THE QUASI-ORBIT SPACE OF CONTINUOUS C*-DYNAMICAL SYSTEMS

BY

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ABSTRACT. Let \((A, G, \alpha)\) be a separable continuous C*-dynamical system. Suppose \(G\) is amenable and \(\alpha\) is free on the dual \(\hat{A}\) of \(A\). Then the quasi-orbit space \((\text{Prim } A/\alpha)\sim\) of the primitive ideal space \(\text{Prim } A\) of \(A\) by \(\alpha\) is homeomorphic to the induced primitive ideal space which is dense in the primitive ideal space \(\text{Prim } C^*(A; \alpha)\) of the C*-crossed product \(C^*(A; \alpha)\) of \(A\) by \(\alpha\).

1. Introduction. The theory of crossed products of operator algebras has been developed by a number of people since von Neumann constructed the examples of factors. Among its various studies, Effros-Hahn [3] topologically characterized under certain conditions the primitive ideal space of C*-crossed products as the quasi-orbit space of transformation groups for separable abelian discrete C*-dynamical systems. Succeedingly, Zeller-Meier [7] generalized their result in separable discrete C*-dynamical systems. However, several different aspects come out in the continuous case. For instance, the original C*-algebra is never imbedded in its crossed product as a sub C*-algebra, therefore it has no associated conditional expectations from the crossed product. Moreover, it is unclear that any element in crossed products can be described as a Fourier expansion in certain fashion.

In this paper, we shall discuss the quasi-orbit space of separable continuous C*-dynamical systems, specifically, given a separable continuous C*-dynamical system \((A, G, \alpha)\). Suppose \(G\) is amenable and \(\alpha\) is free on the dual \(\hat{A}\) of \(A\). Then the quasi-orbit space \((\text{Prim } A/\alpha)\sim\) of the primitive ideal space \(\text{Prim } A\) of \(A\) by \(\alpha\) is homeomorphic to the induced primitive ideal space \(\{(\text{Ind } \rho)^{-1}(0)\}_{\rho \in \hat{A}}\) which is dense in the primitive ideal space \(\text{Prim } C^*(A, \alpha)\) of the C*-crossed product \(C^*(A, \alpha)\) of \(A\) by \(\alpha\). This can be considered as a generalization of Zeller-Meier's result.

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2. Preliminaries. According to Effros-Hahn [3], Dang-Ngoc-Guichardet [2] Takai [5], [6], and Zeller-Meier [7], we shall briefly review several basic notions and fundamental results which will be used later. A triple \((A, G, \alpha)\) of a \(C^*\)-algebra \(A\), a locally compact group \(G\), and a \(*\)-homomorphism \(\alpha\) of \(G\) into \(\text{Aut}(A)\) is called a **continuous \(C^*\)-dynamical system** if \(\alpha\) is pointwise norm continuous, where \(\text{Aut}(A)\) is the set of all \(*\)-automorphisms of \(A\). It is said to be **separable** if \(A\) and \(G\) are separable. Moreover, the action \(\alpha\) is said to be **free** on a subset \(S\) of the representation space \(\text{Rep} A\) of \(A\) if given a \(g \neq e \in G\), and a \(\rho \neq 0 \in S\), any subrepresentation \(\rho' \neq 0\) of \(\rho\) dominates a representation \(\rho'' \neq 0\) with \(\alpha_g \cdot \rho'' \downarrow \rho''\), where \((\alpha_g \cdot \rho'')(a) = \rho'' \circ \alpha_g^{-1}(a)\) for \(a \in A\), \(g \in G\). Let \(\text{Fac} A\) be the set of all factor representations of \(A\). Since any nonzero subrepresentation of \(\rho\) is quasi-equivalent to \(\rho\) if \(\rho \in \text{Fac} A\), freeness of \(\alpha\) on \(S\) for a subset \(S\) of \(\text{Fac} A\) is equivalent to saying that \(\alpha_g \cdot \rho \downarrow \rho\) for every \(\rho \neq 0 \in S\). Furthermore, due to the fact that freeness is congruent with quasi-equivalence (resp. equivalence), it can be defined on the quasi-dual \(\hat{A}\) (resp. the dual \(\hat{A}\)) of \(A\).

Now let \((A, G, \alpha)\) be a continuous \(C^*\)-dynamical system. According to Effros-Hahn [3], and Zeller-Meier [7], the quasi-orbit space \((\text{Prim} A/\alpha)\) of the primitive ideal space \(\text{Prim} A\) of \(A\) by \(\alpha\) is defined by the set of all orbit closures in \(\text{Prim} A\). It can be identified with the quotient of \(\text{Prim} A\) by the following equivalence relation: \(p \sim q\) if and only if \((G \cdot p) = (G \cdot q)\), where \((G \cdot p)\) is the closure of the orbit of \(p \in \text{Prim} A\) by \(\alpha\). Then, it is easily seen that \((\text{Prim} A/\alpha)\) is a \(T_0\)-space with respect to the quotient topology of \(\text{Prim} A\). Define a Borel structure on \((\text{Prim} A/\alpha)\) by the quotient map of \(\text{Prim} A\). Then, it is saturated with the quotient topology. Suppose \(A\) is separable, each element of \((\text{Prim} A/\alpha)\) is Borel since \((\text{Prim} A/\alpha)\) is a \(T_0\)-space (cf. [3]).

The following lemma has the key role in finding a mapping from the primitive ideal space \(\text{Prim} C^*(A; \alpha)\) of the crossed product \(C^*(A; \alpha)\) into \((\text{Prim} A/\alpha)\) :

**Lemma 2.1.** Let \((A, G, \alpha)\) be a separable continuous \(C^*\)-dynamical system. Suppose there exists a projection valued measure on \((\text{Prim} A/\alpha)\) whose values are zero or identity operators. Then it is concentrated in one point (cf. [3]). Notice that it is unnecessary to assume the separability of \(G\) to show the above lemma.

Next, we shall define the \(C^*\)-crossed product (resp. the reduced \(C^*\)-crossed product) \(C^*(A; \alpha)\) (resp. \(C^*_{\text{red}}(A; \alpha)\)) of \(C^*\)-algebra \(A\) by a continuous action \(\alpha\) of a locally compact group as follows: Let \(L^1(A; G)\) be the set of all \(A\)-valued Bochner integrable measurable functions on \(G\) with respect to the left Haar measure.
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\( \alpha_g \) of \( G \) with the \(*\)-algebraic structure given by

\[
(xy)(g) = \int_G x(h) \alpha_h[y(h^{-1}g)] \, dh
\]

and

\[
x^*(g) = \Delta(g)^{-1} \alpha_g[x(g^{-1})]^*,
\]

where \( \Delta \) is the modular function of \( G \). Then, \( C^*(A; \alpha) \) (resp. \( C^r_*(A; \alpha) \)) is the completion of \( L^1(A; \alpha) \) with respect to the \( C^*\)-norm \( \| \cdot \| \) (resp. \( \| \cdot \|_* \)) defined by

\[
\|x\| = \sup \{\|\pi(x)\| : \pi \in \Rep L^1(A; G)\}
\]

and

\[
\|x\|_* = \sup \{\|\Ind \rho(x)\| : \rho \in \Rep A\},
\]

where \( \Ind \rho \) is the representation of \( C^*(A; \alpha) \) induced by \( \rho \) (cf. [5], [6]). It is clear that \( C^r_*(A; \alpha) \) is a quotient algebra of \( C^*(A; \alpha) \). Let us define a positive linear functional \( \tilde{\varphi}_f \) on \( C^r_*(A; \alpha) \) determined by

\[
\tilde{\varphi}_f(x) = \int G \times G f(g^{-1}h) \overline{f(h)} \varphi \circ \alpha_h^{-1}[x(g)] \, dg \, dh
\]

for \( x \in L^1(A; G) \), where \( \varphi \) is in the set \( (A^*)_+ \) of all positive linear functionals on \( A \), and \( f \) is in the \(*\)-algebra \( K(G) \) of all continuous functions on \( G \) with compact support. Then, one can estimate \( \|\Ind \rho(x)\| \) by \( \tilde{\varphi}_f \) as follows:

\[
\|\Ind \rho(x)\| = \sup \left\{ \left( \tilde{\varphi}_f(\psi \star x \star y) \right)^{1/2} : y \in L^1(A; G), f \in K(G), \varphi \in \Omega(y; f) \cap W(\rho) \right\}
\]

where \( \Omega(y; f) = \{ \varphi \in (A^*)_+ : \tilde{\varphi}_f(\psi \star y) \neq 0 \} \), and \( w(\rho) = \{ \varphi \in (A^*)_+ : \pi_\varphi \in \Rep A \text{ is weakly contained in } \rho \} \) (cf. [5]).

We now state a sufficient condition under which \( C^*(A; \alpha) \) is equal to \( C^r_*(A; \alpha) \).

**Lemma 2.2.** Let \((A, G, \alpha)\) be a continuous \( C^*\)-dynamical system. Suppose \( G \) is amenable as a topological group, then we have

\[
C^*(A; \alpha) = C^r_*(A; \alpha) \quad \text{(cf. [5], [6])}.
\]

Moreover, we present a condition under which \( \Ind \rho \) is faithful on \( C^r_*(A; \alpha) \).

**Lemma 2.3.** Let \((A, G, \alpha)\) be as in the preceding, and \( \rho \) in \( \Rep A \). Then the following conditions are equivalent:

(I) \( \Sigma_{g \in G} \alpha_g \cdot \rho \) is faithful on \( A \),

(II) \( \Ind \rho \) is faithful on \( C^r_*(A; \alpha) \) (cf. [5]).
By the above lemma, one knows without difficulty that faithfulness of \( \rho \) implies that of \( \text{Ind} \, \rho \).

Finally, we shall write down a couple of special notations for later use. Let \((A, G, \alpha)\) be a continuous \(C^*\)-dynamical system. Then given \(X \in C^*(A; \alpha), a \in A\), there exists an element \(aX \in C^*(A; \alpha)\), with \((aX)(g) = aX(g)\) if \(X \in L^1(A; G)\). Let \(I\) be an ideal of \(C^*(A; \alpha)\) where "ideal" involves closed and two-sided throughout this paper. Denote by \([A : I]\) the set of all elements \(a\) in \(A\) such that \(aX \in I\) for all \(X \in C^*(A; \alpha)\). It is not evident whether \([A : I]\) is an ideal of \(A\). However, the following observation tells us that it is affirmative:

**Lemma 2.4.** Let \((A, G, \alpha)\) be a continuous \(C^*\)-dynamical system, and \(\pi \in \text{Rep} \, C^*(A; \alpha)\). Suppose \(\rho\) is in \(\text{Cov-rep} \, A\) corresponding to \(\pi\), then we have

\[
[A : \pi^{-1}(0)] = \rho^{-1}(0) \quad (\text{cf. [3]})
\]

where \(\text{Cov-rep} \, A\) is the set of all covariant representations of \(A\), and \(\pi^{-1}(0)\) (resp. \(\rho^{-1}(0)\)) is the kernel of \(\pi\) (resp. \(\rho\)). From this fact, \([A : I]\) for an ideal \(I\) of \(C^*(A; \alpha)\) is an invariant ideal of \(A\) under the action \(\alpha\). (It is called \(\alpha\)-invariant.)

3. The quasi-orbit space of continuous \(C^*\)-dynamical systems. In this section, we shall discuss the topological relation between the quasi-orbit space of continuous \(C^*\)-dynamical systems and the primitive ideal space of \(C^*\)-crossed products under certain conditions. Let \((A, G, \alpha)\) be a separable continuous \(C^*\)-dynamical system. Given a \(\pi\) in \(\text{Fac} \, C^*(A; \alpha)\), there exists a unique \(\rho\) in \(\text{Cov-rep} \, A\) corresponding to \(\pi\). Due to Glimm [4], there exists a unique projection valued measure \(\mu_\rho\) on the Borel structure of \(\text{Prim} \, A\) such that

\[
(3.1) \quad \mu_\rho(\overline{U}) = \text{Proj}[\rho(k(\overline{U}^c))] \mathfrak{h}_\rho
\]

for every open set \(\overline{U}\) in \(\text{Prim} \, A\), where \(\text{Proj}\) means projection and \(\overline{U}^c\) is the complement of \(\overline{U}\) in \(\text{Prim} \, A\), and \(k(\overline{U}^c)\) is the kernel of \(\overline{U}^c\). Moreover, the range of \(\mu_\rho\) is contained in the center \(z_\rho\) of the von Neumann algebra \(\rho(A)^-\) generated by \(\rho(A)\). Since \(\pi\) is primary and \(\rho\) is covariant, the measure \(\mu_\rho\) is ergodic. Namely, the values of \(\alpha\)-invariant Borel sets under \(\mu_\rho\) are zero or identity operators on \(\mathfrak{h}_\rho\). Let \(\eta\) be the natural mapping from \(\text{Prim} \, A\) onto \((\text{Prim} \, A/\alpha)^-\), and let \(\nu_\rho\) be the image of \(\mu_\rho\) by \(\eta\). Then the range of \(\nu_\rho\) is zero or identity operators on \(\mathfrak{h}_\rho\).

By Lemma 2.1, there exists an element \(\eta(p)\) in \((\text{Prim} \, A/\alpha)^-\) at which \(\nu_\rho\) is concentrated. We claim that \((G \cdot p)^-\) is the support \(\text{Supp} \, \mu_\rho\) of \(\mu_\rho\). In fact, since \(\eta^{-1} \circ \eta(p) \subset (G \cdot p)^-\), it follows that \(\text{Supp} \, \mu_\rho \subset (G \cdot p)^-\). By the fact that \(\text{Supp} \, \mu_\rho\) is \(\alpha\)-invariant closed in \(\text{Prim} \, A\), it contains \((G \cdot p)^-\), or must be disjoint from \(\eta^{-1} \circ \eta(P)\). However, the latter case does not occur since \(\mu_\rho\) is concentrated in \(\eta^{-1} \circ \eta(P)\).

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Consequently, one gets \( \text{Supp} \, \mu_p = (G \cdot p)^{-} \). Moreover, it can be verified that \( \text{Supp} \, \mu_p = h(\rho^{-1}(0)) \) where \( h(\rho^{-1}(0)) \) is the hull of \( \rho^{-1}(0) \). Actually, it is clear by definition that \( h(\rho^{-1}(0))^c \) is an open null set with respect to \( \mu_p \). Therefore, \( \text{Supp} \, \mu_p \subset h(\rho^{-1}(0)) \). Suppose \( \text{Supp} \, \mu_p \) is not equal to \( h(\rho^{-1}(0)) \). Then, there exists an open neighborhood \( U \) of an element \( q \) in \( h(\rho^{-1}(0)) \) from which \( \text{Supp} \, \mu_p \) is disjoint. Hence, it follows that \( k(U^c) \subset \rho^{-1}(0) \). Taking hull, one obtains that \( U \) is disjoint from \( h(\rho^{-1}(0)) \), which is a contradiction. Summing up the argument discussed above, we have the following lemma:

**Lemma 3.1.** Let \((A, G, \alpha)\) be a separable continuous C*-dynamical system, and let \( \pi \) be in \( \text{Fac} \, C^*(A; \alpha) \). Suppose \( \rho \) is in \( \text{Cov}-\text{rep} \, A \) corresponding to \( \pi \). Then there exist a unique (up to equivalence) projection valued Borel measure \( \mu_p \) on \( \text{Prim} \, A \), and an element \( P \) in \( \text{Prim} \, A \) such that

\[
\text{Supp} \, \mu_p = (G \cdot p)^{-} = h(\rho^{-1}(0)).
\]

**Remark.** The measure \( \mu_p \) also can be realized in the following way: Let \( \rho = \int_A \rho_x \, d\nu(x) \) be the central decomposition of \( \rho \). Using the canonical mapping \( \phi \) from \( A \) onto \( \text{Prim} \, A \), one gets the image \( \mu \) of \( \nu \) under \( \phi \). It is nothing but \( \mu_p \) up to equivalence.

By Lemmas 2.4 and 3.1, we can define a mapping \( \Phi \) of \( \text{Prim} \, C^*(A; \alpha) \) into \( (\text{Prim} \, A/\alpha)^- \) by

\[
\Phi(\Psi) = h([A : \Psi]).
\]

Then, we have the following:

**Lemma 3.2.** The mapping \( \Phi \) defined above is continuous from \( \text{Prim} \, C^*(A; \alpha) \) with the hull kernel topology into \( (\text{Prim} \, A/\alpha)^- \) with the quotient topology.

**Proof.** Though the proof goes on as in the abelian case, we shall trace it for completeness. Given a closed set \( F \) in \( (\text{Prim} \, A/\alpha)^- \), it suffices to show that there exists a subset \( I \) of \( C^*(A; \alpha) \) such that \( \Phi^{-1}(F) = h(I) \).

Define a set \( I \) by \( \{ax : a \in k(F), x \in C^*(A; \alpha)\} \) where \( F = \eta^{-1}(F) \). Then, this set enjoys the property cited above. In fact, for any \( \Psi \in \Phi^{-1}(F) \), there exists a \( \pi \) in \( \text{Irr} \, C^*(A; \alpha) \) whose kernel is \( \Psi \). Let \( \rho \) be in \( \text{Cov}-\text{rep} \, A \) corresponding to \( \pi \). By Lemma 3.1, there exists a \( P \) in \( \text{Prim} \, A \) such that \( h([A : \Psi]) = (G \cdot p)^{-} \). Since \( \Phi(\Psi) \in F \), it follows by (3.2) that \( p \in \eta^{-1}(F) = F_1 \). Since \( F_1 \) is \( \alpha \)-invariant closed in \( \text{Prim} \, A \), it contains \( (Gp)^{-} \), which means that \( h([A : \Psi]) = F_1 \). Taking the kernel, one gets that \( k(F_1) \subset [A : \Psi] = \rho^{-1}(0) \). Since \( \pi(ax) = \rho(a)\pi(x) \) for \( a \in A \) and \( x \in C^*(A; \alpha) \), it is verified that \( I \subset \pi^{-1}(0) = \Psi \). Thus, one obtains that \( \Psi \in h(I) \). Since one can trace the above argument back until its starting point, one gets the conclusion. Q.E.D.
From now on, let us assume that the action $\alpha$ is free on $\hat{A}$. Then, the next lemma guarantees that $\Phi$ is surjective:

**Lemma 3.3.** Suppose $\alpha$ is free on $\hat{A}$. Then, one has for every $\rho \in \text{Irr } A$,

(i) $\text{Ind } \rho \in \text{Irr } \text{C}^{*}(A; \alpha)$, and

(ii) $\Phi[(\text{Ind } \rho)^{-1}(0)] = (G \cdot \rho^{-1}(0))^{-}$, 

where $\text{Irr}$ means the set of all irreducible representations.

**Proof.** (i) Let $G_{\rho}$ be the stabilizer of the equivalence class $\hat{\rho} \in \hat{A}$ of $\rho$, and let $(\hat{G}, \lambda)$ be in Cov-rep$(A, G)$ corresponding to Ind $\rho$. Let $\hat{U}$ be a strongly measurable cocycle representation of $G_{\rho}$ such that $\hat{U}_{g} \rho(a) \hat{U}_{g} = \alpha_{g} \cdot \rho(a)$ for $a \in A$, $g \in G_{\rho}$. Define $\hat{U}_{g}, g \in G_{\rho}$, by

$$
(\hat{U}_{g} \xi)(h) = (\Delta(\rho))^{\frac{1}{2}} \hat{U}_{g} \xi(hg) \quad \text{for } \xi \in L^{2}(\hat{G}; G), \ h \in G.
$$

Due to Busby-Smith [1], it can be shown that the commutant $\hat{\rho}(A)'$ of $\hat{\rho}$ is generated by the $\hat{U}_{g}, g \in G_{\rho}$, and the diagonalizable algebra. Since $\alpha$ is free on $\hat{A}$, it follows that $\hat{\rho}(A)'$ is the diagonal algebra. Since the only diagonals which commute with $\lambda$ are the constant, one gets the statement (i).

(ii) By definition, it suffices to show that $h(\hat{\rho}^{-1}(0)) = (G \cdot \rho^{-1}(0))^{-}$. Since $\hat{\rho}^{-1}(0) = \bigcap_{g \in G} (\alpha_{g} \cdot \rho)^{-1}(0)$, it follows that

$$
h(\hat{\rho}^{-1}(0)) = h\left(\bigcap_{g \in G} (\alpha_{g} \cdot \rho)^{-1}(0)\right) \supset \left(\bigcup_{g \in G} h[(\alpha_{g} \cdot \rho)^{-1}(0)]\right)^{-},
$$

which implies that $(G \cdot \rho^{-1}(0))^{-} \subset h(\hat{\rho}^{-1}(0))$. On the other hand, let $\mu_{\hat{\rho}}$ be the Borel measure determined by $\hat{\rho}$ as (3.1). Then one gets by simple computation that $\text{Supp } \mu_{\hat{\rho}} \subset (G \cdot \rho^{-1}(0))^{-}$. Thus, the statement (ii) follows by Lemma 3.1.

Q.E.D.

In general, it is not clear whether the mapping $\Phi$ is injective. However, one can get a partial answer under the amenability of $G$. Let $E$ be the set of all $(\text{Ind } \rho)^{-1}(0), \rho \in \text{Irr } A$. Then the following lemma tells us that $\Phi$ is injective on $E$:

**Lemma 3.4.** Let $(A, G, \alpha)$ be a separable continuous $C^{*}$-dynamical system, and let $\alpha$ be free on $\hat{A}$. Suppose $G$ is amenable as a topological group. Then one has for every $\rho \in \text{Irr } A$,

$$(\text{Ind } \rho)^{-1}(0) = \text{C}^{*}(\hat{\rho}^{-1}(0); \alpha)$$

where $\hat{\rho} \in \text{Cov-rep } A$ corresponding to Ind $\rho$.

**Proof.** By definition, one easily gets that $(\text{Ind } \rho)^{-1}(0) \supset \text{C}^{*}(\hat{\rho}^{-1}(0); \alpha)$. To show the converse inclusion, one first observes that $\text{C}^{*}(\hat{\rho}^{-1}(0); \alpha)$ is a primitive ideal of $\text{C}^{*}(A; \alpha)$. In fact, one estimates by Lemmas 2.2 and 2.3 that
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\[ C^*(A; \alpha)/(\text{Ind } \rho)^{-1}(0) \cong (\text{Ind } \rho)[C^*(A; \alpha)] \]

\[ \cong (\text{Ind } \iota)[C^r_\rho(\bar{\rho}(A); \bar{\alpha})] \cong C^*(\bar{\rho}(A); \bar{\alpha}) = C^*(\bar{\rho}(A); \bar{\alpha}) \]

\[ \cong C^*(A/\bar{\rho}^{-1}(0); \bar{\alpha}) \cong C^*(A; \alpha)/C^*(\rho^{-1}(0); \alpha), \]

where \( \bar{\alpha} \) (resp. \( \bar{\alpha} \)) is the continuous action of \( G \) on \( \bar{\rho}(A) \) (resp \( A/\bar{\rho}^{-1}(0) \)) determined by \( \bar{\rho} \) (resp. the natural map of \( A \) onto \( A/\rho^{-1}(0) \)), and \( \iota \) is the trivial automorphism of \( \bar{\rho}(A) \). Therefore, there exists a \( \pi \) in \( \text{Irr } C^*(A; \alpha) \) with \( \pi^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha) \). Since every nonzero vector \( \xi \) in \( B_\pi \) is cyclic for \( \pi \), it suffices to show that \( \pi^{-1}(0) \supset (\text{Ind } \rho)^{-1}(0) \) for \( \pi \) corresponding to \( \omega_\iota \circ \pi \). Let \( (L, \mathcal{V}) \) be in \( \text{Cov-Rep}(A, G) \) associated with \( \pi \). Then, one can see by Lemma 2.4 and \( \pi^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha) \) that

\[ L^{-1}(0) = \{ A : \pi^{-1}(0) \} = \bar{\rho}^{-1}(0). \]

Since \( \bar{\rho}^{-1}(0) = \left( \sum_{g \in G} \alpha_g \cdot \rho \right)^{-1}(0) \), the equality (3.3) means that \( L \) is weakly equivalent to \( \sum_{g \in G} \alpha_g \cdot \rho \). On the other hand, the amenability of \( G \) gives us a sequence \( (f^g_n)_n \) of \( K(G) \) such that \( f^g_n \circ f^g_n \) converges to 1 with respect to compact open topology where \( f^g_n(g) = \frac{1}{n}(g^{-1}) \). Let us take a positive linear functional \( \Psi_n \) on \( C^*(A; \alpha) \) determined by \( \Psi_n(x) = \int_G (f^g_n \circ f^g_n)(g)(L[x(g)]\mathcal{V}(g) \xi \otimes dg \) for \( x \in L^1(A; G) \). Then, it can be shown by [5] that

\[ \Psi_n(y^*x^*xy) \leq \|y\|^2 \|f^g_n \otimes \xi \|^2 \|L(x)(\mathcal{V}(g) \xi)\|^2 \]

for every \( x, y \in C^*(A; \alpha) \), and \( n = 1, 2, \ldots \). Since \( L \) is weakly equivalent to \( \sum_{g \in G} \alpha_g \cdot \rho \), it follows by (2.1) that

\[ \|L(x)(\mathcal{V}(g) \xi)\| = \|L(x)(\mathcal{V}(g) \xi)\| = \|	ext{Ind } \rho(x)\|. \]

Thus, one obtains by (3.4) that

\[ \Psi_n(y^*x^*xy) \leq \|y\|^2 \|f^g_n \otimes \xi \|^2 \|\text{Ind } \rho(x)\|^2 \]

for every \( x, y \in C^*(A; \alpha) \) and \( n = 1, 2, \ldots \). Now suppose \( x \) is in \( (\text{Ind } \rho)^{-1}(0) \). Then, it follows that \( \Psi_n(y^*x^*xy) = 0 \) for all \( y \in C^*(A; \alpha) \) and \( n = 1, 2, \ldots \).
Since $\Psi_n$ converges to $w_{\xi} \circ \pi$ weakly, $w_{\xi} \circ \pi(y \cdot x \cdot xy) = 0$ for all $y \in C^*(A; \alpha)$, which means that $x \in \pi^{-1}_E(0)$. Q.E.D.

By the above lemma, the mapping $\Phi$ is bijective from $E$ onto $(\text{Prim } A/\alpha)^\sim$. Combining all the lemmas obtained above, we shall show the following theorem which is a generalization of the result which is obtained by Effros-Hahn, and Zeller-Meier (cf. [3], [7]):

**Theorem 3.5.** Let $(A, G, \alpha)$ be a separable continuous $C^*$-dynamical system. Suppose $G$ is amenable and $\alpha$ is free on the dual $\hat{A}$ of $A$. Then, the quasi-orbit space $(\text{Prim } A/\alpha)^\sim$ of the primitive ideal space Prim $A$ of $A$ by $\alpha$ is topologically imbedded in the primitive ideal space Prim $C^*(A; \alpha)$ of the $C^*$-crossed product $C^*(A; \alpha)$ of $A$ by $\alpha$.

**Proof.** Using the same notations denoted before, we shall show that $\Phi$ is a closed mapping from $E$ onto $(\text{Prim } A/\alpha)^\sim$. Let $F$ be a closed set in $E$. Define $F_0$ by $\eta^{-1} \Phi(F)$. It has to be shown that $F_0$ is closed in Prim $A$. Take a $p \in F_0$, and an $L \in \text{Prim } A$ with $L^{-1}(0) = p$. To prove $p \in F_0$, it suffices to show that

$$(3.6) \quad (\text{Ind } L)^{-1}(0) \supset \bigcap \{ (\text{Ind } \rho)^{-1}(0) : \text{Ind } \rho \in F \}.$$ 

Since $p \in F_0$, it follows that

$$L^{-1}(0) \supset \bigcap \{ \eta^{-1} [(G \cdot \rho^{-1}(0))] : \text{Ind } \rho \in F \}$$

$$\supset \bigcap \{ (\alpha_g \cdot \rho)^{-1}(0) : \text{Ind } \rho \in F, g \in G \},$$

which implies that $L$ is weakly contained in the set $\{ \alpha_g \cdot \rho : \text{Ind } \rho \in F, g \in G \}$. By the similar method used in the previous lemma, one has that

$$\| (\text{Ind } L)(x) \| \leq \sup \{ \| (\text{Ind } \alpha_g \cdot \rho)(x) \| : \text{Ind } \rho \in F, g \in G \}$$

$$= \sup \{ \| (\text{Ind } \rho)(x) \| : \text{Ind } \rho \in F \}$$

for all $x \in C^*(A; \alpha)$. Therefore, the inclusion (3.8) holds. Thus, the mapping $\Phi$ is a homeomorphism from $E$ onto $(\text{Prim } A/\alpha)^\sim$. Q.E.D.

Since $k(E) = (0)$, it follows by the above theorem that $(\text{Prim } A/\alpha)^\sim$ is dense in $\text{Prim } C^*(A; \alpha)$. Moreover, it is true that $\text{Prim } C^*(A; \alpha) = \bigcup \{ \mathcal{P} : \mathcal{P} \in E \}$. In fact, let $\pi^{-1}(0) \in \text{Prim } C^*(A; \alpha)$ where $\pi \in \text{Irr } C^*(A; \alpha)$. Taking $L \in \text{Cov-rep } A$ corresponding to $\pi$, there exists a $\rho \in \text{Irr } A$ such that $\text{Supp } \mu_L = (G \cdot \rho^{-1}(0))$. Then, it follows that $L^{-1}(0) = \tilde{\rho}^{-1}(0)$. By Lemma 3.4, one has $(\text{Ind } \rho)^{-1}(0) = C^*(\tilde{\rho}^{-1}(0); \alpha)$. Therefore, one obtains that $(\text{Ind } \rho)^{-1}(0) \subset \pi^{-1}(0)$, which means that $\pi^{-1}(0) \in ((\text{Ind } \rho)^{-1}(0))^{-1}$.

Summing up the argument, we get the following:

**Corollary 3.6.** Let $(A, G, \alpha)$ be as the preceding. Then $(\text{Prim } A/\alpha)^\sim$ is dense in $\text{Prim } C^*(A; \alpha)$ and
Prim $C^*(A; \alpha) = \bigcup \{ \Psi^- : \Psi \in (\text{Prim } A/\alpha)^{-}\}$. 

REFERENCES


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