ADJOINT ABELIAN OPERATORS ON $L^p$ AND $C(K)$

BY

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ABSTRACT. An operator $A$ on a Banach space $X$ is said to be adjoint abelian if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on $X$ such that $[Ax, y] = [x, Ay]$ for all $x, y \in X$. In this paper we show that every adjoint abelian operator on $C(K)$ or $L^p(\Omega, \Sigma, \mu)$ $(1 < p < \infty, p \neq 2)$ is a multiple of an isometry whose square is the identity and hence is of the form $Ax(\cdot) = \lambda \alpha(\cdot)(x \circ \phi)(\cdot)$ where $\alpha$ is a scalar valued function with $\alpha(\cdot) \alpha \circ \phi(\cdot) = 1$ and $\phi$ is a homeomorphism of $K$ (or a set isomorphism in the case of $L^p(\Omega, \Sigma, \mu)$) with $\phi \circ \phi = \text{identity (essentially)}$.

1. Introduction. An operator $A$ on a Banach space $X$ is said to be adjoint abelian if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on $X$ such that

$[Ax, y] = [x, Ay]$ for all $x, y \in X$. In this note we show that every adjoint abelian operator on $C(K)$ or $L^p$ $(1 < p < \infty, p \neq 2)$ is a multiple of an isometry and hence is of the form

$Ax(\cdot) = \lambda \alpha(\cdot)(x \circ \phi)(\cdot)$

where $\alpha$ is a scalar valued function with $\alpha(\cdot) \alpha \circ \phi(\cdot) = 1$ and $\phi$ is a homeomorphism of $K$ (or a set isomorphism in the case of $L^p$) with $\phi \circ \phi = \text{identity (essentially)}$.

Our method is to use known characterizations of Hermitian operators on the spaces in question, together with the observation of Stampfli [12] that if $A$ is adjoint abelian then $A^2$ is both Hermitian and adjoint abelian, to show that if $A$ is adjoint abelian then (Theorems 1 and 4)

$A^2 = \rho I$ for some $\rho > 0$.

Furthermore, if (3) holds, then for $\lambda = \sqrt{\rho}$ we have

$\| (\lambda^{-1}A)(x) \|^2 = [\lambda^{-1}Ax, \lambda^{-1}Ax] = (\lambda^{-1})^2 [Ax, Ax]$

$= \rho^{-1} [A^2x, x] = \rho^{-1} [\rho x, x] = [x, x] = \|x\|^2$.

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Hence $X^{-1}A$ is an isometry. The result (2) now follows easily from known characterizations of isometries on $C(K)$ and $L^p$.

The result of the calculation above will be used in both §2 and 3 and we find it convenient to include a formal statement here.

**Lemma 1.** If $A$ is adjoint abelian on a Banach space $X$ and $A^2 = \rho I$ for some $\rho > 0$, then $A$ is a positive multiple of an isometry on $X$.

In [12], Stampfli asked whether every adjoint abelian operator on a weakly complete Banach space is scalar. In [11], it was shown that if $A$ satisfies (3), then $A$ is a scalar operator. Hence by our Theorems 1 and 4, every adjoint abelian operator on $C(K)$ ($K$ compact metric) or $L^p(\Omega, \Sigma, \mu)$ ($\Omega, \Sigma, \mu$ a $\sigma$-finite measure space) is a scalar operator. (A slightly more general such result is given in §4.)

2. Adjoint abelian operators on $C(K)$. Let $K$ be a compact metric space and let $C(K)$ denote the space of continuous complex valued functions on $K$ with the supremum norm. Any semi-inner product $[\cdot , \cdot]$ on $C(K)$ is determined by a mapping $f \mapsto f^*$ of $C(K)$ to its dual, where $f^*(f) = ||f||^2$, $||f^*|| = ||f||$ and $[g, f] = f^*(g)$. For each such $f^* \in C(K)^*$, there exists a regular complex Borel measure $\nu_f$ on the Borel sets of $K$ such that $f^*(g) = \int_K g\, d\nu_f$ [3]. Hence

$$[g, f] = \int_K g\, d\nu_f$$

is the general form of any s.i.p. in $C(K)$. We note here that the measure $\nu_f$ really depends on $f^*$ so that in the notation of (4) we are assuming that the particular mapping $f \mapsto f^*$ has been preassigned.

For $f \in C(K)$, let $P_f = \{t \in K: |f(t)| = ||f||\}$ be called the peak set for $f$.

**Lemma 2.** For $f \in C(K)$, the measure $\nu_f$ as in (4) has the properties that

(i) $\nu_f(P_f) = ||f||$,

(ii) if $f \geq 0$, then $\nu_f(P_f) = ||f||$,

(iii) if $P_f = \{t_0\}$, then $\nu_f(P_f) = \beta(t_0)$.

**Proof.** Suppose $f \in C(K)$ and let $G$ be any open set in $K$ containing $P_f$. Let $\lambda = \sup\{|f(x)|: x \in K \setminus G\}$. Now $\lambda < ||f||$ and if $\nu_f(K \setminus G) > 0$, then

$$||f||^2 = [f, f] = \int_K f\, d\nu_f \leq \lambda \nu_f(K \setminus G) + ||f|| \nu_f(G)$$

$$\leq ||f||(\nu_f(K \setminus G) + \nu_f(G)) = ||f||^2$$

since $\nu_f(K) = ||f^*|| = ||f||$. Hence we must have $\nu_f(K \setminus G) = 0$ and $\nu_f(G) = ||f||$.

Since $\nu_f$ is regular, for $\epsilon > 0$, there exists an open set $G$ such that $P_f \subset G$.
and \( |\nu_f(G)| < |\nu_f(P_f)| + \varepsilon \). We conclude that \( |\nu_f(P_f)| = \|f\| \). If \( B \) is any Borel set such that \( B \cap P_f = \emptyset \), there is a closed set \( F \subset B \) such that \( |\nu_f(B)| < |\nu_f(F)| + \varepsilon \). The argument above shows that \( |\nu_f(K \setminus F)| = 1 \) so that \( |\nu_f(F)| = 0 \). It follows that \( |\nu_f(B)| = 0 \). Hence, for \( f \geq 0 \), we have

\[
\|f\|^2 = \int_K f dv_f = \int_{P_f} f dv_f + \int_{K \setminus P_f} f dv_f = \int_{P_f} f dv_f = \|f\| \nu_f(P_f).
\]

Finally, suppose \( P_f = \{t_0\} \). Then \( \|f\|^2 = \int_K f dv_f = f(t_0) \nu_f(\{t_0\}) \) so that \( |f(t_0)|^2 = f(t_0) \nu_f(\{t_0\}) \). Therefore \( \nu_f(\{t_0\}) = \overline{f}(t_0) \) and the proof is complete.

We observe here that if \( \psi \) is a function which assigns to each \( g \in C(K) \) an element of the peak set \( P_g \), then

\[
[f, g] = \mu(g) \overline{\mu(g)}
\]

defines a s.i.p. on \( C(K) \) which is compatible with the norm. If \( \phi \) is a homeomorphism of \( K \) onto itself with the property that \( \phi \circ \phi \) is the identity on \( K \), then \( \|g \circ \phi\| = \|g\| \) and \( P_{g \circ \phi} = \phi(P_g) \) for all \( g \in C(K) \).

**Lemma 3.** If \( \phi \) is a homeomorphism of the compact metric space \( K \) with the property that \( \phi \circ \phi \) is the identity, then there is a choice function \( \psi_0 \) as in (5) such that

\[
\psi_0(g \circ \phi) = \phi(\psi_0(g))
\]

and

\[
\psi_0(g_1) = \psi_0(g_2) \quad \text{whenever} \quad P_{g_1} = P_{g_2}
\]

for all \( g, g_1, g_2 \in C(K) \).

**Proof.** Let \( C \) be the set of all choice functions on subsets \( C(K) \) with the following properties:

(i) If \( \psi \in C \), then \( D(\psi) \) is the domain of \( \psi \) is of the form \( Y \cup (Y \circ \phi) \) for some subset \( Y \subset C(K) \);

(ii) \( \psi(g \circ \phi) = \phi(\psi(g)) \) for each \( g \in D(\psi) \);

(iii) \( \psi(g_1) = \psi(g_2) \) whenever \( g_1, g_2 \in D(\psi) \) and \( P_{g_1} = P_{g_2} \).

If \( \psi_1, \psi_2 \in C \) define \( \psi_1 \leq \psi_2 \) whenever \( D(\psi_1) \subset D(\psi_2) \) and \( \psi_2(f) = \psi_1(f) \) for \( f \in D(\psi_1) \). Then \( C \) is a nonempty partially ordered set and it may be shown by an argument using Zorn's Lemma that \( C \) contains a maximal element \( \psi_0 \). It is straightforward to show that \( D(\psi_0) = C(K) \) and therefore (ii) and (iii) are the same as (6) and (7).

As we mentioned in the introduction, we will need a characterization of Hermitian operators on \( C(K) \). Sinclair [10] has shown that an operator is Hermitian on \( C(K) \) if and only if it is multiplication by a real valued function
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in $C(K)$. This result has also been obtained in [14] by a different method.

**Theorem 1.** Let $K$ be a compact metric space and suppose $A \neq 0$ is an adjoint abelian operator on $C(K)$. Then there exists a positive constant $\lambda$ such that $A^2 = \lambda I$ (where $I$ is the identity operator).

**Proof.** Since $A$ is adjoint abelian, there exists for each $f \in C(K)$ a regular complex Borel measure $\nu_f$ such that $[g, f] = \int_K g \, d\nu_f$ defines a s.i.p. on $C(K)$ compatible with the norm and such that

\[(8) \quad [Ag, f] = [g, Af] \quad \text{for all } g, f \in C(K).\]

Now $A^2$ must also satisfy (8) and must be Hermitian as well, i.e. $[A^2 f, f]$ is real for all $f \in C(K)$ [12]. Thus by the characterization of Hermitian operators mentioned above, there exists a real valued function $h \in C(K)$ such that $A^2 f = hf$ for all $f \in C(K)$. In fact, $h(t) \geq 0$ for all $t$.

Let $t_0 \in K$. Suppose $|h(t_0)| < \|h\|$ and let $t_n \in P_{h^n}$. By Urysohn’s lemma, there exists $g \in C(K)$ such that $g(t_0) = 1, g(t_n) = \frac{1}{2}(1 + |h(t_0)|/\|h\|) = \lambda_0$ and $g(t) \in (\lambda_0, 1)$ for all $t \in K(t_0, t_n)$. Then

\[\|hg(t_n)\| = \|g(t_n)\| \quad \|g(t_n)\| = (\|h\|/2)(1 + |h(t_0)|/\|h\|)\]

\[= \frac{1}{2}(\|h\| + |h(t_0)|) > |h(t_0)| = \|hg(t_0)\|.\]

Hence $t_0 \not\in P_{h^n}$. Again by Urysohn’s lemma there exists $f \in C(K)$ such that $f(t_0) = 1$ and $f(t) = \|h\|/\|h_g\|$ for $t \in P_{h^n}$. Since $g$ has a singleton peak set, we recall from Lemma 2(iii) that

\[[A^2 f, g] = [hf, g] = \int_K hf \, dv_g = hf(t_0)g(t_0) = hf(t_0) = h(t_0).\]

Moreover,

\[[f, A^2 g] = [f, hg] = \int_K f \, dv_{hg} = \int_{P_{h^n}} f \, dv_{hg}\]

\[= \frac{\|h\|}{\|hg\|} \nu_{hg}(P_{h^n}) = \frac{\|h\|}{\|hg\|} \|hg\| = \|h\|\]

since $\nu_{hg}(P_{h^n}) = \|hg\|$ by Lemma 2(ii). Therefore $[A^2 f, g] \neq [f, A^2 g]$ which is a contradiction. We conclude that $h$ is constant; indeed, $A^2 f = hf$ for all $f \in C(K)$.

**Theorem 2.** Let $K$ be a compact metric space and $A$ a nonzero operator on $C(K)$. Then $A$ is adjoint abelian if and only if there exists a homeomorphism $\phi$ on $K$, a positive constant $\lambda$ and a unimodular function $\alpha \in C(K)$ such that for every $f \in C(K)$,
(9) \[ Af(t) = \lambda \alpha(t)f \circ \phi(t), \quad t \in K, \]

where

(i) \((\phi \circ \phi)(t) = t \) for all \(t \in K,\) and
(ii) \(\alpha(t)\alpha(\phi(t)) = 1 \) for all \(t \in K.\)

**Proof.** Let us first prove the sufficiency of the conditions.

Let a s.i.p. be given as in (5) where the associated choice function \(\psi\) satisfies (6) and (7). Then

\[ [Af, g] = \lambda \alpha(\psi(g))f \circ \phi(\psi(g))\overline{\phi(\psi(g))}, \]

and since \(P_A g = P_{g \circ \phi}\) we have

\[ \psi(Ag) = \psi(g \circ \phi) = \phi(\psi(g)). \]

By using (10) along with (9), (i) and (ii), we may then obtain

\[ [f, Ag] = f(\psi(Ag))\overline{\psi(\psi(g))} = \lambda f \circ \phi(\psi(g))\overline{\phi(\psi(g))} = \lambda f \circ \phi(\psi(g)) = [Af, g]. \]

On the other hand, suppose \(A\) is adjoint abelian. By Theorem 1, \(A^2 = \rho I\) for some \(\rho > 0\) and by Lemma 1, \(A = \lambda U\) for some isometry \(U\) on \(C(K)\). By the Banach-Stone Theorem [3, p. 442], there exists a unimodular function \(\alpha\) and a homeomorphism \(\phi\) on \(K\) such that \(Uf(\cdot) = \alpha(\cdot)f \circ \phi(\cdot)\) for all \(t \in C(K)\). Thus (9) is satisfied.

Since \(A^2 = \lambda^2 I\), we have

\[ \lambda^2 f(t) = (A^2 f)(t) = A(\lambda \alpha f \circ \phi)(t) \]

\[ = \lambda^2 \alpha(t)\alpha(\phi(t))f(\phi \circ \phi(t)) \quad \text{for all } t \in K \text{ and } f \in C(K). \]

If we take \(f \equiv 1\) in (11) we obtain

\[ 1 = \alpha(t)\alpha(\phi(t)) \quad \text{for all } t \in K. \]

From this and (11) we get

\[ f(t) = f(\phi \circ \phi(t)) \quad \text{for all } f \in C(K) \text{ and } t \in K. \]

It follows readily that \((\phi \circ \phi)(t) = t\) for all \(t \in K\) which establishes (i) and concludes the proof of the theorem.

We remark that in the case of \(X = [0, 1]\), for example, there are only two choices for \(\phi\)

\[ \phi(t) = t \quad \text{and} \quad \phi(t) = 1 - t. \]

3. **Adjoint abelian operators on \(L^p\).** In this section we characterize the adjoint abelian operators on \(L^p(\Omega, \Sigma, \mu)\) where \((\Omega, \Sigma, \mu)\) is a \(\sigma\)-finite measure
space. For this we first need a characterization of Hermitian operators on these spaces. In the case that $(\Omega, \Sigma, \mu)$ is nonatomic the result is given in [8].

A s.i.p. compatible with the norm in $L^p(\Omega, \Sigma, \mu)$ for $1 \leq p < \infty$, $p \neq 2$, is given by

$$[f, g] = \|g\| \int_\Omega f \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{sgn} \ g. \tag{13}$$

If $p > 1$, this s.i.p. is unique, but for the characterization of Hermitians any s.i.p. compatible with the norm will suffice.

**Lemma 4.** Let $f_1, f_2$ be real valued functions in $L^p(\Omega, \Sigma, \mu)$, $1 \leq p < \infty$, $p \neq 2$, with essentially disjoint supports $\Omega_1$ and $\Omega_2$ respectively. Then

$$\int_\Omega (Hf_2)f_1 |f_2|^{p-1} \text{sgn} \ f_1 = \int_\Omega (Hf_1)f_2 |f_1|^{p-1} \text{sgn} \ f_2$$

for every Hermitian operator $H$ on $L^p$.

**Proof.** The result follows immediately from a result of Tam [13] and the fact that if $H$ is Hermitian, $[H(f_1 + e^{i\theta}f_2), (f_1 + e^{i\theta}f_2)]$ is real for every real value of $\theta$.

**Corollary 1.** If $\Omega_1, \Omega_2 \in \Sigma$ with $\mu(\Omega_1 \cap \Omega_2) = 0$, and $\chi_1, \chi_2$ are the associated characteristic functions, then

$$\int_{\Omega_1} H\chi_2 = \int_{\Omega_2} H\chi_1.$$

**Theorem 3.** Let $H$ be a Hermitian operator on $L^p(\Omega, \Sigma, \mu)$ with $1 \leq p < \infty$, $p \neq 2$. Then $H$ is Hermitian if and only if there exists a real valued function $h \in L^\infty(\Omega, \Sigma, \mu)$ such that $Hf = hf$ a.e. for every $f \in L^p$.

**Proof.** The proof will be given for a finite measure space since the extension to $\sigma$-finite measure spaces follows exactly as indicated in §6 of Lumer's paper [8].

Let $\Omega_1 \in \Sigma$ and $\chi_1$ be its characteristic function. Suppose $H\chi_1 \neq 0$ a.e. on $\Omega \setminus \Omega_1$. Then there exists a measurable set $\Omega_2 \subset \Omega \setminus \Omega_1$ with positive measure such that

$$\int_{\Omega_2} H\chi_1 \neq 0. \tag{14}$$

Let $\chi_2$ be the characteristic function of $\Omega_2$ and $f_1 = \alpha \chi_1, f_2 = \chi_2$ with $\alpha > 1$. Applying Lemma 4 we obtain

$$\int_{\Omega_1} (H\chi_2)\alpha^{p-1} = \alpha \int_{\Omega_2} H\chi_1. \tag{15}$$

It now follows from Corollary 1 and (15) that

$$(\alpha^{p-1} - \alpha)\int_{\Omega_2} H\chi_1 = 0 \tag{16}$$
which contradicts (14). Hence, $Hx_1 = 0$ a.e. on $\Omega \setminus \Omega_1$ and the proof now follows exactly as the proof of Theorem 9 in [8].

**Theorem 4.** Let $A$ be adjoint abelian on $L^p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$, $p \neq 2$. Then there exists a positive constant $\rho$ such that $A^2 = \rho I$.

**Proof.** As we have previously observed, $A^2$ is Hermitian as well as adjoint abelian. Hence by Theorem 3 there exists a real $L^\infty$ function $h$ such that $A^2f = hf$ for every $f \in L^p$ and where $h(t) \geq 0$ a.e. on $\Omega (\langle hf, f \rangle = \langle Af, f \rangle \geq 0)$. Furthermore, $A^2$ is adjoint abelian; thus

$$
\langle hf, g \rangle = \langle f, hg \rangle \quad \text{for all } f, g \in L^p.
$$

From the combination of (13) and (17) we obtain

$$
\int_\Omega f \ sgn \ g \left[ |g| h \left( \frac{|g|}{|f|} \right)^{p-1} - \|hg\| \left( \frac{|hg|}{|f|} \right)^{p-1} \ sgn \ h \right] = 0
$$

for all $f, g \in L^p$. For a given $f, g$ we may replace $f$ by an appropriate product of the form $e^{ia(t)}f(t)$ so that (18) holds with the integrand replaced by its absolute value. Hence,

$$
\|f\|_p \ |\langle g, f \rangle|_p^{-1} |h|_p^{p-2} - \|hg\|_p^{p-2} |h|_p^{p-2} \ sgn \ h = 0 \quad \text{a.e.}
$$

Let $Z(k) = \{ t \in \Omega : k(t) = 0 \}$ for any function $k$ on $\Omega$. It follows from (19) that

$$
|f|_p |g|_p^{p-1} |h|_p^{p-2} - \|hg\|_p^{p-2} |h|_p^{p-2} = 0 \quad \text{a.e.}
$$

on $\Omega \setminus Z(h)$. For any $g \in L^p$, we have (taking $f = g$)

$$
|f|_p |g|_p^{p-2} - \|hg\|_p^{p-2} |h|_p^{p-2} = 0 \quad \text{a.e.}
$$

on $\Omega \setminus (Z(h) \cup Z(g))$. Therefore

$$
h(t) = \|hg\|_p^{p-2} |h|_p^{p-2} \quad \text{a.e.}
$$

on $\Omega \setminus (Z(g) \cup Z(h))$ for all $g \in L^p$. It follows that $h$ is constant a.e. in $\Gamma \setminus Z(h)$ for every $\Gamma \in \Sigma$ with $\mu(\Gamma) < \infty$. Since $\Omega$ is $\sigma$-finite, it follows that $h$ is constant a.e. on $\Omega \setminus Z(h)$. In fact from (22) we must have

$$
h = \|hg\|_p^{p-2} |h|_p^{p-2} = \lambda \quad \text{a.e.}
$$

on $\Omega \setminus Z(h)$ for every $g \in L^p$. The proof of the theorem will be complete if we can show that $\mu(Z(h)) = 0$.

Let $F \subseteq Z(h)$ with $F \subseteq \Sigma$; $\mu(F) < \infty$ and $G \subseteq \Omega \setminus Z(h)$ with $\sigma < \mu(G) < \infty$. Such sets $F, G$ must exist; otherwise $Z(h)$ and $\Omega \setminus Z(h)$ would be atoms of infinite measure which is impossible since the measure space is $\sigma$-finite. Let $g = \chi_G + \chi_F$ so that $\|g\|_p^p = \mu(G) + \mu(F)$. Now
It now follows that $\mu(Z(h)) = 0$.

We may use Lamperti's characterization of onto isometries of $L^p$ to obtain the description of adjoint abelian operators on $L^p$ announced in the introduction. Let us recall the notation and the theorem of Lampertii which we shall need [6].

A regular set isomorphism of the measure space $(\Omega, \Sigma, \mu)$ will mean a mapping $T$ of $\Sigma$ into $\Sigma$ defined modulo sets of measure zero satisfying $T(\Omega \setminus F) = T(\Omega) \setminus TF$, $T(\bigcup F_n) = \bigcup TF_n$ disjoint $F_n$, and $\mu(TF) = 0$ if and only if $\mu(F) = 0$. For any measurable function $f$ on $(\Omega, \Sigma, \mu)$ we will write $f \circ T$ to be the function obtained from a limit of simple functions where by definition, $\chi_E \circ T = \chi_{TE}$ for each $E \in \Sigma$. Lamperti [6] proved that if $U$ is an isometry on $L^p$, then there exists a regular set isomorphism $T$ and a function $\alpha(t)$ such that

$$Uf(t) = \alpha(t)f \circ T(t) \quad \text{a.e.}$$

and

$$\int_{TE} |\alpha|^p \, d\mu = \mu(E) \quad \text{for } E \in \Sigma.$$

Conversely, if $U$ satisfies (24) and $\alpha$ satisfies (25), then $U$ is an isometry of $L^p$.

**Theorem 5.** If $1 < p < \infty$, $p \neq 2$, and $A$ is a nonzero operator on $L^p$, then $A$ is adjoint abelian if and only if there exists a regular set isomorphism $T$, a measurable function $\alpha$ and a real number $\lambda$ such that

$$Af(t) = \lambda \alpha(t)f \circ T(t) \quad \text{a.e. for } f \in L^p$$

where

$$\alpha(t)\alpha \circ T(t) = 1 \quad \text{a.e.,}$$

$$\int_{TE} |\alpha|^p \, d\mu = \mu(E) \quad \text{for } E \in \Sigma,$$

$$T \circ T(E) = E \quad \text{(modulo sets of measure zero)}.$$
It follows from this that $T \circ T(E) = E$ modulo a set of measure zero and $\alpha(\tau) \circ T(\tau) = 1$ a.e. on sets of finite measure. The extension to sets of infinite measure follows readily from the $\sigma$-finiteness of $\Omega$, and (27), (29) are established.

Next suppose (26), (27), (28), and (29) are satisfied by $A$, $\alpha$, $T$, $\lambda$. If $E \in \Sigma$, we have

$$
\int_{\Omega} \chi_{E} \circ T = \int_{\Omega} \chi_{TE} = \mu(TE)
$$

$$
= \int_{T \circ T(E)} |\alpha|^p \quad \text{by (28)}
$$

$$
= \int_{E} |\alpha|^p \quad \text{by (29)}
$$

$$
= \int_{\Omega} |\alpha|^p \chi_{E}.
$$

In the same way, it can be shown that if $f$ is a simple function with support of finite measure then

$$
\int_{\Omega} f \circ T = \int_{\Omega} |\alpha|^p f
$$

and finally, if $|\alpha|^p f$ is integrable, then $f \circ T$ is integrable and

$$
(31) \quad \int_{\Omega} f \circ T = \int_{\Omega} |\alpha|^p f \quad \text{for all } f \in L^p
$$

since any measurable $f$ is the a.e. limit of a sequence of simple functions with finite support [9, p. 224].

Now suppose $f, g \in L^p$. Then using the given conditions along with (31) and the fact that $T$ distributes across products, we obtain

$$
[Af, g] = \lambda \|g\| \int \alpha(f \circ T) \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{sgn } g
$$

$$
= \lambda \|g\| \int \alpha f \circ T |\alpha|^{p-1} |\alpha \circ T|^{p-1} \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{sgn } \text{sgn}(\alpha \circ T) \text{sgn } g
$$

$$
= \lambda \|g\| \int |\alpha|^p (f \circ T) |\alpha \circ T|^{p-1} \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{sgn } (\alpha \circ T) \text{sgn } g
$$

$$
= \|Ag\| \int (f \circ T \circ T) |\alpha \circ T \circ T|^{p-1} \left( \frac{|g \circ T|}{\|g\|} \right)^{p-1}
$$

$$
\cdot \frac{\lambda}{|\alpha|} \text{sgn } (\alpha \circ T \circ T) \text{sgn } (g \circ T)
$$

$$
= \|Ag\| \int f \left( \frac{|\alpha|^p |\alpha|^p |g \circ T|^{p-1}}{|\alpha|^p \mathbb{R}^p |g|^p} \right) \frac{\lambda}{|\alpha|} \text{sgn } \text{sgn } (g \circ T)
$$

$$
= \|Ag\| \int f \left( \frac{|Ag|}{\|Ag\|} \right)^{p-1} \text{sgn } Ag
$$

and $A$ is adjoint abelian.
A characterization of adjoint abelian operators on $L^p$ which is included in Theorem 5, has been obtained previously in [1] and [4].

The results above are related to some recent work of Byrne and Sullivan [2] on contractive projections on $L^p$. A projection $P$ on $L^p$ is called contractive if $\|P\| = 1$. An isometry $U$ with the property that $U^2 = I$ is called a reflection. In [2], it is proved that $P$ and $I - P$ are contractive if and only if $P = (I + U)/2$ for some reflection $U$. The next two corollaries are immediate from Theorems 4 and 5 and the work in [2].

**Corollary 2.** A nonzero operator $A$ on $L^p$ is adjoint abelian if and only if $A$ is a real multiple of a reflection.

**Corollary 3.** Both $P$ and $I - P$ are contractive projections on $L^p$ if and only if there is a real number $\lambda$ and an adjoint abelian operator $A$ such that $P = (I + \lambda A)/2$.

Stampfli [12] has proved that an operator $B$ on a weakly complete Banach space has a proper invariant subspace if it commutes with an adjoint abelian operator $A$ where $A \neq \lambda I$.

**Corollary 4.** Let $B$ be a bounded operator $L^p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$, $p \neq 2$. If there exists a regular, measure preserving set isomorphism $T$ such that $T$ is not the identity, $T \circ T(E) = E$ modulo sets of measure zero for all $E \in \Sigma$, and

$$B(f \circ T) = Bf \circ T \quad \text{for all } f \in L^p,$$

then $B$ has a proper invariant subspace.

**Proof.** If we define $A$ on $L^p$ by $Af = f \circ T$, then (26), (27), (28), and (29) are satisfied by taking $\alpha = 1$. Hence $A$ is adjoint abelian and (32) is simply the condition that $A$ commutes with $B$.

4. Adjoint abelian operators and isometries. We have shown that adjoint abelian operators on $C(K)$ and $L^p(\Omega, \Sigma, \mu)$ are multiples of isometries. This is also the case for adjoint abelian operators on certain spaces of class $S$ discussed in [4]. The next theorem characterizes the types of isometries which can give rise to adjoint abelian operators in this manner.

**Theorem 6.** Let $U$ be an isometry on the Banach space $X$ and $\lambda$ a real scalar. The operator $A = \lambda U$ is adjoint abelian if and only if $U^2 = I$.

**Proof.** Let $U$ be an isometry with $U^2 = I$. By a theorem of Koehler and Rosenthal [5] there exists a s.i.p $[\cdot, \cdot]$ compatible with the norm such that

$$[Ux, Uy] = [x, y] \quad \text{for all } x, y \in X.$$
Hence,
\[
[\lambda Ux, y] = [\lambda Ux, U^2y] \\
= [x, Uy] \quad \text{from (33)} \\
= [x, \lambda Uy].
\]

Next suppose \( A = \lambda U \) is adjoint abelian and \( U \) is an isometry. Then for every \( x \in X \)
\[(34) \quad [(\lambda U)^2 x, x] = [\lambda Ux, \lambda Ux] = \|\lambda Ux\|^2 = \lambda^2 \|x\|^2.\]
It follows from (34) that
\[(35) \quad [(U^2 - I)x, x] = 0 \quad \text{for every } x.\]
By Theorem 5 of [7] we conclude that \( U^2 = I \).

One could give slightly different proofs of the "sufficiency" parts of Theorems 2 and 5 by using Theorem 6, showing that the given conditions characterize reflections.

From Theorem 6 above and Theorem 1 of [11] the next corollary is immediate.

**Corollary 5.** Every adjoint abelian operator which is a multiple of an isometry is necessarily a scalar operator.

In particular, as mentioned in the introduction, every adjoint abelian operator on \( C(K) \) or \( L^p \) is scalar.

In conclusion, we raise the following question: On what Banach spaces is every adjoint abelian operator a real multiple of an isometry?

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**BIBLIOGRAPHY**


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