FINITE GROUPS AS ISOMETRY GROUPS

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ABSTRACT. We show that given any finite group $G$ of cardinality $k + 1$, there is a Riemannian sphere $S^{k-1}$ (imbeddable isometrically as a hypersurface in $\mathbb{R}^k$) such that its full isometry group is isomorphic to $G$. We also show the existence of a finite metric space of cardinality $k(k + 1)$ whose full isometry group is isomorphic to $G$.

Let $G$ be a finite group of $k + 1$ elements $\{1, g_1, \ldots, g_k\}$.

THEOREM. There exists a Riemannian metric on the sphere $S^{k-1}$ such that the isometry group is isomorphic to $G$.

Proof. Label the $k + 1$ vertices of a regular $k$-simplex $\Delta_k$ by the names $1, g_1, \ldots, g_k$ of the elements of $G$. Assume $\Delta_k$ to be inscribed in a standard $S^{k-1}$ sitting in $\mathbb{R}^k$ as usual. $T_y(S^{k-1})$ denotes the tangent space at $y$.

Now in $T_1(S^{k-1})$ pick an orthonormal frame $(v_1, \ldots, v_{k-1})$. Pick $\varepsilon > 0$ small and let

$$w_i = \varepsilon(1 + (i - 1)/4k^2)v_i, \quad 1 \leq i \leq k - 1.$$ 

Let

$$Q = \{\exp_1(w_i) | 1 \leq i \leq k - 1\} \cup \{\exp_1(0)\} \cup \{w_1/10\}.$$ 

$\exp_1$ is the exponential map $\exp_1: T_1(S^{k-1}) \rightarrow S^{k-1}$.

Think of $G$ as acting on $S^{k-1}$ by the isometries induced from the permutation representation on the vertices of $\Delta_k$. Let $X = \{gQ | g \in G\}$.

PROPOSITION. With the induced metric from $\mathbb{R}^k$, the metric space $X$ has its group of isometries isomorphic to $G$.

Proof. Clearly $G$ acts as a group of isometries of $X$, since $X = h(gQ | g \in G) = \{hgQ | g \in G\} = \{gQ | g \in G\} = X$.

Conversely, any isometry of $X$ must take the point 1 to some point $g$, since the points $g$ are characterized by being the only points in $X$ having their
two nearest neighbors at distance of \( \epsilon/10 \) and \( \epsilon \) respectively. Once we know that \( 1 \mapsto g \), the configuration \( gQ \) determines the image of the frame \( (w_1, \ldots, w_{k-1}) \) at 1, and hence determines the unique isometry of \( X \) defined by the element \( g \in G \). Of course \( \epsilon \) must be chosen small enough so that the configurations \( gQ, g \in G \) do not "interfere" with one another.

Now we add bumps to \( S^{k-1} \) at the points of \( X \) using scalar multiplication in \( \mathbb{R}^k \). Let

\[
\delta = (1/3) \min \{ \text{dist}_{s^{k-1}}(x, y) | x, y \in X \}.
\]

Let \( f: [0, \delta] \rightarrow \mathbb{R} \) be a smooth function satisfying

(a) \( f(s) = 100, \ 0 < s < \delta/2 \),
(b) \( f(\delta) = 1; f^{(k)}(\delta) = 0, k = 1, 2, \ldots \),
(c) \( f^{(k)}(\delta/2) = 0, k = 1, 2, \ldots \), and
(d) \( f'(s) < 0 \) if \( \delta/2 < s < \delta \).

Now for each point \( x \in X \) we remove the disk \( \exp_x(D_\delta) \) from \( S^{k-1} \) and replace it by the point set \( B_x = \{ f(|v|) \exp_x(v) | v \in D_\delta \} \), where \( D_\delta \) is the \( (\delta) \)-disk about the origin of \( T_x(S^{k-1}) \). Clearly the set \( S^{k-1} - \bigcup_{x \in X} \exp_x(D_\delta) \bigcup \bigcup_{x \in X} B_x \) is a smooth \( S^{k-1} \) imbedded in \( \mathbb{R}^k \). We give it the induced Riemannian metric from \( \mathbb{R}^k \) and denote it by \( M \).

**Claim:** \( \text{Isom}(M) \approx G \).

**Proof.** First we notice that the points of \( 100 \cdot X \subset M \) must be taken to themselves by any isometry \( I \) of \( M \), by the choice of the function \( f \). Clearly the same arguments above for \( X \) hold for \( 100 \cdot X \), hence the isometry \( I: M \rightarrow M \) restricted to \( 100 \cdot X \) comes from the action of \( G \).

Let us now consider the "bump" \( B_1 \) above the point 1. Let us define for \( r \geq 0, \ S_r = \{ f(r) \cdot \exp_1(v) | |v| = r, v \in T_1(S^{k-1}) \} \). In other words, \( S_r \) is the \((k - 2)\)-sphere of \( B_1 \) lying above the \((k - 2)\)-sphere about 1 of radius \( r \), for \( 0 < r \leq \delta \), and for \( r = 0 \) we set \( S_0 = p \), the peak point of \( B_1 \).

Now it is easy to show that the orthogonal trajectories of the \( S_r \)'s are geodesics of \( M \) and as such must be preserved under any isometry taking \( p \) to \( p \).

Thus any isometry \( I \) of \( M \) which takes \( p \) to \( p \) (and which must thus leave all points of \( 100 \cdot X \) fixed) must be a "rotation" on all of \( B_1 \), determined by \( I \mid_{S_r} \), carrying each \( S_r \) into itself by the "same" element of \( O(k - 2) \). Similarly, this \( I \) must rotate each bump \( B_x, x \in X \).

Also this rotation must extend past the boundary of the bumps for some ways, so we can easily extend \( I \mid_{(M - \bigcup_x B_x)} \) to an isometry \( \tilde{I} \) of \( S^{k-1} \) to itself, by simply "coning" \( I \) over \( \exp_x(D_\delta), x \in X \). Clearly we will have \( \tilde{I}(x) = x \) for \( x \in X \), and it follows easily that \( \tilde{I}: S^{k-1} \rightarrow S^{k-1} \) is the identity. Hence \( I: M \rightarrow M \) must have been the identity.
Now it is clear that for each \( g \in G \) there is one isometry of \( M \) determined by the action of \( g \) on \( S^{k-1} \), extended to \( \mathbb{R}^k \), restricted to \( M \). Now if there is another isometry \( I : M \rightarrow M \) such that \( I \mid X = g \mid X \), then \( I \circ g^{-1} : M \rightarrow M \) must leave points of \( X \) fixed, so by the above discussion must be the identity. This establishes \( \text{Isom}(M) \cong G \).

**Corollary.** Any finite group \( G \) is isomorphic to the (full) isometry group of a finite subset \( X_G \) of euclidean space. If \( \text{card}(G) = k \) then the \( X_G \) can be found with \( \text{card}(X_G) = k^2 - k \) in euclidean space of dimension \( k - 1 \).

**Proof.** Simply take \( X_G = X \) in the proof of the Theorem, and count (noting that we initially took \( \text{card}(G) = k + 1 \)).

**Remark.** Further considerations can very likely reduce the necessary cardinality for \( X_G \) to \( k(k - 3) \). The various numbers
\[
d = \min\{\text{card}(X) \mid G \cong \text{Isom}(X)\} \quad \text{and} \quad e = \min\{N \mid G \text{ has a faithful representation into } O(N)\}
\]
seem to be interesting invariants of a finite group \( G \).