangle$ be stable. Then $S^{-1}\mathrm{R} = \{r/s | r \text{ in } R, s \text{ in } S\}$ where $r_1/s_1 = r_2/s_2$ if $r_1 s_2 = s_1 r_2$ since S has no zero divisors. Further, $S^{-1}R$ is a ring under the usual operations. The injective ring morphism $r \to r/1$ embeds R as a subring of $S^{-1}R$.

Let V be a free R-module with basis $\{b_1, \ldots, b_n\}$. Then $S^{-1}V = \{v/s | v$ in V, s in $S\}$ where $m_1/s_1 = m_2/s_2$ if $s_1 m_2 = s_2 m_1$ is a free $S^{-1}R$-module with basis $\{b_1/1, \ldots, b_n/1\}$ under the natural operations. We consider $V \subset S^{-1}V$ by $v \to v/1$. If σ: $V \to V$ is an R-morphism, then σ extends to an $S^{-1}R$-morphism $\bar{\sigma}$: $S^{-1}V \to S^{-1}V$ by $\bar{\sigma}(v/s) = \sigma(v)/s$.

We adopt the following notation and assumptions for the remainder of this paper:

(a) $\langle R, S\rangle$ is a stable pair,

(b) V is a free R-module of R-dimension $n \geqslant 2$,

(c) $\bar{R}$ denotes $S^{-1}R$,

(d) $\bar{V}$ denotes $S^{-1}V$.

The dual module of $\bar{V}$ will be denoted by $\bar{V}^*$, the $\bar{R}$-morphisms of $\bar{V}$ by $\mathrm{End}(\bar{V})$, and the invertible $\bar{R}$-morphisms of $\bar{V}$ by $GL(\bar{V})$ (the *general linear group*). The *special linear group* $SL(\bar{V}) = \{\sigma \text{ in } GL(\bar{V}) | \det(\sigma) = 1\}$. Occassionally we subscript this notation to stress the scalar ring, e.g. $GL_{\bar{R}}(\bar{V})$ rather than $GL(\bar{V})$.

Let φ: $\bar{V} \to \bar{R}$ be a surjective $\bar{R}$-morphism. Then $\bar{V}$ splits as $\bar{V} \simeq \ker(\varphi) \oplus \bar{R}$. Since projective modules over $\bar{R}$ are free, $H = \ker(\varphi)$ is a free $\bar{R}$-submodule of $\bar{V}$ of dimension $n - 1$, i.e. H is a *hyperplane*. If a is in $\bar{V}$ and $\varphi a = 0$ define $\tau_{a,\varphi}$: $\bar{V} \to \bar{V}$ by $\tau_{a,\varphi}(x) = x + \varphi(x)a$. The mapping $\tau_{a,\varphi}$ is called a *transvection* with *vector* a and *hyperplane* H.

(II.1). LEMMA. (a) $\tau_{a,\varphi} = 1$ *if and only if* $a = 0$.

(b) $\sigma\tau_{a,\varphi}\sigma^{-1} = \tau_{\sigma a,\varphi\sigma^{-1}}$ *for all* σ *in* $GL(\bar{V})$.

(c) $\tau_{a,\varphi}\tau_{b,\varphi} = \tau_{a+b,\varphi}$.

An element a of $\bar{V}$ is called *unimodular* if the $\bar{R}$-submodule $\bar{R}a$ of $\bar{V}$ is a direct summand of $\bar{V}$ (thus $\bar{R}a$ is free and a is an element of a basis for $\bar{V}$).

(II.2). LEMMA. *For a in $\bar{V}$ the following are equivalent*:

(a) *$\bar{R}a$ is a direct summand of $\bar{V}$.*

(b) *There is a* ϕ *in* $\bar{V}^*$ *with* $\phi(a) = 1$.

(c) *The map* $\bar{R} \to \bar{V}$ *by* $r \to ra$ *is split injective.*

(d) *If* $a = \alpha_1 b_1 + \cdots + \alpha_n b_n$ *for a basis* $\{b_1, \ldots, b_n\}$ *of* $\bar{V}$ *then* $(\alpha_1, \ldots, \alpha_n) = \bar{R}$.

(e) *The element* a *may be extended to a basis of* $\bar{V}$.

If $\tau_{a,\varphi} = \tau_{b,\psi}$ where a is unimodular, then $\varphi(x)a = \psi(x)b$ for all x in $\bar{V}$. Since φ and ψ are surjective, there exist s and t in $\bar{R}$ with $a = sb$ and $b = ta$. Thus $a = sta$ and $st = 1$ since a is unimodular. Hence b is unimodular. Finally $\varphi(x) = t\psi(x)$ for all x in $\bar{V}$.

(II.3). LEMMA. *Let* a *be unimodular and* $\tau_{a,\varphi} \neq 1$. *Then* $\tau_{a,\varphi} = \tau_{b,\psi}$ *if and only if there is a unit* t *with* $a = t^{-1}b$ *and* $\varphi = t\psi$.

Suppose $\tau_{a,\varphi}$ has hyperplane H with basis $b_1, \ldots, b_{n-1}$ where $b_1, \ldots, b_n$ is a basis for $\bar{V}$. Then $\tau_{a,\varphi}$ is the identity on H and $\tau_{a,\varphi}(b_n) - b_n$ is in H. Thus, if $\tau_{a,\varphi}(b_n) = \alpha_1 b_1 + \cdots + \alpha_{n-1}b_{n-1} + b_n$ then τ has a matrix representation of the form

$$\begin{bmatrix} 1 & 0 & \cdot & & \cdot & & \cdot\ \alpha_1 \\ 0 & 1 & & & & & \cdot \\ & & \cdot & & & & \\ \cdot & & & & & & \cdot \\ & & & \cdot & & & \\ \cdot & & & & & & \cdot \\ & & & & \cdot & & \\ \cdot & & & & & 1 & \alpha_{n-1} \\ 0 & & & & & 0 & 1 \end{bmatrix}$$

and the determinant $\det(\tau) = 1$. This, together with (II.1)(b) implies that the group $T(\bar{V})$ of transvections in $GL(\bar{V})$ is a normal subgroup of $SL(\bar{V})$.

Let $B = \{b_1, \ldots, b_n\}$ be a basis for $\bar{V}$. When we interpret the above groups as groups of matrices relative to a basis, we write $GL_n(\bar{R})$ for $GL(\bar{V})$, $SL_n(\bar{R})$ for $SL(\bar{V})$ and $T_n(\bar{R})$ for $T(\bar{V})$. Let B have a dual basis $\{b_1^*, \ldots, b_n^*\} = B^*$. By an *elementary transvection* relative to B we mean a transvection of the form $\tau_{\lambda b_i, b_j^*}$. The group generated by elementary transvections *relative* to B is denoted by $E_B(\bar{V})$ (or $E_n(\bar{R})$ when the basis is fixed).

A *radiation* on $\bar{V}$ is an R-morphism σ_r where $\sigma_r(x) = rx$ for all x in $\bar{V}$. Let $RL(\bar{V})$ ($RL_n(\bar{R})$ for a specified basis) denote the group of radiations.

Finally, for a given basis $B = \{b_1, \ldots, b_n\}$ a *dilation* is an R-morphism $d_\lambda \colon \bar{V} \to \bar{V}$ where $d_\lambda(b_i) = b_i$ for $i \neq n$ and $d_\lambda(b_n) = \lambda b_n$. Let $GE_B(\bar{V})$ be generated by $E_B(\bar{V})$ and $D_B(\bar{V})$. To place this in perspective we record the following fact.

(II.4). THEOREM. (a) *If R is a Euclidean domain, a local ring or a semi-local ring and $S = R^*$ then $GL(\bar{V}) = GE_B(\bar{V})$ for any basis B with $|B| \geqslant 2$ (Bass* [2, V]).

(b) *If R is a Dedekind domain whose quotient field is a finite extension of the rationals and $S = R - \{0\}$ then $GL(V) = GE_B(V)$ for $|B| \geqslant 3$ ($GL(V)$ defined below) (Bass* [2, VI]).

If we fix a basis B of $\bar{V}$ then there is a natural bijection between the set of free $\bar{R}$-bases of $\bar{V}$ and $GL(\bar{V})$ under $\sigma \to B'$ where $B' = \sigma B$. Thus we may index $E_B(\bar{V})$ by σ in $GL(\bar{V})$. To illustrate our discussion

(**)

$$\begin{tikzcd}
& & GL(\bar{V}) \arrow[dl, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
& GE_\sigma(\bar{V}) \arrow[ddl, dash] \arrow[ddr, dash] & & SL(\bar{V}) \arrow[d, dash] & RL(\bar{V}) \\
& & & T(\bar{V}) \arrow[dl, dash] & \\
D_\sigma(\bar{V}) & & E_\sigma(\bar{V}) & &
\end{tikzcd}$$

(for each σ in $GL(\bar{V})$).

The general linear group $GL(\bar{V})$ under conjugation acts as a transitive transformation group on $\{E_\sigma(\bar{V}) | \sigma \text{ in } GL(\bar{V})\}$. If the orbit of $\{E_\sigma(\bar{V})\}$ consists of only one element, then $E_\sigma(\bar{V}) = E_\beta(\bar{V})$ for all σ and β. Thus $E_\sigma(\bar{V})$ is normal in $GL(\bar{V})$ and $T(\bar{V}) = E_\sigma(\bar{V})$. In general, $T(\bar{V}) = \bigcup_\sigma E_\sigma(\bar{V})$.

Let M be an R-submodule of $\bar{V}$. Define

$$GL(M) = \{\sigma \text{ in } GL(\bar{V}) | \sigma M = M\}, \qquad SL(M) = GL(M) \cap SL(\bar{V}),$$

$$T(M) = GL(M) \cap T(\bar{V}), \qquad RL(M) = GL(M) \cap RL(\bar{V}).$$

Our principal concern is when $M = V$. The main purpose of this paper is a description of the group isomorphisms Λ: $E_\sigma(V) \to GL(V)$. If $GE_\sigma(V) = GL(V)$ and Λ is an automorphism of $GL(V)$, we are then able to completely characterize the form of Λ.

(III). **Automorphisms of** $\mathrm{End}_R(V)$. The ring automorphisms of $\mathrm{End}_R(V)$ are well known when $R = \bar{R}$ and hence $V = \bar{V}$. This short section illustrates the theory for stable pairs.

A set $\{F_{ij}\}$, $1 \leqslant i, j \leqslant n$, of elements of $\mathrm{End}(\bar{V})$ is a set of *matrix units* if

(a) $F_{ij} = F_{lm}$ only if $i = l$, $j = m$.

(b) $F_{ij}F_{lm} = \delta_{jl}F_{im}$ (δ = Kronecker delta).

(c) $\Sigma_i F_{ii} = I$ (I = identity on $\bar{V}$).

Observe $GL(\bar{V})$ carries sets of matrix units into sets of matrix units under conjugation. If $B = \{b_1, \ldots, b_n\}$ is a basis for $\bar{V}$ then $\{E_{ij}\}$ given by $E_{kj}b_i = \delta_{ji}b_k$ is a set of matrix units called the *standard* matrix units relative to B.

(III.1). THEOREM. *The general linear group $GL(\bar{V})$ is transitive under conjugation on the sets of matrix units in* $\mathrm{End}(\bar{V})$, *i.e. if $\{F_{ij}\}$ and $\{G_{ij}\}$ are two sets of matrix units then there is a σ in $GL(\bar{V})$ with $\sigma F_{ij}\sigma^{-1} = E_{ij}$, $1 \leqslant i, j \leqslant n$.*

PROOF. Let $\{b_1, \ldots, b_n\}$ be a basis of $\bar{V}$ and $\{E_{ij}\}$ the standard matrix units relative to this basis. Let $\{F_{ij}\}$ be an arbitrary set of matrix units in $\mathrm{End}(\bar{V})$. By (b) and (c) of the definition of matrix units, it is easy to see that $\bar{V} = F_{11}\bar{V} \oplus \cdots \oplus F_{n1}\bar{V}$. But each $F_{i1}\bar{V}$ is projective, hence free. By uniqueness of dimension of free modules over commutative rings we have $\dim(F_{i1}\bar{V}) = 1$ for $1 \leqslant i \leqslant n$.

Let $L_i = F_{i1}\bar{V}$ then $F_{ji}|_{L_i}\colon L_i \to L_j$ and $F_{ji}|_{L_i} \circ F_{ij}|_{L_j} =$ identity. Thus $F_{ji}|_{L_i}\colon L_i \to L_j$ is an $\bar{R}$-isomorphism.

Let $L_1 = Rb$. Then $F_{j1}b$ is an $\bar{R}$-basis for L_j. Let $c_1 = b$ and $c_j = F_{j1}b$ for $j \geqslant 2$. We have $\{c_1, \ldots, c_n\}$ an $\bar{R}$-basis for $\bar{V}$. Now define σ in $GL(\bar{V})$ by $\sigma(c_i) = b_i$ for $1 \leqslant i \leqslant n$.

Consider

$$(\sigma F_{ij}\sigma^{-1})(b_k) = \sigma F_{ij}c_k = \sigma F_{ij}F_{k1}b = \sigma(\delta_{jk}F_{i1}b) = \delta_{jk}\sigma(c_i) = \delta_{jk}b_i = E_{ij}(b_k)$$

finishing the proof.

(III.2). THEOREM. *Let Λ: $\mathrm{End}_R(V) \to \mathrm{End}_R(V)$ be a ring automorphism. Then there is a ring automorphism σ: $R \to R$ and a β in $GL_{\bar{R}}(\bar{V})$ such that $\Lambda(\alpha) = \beta^{-1}\alpha^{\sigma}\beta$ for all α in* $\mathrm{End}(V)$ *where if $\alpha = [\alpha_{ij}]$ then $\alpha^{\sigma} = [\sigma(\alpha_{ij})]$.*

PROOF. Let σ be induced by the restriction of Λ to the center of $\mathrm{End}(V)$. Select a basis with standard matrix units $\{E_{ij}\}$. Then $\alpha = \Sigma_{i,j}\,\alpha_{ij}E_{ij}$ and

$$\Lambda(\alpha) = \sum \Lambda(\alpha_{ij}E_{ij}) = \sum \Lambda(\alpha_{ij}I)\Lambda E_{ij} = \sum \sigma(\alpha_{ij})\Lambda E_{ij}.$$

But, if $F_{ij} = \Lambda E_{ij}$ then $\{F_{ij}\}$ is a set of matrix units. By (III.1) $\beta F_{ij}\beta^{-1} = E_{ij}$ for some β in $GL_{\bar{R}}(\bar{V})$. Thus

$$\Lambda(\alpha) = \sum \sigma(\alpha_{ij})\beta^{-1}E_{ij}\beta = \beta^{-1}\left(\sum \sigma(\alpha_{ij})E_{ij}\right)\beta = \beta^{-1}\alpha^{\sigma}\beta.$$

We first make some remarks on the hypothesis that $\langle R, S\rangle$ be stable, in particular, that $S^{-1}R = \bar{R}$ have all projectives free. This was utilized to assure $F_{11}\bar{V} \simeq \bar{R}$ in the proof of (III.1) which was the key to the above theorem.

Let $L = F_{11}\bar{V}$. Without the hypothesis on $\bar{R}$, L is a projective module of rank 1, i.e. each localization L_P at a prime P of $\bar{R}$ is free of dimension 1. Recall $\mathrm{Pic}(\bar{R})$ is the projective class group of isomorphism classes of projective rank 1 $\bar{R}$-modules where if $[L_1]$ and $[L_2]$ represent the classes of L_1 and L_2 then $[L_1] \circ [L_2] = [L_1 \otimes_{\bar{R}} L_2]$ is the group operation with identity $[\bar{R}]$.

Since $F_{11}\bar{V} \simeq F_{i1}\bar{V}$, $\bar{V} \simeq L \oplus \cdots \oplus L$ (n-summands). Now compute the nth exterior product of $\bar{V}$. Since $\bar{V}$ is free of dimension n, $\Lambda^n(\bar{V}) \simeq \bar{R}$. But $\Lambda^n(L \oplus \cdots \oplus L) = L \otimes \cdots \otimes L$. Thus the order of $[L]$ divides n in $\mathrm{Pic}(\bar{R})$. Thus to force $L \simeq \bar{R}$ we may replace the hypothesis that $\bar{R}$ has all projective modules free by the hypothesis that $\mathrm{Pic}(\bar{R})$ has no nontrivial elements of order dividing n [see [3, pp. 152–153, Exercise #21]]. It is not obvious that this weaker hypothesis is sufficient to obtain the automorphisms for the general linear group.

We now examine the "inner" part of Λ in (III.2). For σ in $GL(\bar{V})$ let Φ_σ denote the inner-automorphism $\alpha \to \sigma^{-1}\alpha\sigma$ of $\mathrm{End}_{\bar{R}}(\bar{V})$. The question is, "For what σ does $\Phi_\sigma(\mathrm{End}_R(V)) \subseteq \mathrm{End}_R(V)$?"

Let W be an R-submodule of $\bar{V}$. We say that σ in $GL(\bar{V})$ is *on* W if $\sigma W = W$. Recall an R-submodule T of $\bar{R} = S^{-1}R$ is

(a) *fractional* if there is a λ in R with $\lambda T \subseteq R$;

(b) *invertible* if there is an R-submodule T' of $\bar{R}$ with $TT' = R$. Here T' is called the *inverse* of T and written T^{-1}.

Note a finitely generated R-submodule of $\bar{R}$ is fractional.

Suppose σ in $GL(\bar{V})$ is on W a finitely generated R-submodule of $\bar{V}$. If T is an R-submodule of $\bar{R}$ then $\sigma(TW) \subseteq TW$. Similarly $\sigma^{-1}(TW) \subseteq TW$. Thus $\sigma(TW) = TW$ and $GL_R(W) \subseteq GL_R(TW)$. On the other hand, if T is invertible then $GL_R(TW) \subseteq GL_R(T^{-1}TW) \subseteq GL_R(W)$. Thus for T invertible $GL_R(TW) = GL_R(W)$. Therefore, if σ in $GL_{\bar{R}}(\bar{V})$ is such that $\sigma V = TV$ where T is an invertible R-submodule of $\bar{R}$ then since $GL_R(TV) = GL_R(V)$ we have $\sigma^{-1}\alpha\sigma$ in $\mathrm{End}_R(V)$ for all α in $\mathrm{End}_R(V)$. That is, $\Phi_\sigma(\mathrm{End}(V)) \subseteq \mathrm{End}(V)$.

We now show the converse is true. Suppose σ is in $GL(\bar{V})$ and $\Phi_\sigma(\mathrm{End}(V)) \subseteq \mathrm{End}(V)$. Let $\{b_1, \ldots, b_n\}$ be an R-basis for V. Let x be unimodular in $\bar{V}$. The *coefficient* of x is $A_x = \{\alpha \text{ in } \bar{R} \mid \alpha x \text{ is in } V\}$. Observe

(a) A_x is an R-submodule of $\bar{R}$,

(b) A_x is nondegenerate, i.e. $\bar{R}A_x = \bar{R}$,

(c) if σ is invertible in $GL(V)$, then $A_x = A_{\sigma x}$.

Define E_{ij}: $V \to V$ by $E_{ij}(b_k) = \delta_{ij}b_i$ (standard matrix units). Then E_{i1}: $b_1 \to b_i$ determines an R-isomorphism $Rb_1 \to Rb_i$ satisfying $\alpha b_i = E_{i1}(\alpha b_1)$ for all α in $\bar{R}$ with inverse $E_{1i}|_{Rb_i}$ (as in the proof of (III.1)).

We have $\Phi_\sigma(E_{ij}) = \sigma^{-1}E_{ij}\sigma$ in $\mathrm{End}(V)$ and $\sigma^{-1}E_{ij}\sigma|_{R\sigma^{-1}b_1}$: $R\sigma^{-1}b_1 \to R\sigma^{-1}b_i$ is an R-isomorphism preserving scalar multiplication by elements in R. Thus $A_{\sigma^{-1}b_1} = A_{\sigma^{-1}b_i}$.

Set $B = A_{\sigma^{-1}b_1}$. Then $B\sigma^{-1}b_1 + \cdots + B\sigma^{-1}b_n \subset V$. We claim this is an equality. It suffices to show for each j that b_j is in $B\sigma^{-1}b_1 + \cdots + B\sigma^{-1}b_n$. Let $\sigma b_j = \Sigma_i\, \alpha_{ij}b_i$ where α_{ij} are in $\bar{R}$. Then since $b_j = \sigma^{-1}(\sigma b_j) = \Sigma_i\, \alpha_{ij}\sigma^{-1}b_i$, it is sufficient to show each α_{ij} is in B. If each α_{ij} is in B then $\sigma V = BV$.

Fix i and j. Then $\Phi_\sigma(E_{1i}) = \sigma E_{1i}\sigma^{-1}$ is in $\mathrm{End}(V)$. Thus

$$\Phi_\sigma(E_{1i})(b_j) = (\sigma E_{1i}\sigma^{-1})\left(\sum_k \alpha_{kj}\sigma^{-1}b_k\right) = \alpha_{ij}\sigma b_1$$

is in V. Hence α_{ij} is in $B = A_{\sigma b_1}$. Thus $B\sigma^{-1}b_1 + \cdots + B\sigma^{-1}b_n = V$. It is easy to see the sum is direct. Thus $V \simeq B \oplus \cdots \oplus B$ (n summands). Thus B is a projective R-module. Hence by [3, p. 117, Theorem 4] B is invertible.

(III.3). THEOREM. *Let σ be in $GL(\bar{V})$. Then the inner-automorphism Φ_σ carries* $\mathrm{End}_R(V)$ *into* $\mathrm{End}_R(V)$ *if and only if $\sigma V = TV$ where T is an invertible R-submodule of $\bar{R}$.*

We now examine the ring automorphism $\beta: R \to R$ in (III.2) and its relation to S. The set S is *fully invariant* under $\mathrm{Aut}(R)$ if $\sigma(S) = S$ for σ in $\mathrm{Aut}(R)$. Stable pairs $\langle R, S\rangle$ where this occurs are, for example:

(a) R = domain, S = nonzero divisors.

(b) R = local, connected semilocal or a principal ideal domain, S = units of R.

If S is fully invariant under $\mathrm{Aut}(R)$ then σ in $\mathrm{Aut}(R)$ extends to $\bar{\sigma}$ in $\mathrm{Aut}(\bar{R})$ by $\bar{\sigma}(r/s) = \sigma(r)/\sigma(s)$.

On the other hand, if $\bar{\sigma}$ is an automorphism of $\bar{R}$ then the restriction map $\bar{\sigma} \to \bar{\sigma}|_R$ determines an injective ring morphism $\bar{\sigma}|_R: R \to \bar{R}$. Let $I(R)$ denote the stablizer of R under $|_R$, i.e. $I(R) = \{\bar{\sigma}$ in $\mathrm{Aut}(\bar{R}) \mid \bar{\sigma}|_R$ is in $\mathrm{Aut}(R)\}$. If $\Omega = \bigcup_\beta \beta(R)$ (β extends over $\mathrm{Aut}(\bar{R})$), then $\mathrm{Aut}(\bar{R})$ is transitive on Ω and $|\Omega| = [\mathrm{Aut}(\bar{R}): I(\bar{R})]$. The $\ker(\,|_R) = \{\bar{\sigma}$ in $\mathrm{Aut}(\bar{R}) \mid \bar{\sigma}|_R = \text{identity}\} = \mathrm{Aut}_R(\bar{R})$.

But $\bar{R} = S^{-1}R$. Thus if $\bar{\sigma}|_R$ = identity then $\bar{\sigma} = 1$ ($r = (r/s)s$ implies $\bar{\sigma}(r) = \bar{\sigma}(r/s)\bar{\sigma}(s)$. Hence if $\bar{\sigma}|_R = 1, r = \bar{\sigma}(r/s)s$ and $\bar{\sigma} = 1$). Hence $\mathrm{Aut}_R(\bar{R}) = 1$.

Returning to $\Lambda(\alpha) = \beta^{-1}\alpha^\sigma\beta$, suppose S is fully invariant. Let $GI(V) = \{\beta$ in $GL(\bar{V}) \mid \beta V = TV$ and T invertible$\}$.

Note center$(GL(\bar{V})) \leqslant GI(V)$. Set $PGI(V) = GI(V)/\text{center}(GL(\bar{V}))$. The automorphism Λ may be identified with the pair $\langle\beta, \sigma\rangle$ where β is in $GI(V)$ and σ is in $\mathrm{Aut}(R) = I(R) \leqslant \mathrm{Aut}(\bar{R})$. Then $\langle\beta_1, \sigma_1\rangle = \langle\beta_2, \sigma_2\rangle$ if and only if $\beta_1 = u\beta_2$ for u a unit in $\bar{R}$ and $\sigma_1 = \sigma_2$. The following result is now straightforward.

(III.5). THEOREM. *If S is fully invariant under* $\mathrm{Aut}(R)$ *then* $\mathrm{Aut}(\mathrm{End}(V))$ *is the semidirect product of $PGI(V)$ and $I(R)$* = $\mathrm{Aut}(R)$.

(IV). **Automorphisms of $GL(V)$.** In this section we include in our basic hypothesis on $\langle R, S\rangle$, V and $\bar{V}$ as stated in (II) the following assumptions:

(a) 2 is a unit in R,

(b) $\dim(V) \geqslant 3$,

(c) G denotes a subgroup of $GL(V)$ satisfying $E_B(V) \subseteq G \subseteq GL(V)$ for some basis B. For example, G may be $E_B(V)$, $T(V)$, $SL(V)$ or $GL(V)$. We let $B = \{b_1, \ldots, b_n\}$.

We let Λ be an automorphism of G. For the basis $B = \{b_1, \ldots, b_n\}$ of V

our purpose is to determine the action of Λ on the group $E_B(V)$ of elementary matrices in G. In certain cases this permits the description of Λ on G.

It was noted in the last section that the hypothesis on $\bar{R}$ was only necessary to force the images of elementary matrix units to behave properly. Once the images of involutions in this section have been satisfactorily described the remaining theorems and proofs require only that R be a connected (i.e. has only trivial idempotents) ring with 2 a unit.

If $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ is a basis of $\bar{V}$ and $\underline{P} = \{\rho, \sigma, \ldots\}$ is a set of $\bar{R}$-morphisms of $\bar{V}$, we write $(\underline{P})_{\bar{B}}$ for the elements of $\underline{P}$ represented as matrices relative to the basis $\bar{B}$. If $\underline{P} = \{\rho\}$ we write $(\rho)_{\bar{B}}$ for $(\underline{P})_{\bar{B}}$.

An element σ in $\mathrm{End}(\bar{V})$ is an *involution* if $\sigma^2 = 1$. If σ is an involution, we may associate to σ a *positive* and *negative eigen-module*:

$$P(\sigma) = \{x \text{ in } \bar{V} | \sigma x = x\}, \qquad N(\sigma) = \{x \text{ in } \bar{V} | \sigma x = -x\}.$$

Clearly $N(\sigma) \cap P(\sigma) = 0$ since 2 is a unit. Further, for x in $\bar{V}$, $x = 1/2(x - \sigma(x)) + 1/2(x + \sigma(x))$, i.e. $\bar{V} = P(\sigma) + N(\sigma)$. Hence $\bar{V} = P(\sigma) \oplus N(\sigma)$ and since $\bar{R}$ has all its projective modules free, $P(\sigma)$ and $N(\sigma)$ are free. Thus we may select a basis for $P(\sigma)$ and $N(\sigma)$ so that σ has a matrix of the form

$$\begin{bmatrix} I_{n-t} & 0 \\ 0 & -I_t \end{bmatrix}$$

where $t = \dim(N(\sigma))$, $n - t = \dim(P(\sigma))$ and I_s denotes an s by s identity matrix.

(IV.1). LEMMA. *Let σ be an involution in $GL(\bar{V})$. Then there is a basis $\bar{B}$ of $\bar{V}$ such that $(\sigma)_{\bar{B}} = I_{n-t} \oplus (-I_t)$.*

An involution having an associated matrix relative to some basis of the form $I_{n-t} \oplus (-I_t)$ is said to be of *type* $(t, n - t)$.

Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ denote a basis of $\bar{V}$. Let $\Sigma_{\bar{B}}$ denote the set of elements σ in $GL(\bar{V})$ such that $\sigma(\bar{b}_i) = \pm\bar{b}_i$ for $1 \leqslant i \leqslant n$. An element of $\Sigma_{\bar{B}}$ is an involution and any two elements of $\Sigma_{\bar{B}}$ commute.

(IV.2). LEMMA. *Let $\{\sigma_i\}_{i=1}^r$ be a collection of pairwise commuting involutions in $GL(\bar{V})$. Then there is a basis $\bar{B}$ of $\bar{V}$ such that σ_i is in $\Sigma_{\bar{B}}$ for $1 \leqslant i \leqslant r$.*

PROOF. The proof is a straightforward induction on the cardinality r of $\{\sigma_i\}_{i=1}^r$.

(IV.3). COROLLARY. (a) *There are at most $\binom{n}{t}$ elements in any collection of pairwise commuting involutions of type $(t, n - t)$.*

(b) *In any set of pairwise commuting involutions there are at most 2^n elements.*

Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be any basis of $\bar{V}$. Define ψ_{ij} $(i < j)$ in $E_{\bar{B}}(\bar{V})$ by $\psi_{ij}(\bar{b}_k) = -\bar{b}_k$ if $k = i, j$ and $\psi_{ij}(\bar{b}_k) = \bar{b}_k$ if $k \neq i, j$. It should be observed that the family ψ_{ij} $(i < j)$ are actually elements of $E_{\bar{B}}(\bar{V})$. To see this note $\psi_{ij} = (\tau_{\bar{b}_i, \bar{b}_j^*} \tau_{-2\bar{b}_j, \bar{b}_i^*})^2$ where the τ's are elementary transvections defined in (II). Then $\{\psi_{ij} | 1 \leqslant i < j \leqslant n\}$ is a set of $\binom{n}{2}$ pairwise commuting involutions of type $(2, n-2)$ in $\Sigma_{\bar{B}}$. Further, $\psi_{ij}\psi_{jk} = \psi_{ik}$. We denote the above set $\{\psi_{ij}\}$ by $\Psi_{\bar{B}}$ for the basis $\bar{B}$. Finally, we note the ψ_{ij} are conjugate under $E_{\bar{B}}(\bar{V})$. For example, if $\sigma = \tau_{-\bar{b}_j, \bar{b}_i^*} \tau_{\bar{b}_i, \bar{b}_j^*} \tau_{-\bar{b}_j, \bar{b}_i^*}$ then $\sigma\psi_{is}\sigma^{-1} = \psi_{js}$ $(s \neq i, j)$.

(IV.4). THEOREM. *Let Λ: $G \to G$ be a group automorphism. Then there is a basis $\bar{B}$ of $\bar{V}$ with $\Lambda\Psi_B = \Psi_{\bar{B}}$. In particular, if $\Psi_B = \{\psi_{ij}\}$ and $\Psi_{\bar{B}} = \{\bar{\psi}_{ij}\}$ then $\Lambda\psi_{ij} = \bar{\psi}_{ij}$ for all i and j.*

PROOF. For the basis B, let $\Psi_B = \{\psi_{ij}\}$. Then $\Lambda\Psi_B = \{\Lambda\psi_{ij}\}$ is a set of $\binom{n}{2}$ pairwise commuting, distinct, conjugate involutions. Thus, all of $\Lambda\psi_{ij}$ are of the same type, say $(t, n-t)$. The proof proceeds as [16, Proof of Theorem 5, pp. 167–169]. A little additional care must be employed in step (5) of the above proof to index the involutions more carefully before applying a permutation matrix. The trick is that the $(2, n-2)$ involutions $\{\psi_{i,i+1} | 1 \leqslant i \leqslant n-1\}$ generate Σ_B and consequently their images under Λ will generate $\Lambda(\Sigma_B)$.

Before we continue the proof we remark on the method we will use. Historically, three major approaches or techniques have been developed to achieve a characterization of the automorphisms: (a) The method of involution, e.g. [6], (b) The O'Meara school or the method of residual spaces, e.g. [12], (c) The matrix approach or the Chinese school, e.g. [14], [16]. The approaches (a) (which occurred first) and (b) (which is currently producing wide-ranging results for the various classical linear groups and their congruence subgroups over domains) use the given automorphism to establish a projectivity and then employ the Fundamental Theorem of Projective Geometry to create a semilinear isomorphism which is used to create the form of Λ. The approach (c) is matrix theoretic and does not invoke the Fundamental Theorem (indeed, from an overview the arguments appear to reprove the Theorem in each case. However, it is difficult to identify where the Theorem actually appears in the proofs). The advantage of this highly computational argument is that in dealing with matrices and their elements, it is easier to allow a greater variety in the choices of the scalar ring, e.g. allow zero divisors. This motivated the work in [14] and the current paper. On the other hand, the arguments of this paper are basically the same as [14] and [16] with only additional care to show the proofs apply in a more general setting. Thus, for the next several results we will state the definitions and theorems and point out where the proof (or a variation) has already occurred and may be adapted to our context. Finally, the hypothesis on $\bar{R}$ was utilized to assure proper behavior for the splittings of $\bar{V}$ induced by involutions. For the next several results, one needs only that R have trivial idempotents and 2 a unit.

(IV.5). LEMMA. *If a, b, c and d are in R and*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = -I \quad \text{and} \quad \begin{bmatrix} a & -b \\ c & -d \end{bmatrix}^2 = I$$

then $a = d = 0$ and $c = -b^{-1}$.

PROOF. Modify slightly [14, Lemma 3.3] or [16, Lemma 3].

(IV.6). LEMMA. *The polynomial $X^2 - 1$ has only ± 1 as its zeros in $\bar{R}$.*

PROOF. The ring R has trivial idempotents and 2 is a unit. Thus $X^2 - 1$ is a separable polynomial.

Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be any basis of $\bar{V}$. We let $H_{\bar{B}}$ denote the set of "permutations" in $E_{\bar{B}}(\bar{V})$ given by

$$\eta_{i,i+1}(\bar{b}_k) = \bar{b}_k \quad \text{for } k \neq i, i+1,$$

$$\eta_{i,i+1}(\bar{b}_i) = -\bar{b}_{i+1}, \qquad \eta_{i,i+1}(\bar{b}_{i+1}) = \bar{b}_i$$

where $1 \leqslant i \leqslant n-1$. To see that the $\eta_{i,i+1}$ are in $E_{\bar{B}}(\bar{V})$, let $j = i+1$ and note $\eta_{i,j} = \tau_{-\bar{b}_i,\bar{b}_j^*}\tau_{\bar{b}_j,\bar{b}_i^*}\tau_{-\bar{b}_i,\bar{b}_j^*}$.

If δ is the dilation $\delta(\bar{b}_i) = \lambda_i\bar{b}_i$, $1 \leqslant i \leqslant n$, then $\delta\eta_{i,i+1} = \eta_{i,i+1}\delta$ if and only if $\lambda_i = \lambda_{i+1}$.

(IV.7). THEOREM. *Let Λ: $G \to G$ be a group automorphism. Then there is a basis $\bar{B}$ of $\bar{V}$ with*

(a) $\Lambda\Psi_B = \Psi_{\bar{B}}$,

(b) $\Lambda H_B = H_{\bar{B}}$.

In particular, as in (IV.4), $\Lambda\psi_{ij} = \bar{\psi}_{ij}$ *and* $\Lambda\eta_{i,i+1} = \bar{\eta}_{i,i+1}$.

PROOF. Modify [14, Theorem 3.4] or the portion of [16, Theorem 6] from equation (4.1) to equation (4.3). Do not be concerned about our lack of reference to the case $e = -1$ mentioned in both proofs. In the next stage one shows the case $e = -1$ does not occur, e.g. see last 3 lines of [14, p. 384].

Let $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ be any basis for V with dual basis $\bar{B}^* = \{\bar{b}_1, \ldots, \bar{b}_n^*\}$. Recall $E_{\bar{B}}(V)$ is the group generated by elementary transvections $\tau_{\lambda\bar{b}_i,\bar{b}_j^*}$ (λ in R) relative to $\bar{B}$.

The next step is to examine the image of $\tau_{\lambda b_i,b_j^*}$ when $\lambda = 1$ under the action of Λ. This is the most difficult computational step in characterizing the automorphisms.

(IV.8). THEOREM. *Let Λ: $G \to G$ be a group automorphism. Then there* *is a basis* $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ *of* $\bar{V}$ *such that*

$$\Lambda(\tau_{b_i, b_j^*}) = \tau_{\bar{b}_i, \bar{b}_j^*} \quad \textit{for all } i \textit{ and } j,$$

or

$$\Lambda(\tau_{b_i}, \tau_{b_j^*}) = \tau_{-\bar{b}_j, \bar{b}_i^*} \quad \textit{for all } i \textit{ and } j.$$

PROOF. This proof follows [14, Theorem 3.5] or [16, Theorem 6, from line (4.3) to §5]. The modifications that are necessary consist of a more careful examination of the identities

(a) $\Lambda(\tau\psi_{23})^2 = I$,

(b) $\Lambda(\eta_{12}\tau)^3 = I$,

(c) $\Lambda(\eta_{23}^{-1}\tau\eta_{23}) = \Lambda\tau_{b_1, b_3^*}$,

where $\tau = \tau_{b_1, b_2^*}$ (see (2), (3), (4) of [14] or line 7, line 12, line 18 on p. 171 of [16]). In both previous papers these identities were only partially applied. Carefully writing out the equations, say in [14, Theorem 3.5] and using (a)(b)(c) the proof will follow.

We can now determine the action of an automorphism of G on the group $E_B(V)$ generated by elementary transvections relative to the basis B.

Let α be in $GL(\bar{V})$ and $\bar{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}$ by any basis for $\bar{V}$. Suppose $\alpha(\bar{b}_i) = \Sigma_j\, \alpha_{ji}\bar{b}_j$. Then recall the matrix of α relative to $\bar{B}$, i.e. $[\alpha_{ji}]$, is denoted by $(\alpha)_{\bar{B}}$. We let α^* denote the element of $GL(\bar{V})$ resulting from the transpose and the inverse applied to α, i.e. $\alpha^* = (\alpha^{-1})^t$. Finally, if σ is a ring morphism of $\bar{R}$ (or of R) then α^σ is the element of $GL(\bar{V})$ given by $\alpha^\sigma(\bar{b}_i) = \Sigma_j\, \sigma(\alpha_{ji})\bar{b}_j$.

(IV.9). THEOREM. *Let* Λ: $G \to G$ *be a group automorphism. Then* $\Lambda|_{E_B(V)}$: $E_B(V) \to G$ *is an injective group morphism whose action on* α *in* $E_B(V)$ *is described as follows: There is an injective ring morphism* σ: $R \to \bar{R}$ *and a basis* $\bar{B}$ *of* $\bar{V}$ *such that either*

$$(\Lambda(\alpha))_{\bar{B}} = [(\alpha)_B]^\sigma \quad \textit{for all } \alpha \textit{ in } E_B(V)$$

or

$$(\Lambda(\alpha))_{\bar{B}} = [(\alpha)_B]^{*\sigma} \quad \textit{for all } \alpha \textit{ in } E_B(V).$$

PROOF. Modify [14, Theorem 3.6], by replacing $SL_n(R)$ in that statement by $E_B(V)$. Also, in this case, σ is only injective.

Let B and $\bar{B}$ be bases of $\bar{V}$. Suppose v and w are $\bar{R}$-morphisms of $\bar{V}$ and $(v)_B = (w)_{\bar{B}}$. This situation may be thought of as follows: The group of invertible matrices $GL_n(\bar{R})$ acts under conjugation as a transitive transformation group on $\theta_v = \{(v)_B | B \text{ basis of } \bar{V}\}$ which is the set of matrix representations of v relative to bases of $\bar{V}$. To say that $(v)_B = (w)_{\bar{B}}$ means that $\theta_v \cap \theta_w \neq \emptyset$. Thus, by transitivity $\theta_v = \theta_w$. More concretely, suppose $\rho B = \bar{B}$. Then, for matrices, $(w)_{\bar{B}} = (\rho)(w)_B(\rho^{-1}) = (\rho w \rho^{-1})_B$. Hence, $(v)_B = (w)_{\bar{B}} = (\rho w \rho^{-1})_B$ and $v = \rho w \rho^{-1}$ since they agree on B.

Thus, the above theorem may be stated as follows: $\Lambda(\alpha) = \rho^{-1}\alpha^{\sigma}\rho$ for all α in $E_B(V)$ or $\Lambda(\alpha) = \rho^{-1}\alpha^{*\sigma}\beta$ for all α in $E_B(V)$ where $\rho B = \bar{B}$.

Since Λ: $G \to G$ we may again ask if this imposes restraints on the "inner" portion of the automorphism. The proof of (III.3) carries through essentially unchanged except that the matrix units E_{ji} appearing in the proof need to be replaced by Σ_{ji} satisfying the following: If $B = \{b_1, \ldots, b_n\}$ is a basis of V with dual basis $B^* = \{b_1^*, \ldots, b_n^*\}$ then Σ_{ji} is in $GL(\bar{V})$ and $\Sigma_{ji}(b_i) = b_j$. (For example, take $\Sigma_{ji} = \tau_{-b_i,b_j^*}\tau_{b_j,b_i^*}$.) The technique is the same as O'Meara [11, pp. 94–95, Theorem 5.5B]. Thus, for the automorphism as described in the previous paragraph, $\rho V = TV$ where T is an invertible R-submodule of $\bar{R}$.

Since $\rho V = TV$ where T is an invertible R-submodule of $\bar{R}$, the inner automorphism Φ_ρ given by $\Phi_\rho(\alpha) = \rho\alpha\rho^{-1}$ carries $GL(V)$ into itself. Thus $(\Phi_\rho \circ \Lambda)(G) \subseteq GL(V)$. But $(\Phi_\rho \circ \Lambda)(\alpha) = \alpha^\sigma$. Hence $\mathrm{Im}(\sigma) \subseteq R$. For example, for the first case if λ is in R and $\tau = \tau_{\lambda b_i, b_j^*}$ then $(\Phi_\rho \circ \Lambda)(\tau) = \tau^\sigma = \tau_{\sigma(\lambda)b_i,b_j^*}$ is in $GL(V)$. Hence $\sigma(\lambda)$ is in R. Using Λ^{-1} on G we produce in a similar fashion an injective ring morphism σ^Λ: $R \to R$ satisfying $\sigma^\Lambda\sigma = 1 = \sigma^\Lambda\sigma$. Thus, σ is a ring automorphism of R.

We now restate (IV.9). Observe that by our proofs ρ depends on the basis B. We discuss this dependence below.

(VI.10). THEOREM. *Let* Λ: $G \to G$ *be a group automorphism. Then there is a* ρ *in* $GL(\bar{V})$ *and a ring automorphism* σ *of* R *such that either* $\Lambda(\alpha) = \rho^{-1}\alpha^\sigma\rho$ *for all* α *in* $E_B(V)$ *or* $\Lambda(\alpha) = \rho^{-1}\alpha^{*\sigma}\rho$ *for all* α *in* $E_B(V)$. *Further,* $\rho V = TV$ *for some invertible R-submodule of* $\bar{R}$.

The above results describe the action of Λ when restricted to $E_B(V)$. Let $B = \{b_1, \ldots, b_n\}$ and suppose $C = \{c_1, c_2, \ldots, c_n\}$ is another basis for V with $\eta B = C$ where $\eta(b_i) = c_i$. If $\tau = \tau_{\lambda b_i, b_j^*}$ is in $E_B(V)$ with λ in R, then $\eta\tau\eta^{-1} = \tau_{\lambda\eta b_i, b_j^*\eta^{-1}} = \tau_{\lambda c_i, c_j^*}$. Thus $\eta E_B(V)\eta^{-1} = E_C(V)$.

Let $\tau_1 = \eta\tau\eta^{-1}$. We have $\Lambda(\tau) = \rho^{-1}\tau^\sigma\rho$ or $\Lambda(\tau) = \rho^{-1}\tau^{*\sigma}\rho$. Assume $\Lambda(\tau) = \rho^{-1}\tau^\sigma\rho$. (The second case is handled in a similar fashion.) Then

$$\begin{aligned}\Lambda(\tau_1) = \Lambda(\eta\tau\eta^{-1}) &= \Lambda(\eta)\Lambda(\tau)\Lambda(\eta)^{-1} = \Lambda(\eta)\rho^{-1}\tau^\sigma\rho\Lambda(\eta)^{-1}\\ &= \Lambda(\eta)\rho^{-1}(\eta^{-1}\tau_1\eta)^\sigma\rho\Lambda(\eta)^{-1} = \Lambda(\eta)\rho^{-1}(\eta^\sigma)^{-1}\tau_1^\sigma\eta^\sigma\rho\Lambda(\eta)^{-1}\\ &= [\eta^\sigma\rho\Lambda(\eta^{-1})]^{-1}\tau_1^\sigma[\eta^\sigma\rho\Lambda(\eta^{-1})].\end{aligned}$$

Set $\rho_\eta = \eta^\sigma\rho\Lambda(\eta^{-1})$. Thus $\Lambda(\eta) = \rho_\eta^{-1}\eta^\sigma\rho$ for all η in $GL(V)$. The best situation occurs when $\rho_\eta = \rho$. We single out this situation with a definition.

A pair $\langle R, S\rangle$ is said to be *strongly stable* if

(a) $\langle R, S\rangle$ is stable,

(b) $T(V) = E_B(V)$.

(Recall $T(V)$ denotes the group of transvections, see (II).) This situation occurs most often when $SL(V) = T(V) = E_B(V)$. Theorem (II.4) provides several examples of strongly stable rings where $SL(V) = E_B(V)$. Also, G. Cooke [5] recently generalized the Euclidean algorithm to a "k-stage" algorithm for domains. Cooke shows that a domain R possessing a k-stage algorithm is a unique factorization domain (but not necessarily a principal ideal domain) and has the property that $SL(V) = E_B(V)$. On the other hand, Bass [1] recently exhibited a principal ideal domain R for which the K-group $K_1(R)$ is not isomorphic to the units R^* under the determinant map. Thus, $SL(V)$ is not generated in Bass' example by the group of elementary matrices relative to a fixed basis.

(IV.11). LEMMA. *Let a in $\bar{V}$ be unimodular and φ in $\bar{V}^*$ be surjective with $\varphi a = 0$. Then there is a nonzero divisor λ with $\tau_{\lambda a,\varphi}$ in $T(V)$.*

PROOF. Let $\bar{x}_1, \ldots, \bar{x}_n$ be a basis for $\bar{V}$ and $\bar{\varphi}_1, \ldots, \bar{\varphi}_n$ be a basis for $\bar{V}^*$ with $a = \bar{x}_1$ and $\varphi = \bar{\varphi}_n$.

Let $j \neq i$. The image $\bar{\varphi}_j(V)$ is a fractional R-submodule of $\bar{R}$. Let $A_i = \{r$ in $\bar{R} | r\bar{x}_i$ in $V\}$ (coefficient of $\bar{x}_i$). Then for $\bar{\lambda}$ a nonzero divisor in S we have $\bar{\lambda}(\bar{\varphi}_j(V)) \subset R$. Further we may take the same $\bar{\lambda}$ for all $\bar{\varphi}_j$, $j \neq i$. Now let β be a nonzero divisor in A_i. Set $\lambda = \bar{\lambda}\beta$. Then $\lambda\bar{\varphi}_j(V)\bar{x}_i \subset A_i\bar{x}_i \subset V$. Hence $\tau_{\lambda\bar{x}_i,\bar{\varphi}_j}(V) \subseteq V$.

(IV.12). THEOREM. *Let $\langle R, S\rangle$ be a strongly stable pair. Let $\Lambda\colon G \to G$ be a group automorphism where $SL(V) \leqslant G \leqslant GL(V)$. Then $\Lambda(\alpha) = \chi(\alpha)\rho^{-1}\alpha^\sigma\rho$ for all α in G or $\Lambda(\alpha) = \chi(\alpha)\rho^{-1}\alpha^{*\sigma}\rho$ for all α in G, where*

(a) *$\sigma\colon R \to R$ is a ring automorphism,*

(b) *ρ is in $GL(\bar{V})$ and $\rho V = TV$ where T is an invertible R-submodule of $\bar{R}$,*

(c) *$\chi\colon G \to$ center(G) is a group morphism.*

PROOF. Let B be a basis of V. We assume Λ has the form $\Lambda(\alpha) = \rho^{-1}\alpha^\sigma\rho$ for all α in $E_B(V)$. (The second case is handled in a similar manner.) Since $\langle R, S\rangle$ is strongly stable $T(V) = E_B(V)$ and $\Lambda(\alpha) = \rho^{-1}\alpha^\sigma\rho$ for all α in $T(V)$.

Define $\Phi\colon G \to G$ by $\Phi(\alpha) = [\rho\Lambda(\sigma)\rho^{-1}]^{\sigma^{-1}}$. Hence $\Phi(\alpha) = \alpha$ for all α in $T(V)$.

Let $L = \bar{R}a$ be a line in $\bar{V}$. Let φ be surjective in $\bar{V}^*$ with $\varphi a = 0$. Then $\tau_{a,\varphi}$ is in $T(\bar{V})$. By the lemma there is a λ in S with $\tau_{\lambda a,\varphi}$ in $T(V)$.

Let α be in G. Then $\alpha\tau_{\lambda a,\varphi}\alpha^{-1} = \tau_{\lambda\alpha a,\varphi\alpha^{-1}}$ is a transvection in $T(V)$. By definition of Φ, $\Phi(\tau_{\lambda a,\varphi}) = \tau_{\lambda a,\varphi}$ and $\Phi(\tau_{\lambda\alpha a,\varphi\alpha^{-1}}) = \tau_{\lambda\alpha a,\varphi\alpha^{-1}}$. The latter transvection has line αL where $L = \bar{R}a$. On the other hand,

$$\Phi(\alpha\tau_{\lambda a,\varphi}\alpha^{-1}) = (\Phi\alpha)(\Phi\tau_{\lambda a,\varphi})(\Phi\alpha^{-1}) = \tau_{\lambda(\Phi\alpha)a,\varphi(\Phi\alpha)^{-1}}$$

which has line $\Phi(\alpha)L$. Thus $\alpha L = \Phi(\alpha)L$, i.e. $\alpha^{-1}\Phi(\alpha)\colon L \to L$ fixes lines. Thus $\alpha^{-1}\Phi(\alpha)$ is in $RL(\bar{V})$-scalar multiplications. Set $\alpha^{-1}\Phi(\alpha) = \chi(\alpha)$. It is straightforward to verify that $\chi(\alpha)$ is in $RL(V)$ and $\alpha \to \chi(\alpha)$ is a group morphism $G \to$ center(G).

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