AN ASYMPTOTIC FORMULA FOR AN INTEGRAL
IN STARLIKE FUNCTION THEORY

BY

R. R. LONDON AND D. K. THOMAS

ABSTRACT. The paper is concerned with the integral

\[ H = \int_0^{2\pi} |f(e^{i\theta})|^a |F(\Re F(\theta))|^k \, d\theta \]

in which \( f \) is a function regular and starlike in the unit disc, \( F = zf'/f \), and the parameters \( a, r, k \) are real. A study of \( H \) is of interest since various well-known integrals in the theory, such as the length of \( f(|z| = r) \), the area of \( f(|z| < r) \), and the integral means of \( f \), are essentially obtained from it by suitably choosing the parameters. An asymptotic formula, valid as \( r \to 1 \), is obtained for \( H \) when \( f \) is a starlike function of positive order \( \alpha \), and the parameters satisfy \( \alpha \sigma + r + \kappa > 1, r + \kappa > 0, \kappa > 0, \sigma > 0 \). Several easy applications of this result are made; some to obtaining old results, two others in proving conjectures of Holland and Thomas.

1. Introduction. Let a function \( f \) be regular in the open unit disc \( D \), and such that \( f(0) = 0, f'(0) = 1 \). Suppose a function \( F \) exists, regular in \( D \), and of positive real part, for which

\[ F(z) = zf'(z)/f(z) \quad (0 < |z| < 1), \quad F(0) = 1. \]

Then \( f \) is called starlike. It is well known that a starlike function is univalent, and maps \( D \) onto a set starshaped with respect to the origin.

Suppose now that \( f \) is starlike, and let (1.1) define \( F \). We consider for \( 0 < r < 1 \) and real \( \sigma, \tau, \) and \( \kappa \), the integral

\[ H(\sigma, \tau, \kappa) = \int_0^{2\pi} |f(re^{i\theta})|^a |F(re^{i\theta})|^\tau (\Re F(re^{i\theta}))^k \, d\theta. \]

With various choices of \( \sigma, \tau, \) and \( \kappa \), this integral is well known in starlike (and univalent) function theory. For example,

\[ H(1, 1, 0) = \int_0^{2\pi} |f(re^{i\theta})|f(re^{i\theta})|d\theta = \int_0^{2\pi} r |f'(re^{i\theta})| d\theta \]

is the length of \( f(|z| = r) \), and

\[ H(2, 0, 1) = 2 \int_0^{2\pi} |f(re^{i\theta})|^2 \Re F(re^{i\theta}) \, d\theta \]

is the area of \( f(|z| < r) \).
where \( A(r, f) \) is the area of \( f(\{z:\ |z| \leq r\}) \). Also, for \( \lambda > 0 \),

\[
H(r, \lambda, 0, 0) = \int_0^{2\pi} f(re^{i\theta})^\lambda \, d\theta = 2\pi I(r, \lambda, f),
\]
and

\[
H(r, \lambda, \lambda, 0) = \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda \, d\theta
\]

(1.3)

\[
= \int_0^{2\pi} r^\lambda |f'(re^{i\theta})|^\lambda \, d\theta = 2\pi r^\lambda J(r, \lambda, f),
\]

where \((J(r, \lambda, f))^{1/\lambda}\) and \((J(r, \lambda, f))^{1/\lambda}\) are integral means of \( f \) and \( f' \) respectively. In the present paper we suppose \( f \) to have a certain minimal growth, and find, for suitable \( \sigma, \tau, \) and \( \kappa \), as \( r \) tends to one, an asymptotic formula for \( H(r, \sigma, \tau, \kappa) \).

A few remarks and definitions precede the statement of this result. From a classical theorem on regular functions of positive real part, and the relationship (1.1), we find for a starlike function \( f \) the well-known representation

\[
f(z) = z \exp \left( -\int_0^{2\pi} \log(1 - re^{-e^{i\theta}}) \, d\mu(\theta) \right) \quad (z \in D),
\]

where \( \mu \) is nondecreasing on \([0, 2\pi]\) and \( \int_0^{2\pi} d\mu(\theta) = 2 \). Any such function \( \mu \) satisfying (1.5) is continuous apart from jumps of height at most \( \int_0^{2\pi} d\mu(\theta) \). Pommerenke has shown (implicitly) in [4] that

\[
\Delta(\varphi, f) = \lim_{r \to 0} \frac{\log |f(re^{i\varphi})|}{\log(1 - r)}
\]

is the jump of \( \mu \) at \( \varphi \) if \( 0 < \varphi < 2\pi \), and the sum of the jumps at 0 and \( 2\pi \) when \( \varphi = 0 \) or \( 2\pi \). For \( \alpha(f) = \sup_{\varphi} \Delta(\varphi, f) \) and \( M(r) = \max_{|z|=r} |f(z)| \) (\( 0 < r < 1 \)) he has shown

\[
\alpha(f) = \lim_{r \to 1} \frac{\log M(r, f)}{-\log(1 - r)}.
\]

(1.6)

We call \( \alpha(f) \) the order of \( f \), and \( \Delta(\varphi, f) \) the radial order of \( f \) on \( \{re^{i\varphi}\} \).

Starlike functions of positive order are the main concern of the present paper. We shall, in the following, make implicit use of the fact that for such a function \( f \), \( \{\varphi: \Delta(\varphi, f) > 0\} \) is countable; and for \( 0 < c \ll \alpha \), \( \{\varphi: \Delta(\varphi, f) \geq c\} \) is finite nonempty.

Our result is as follows:

**Theorem 1.** Let \( f \) be a starlike function of positive order \( \alpha \), and denote by \( \varphi_1, \ldots, \varphi_N \) the values of \( \varphi \) in \([0, 2\pi]\) for which \( \alpha \) is the radial order of \( f \) on \( \{re^{i\varphi}\} \). Then if \( \sigma > 0, \kappa \gg 0, \tau + \kappa \gg 0, \) and \( \alpha \sigma + \tau + \kappa > 1 \),

\[
H(r, \sigma, \tau, \kappa) \sim \alpha^{\tau + \kappa} C(\alpha \sigma + \tau + 2\kappa)(1 - r)^{-\tau - \kappa} \sum_{\nu=1}^{N} |f(re^{i\varphi_\nu})|^\sigma,
\]
as \( r \to 1 \), where, for \( x > 1 \), \( C(x) = \int_{-\infty}^{\infty} dt/(1 + t^2)^{x} = \Gamma(\frac{1}{2}x - \frac{1}{2})\Gamma(\frac{1}{2})(\frac{1}{2}x)\).

Theorem 1 has some interesting applications. Let \( f \) be starlike of order \( \alpha \).

Recalling (1.2) to (1.4), and using, with Theorem 1, the relationship

\[
M(r, f) \sim \max(|f(re^{i\theta_1})|, \ldots, |f(re^{i\theta_N})|) \quad (\alpha > 0)
\]

(proved at the end of §2) we obtain for \( \alpha > 0 \):

\[
\liminf_{r \to 1} \frac{A(r, f)}{M^2(r, f)} \geq \frac{1}{2} \alpha C(2\alpha + 2), \quad \limsup_{r \to 1} \frac{A(r, f)}{M^2(r, f)} \leq \frac{1}{2} N\alpha C(2\alpha + 2),
\]

for \( \alpha > 0 \), \( \alpha \lambda > 1 \):

\[
\liminf_{r \to 1} \frac{I(r, \lambda, f)}{(1 - r)M^\lambda(r, f)} \geq C(\alpha \lambda), \quad \limsup_{r \to 1} \frac{I(r, \lambda, f)}{(1 - r)M^\lambda(r, f)} \leq N\alpha C(\alpha \lambda),
\]

and for \( \alpha > 0 \), \( (1 + \alpha)\lambda > 1 \):

\[
\liminf_{r \to 1} \frac{J(r, \lambda, f)}{(1 - r)^{1-\lambda}M^\lambda(r, f)} \geq \alpha^\lambda C(\alpha \lambda + \lambda), \quad \limsup_{r \to 1} \frac{J(r, \lambda, f)}{(1 - r)^{1-\lambda}M^\lambda(r, f)} \leq N\alpha^\lambda C(\alpha \lambda + \lambda).
\]

These are all results of Sheil-Small [5].

From (1.2), \( A(r, f) = \frac{1}{2}H(r, 2, 0, 1) \) and \( A'(r, f) = H(r, 2, 2, 0)/r \); so for \( \alpha > 0 \) Theorem 1 also yields

\[
\lim_{r \to 1} \frac{(1 - r)A'(r, f)}{A(r, f)} = 2\alpha.
\]

A proof of this result, and a simple proof of (1.8) in the case \( \alpha = 0 \), are to be found in [3]. We are also able to prove, using Theorem 1, that, for \( \alpha \lambda > 1 \),

\[
\lim_{r \to 1} \frac{(1 - r)I'(r, \lambda, f)}{I(r, \lambda, f)} = \alpha \lambda - 1,
\]

a result conjectured in [2]. In fact, once we have noticed, from (1.3), that, for any \( \lambda > 0 \), \( I(r, \lambda, f) = H(r, \lambda, 0, 0)/2\pi \), and

\[
I'(r, \lambda, f) = \frac{\lambda}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\lambda Re F(re^{i\theta}) d\theta = \frac{\lambda}{2\pi} H(r, \lambda, 0, 1),
\]

the proof via Theorem 1 follows simply on noting

\[
\alpha \lambda C(\alpha \lambda + 2) = (\alpha \lambda - 1)C(\alpha \lambda), \quad \text{for } \alpha \lambda > 1.
\]

Another conjecture in [2] is that, for \( (1 + \alpha)\lambda > 1 \),

\[
\lim_{r \to 1} \frac{(1 - r)J'(r, \lambda, f)}{J(r, \lambda, f)} = (1 + \alpha)\lambda - 1.
\]

A corollary of Theorem 1 yields a proof of (1.10) when \( (1 + \alpha)\lambda > 1 \), \( \alpha > 0 \), as we shall see in §4.

Our proof of Theorem 1 begins in §2, where some preliminary results are
obtained, and is completed in §3. In all that follows we assume that $0 < r < 1$, and that $\theta$ is a real number. Also that the $o$, $O$, and $\sim$ notations refer to behaviour as $r$ tends to one. The term $r$ near one means all values of $r$ in $(\eta, 1)$, for some $\eta$ in $(0, 1)$.

2. Preliminaries. In this section we prove a number of results on what are essentially powers of starlike functions. A function $g$ will be called star-powered whenever

$$(2.1) \quad g(z) = z \exp \left( - \int_0^{2\pi} \log(1 - e^{-i\theta}z) d\nu(t) \right) \quad (z \in D),$$

for some nondecreasing function $\nu$ on $[0, 2\pi]$. For such a $g$, we define as for a starlike function $g$ the terms order of $g$ and radial order of $g$ on $\{re^{i\theta}\}$.

The results we now prove are directed towards finding for a star-powered function $g$, and the function

$$(2.2) \quad G: G(z) = zg'(z)/g(z) \quad (z \in D\setminus\{0\}), \quad G(0) = 1$$

information about behaviour on various subsets of $D$.

**Lemma 2.1.** Let $g$ be a star-powered function of positive order $\beta$, and in the above notation, put $K = \int_0^{2\pi} d\nu(t)$. Then

(i) for $\epsilon > 0$, and $r$ near one, $M(r, g) < (1 - r)^{-\beta - \epsilon}$;

(ii) with $G$ defined by (2.2), $M(r, G) < 1 + Kr(1 - r)^{-1}$.

This is a well-known result when $g$ is starlike [4], [3] and the extension to star-powered functions is simple enough to omit.

**Lemma 2.2.** Let $g$ be a star-powered function of positive order $\beta$, and denote by $\psi_1, \ldots, \psi_p$ the values of $\psi$ in $[0, 2\pi)$ for which $\beta$ is the radial order of $f$ on $\{re^{i\theta}\}$. Put

$$T(r) = \{0, 2\pi\} \setminus \bigcup_{k=1}^p \{\theta \mod 2\pi: |\theta - \psi_k| < l(r)\},$$

where $l(r) = (-\log 1 - r)^{-1}$, and let $\gamma$ be the largest radial order of $g$ less than $\beta$. Then for any positive $\epsilon$, and $r$ near one,

$$\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1 - r)^{\gamma - \epsilon}.$$

**Proof.** The function $h$,

$$h(z) = g(z) \prod_{k=1}^p (1 - ze^{-i\psi_k})^\beta,$$

is star-powered and has order $\gamma$, so by Lemma 2.1, for $\epsilon > 0$, and $r$ near one, $M(r, h) < (1 - r)^{-\gamma - \frac{3}{2}\epsilon}$. Also, for $r$ near one, we have uniformly when $\theta \in T(r)$

$$\prod_{k=1}^p |1 - re^{i(\theta - \psi_k)}|^{-\beta} \leq |1 - re^{il(r)}|^{-\beta p} \leq (\frac{4}{U}\frac{l^2(r)}{r})^{-\frac{1}{2}\beta p}.$$
Hence
\[
\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1-r)^{-\gamma'-\beta}(\log(1-r))^{\beta P},
\]
from which the stated result follows.

**Lemma 2.3.** Let \( g \) be a star-powered function of positive radial order \( \beta \) on \( \{re^{i\theta}\} \) \((0 \leq \varphi < 2\pi)\). Suppose that \( \lambda \) is a positive number, and \( \delta \) a positive function on \((0, 1)\) for which \( \lambda(1-r) < \delta(r) = o(1) \). Then
\[
|g(re^{i\theta})| < (1 + \lambda^2 + o(1))^{-\frac{1}{2}} |g(re^{i\theta})|
\]
uniformly when \( \lambda(1-r) < |\theta - \varphi| < \delta(r) \).

**Proof.** Let
\[
g(z) = z \exp \left(-\int_0^{e^{-i\varphi}} \log(1 - e^{-it}z) \, dt\right) \quad (z \in D)
\]
then, by an elementary argument,
\[
|g(re^{i\theta})| < (1 + o(1))r|h(re^{i\theta})|(1-r)^{-\beta}
\left(1 + \left(\frac{\theta - \varphi}{1-r}\right)^2\right)^{-\frac{1}{2}}
\]
uniformly when \( |\theta - \varphi| < \delta(r) \). For \( h \) we shall prove
\[
h(re^{i\theta}) < |h(re^{i\theta})| \exp \left(o(1) \int_0^{\theta - \varphi/2} (1-r)^{-\frac{1}{2}} (1 + t^2)^{-\frac{1}{2}} \, dt\right)
\]
uniformly when \( \lambda(1-r) < |\theta - \varphi| < \delta(r) \). The lemma then follows by combining (2.3) and (2.4) to form
\[
|g(re^{i\theta})| < (1 + o(1))|g(re^{i\theta})|
\left(1 + \left(\frac{\theta - \varphi}{1-r}\right)^2\right)^{-\frac{1}{2}}
\cdot \exp \left(o(1) \int_0^{\theta - \varphi/2} (1-r)^{-\frac{1}{2}} (1 + t^2)^{-\frac{1}{2}} \, dt\right),
\]
valid uniformly for \( \lambda(1-r) < |\theta - \varphi| < \delta(r) \), and by noting that for any suitably small positive \( \epsilon \)
\[
y_\epsilon: y_\epsilon(x) = (1 + x^2)^{-\frac{1}{2}} \exp \left(e \int_0^{\frac{\gamma x}{2}} (1 + t^2)^{-\frac{1}{2}} \, dt\right) \quad (x > \lambda)
\]
is a decreasing function.

For the proof of (2.5), let \( \eta \) be a positive function on \((0, 1)\) for which
\[\eta(r) = o(1), \text{ yet } \delta(r) = o(1)1 - re^{i\eta(r)}\], and put
\[
E(r) = \{\theta \mod 2\pi: |\theta - \varphi| < \eta(r)\}, \quad E'(r) = [0, 2\pi] \setminus E(r)
\]
Write
\[
g(z) = z \exp \left(-\int_0^{2\pi} \log(1 - e^{-it}z) \, dt\right) \quad (z \in D),
\]
then from (2.3) we deduce
\[
h(z) = \exp \left(-\int_0^{2\pi} \log(1 - e^{-it}z) \, dt\right) \quad (z \in D)
\]
where

\[ \tau(t) = \nu(t) = \begin{cases} \beta, & \varphi < t \leq 2\pi, \\ 0, & 0 \leq t \leq \varphi. \end{cases} \]

We have, for any real \( t, \)

\[
\log |1 - r e^{i(\varphi - t)}| - \log |1 - r e^{i(\theta - t)}| = \text{Re} \int_{\frac{\theta - t}{\varphi - t}}^{\frac{\beta - t}{\varphi - t}} \left( \log \frac{1}{1 - re^{iu}} \right) \, du
\]

\[
\leq \max_{0 < r < 2\pi} \int_{0}^{\frac{\varphi - \theta}{\varphi - t}} \frac{du}{|1 - re^{iu}|} = 2 \int_{0}^{\frac{\varphi - \theta}{\varphi - t}} \frac{du}{|1 - re^{iu}|}.
\]

So with the aid of (2.7) we obtain

\[
\int_{E(r)} (\log |1 - r e^{i(\varphi - t)}| - \log |1 - r e^{i(\theta - t)}|) \, d\tau(t)
\]

\[
\leq 2 \left( \int_{E(r)} d\tau(t) \right) \left( \int_{0}^{\frac{\varphi - \theta}{\varphi - t}} \frac{du}{|1 - re^{iu}|} \right)
\]

\[
= o(1) \int_{0}^{\frac{\varphi - \theta}{\varphi - t}} \frac{du}{|1 - re^{iu}|} = o(1) \int_{0}^{\frac{\varphi - \theta}{\varphi - t}} \frac{dv}{|1 - re^{iu}|} = o(1)
\]

uniformly when \(|\theta - \varphi| < \delta(r)\). Now observe that, by the choice of \( \eta, \)

\[
\left| \frac{1 - r e^{i(\theta - t)}}{1 - r e^{i(\varphi - t)}} - 1 \right| \leq \left| \frac{e^{i(\theta - \varphi)} - 1}{1 - re^{i(\varphi - t)}} \right| = o(1)
\]

uniformly for \( t \in E' \) and \(|\theta - \varphi| < \delta(r)\). Hence

\[
\int_{E'(r)} (\log |1 - r e^{i(\varphi - t)}| - \log |1 - r e^{i(\theta - t)}|) \, d\tau(t) \leq o(1) \int_{0}^{2\pi} d\tau(t) = o(1)
\]

uniformly when \(|\theta - \varphi| < \delta(r)\). Combining (2.8) and (2.9), and using (2.6) we obtain (2.5) uniformly for \( \lambda(1 - r) < |\theta - \varphi| < \delta(r)\).

**Lemma 2.4.** Let \( g \) be a star-powered function of positive radial order \( \beta \) on \( \{re^{i\varphi} \} \) \((0 \leq \varphi < 2\pi)\), and define \( G \) by (2.2). Then for \( 0 < c < C \)

(i) \[ |G(re^{i\varphi})| \sim \beta(1 - r)^{-1} \left( 1 + \left( \frac{\theta - \varphi}{1 - r} \right)^2 \right)^{-\frac{1}{2}}, \]

and

\[ \text{Re} \, G(re^{i\varphi}) \sim \beta(1 - r)^{-1} \left( 1 + \left( \frac{\theta - \varphi}{1 - r} \right)^2 \right)^{-1}, \]

uniformly when \(|\theta - \varphi| < C(1 - r)\).

(ii) \[ \text{Im} \, G(re^{i\varphi}) \sim \beta(\theta - \varphi)(1 - r)^{-2} \left( 1 + \left( \frac{\theta - \varphi}{1 - r} \right)^2 \right)^{-1} \]
uniformly when \( c(1-r) < |\theta - \phi| < C(1-r) \).

(iii) \[ |g(re^{i\theta})| \sim |g(re^{i\phi})| \left( 1 + \left( \frac{\theta - \phi}{1-r} \right)^2 \right)^{-\frac{1}{2}} \]

uniformly when \( |\theta - \phi| < C(1-r) \).

**Proof.** Parts (i) and (ii) are easily proved if \( G(z) = \beta z/(1-z) \), \( \varphi = 0 \), so in deriving (i) and (ii) we shall prove no more than

\[
G(re^{i\theta}) - \frac{\beta re^{i(\theta-\varphi)}}{1 - re^{i(\theta-\varphi)}} = o(1) \frac{1}{1-r}
\]

uniformly when \( |\theta - \varphi| < \delta(r) \), where \( \delta \) is any positive \( o(1) \) function on \((0, 1)\).

We have, from (2.1) and (2.2), for some nondecreasing function \( \nu \),

\[
G(z) = 1 + \int_0^{2\pi} \frac{e^{-it}z}{1-e^{-it}z} \, dv(t) \quad (z \in D)
\]

and this we rewrite in terms of

\[
\tau(t) = \nu(t) - \begin{cases} \beta, & \varphi < t \leq 2\pi, \\ 0, & 0 \leq t \leq \varphi, \end{cases}
\]

as

\[
G(re^{i\theta}) - \frac{\beta re^{i(\theta-\varphi)}}{1 - re^{i(\theta-\varphi)}} = 1 + \int_0^{2\pi} \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} \, d\tau(t).
\]

Let \( \eta \) and \( \epsilon \) be positive functions on \((0, 1)\) for which \( \epsilon(r) = \delta(r) + \eta(r) = o(1) \), and \( 1-r = o(1)1|1-re^{i(\theta-r)}| \). Put

\[
P(r) = \{ \theta \mod 2\pi : |\theta - \varphi| < \epsilon(r) \}, \quad Q(r) = [0, 2\pi] \setminus P(r),
\]

and consider now only values of \( r \) for which \( Q(r) \) is nonempty. Then from (2.11) we see that

\[
\int_{P(r)} \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} \, d\tau(t) \leq \frac{1}{1-r} \int_{P(r)} d\tau(t) = o(1) \frac{1}{1-r}
\]

uniformly for real \( \theta \). Moreover, for \( |\theta - \varphi| < \delta(r) \) and \( t \in Q(r) \),

\[ \eta(r) \leq |\theta - t| \leq 2\pi - \eta(r) \], so we also have, uniformly for \( |\theta - \varphi| < \delta(r) \),

\[
\int_{Q(r)} \frac{re^{i(\theta-t)}}{1-re^{i(\theta-t)}} \, d\tau(t) \leq \frac{1}{1-re^{\eta(r)}} \int_{Q(r)} d\tau(t) = o(1) \frac{1}{1-r},
\]

by the choice of \( \eta \). Now combining (2.12), (2.13) and (2.14), the required estimate (2.10) is easily obtained.

We next derive (iii). Using the identity

\[
\frac{\partial}{\partial \theta} \log |g(re^{i\theta})| = - \text{Im} \, G(re^{i\theta})
\]
and (ii) of this lemma, we have, uniformly for \(c(1 - r) < \theta - \varphi < C(1 - r)\),

\[
\log \left| \frac{g(re^{it\theta})}{g(re^{i(\theta + c(1-r))})} \right| \sim -\beta \int_{\theta}^{\varphi + c(1-r)} \frac{t - \varphi}{(1 - r)^2 + (t - \varphi)^2} \, dt
\]

\[
= -\beta \int_{c(1-r)}^{\theta - \varphi} \frac{u}{(1 - r)^2 + u^2} \, du = -\beta \int_{c}^{(\theta - \varphi)/1-r} \frac{t}{1 + t^2} \, dt
\]

\[
= -\beta \frac{1}{2} \log \left( 1 + \left(\frac{(\theta - \varphi)/(1-r)}{1 + c^2}\right)^2 \right).
\]

From this, and a similar argument, we deduce

\[
\left| \frac{g(re^{i\theta})}{g(re^{i(\theta + c(1-r))})} \right| \sim \left( \frac{1 + c^2}{1 + ((\theta - \varphi)/(1-r))^2} \right)^{\frac{\beta}{2}},
\]

and

\[
\left| \frac{g(re^{i\theta})}{g(re^{i(\theta - c(1-r))})} \right| \sim \left( \frac{1 + c^2}{1 + ((\theta - \varphi)/(1-r))^2} \right)^{\frac{\beta}{2}}
\]

valid uniformly for \(c(1 - r) < \theta - \varphi < C(1 - r)\), and \(-C(1 - r) < \theta - \varphi < -c(1 - r)\) respectively. When \(|\theta - \varphi| \leq c(1 - r)\) we have, using (2.15) again and Lemma 2.1(ii),

\[
\left| \log \frac{g(re^{i\theta})}{g(re^{i\theta'})} \right| = \int_{\theta}^{\theta'} \text{Im} \, G(re^{it}) \, dt \leq c(1-r) \left( 1 + \frac{Kr}{1-r} \right),
\]

from which

\[
e^{-c(1-r+Kr)} \leq \left| \frac{g(re^{i\theta})}{g(re^{i\theta'})} \right| \leq e^{c(1-r+Kr)},
\]

and this modifies trivially to

\[
e^{-c(1-r+Kr)} \left( \frac{1}{1 + ((\theta - \varphi)/(1-r))^2} \right)^{\frac{\beta}{2}} \leq \left| \frac{g(re^{i\theta})}{g(re^{i\theta'})} \right| \leq e^{c(1-r+Kr)} \left( \frac{1 + c^2}{1 + ((\theta - \varphi)/(1-r))^2} \right)^{\frac{\beta}{2}}.
\]

Since \(c\) is an arbitrary positive number, the last two results imply (iii).

To conclude this sequence of lemmas we shall prove a result assumed in §1.

**Lemma 2.5.** Let \(g\) be a star-powered function of positive order \(\beta\), and denote by \(\psi_1, \ldots, \psi_p\) the values of \(\psi\) in \([0, 2\pi)\) for which \(\beta\) is the radial order of \(f\) on \(\{re^{i\psi}\}\). Then
\[ M(r, g) \sim \max(|g(re^{i\psi_1})|, \ldots, |g(re^{i\psi_p})|). \]

**Proof.** Let
\[ T(r) = [0, 2\pi] \setminus \bigcup_{k=1}^{p} \{ \theta \mod 2\pi: |\theta - \psi_k| < l(r) \}, \]
then from Lemma 2.2, and the inequalities
\[ |\log z| > r(1 + r)^{\nu(2\pi) - \nu(0)}|1 - ze^{-i\psi_k}k^{-\beta}| \quad (|z| = r, k = 1, \ldots, p), \]
which follows easily from (2.1), we have
\[ (2.16) \sup_{\theta \in T(r)} |g(re^{i\theta})| = o(1)M(r, g). \]

Now using Lemma 2.3, we see via (2.16) that if \( \epsilon > 0 \) and
\[ W(r, \epsilon) = [0, 2\pi] \setminus \bigcup_{k=1}^{p} \{ \theta \mod 2\pi: |\theta - \psi_k| < \epsilon(1 - r) \}, \]
then, for \( r \) near one,
\[ \sup_{\theta \in W(r, \epsilon)} |g(re^{i\theta})| < (1 + \frac{\epsilon^2}{2})^{-\frac{1}{\beta}}M(r, g). \]
So \( |g(re^{i\eta(r)})| = M(r, g) \) where, for the same \( \epsilon \) and \( r \), \( |\eta(r) - \psi_k| \leq \epsilon(1 - r) \) and \( k = k(r) \in \{1, \ldots, N\} \). Since \( \epsilon \) is an arbitrary positive number we deduce from Lemma 2.4(iii) that
\[ M(r, g) < (1 + o(1))|g(re^{i\psi_k})| \]
where \( k = k(r) \). Hence
\[ M(r, g) < (1 + o(1))\max(|g(re^{i\psi_1})|, \ldots, |g(re^{i\psi_p})|), \]
and obviously this completes the proof.

3. **Proof of Theorem 1.** Let \( f \) be a starlike function of positive order \( \alpha \), and denote by \( \varphi_1, \ldots, \varphi_N \) the values of \( \varphi \) in \([0, 2\pi)\) for which \( \alpha \) is the radial order of \( f \) on \( \{re^{i\varphi}\} \). Let \( l(r) = (-\log(1 - r))^{-1} \), and put
\[ U_k(r) = \{ \theta: |\theta - \varphi_k| < l(r) \} \quad (k = 1 \text{ to } N), \]
\[ T(r) = [0, 2\pi] \setminus \bigcup_{k=1}^{N} \{ \theta \mod 2\pi: \theta \in U_k(r) \}. \]
Then, for real \( \alpha, \tau \) and \( \kappa \), and \( r \) near one, \( H(r, \alpha, \tau, \kappa) = \Sigma_{k=1}^{N} X_k + Y \), where
\[ X_k = \int_{U_k(r)} |f|^{\alpha}|F|^{\tau}(\text{Re } F)^{\kappa}d\theta, \quad Y = \int_{T(r)} |f|^{\alpha}|F|^{\tau}(\text{Re } F)^{\kappa}d\theta. \]
When \( \alpha \sigma + \tau + \kappa > 1, \tau + \kappa > 0, \kappa > 0, \sigma > 0 \), we shall find for each \( X_k \) the asymptotic formula
\[ (3.1) \quad X_k \sim \alpha^{r + \kappa}C(\alpha \sigma + \tau + 2\kappa)(1 - r)^{1-\tau-\kappa}|f(re^{i\varphi_k})|^{\alpha}, \]
and for $Y$ the estimate
\begin{equation}
Y = o(1)(1 - r)^{-\alpha \sigma - r - \kappa}.
\end{equation}

Since, by the representation (1.5), $|f(re^{i\phi_k})| \geq r(1 + r)^{-2}(1 - r)^{-\alpha}$, we then have
$Y = o(1)\sum_{k=1}^{N}X_k$; and consequently $H(r, \sigma, \tau, \kappa) \sim \sum_{k=1}^{N}X_k$. Our proof of
(3.1) is in §3.1, and that of (3.2) in §3.2.

3.1. Denote by $\phi$ any one of the $\phi_k$, and by $U(r)$ the corresponding $U_k(r)$. We have to prove
\begin{equation}
\int_{U(r)} |f(re^{i\theta})|^\sigma |F(re^{i\theta})|^\tau (\text{Re} F(re^{i\theta}))^\kappa \, d\theta
\end{equation}
for $\alpha \sigma + \tau + \kappa > 1$, $\tau + \kappa > 0$, $\kappa > 0$, $\sigma > 0$. Let
\begin{equation}
\psi(r, x) = \{ \theta : |\theta - \phi| < x(1 - r) \} \quad (x > 0)
\end{equation}
and write
\begin{equation}
\int_{U(r)} = \int_{V(r,x)} + \int_{U(r) \setminus V(r,x)} = I_1 + I_2,
\end{equation}
say, where the missing integrand is that in (3.3). For $I_1$ we have, by Lemma
2.4, for real $\sigma$, $\tau$, and $\kappa$, and for $x > 0$,
\begin{equation}
I_1 \sim \alpha^{\tau + \kappa}(1 - r)^{1 - \tau - \kappa}|f(re^{i\phi})|^\sigma \int_{V(r,x)} \left(1 + \left(\theta - \phi \right)^{2\tau - 2}\right)^{-\frac{1}{2}(\alpha \sigma + \tau + 2\kappa)} \, d\theta
\end{equation}
\begin{equation}
= \alpha^{\tau + \kappa}(1 - r)^{1 - \tau - \kappa}|f(re^{i\phi})|^\sigma \int_{-\infty}^{\infty} \left(1 + t^2\right)^{-\frac{1}{2}(\alpha \sigma + \tau + 2\kappa)} \, dt.
\end{equation}

For $I_2$ consider first the case $\tau + \kappa \neq 0$. Let $p$ and $q$ be chosen so that $\alpha \sigma p > 1$, $(\tau + \kappa)q > 1$, $p^{-1} + q^{-1} = 1$, $p > 1$; it is easy to verify that this is possible.
Then, using the inequality $\kappa > 0$, and Hölder’s inequality, we obtain
\begin{equation}
I_2 \leq \int_{U \setminus V} |f|^p |F|^\tau + \kappa \, d\theta \leq \left(\int_{U \setminus V} |f|^p \right)^{1/p} \left(\int_{0}^{2\pi} |F|^{\tau + \kappa} q \right)^{1/q},
\end{equation}
where $U \equiv U(r)$, $V \equiv V(r, x)$. With $I(\lambda) = \int_{U \setminus V} |f|^\lambda$, and use of Hayman’s well-
known estimate [1], applicable here since $(\tau + \kappa)q > 1$, this becomes
\begin{equation}
I_2 = O(1)(1 - r)^{(1/q) - \tau - \kappa}(I(\alpha \sigma + 1))^{1/p}.
\end{equation}
To deal with $I$ we put
\begin{equation}
g(z) = z(1 - ze^{-i\phi})^{\frac{1}{2}(\alpha \lambda + 1)}(f(z)/z)^{\lambda} \quad (z \in D \setminus \{0\})
\end{equation}
where $\alpha \lambda > 1$, and write
\begin{equation}
r^{1-\lambda}I(\lambda) = \int_{U \setminus V} |g(re^{i\theta})| \, d\theta = \int_{\psi - i(r)} \phi + i(r) \, d\theta = J_1 + J_2.
\end{equation}
On \(\{re^{i\theta}\} \) \(g\) has radial order \(1/2(\alpha\lambda - 1) > 0\), so the lemmas of \(\S 2\) apply. From Lemma 2.4(ii) and (2.15), we see that, for \(r\) near one, \(|g(\theta)|\) increases throughout the interval \((\varphi - x(1 - r), \varphi - x(1 - r))\). So on applying Lemma 2.3 to \(|g(\theta)|\) for \(\theta \in (\varphi - l(1 - r), \varphi - x(1 - r))\) we have \(r > r_0(\xi)\) that in \((\varphi - l(1 - r), \varphi - x(1 - r))\), \(|g(\theta)| < |g(\theta - x(1 - r))|/\). This result applied to \(J_1\), and a similar one applied to \(J_2\), give for \(r > r_0(\xi)\)

\[
r_1^{-\lambda}I(\lambda) \leq \max_{\lambda}\{g(re^{i(\varphi - x(1 - r))})\} \int_{\lambda \setminus V} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta.
\]

Now using Lemma 2.4(iii) we obtain, for \(r > r_0(\xi)\),

\[
I(\lambda) < |g(re^{i\varphi})| \int_{\lambda \setminus V} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta.
\]

For \(\epsilon > 0\), \(x\) suitably large, and \(r\) near one, it is easy to prove that

\[
\int_{\lambda \setminus V} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta < \epsilon \int_{\lambda \setminus V} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta
\]

by making the substitution \(\theta = \varphi = t(1 - r)\) and using \(\alpha\lambda > 1\). Moreover, for any given positive \(x\), and \(r\) near one,

\[
\int_{\lambda \setminus V} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta < \int_{0}^{2\pi} |1 - re^{i(\varphi - \varphi)}|^{-1/2(\alpha\lambda + 1)} d\theta
\]

< \(A(1 - r)^{-1/2(\alpha\lambda - 1)}\)

where \(A\) denotes an absolute constant. So when \(x > x_0(\epsilon)\) and \(r > r_0(\epsilon)\) we have

\[
I(\lambda) < A(1 - r)^{1 - \tau - \kappa} |f(re^{i\varphi})|^{\sigma}.
\]

(3.7)

\[
I(\lambda) < A(1 - r)^{1 - \tau - \kappa} |f(re^{i\varphi})|^{\sigma}.
\]

(3.8)

To prove (3.8) when \(\tau + \kappa = 0\) we first deduce \(I_2 < I(\sigma)\) using \(\kappa > 0\), and then note that (3.7) applies with \(\lambda = \sigma\) since \(\alpha\sigma = \alpha\tau + \tau + \kappa > 1\). Now using (3.4), (3.5) and (3.8) we easily obtain (3.3).

3.2. We have to prove that for \(\alpha\sigma + \tau + \kappa > 1\), \(\tau + \kappa > 0\), \(\kappa > 0\), \(\sigma > 0\),

\[
\int_{T(\tau)} |f(re^{i\varphi})|^{\sigma} |F(re^{i\varphi})|^\tau (Re F(re^{i\varphi}))^\kappa d\theta = o(1)(1 - r)^{-1 - \alpha\sigma - \tau - \kappa}.
\]

(3.9)

Assume \(\tau + \kappa > 0\) so that \(p\) and \(q\) exist satisfying \(\alpha\sigma p > 1\), \((\tau + \kappa)q > 1\), \(p^{-1} + q^{-1} = 1\), \(p > 1\). Then, since \(\kappa > 0\), we have by Hölder's inequality

\[
\int_{T(\tau)} |f|^\sigma |F|^\tau (Re F)^\kappa \leq \left(\int_{T(\tau)} |f|^\sigma p \right)^{1/p} \left(\int_{0}^{2\pi} |F|^{(\tau + \kappa)q} \right)^{1/q}
\]

(3.10)

\[
= o(1)(1 - r)^{1/q - \tau - \kappa} \left(\int_{T(\tau)} |f|^\sigma p \right)^{1/p},
\]

where \(A\) denotes an absolute constant. So when \(x > x_0(\epsilon)\) and \(r > r_0(\epsilon)\) we have
where we have used Hayman's result [1], applicable since \((\tau + \kappa)q > 1\).

We now let \(\lambda = \varphi_0\), so that \(\alpha \lambda > 1\), and consider \(\int_{T(r)} |f|^{\lambda} d\theta\). Denote by \(\varphi_1, \ldots, \varphi_N, \ldots\) the sequence of \(\varphi\) for which on \(\{re^{i\varphi}\}\) the radial order of \(f\) is positive, and by \(\alpha_k\) the radial order of \(f\) on \(\{re^{i\varphi_k}\}\). Then, since \(\alpha \lambda > 1\), we define a star-powered function \(g\) by

\[
g(z) = z \left( \frac{f(z)}{z} \right)^{\lambda} \prod_{0 < \gamma_k < 1} \left( 1 - ze^{-i\varphi_k} \right)^{\gamma_k + \xi}
\]

for \(z \in D \setminus \{0\}\)

where \(\gamma_k = \alpha_k \lambda - \alpha \lambda + 1\), \(\xi = \min(\alpha \lambda - 1, 1 - \gamma_k)\).

Any radial order of \(g\) less than the order is also less than \(\alpha \lambda - 1\). Thus, by Lemma 2.2, for some positive \(\delta\),

\[
\sup_{\theta \in T(r)} |g(re^{i\theta})| = O(1)(1 - r)^{-\alpha \lambda + \delta}.
\]

Hence

\[
\int_{T(r)} |f|^\lambda d\theta = O(1)(1 - r)^{-\alpha \lambda + \delta} \int_0^{2\pi} \prod_{0 < \gamma_k < 1} |1 - re^{i(\theta - \varphi_k)}|^{-\gamma_k - \xi} d\theta
\]

\[
= O(1)(1 - r)^{-\alpha \lambda + \delta} \int_0^{2\pi} |1 - re^{i\theta}|^{-\gamma_k} d\theta,
\]

since \(\gamma_k + \xi \leq 1\), and the points \(\varphi_k\) for which \(0 < \gamma_k < 1\) are distinct and finite in number. The last integral is \(O(\log 1/(1 - r))\), so

\[
(3.11) \quad \int_{T(r)} |f|^\lambda d\theta = o(1)(1 - r)^{-\alpha \lambda}
\]

and (3.9) follows easily from this (with \(\lambda = \varphi_0\)) and (3.10). To prove (3.9) when \(\tau + \kappa = 0\), we first use \(\kappa > 0\) to deduce

\[
\int_{T(r)} |f|^\sigma |F|^\tau |(Re F)^\lambda| d\theta \leq \int_{T(r)} |f|^\sigma d\theta,
\]

and then note that (3.11) applies with \(\lambda = \sigma\), since \(\alpha \sigma = \alpha \sigma + \tau + \kappa > 1\).

4. Proof of a conjecture by Holland and Thomas. With the aid of a corollary to Theorem 1, we can outline a proof of the conjecture (1.10) in the case \(\alpha > 0\), that is (in the notation of (1.10))

\[
(4.1) \quad \lim_{r \to 1} \frac{(1 - r)J'(r, \lambda)}{J(r, \lambda)} = (1 + \alpha)\lambda - 1, \quad \text{for } \alpha > 0, (1 + \alpha)\lambda > 1
\]

where

\[
J(r, \lambda) = \frac{1}{2\pi r^\lambda} \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda d\theta.
\]

In proving a similar result for \(I\) (also conjectured by Holland and Thomas), we used Theorem 1 to represent both \(I\) and \(I'\). The same approach will suffice for \(J\) but not for \(J'\) since we have

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(4.3) \[ J'(r, \lambda) = \frac{\lambda}{2\pi r^\lambda + 1} \int_0^{2\pi} |f(re^{i\theta})F(re^{i\theta})|^\lambda (\text{Re} F(re^{i\theta}) + \text{Re} F(re^{i\theta}) - 1) \, d\theta \]

where

\[ F(z) = zF'(z)/F(z) \quad (z \in D). \]

However, when \((1 + \alpha)\lambda > 1\), a representation of \(J'\) can be squeezed out from the proof of Theorem 1.

The following results on \(F\) are needed:

(i) If \(f\) has positive radial order on \(\{re^{i\theta}\}\)

\[ \text{Re} F(re^{i\theta}) \approx \left(1 + \left(\frac{\theta - \phi}{1 - r}\right)^2\right)^{\alpha/2} (1 - r)^{-1} \]

uniformly when \(0 < \phi < O(1)(1 - r)\).

(ii) \[ |F(z)| \ll 2r/(1 - r^2), \quad (|z| = r). \]

(i) follows from an argument similar to that in Lemma 2.4(i); for (ii) see [3]. Next we have the

**Corollary to Theorem 1.** Let \(f, \varphi_1, \ldots, \varphi_N\) be given as in Theorem 1, and suppose that \(\sigma > 0, \tau \geq 0, \alpha \sigma + \tau > 1\). If \(\Phi\) is any real function on \(D\) for which

(i) \[ \Phi(re^{i\theta}) \sim \left(1 + \left(\frac{\theta - \phi_k}{1 - r}\right)^2\right)^{\alpha/2} (1 - r)^{-1} \]

uniformly when \(|\theta - \phi_k| < O(1)(1 - r)\), and

(ii) \[ |\Phi(z)| = O(1)(1 - r)^{-1} \quad (|z| = r), \]

then

\[ \int_0^{2\pi} |f(re^{i\theta})|^\sigma |F(re^{i\theta})|^{\tau} \Phi(re^{i\theta}) \, d\theta \sim \alpha^\tau C(\alpha \sigma + \tau + 1)(1 - r)^{-\tau} \sum_{k=1}^N |f(re^{i\theta})|^\sigma. \]

Clearly we may take \(\Phi = \text{Re} F\) in the corollary. In this way we find a representation for

\[ \int_0^{2\pi} |f(re^{i\theta})|^\lambda F(re^{i\theta}) |\text{Re} F(re^{i\theta})| \, d\theta \]

when \(\alpha > 0, (1 + \alpha)\lambda > 1\). Theorem 1 supplies representations for

\[ \int_0^{2\pi} |f(re^{i\theta})|^\lambda F(re^{i\theta}) |\text{Re} F(re^{i\theta})| \, d\theta \]

when \(\alpha > 0, \lambda > 0\), and for \(J(r, \lambda)\) when \(\alpha > 0, (1 + \alpha)\lambda > 1\). With these results it is not difficult to obtain (4.1) via (4.2) and (4.3).

**REFERENCES**


DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY COLLEGE OF SWANSEA, SWANSEA SA 2 8PP, WALES