FINITE GROUPS AS ISOMETRY GROUPS

By

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ABSTRACT. We show that given any finite group $G$ of cardinality $k + 1$, there is a Riemannian sphere $S^{k-1}$ (imbeddable isometrically as a hypersurface in $R^k$) such that its full isometry group is isomorphic to $G$. We also show the existence of a finite metric space of cardinality $k(k + 1)$ whose full isometry group is isomorphic to $G$.

Let $G$ be a finite group of $k + 1$ elements $\{1, g_1, \ldots, g_k\}$.

THEOREM. There exists a Riemannian metric on the sphere $S^{k-1}$ such that the isometry group is isomorphic to $G$.

Proof. Label the $k + 1$ vertices of a regular $k$-simplex $\Delta_k$ by the names $1, g_1, \ldots, g_k$ of the elements of $G$. Assume $\Delta_k$ to be inscribed in a standard $S^{k-1}$ sitting in $R^k$ as usual. $T_y(S^{k-1})$ denotes the tangent space at $y$.

Now in $T_1(S^{k-1})$ pick an orthonormal frame $(v_1, \ldots, v_{k-1})$. Pick $\epsilon > 0$ small and let

$$w_i = \epsilon(1 + (i - 1)/4k^2)v_i, \quad 1 \leq i \leq k - 1.$$ 

Let

$$Q = \{\exp_1(w_i) \mid 1 \leq i \leq k - 1\} \cup \{\exp_1(0)\} \cup \{w_1/10\}.$$ 

$\exp_1$ is the exponential map $\exp_1 : T_1(S^{k-1}) \rightarrow S^{k-1}$.

Think of $G$ as acting on $S^{k-1}$ by the isometries induced from the permutation representation on the vertices of $\Delta_k$. Let $X = \{gQ \mid g \in G\}$.

PROPOSITION. With the induced metric from $R^k$, the metric space $X$ has its group of isometries isomorphic to $G$.

Proof. Clearly $G$ acts as a group of isometries of $X$, since $X = \{h(gQ) \mid g \in G\} = \{hgQ \mid g \in G\} = \{gQ \mid g \in G\} = X$.

Conversely, any isometry of $X$ must take the point 1 to some point $g$, since the points $g$ are characterized by being the only points in $X$ having their
two nearest neighbors at distance of $\epsilon/10$ and $\epsilon$ respectively. Once we know that $1 \mapsto g$, the configuration $gQ$ determines the image of the frame $(w_1, \ldots, w_{k-1})$ at 1, and hence determines the unique isometry of $X$ defined by the element $g \in G$. Of course $\epsilon$ must be chosen small enough so that the configurations $gQ,g \in G$ do not "interfere" with one another.

Now we add bumps to $S^{k-1}$ at the points of $X$ using scalar multiplication in $\mathbb{R}^k$. Let

$$\delta = (1/3)\min\{\text{dist}_{S^{k-1}}(x, y) | x, y \in X\}.$$ 

Let $f: [0, \delta] \rightarrow \mathbb{R}$ be a smooth function satisfying

(a) $f(s) = 100, 0 \leq s \leq \delta/2$,
(b) $f(\delta) = 1; f^{(k)}(\delta) = 0, k = 1, 2, \ldots$,
(c) $f^{(k)}(\delta/2) = 0, k = 1, 2, \ldots$ and
(d) $f'(s) < 0$ if $\delta/2 < s < \delta$.

Now for each point $x \in X$ we remove the disk $\exp_x(D_{\delta})$ from $S^{k-1}$ and replace it by the point set $B_x = \{f(|v|)\exp_x(v) | v \in D_{\delta}\}$, where $D_{\delta}$ is the ($\delta$)-disk about the origin of $T_x(S^{k-1})$. Clearly the set $S^{k-1} - \bigcup_{x \in X} \exp_x(D_{\delta}) \cup \bigcup_{x \in X} B_x$ is a smooth $S^{k-1}$ imbedded in $\mathbb{R}^k$. We give it the induced Riemannian metric from $\mathbb{R}^k$ and denote it by $M$.

**Claim:** $\text{Isom}(M) \cong G$.

**Proof.** First we notice that the points of $100 \cdot X \subset M$ must be taken to themselves by any isometry $I$ of $M$, by the choice of the function $f$. Clearly the same arguments above for $X$ hold for $100 \cdot X$, hence the isometry $I: M \rightarrow M$ restricted to $100 \cdot X$ comes from the action of $G$.

Let us now consider the "bump" $B_1$ above the point 1. Let us define for $r \geq 0$, $S_r = \{f(r) \cdot \exp_u(v) | |u| = r, v \in T_u(S^{k-1})\}$. In other words, $S_r$ is the $(k-2)$-sphere of $B_1$ lying above the $(k-2)$-sphere about 1 of radius $r$, for $0 < r \leq \delta$, and for $r = 0$ we set $S_0 = p$, the peak point of $B_1$.

Now it is easy to show that the orthogonal trajectories of the $S_r$'s are geodesics of $M$ and as such must be preserved under any isometry taking $p$ to $p$.

Thus any isometry $I$ of $M$ which takes $p$ to $p$ (and which must thus leave all points of $100 \cdot X$ fixed) must be a "rotation" on all of $B_1$, determined by $I | \partial B_1$, carrying each $S_r$ into itself by the "same" element of $O(k-2)$. Similarly, this $I$ must rotate each bump $B_x, x \in X$.

Also this rotation must extend past the boundary of the bumps for some ways, so we can easily extend $I | (M - \bigcup_x B_x)$ to an isometry $\widetilde{I}$ of $S^{k-1}$ to itself, by simply "coning" $I$ over $\exp_x(D_{\delta}), x \in X$. Clearly we will have $\widetilde{I}(x) = x$ for $x \in X$, and it follows easily that $\widetilde{I}: S^{k-1} \rightarrow S^{k-1}$ is the identity. Hence $I: M \rightarrow M$ must have been the identity.
Now it is clear that for each \( g \in G \) there is one isometry of \( M \) determined by the action of \( g \) on \( S^{k-1} \), extended to \( \mathbb{R}^k \), restricted to \( M \). Now if there is another isometry \( I : M \to M \) such that \( I \cdot X = g \cdot X \), then \( I \circ g^{-1} : M \to M \) must leave points of \( X \) fixed, so by the above discussion must be the identity. This establishes \( \text{Isom}(M) \approx G \).

**Corollary.** Any finite group \( G \) is isomorphic to the (full) isometry group of a finite subset \( X_G \) of euclidean space. If \( \text{card}(G) = k \) then the \( X_G \) can be found with \( \text{card}(X_G) = k^2 - k \) in euclidean space of dimension \( k - 1 \).

**Proof.** Simply take \( X_G = X \) in the proof of the Theorem, and count (noting that we initially took \( \text{card}(G) = k + 1 \)).

**Remark.** Further considerations can very likely reduce the necessary cardinality for \( X_G \) to \( k(k - 3) \). The various numbers
\[
d = \min\{\text{card}(X) | G \approx \text{Isom}(X)\}
\]
and
\[
e = \min\{N | G \text{ has a faithful representation into } O(N)\}
\]
seem to be interesting invariants of a finite group \( G \).

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