ADJOINT ABELIAN OPERATORS ON $L^p$ AND $C(K)$

BY

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ABSTRACT. An operator $A$ on a Banach space $X$ is said to be adjoint abelian if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on $X$ such that $[Ax, y] = [x, Ay]$ for all $x, y \in X$. In this paper we show that every adjoint abelian operator on $C(K)$ or $L^p(\Omega, \Sigma, \mu)$ ($1 < p < \infty$, $p \neq 2$) is a multiple of an isometry whose square is the identity and hence is of the form $Ax(\cdot) = \lambda \alpha(\cdot)(x \circ \phi)(\cdot)$ where $\alpha$ is a scalar valued function with $\alpha(\cdot) \alpha \circ \phi(\cdot) = 1$ and $\phi$ is a homeomorphism of $K$ (or a set isomorphism in case of $L^p(\Omega, \Sigma, \mu)$) with $\phi \circ \phi = \text{id}$ (essentially).

1. Introduction. An operator $A$ on a Banach space $X$ is said to be adjoint abelian if there is a semi-inner product $[\cdot, \cdot]$ consistent with the norm on $X$ such that

$$[Ax, y] = [x, Ay]$$

for all $x, y \in X$. In this note we show that every adjoint abelian operator on $C(K)$ or $L^p$ ($1 < p < \infty$, $p \neq 2$) is a multiple of an isometry and hence is of the form

$$Ax(\cdot) = \lambda \alpha(\cdot)(x \circ \phi)(\cdot)$$

where $\alpha$ is a scalar valued function with $\alpha(\cdot) \alpha \circ \phi(\cdot) = 1$ and $\phi$ is a homeomorphism of $K$ (or a set isomorphism in the case of $L^p$) with $\phi \circ \phi = \text{id}$ (essentially).

Our method is to use known characterizations of Hermitian operators on the spaces in question, together with the observation of Stampfli [12] that if $A$ is adjoint abelian then $A^2$ is both Hermitian and adjoint abelian, to show that if $A$ is adjoint abelian then (Theorems 1 and 4)

$$A^2 = \rho I \quad \text{for some } \rho > 0.$$  

Furthermore, if (3) holds, then for $\lambda = \sqrt{\rho}$ we have

$$\|\lambda^{-1}A(x)\|^2 = [\lambda^{-1}Ax, \lambda^{-1}Ax] = (\lambda^{-1})^2 [Ax, Ax]$$

$$= \rho^{-1} [A^2x, x] = \rho^{-1} [\alpha x, x] = [x, x] = \|x\|^2.$$
Hence $\lambda^{-1}A$ is an isometry. The result (2) now follows easily from known characterizations of isometries on $C(K)$ and $L^p$.

The result of the calculation above will be used in both §§2 and 3 and we find it convenient to include a formal statement here.

**Lemma 1.** If $A$ is adjoint abelian on a Banach space $X$ and $A^2 = \rho I$ for some $\rho > 0$, then $A$ is a positive multiple of an isometry on $X$.

In [12], Stampfli asked whether every adjoint abelian operator on a weakly complete Banach space is scalar. In [11], it was shown that if $A$ satisfies (3), then $A$ is a scalar operator. Hence by our Theorems 1 and 4, every adjoint abelian operator on $C(K)$ ($K$ compact metric) or $L^p(\Omega, \Sigma, \mu)$ ($\mu$ a $\sigma$-finite measure space) is a scalar operator. (A slightly more general such result is given in §4.)

2. Adjoint abelian operators on $C(K)$. Let $K$ be a compact metric space and let $C(K)$ denote the space of continuous complex valued functions on $K$ with the supremum norm. Any semi-inner product $[\cdot, \cdot]$ on $C(K)$ is determined by a mapping $f \rightarrow f^*$ of $C(K)$ to its dual, where $f^*(f) = \|f\|^2$, $\|f^*\| = \|f\|$ and $[g, f] = f^*(g)$. For each such $f^* \in (C(K))^*$, there exists a regular complex Borel measure $\nu_f$ on the Borel sets of $K$ such that $f^*(g) = \int_K g \, d\nu_f$ [3]. Hence

\[ [g, f] = \int_K g \, d\nu_f \]

is the general form of any s.i.p. in $C(K)$. We note here that the measure $\nu_f$ really depends on $f^*$ so that in the notation of (4) we are assuming that the particular mapping $f \rightarrow f^*$ has been preassigned.

For $f \in C(K)$, let $P_f = \{t \in K : |f(t)| = \|f\|\}$ be called the peak set for $f$.

**Lemma 2.** For $f \in C(K)$, the measure $\nu_f$ as in (4) has the properties that

(i) $\nu_f(P_f) = \|f\|$
(ii) if $f \geq 0$, then $\nu_f(P_f) = \|f\|$
(iii) if $P_f = \{t_0\}$, then $\nu_f(P_f) = f(t_0)$.

**Proof.** Suppose $f \in C(K)$ and let $G$ be any open set in $K$ containing $P_f$. Let $\lambda = \sup\{|f(x)| : x \in K \setminus G\}$. Now $\lambda < \|f\|$ and if $\nu_f(K \setminus G) > 0$, then

\[ \|f\|^2 = [f, f] = \int_K f \, d\nu_f \leq \lambda \nu_f(K \setminus G) + \|f\| \nu_f(G) \]

\[ < \|f\| (\nu_f(K \setminus G) + \nu_f(G)) = \|f\|^2 \]

since $\nu_f(K) = \|f^*\| = \|f\|$. Hence we must have $\nu_f(K \setminus G) = 0$ and $\nu_f(G) = \|f\|$. Since $\nu_f$ is regular, for $\epsilon > 0$, there exists an open set $G$ such that $P_f \subset G$.
and \(|\nu_f(G)| < |\nu_f(P_f)| + \epsilon\). We conclude that \(|\nu_f(P_f)| = \|f\|\). If \(B\) is any Borel set such that \(B \cap P_f = \emptyset\), there is a closed set \(F \subset B\) such that \(|\nu_f(B)| < |\nu_f(F)| + \epsilon\). The argument above shows that \(|\nu_f(K \setminus F)| = 1\) so that \(|\nu_f(F)| = 0\). It follows that \(|\nu_f(B)| = 0\). Hence, for \(f \geq 0\), we have

\[
\|f\|^2 = \int_K f d\nu_f = \int_{P_f} f d\nu_f + \int_{K \setminus P_f} f d\nu_f = \int_{P_f} f d\nu_f = \|f\| \nu_f(P_f).
\]

Finally, suppose \(P_f = \{t_0\}\). Then \(\|f\|^2 = \int_K f d\nu_f = f(t_0) \nu_f(\{t_0\})\) so that \(|f(t_0)|^2 = f(t_0) \nu_f(\{t_0\})\). Therefore \(\nu_f(\{t_0\}) = \overline{f}(t_0)\) and the proof is complete.

We observe here that if \(\psi\) is a function which assigns to each \(g \in C(K)\) an element of the peak set \(P_g\), then

\[
[f, g] = \mathcal{M}(f(g), g)
\]

defines a s.i.p. on \(C(K)\) which is compatible with the norm. If \(\phi\) is a homeomorphism of \(K\) onto itself with the property that \(\phi \circ \phi\) is the identity on \(K\), then \(\|g \circ \phi\| = \|g\|\) and \(P_{g \circ \phi} = \phi(P_g)\) for all \(g \in C(K)\).

**Lemma 3.** If \(\phi\) is a homeomorphism of the compact metric space \(K\) with the property that \(\phi \circ \phi\) is the identity, then there is a choice function \(\psi_0\) as in (5) such that

\[
\psi_0(g \circ \phi) = \phi(\psi_0(g))
\]

and

\[
\psi_0(g_1) = \psi_0(g_2) \quad \text{whenever} \quad P_{g_1} = P_{g_2}
\]

for all \(g, g_1, g_2 \in C(K)\).

**Proof.** Let \(\mathcal{C}\) be the set of all choice functions on subsets \(C(K)\) with the following properties:

(i) If \(\psi \in \mathcal{C}\), then \(\mathcal{D}(\psi) = \text{domain of } \psi\) is of the form \(Y \cup (Y \circ \phi)\) for some subset \(Y \subset C(K)\);

(ii) \(\psi(g \circ \phi) = \phi(\psi(g))\) for each \(g \in \mathcal{D}(\psi)\);

(iii) \(\psi(g_1) = \psi(g_2)\) whenever \(g_1, g_2 \in \mathcal{D}(\psi)\) and \(P_{g_1} = P_{g_2}\).

If \(\psi_1, \psi_2 \in \mathcal{C}\) define \(\psi_1 \leq \psi_2\) whenever \(\mathcal{D}(\psi_1) \subset \mathcal{D}(\psi_2)\) and \(\psi_2(f) = \psi_1(f)\) for \(f \in \mathcal{D}(\psi_1)\). Then \(\mathcal{C}\) is a nonempty partially ordered set and it may be shown by an argument using Zorn's Lemma that \(\mathcal{C}\) contains a maximal element \(\psi_0\). It is straightforward to show that \(\mathcal{D}(\psi_0) = C(K)\) and therefore (ii) and (iii) are the same as (6) and (7).

As we mentioned in the introduction, we will need a characterization of Hermitian operators on \(C(K)\). Sinclair [10] has shown that an operator is Hermitian on \(C(K)\) if and only if it is multiplication by a real valued function.
Theorem 1. Let $K$ be a compact metric space and suppose $A \neq 0$ is an adjoint abelian operator on $C(K)$. Then there exists a positive constant $\lambda$ such that $A^2 = \lambda I$ (where $I$ is the identity operator).

Proof. Since $A$ is adjoint abelian, there exists for each $f \in C(K)$ a regular complex Borel measure $\mu_f$ such that $\langle g, f \rangle = \int_K g \, d\mu_f$ defines a s.i.p. on $C(K)$ compatible with the norm and such that

$$[Ag, f] = [g, Af] \quad \text{for all } g, f \in C(K).$$

Now $A^2$ must also satisfy (8) and must be Hermitian as well, i.e. $[A^2 f, f]$ is real for all $f \in C(K)$ [12]. Thus by the characterization of Hermitian operators mentioned above, there exists a real valued function $h \in C(K)$ such that $A^2 f = hf$ for all $f \in C(K)$. In fact, $h(t) \geq 0$ for all $t$.

Let $t_0 \in K$. Suppose $|h(t_0)| < \|h\|$ and let $t_n \in P_{h^*}$. By Urysohn's lemma, there exists $g \in C(K)$ such that $g(t_0) = 1$, $g(t_n) = \frac{1}{2}(1 + |h(t_0)|/\|h\|) = \lambda_0$ and $g(t) \in (\lambda_0, 1)$ for all $t \in K \setminus \{t_0, t_n\}$. Then

$$|h(t_n)| = |h(t_0)| |g(t_n)| = (\|h\|/2)(1 + |h(t_0)|/\|h\|) = \frac{1}{2}(\|h\| + |h(t_0)|) > |h(t_0)| = |h(t_0)|.$$

Hence $t_0 \notin P_{h^*}$. Again by Urysohn's lemma there exists $f \in C(K)$ such that $f(t) = 1$ and $f(t) = \|h\|/\|h_g\|$ for $t \in P_{h^*}$. Since $g$ has a singleton peak set, we recall from Lemma 2(iii) that

$$[A^2 f, g] = [hf, g] = \int_K hf \, dv_g = hf(t_0)g(t_0) = hf(t_0) = h(t_0).$$

Moreover,

$$[f, A^2 g] = [f, hg] = \int_K f \, dv_{hg} = \int_{P_{h^*}} f \, dv_{hg} = \frac{\|h\|}{\|h_g\|} \|hg\| = \|h\|$$

since $\nu_{hg}(P_{h^*}) = \|hg\|$ by Lemma 2(ii). Therefore $[A^2 f, g] \neq [f, A^2 g]$ which is a contradiction. We conclude that $h$ is constant; indeed, $A^2 f = hf$ for all $f \in C(K)$.

Theorem 2. Let $K$ be a compact metric space and $A$ a nonzero operator on $C(K)$. Then $A$ is adjoint abelian if and only if there exists a homeomorphism $\phi$ on $K$, a positive constant $\lambda$ and a unimodular function $\alpha \in C(K)$ such that for every $f \in C(K)$,
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(9) \[ Af(t) = \lambda \alpha(t) f \circ \phi(t), \quad t \in K, \]

where

(i) \((\phi \circ \phi)(t) = t\) for all \(t \in K\), and

(ii) \(\alpha(t) \alpha(\phi(t)) = 1\) for all \(t \in K\).

**Proof.** Let us first prove the sufficiency of the conditions.

Let a s.i.p. be given as in (5) where the associated choice function \(\psi\) satisfies (6) and (7). Then

\[ [Af, g] = \lambda \alpha(\psi(g)) f \circ \phi(\psi(g)) \overline{\phi(\psi(g))}, \]

and since \(P_A g = P_{g \circ \phi}\) we have

(10) \[ \psi(Ag) = \psi(g \circ \phi) = \phi(\psi(g)). \]

By using (10) along with (9), (i) and (ii), we may then obtain

\[ [f, Ag] = f(\psi(Ag)) \overline{\psi(\psi(g))} \alpha(\psi(g)) \overline{\phi(\psi(g))} = \lambda f \circ \phi(\psi(g)) \overline{\phi(\psi(g))} = [Af, g]. \]

On the other hand, suppose \(A\) is adjoint abelian. By Theorem 1, \(A^2 = \rho I\) for some \(\rho > 0\) and by Lemma 1, \(A = \lambda U\) for some isometry \(U\) on \(C(K)\). By the Banach-Stone Theorem [3, p. 442], there exists a unimodular function \(\alpha\) and a homeomorphism \(\phi\) on \(K\) such that \(Uf(\cdot) = \alpha(\cdot) f \circ \phi(\cdot)\) for all \(f \in C(K)\). Thus (9) is satisfied.

Since \(A^2 = \lambda^2 I\), we have

(11) \[ \lambda^2 f(t) = (A^2 f)(t) = A(\lambda \alpha f \circ \phi)(t) = \lambda^2 \alpha(t) \alpha(\phi(t)) f(\phi \circ \phi(t)) \quad \text{for all } t \in K \text{ and } f \in C(K). \]

If we take \(f = 1\) in (11) we obtain

\[ 1 = \lambda \alpha(t) \alpha(\phi(t)) \quad \text{for all } t \in K. \]

From this and (11) we get

(12) \[ f(t) = f(\phi \circ \phi(t)) \quad \text{for all } f \in C(K) \text{ and } t \in K. \]

It follows readily that \((\phi \circ \phi)(t) = t\) for all \(t \in K\) which establishes (i) and concludes the proof of the theorem.

We remark that in the case of \(X = [0, 1]\), for example, there are only two choices for \(\phi\)

\[ \phi(t) = t \quad \text{and} \quad \phi(t) = 1 - t. \]

3. **Adjoint abelian operators on $L^p$.** In this section we characterize the adjoint abelian operators on \(L^p(\Omega, \Sigma, \mu)\) where \((\Omega, \Sigma, \mu)\) is a \(\sigma\)-finite measure.
space. For this we first need a characterization of Hermitian operators on these spaces. In the case that \((\Omega, \Sigma, \mu)\) is nonatomic the result is given in [8].

A s.i.p. compatible with the norm in \(L^p(\Omega, \Sigma, \mu)\) for \(1 < p < \infty, p \neq 2\), is given by

\[
[f, g] = \|g\| \int_{\Omega} f \left(\frac{|g|}{\|g\|}\right)^{p-1} \text{sgn} \ g.
\]

If \(p > 1\), this s.i.p. is unique, but for the characterization of Hermitians any s.i.p. compatible with the norm will suffice.

**Lemma 4.** Let \(f_1, f_2\) be real valued functions in \(L^p(\Omega, \Sigma, \mu)\), \(1 < p < \infty\), \(p \neq 2\), with essentially disjoint supports \(\Omega_1\) and \(\Omega_2\) respectively. Then

\[
\int_{\Omega} (Hf_2)|f_1|^{p-1} \text{sgn} \ f_1 = \int_{\Omega} (Hf_1)|f_2|^{p-1} \text{sgn} \ f_2
\]

for every Hermitian operator \(H\) on \(L^p\).

**Proof.** The result follows immediately from a result of Tam [13] and the fact that if \(H\) is Hermitian, \([H(f_1 + e^{i\theta}f_2), (f_1 + e^{i\theta}f_2)]\) is real for every real value of \(\theta\).

**Corollary 1.** If \(\Omega_1, \Omega_2 \in \Sigma\) with \(\mu(\Omega_1 \cap \Omega_2) = 0\), and \(\chi_1, \chi_2\) are the associated characteristic functions, then

\[
\int_{\Omega_1} H\chi_2 = \int_{\Omega_2} H\chi_1.
\]

**Theorem 3.** Let \(H\) be a Hermitian operator on \(L^p(\Omega, \Sigma, \mu)\) with \(1 < p < \infty, p \neq 2\). Then \(H\) is Hermitian if and only if there exists a real valued function \(h \in L^\infty(\Omega, \Sigma, \mu)\) such that \(Hf = hf\) a.e. for every \(f \in L^p\).

**Proof.** The proof will be given for a finite measure space since the extension to \(\sigma\)-finite measure spaces follows exactly as indicated in §6 of Lumer’s paper [8].

Let \(\Omega_1 \in \Sigma\) and \(\chi_1\) be its characteristic function. Suppose \(H\chi_1 \neq 0\) a.e. on \(\Omega \setminus \Omega_1\). Then there exists a measurable set \(\Omega_2 \subset \Omega \setminus \Omega_1\) with positive measure such that

\[
\int_{\Omega_2} H\chi_1 \neq 0.
\]

Let \(\chi_2\) be the characteristic function of \(\Omega_2\) and \(f_1 = \alpha \chi_1, f_2 = \chi_2\) with \(\alpha > 1\). Applying Lemma 4 we obtain

\[
\int_{\Omega_1} (H\chi_2)\alpha^{p-1} = \alpha \int_{\Omega_2} H\chi_1.
\]

It now follows from Corollary 1 and (15) that

\[
(a^{p-1} - \alpha)\int_{\Omega_2} H\chi_1 = 0
\]
which contradicts (14). Hence, \( H_x \equiv 0 \) a.e. on \( \Omega \setminus \Omega_1 \) and the proof now follows exactly as the proof of Theorem 9 in [8].

**Theorem 4.** Let \( A \) be adjoint abelian on \( L^p(\Omega, \Sigma, \mu) \) where \( 1 < p < \infty \), \( p \neq 2 \). Then there exists a positive constant \( \rho \) such that \( A^2 = \rho I \).

**Proof.** As we have previously observed, \( A^2 \) is Hermitian as well as adjoint abelian. Hence by Theorem 3 there exists a real \( L^\infty \) function \( h \) such that \( A^2f = hf \) for every \( f \in L^p \) and where \( h(t) \geq 0 \) a.e. on \( \Omega \) \( ([hf, f] = [Af, Af] \geq 0) \). Furthermore, \( A^2 \) is adjoint abelian; thus

\[
[hf, g] = [f, hg] \quad \text{for all } f, g \in L^p.
\]

From the combination of (13) and (17) we obtain

\[
\int \Omega f \sgn g \left[ \frac{[f, g]}{[g, g]} \right]^{p-1} - \|hg\| \left( \frac{[h,f]}{[g, g]} \right)^{p-1} \sgn h = 0
\]

for all \( f, g \in L^p \). For a given \( f, g \) we may replace \( f \) by an appropriate product of the form \( e^{ia(t)}f(t) \) so that (18) holds with the integrand replaced by its absolute value. Hence,

\[
\int \Omega |f| |g|^{p-1} \|h\|^{p-2} - \|hg\| \|hg\|^{p-2} \sgn h | = 0 \quad \text{a.e.}
\]

Let \( Z(k) = \{ t \in \Omega : k(t) = 0 \} \) for any function \( k \) on \( \Omega \). It follows from (19) that

\[
|f| |g|^{p-1} \|h\|^{p-2} - h^{p-2} / \|hg\|^{p-2} | = 0 \quad \text{a.e.}
\]

on \( \Omega \setminus Z(h) \). For any \( g \in L^p \), we have (taking \( f = g \))

\[
|f| |g|^{p-1} \|h\|^{p-2} - h^{p-2} / \|hg\|^{p-2} | = 0 \quad \text{a.e.}
\]

on \( \Omega \setminus (Z(h) \cup Z(g)) \). Therefore

\[
h(t) = \|hg\| / \|g\| \quad \text{a.e.}
\]

on \( \Omega \setminus (Z(g) \cup Z(h)) \) for all \( g \in L^p \). It follows that \( h \) is constant a.e. in \( \Gamma \setminus Z(h) \) for every \( \Gamma \in \Sigma \) with \( \mu(\Gamma) < \infty \). Since \( \Omega \) is \( \sigma \)-finite, it follows that \( h \) is constant a.e. on \( \Omega \setminus Z(h) \). In fact from (22) we must have

\[
h = \|gh\| / \|g\| = \lambda \quad \text{a.e.}
\]

on \( \Omega \setminus Z(h) \) for every \( g \in L^p \). The proof of the theorem will be complete if we can show that \( \mu(Z(h)) = 0 \).

Let \( F \subset Z(h) \) with \( F \in \Sigma ; \mu(F) < \infty \) and \( G \subset \Omega \setminus Z(h) \) with \( \sigma < \mu(G) < \infty \). Such sets \( F, G \) must exist; otherwise \( Z(h) \) and \( \Omega \setminus Z(h) \) would be atoms of infinite measure which is impossible since the measure space is \( \sigma \)-finite. Let \( g = \chi_G + \chi_F \) so that \( \|g\|^p = \mu(G) + \mu(F) \). Now
\[ gh = h \chi_G = \lambda \chi_G \quad \text{and} \quad \lambda^p = \frac{\|gh\|^p}{\|g\|^p} = \frac{\chi_G^p \mu(G)}{\mu(G) + \mu(F)} \quad \text{by (23).} \]

It now follows that \( \mu(Z(h)) = 0 \).

We may use Lamperti's characterization of onto isometries of \( L^p \) to obtain the description of adjoint abelian operators on \( L^p \) announced in the introduction. Let us recall the notation and the theorem of Lamperti which we shall need [6].

A regular set isomorphism of the measure space \((\Omega, \Sigma, \mu)\) will mean a mapping \( T \) of \( \Sigma \) into itself defined modulo sets of measure zero satisfying \( T(\Omega \setminus F) = T(\Omega) \setminus TF, T(\bigcup F_n) = \bigcup TF_n \) disjoint \( F_n \), and \( \mu(TF) = 0 \) if and only if \( \mu(F) = 0 \). For any measurable function \( f \) on \((\Omega, \Sigma, \mu)\) we will write \( f \circ T \) to be the function obtained from a limit of simple functions where by definition, \( \chi_E \circ T = \chi_{TE} \) for each \( E \in \Sigma \). Lamperti [6] proved that if \( U \) is an isometry on \( L^p \), then there exists a regular set isomorphism \( T \) and a function \( \alpha(t) \) such that

\[ Uf(t) = \alpha(t)f \circ T(t) \quad \text{a.e.} \]

and

\[ \int_{TE} |\alpha|^p d\mu = \mu(E) \quad \text{for } E \in \Sigma. \]

Conversely, if \( U \) satisfies (24) and \( \alpha \) satisfies (25), then \( U \) is an isometry of \( L^p \).

**THEOREM 5.** If \( 1 < p < \infty, p \neq 2, \) and \( A \) is a nonzero operator on \( L^p \), then \( A \) is adjoint abelian if and only if there exists a regular set isomorphism \( T \), a measurable function \( \alpha \) and a real number \( \lambda \) such that

\[ Af(t) = \lambda \alpha(t)f \circ T(t) \quad \text{a.e. for } f \in L^p \]

where

\[ \alpha(t) \alpha \circ T(t) = 1 \quad \text{a.e.,} \]

\[ \int_{TE} |\alpha|^p d\mu = \mu(E) \quad \text{for } E \in \Sigma, \]

\[ T \circ T(E) = E \quad \text{(modulo sets of measure zero).} \]

**PROOF.** If \( A \) is adjoint abelian, then by Theorem 4 and Lemma 1, \( A = \lambda U \) where \( U \) is an isometry and \( \lambda \) is real. By Lamperti's theorem, there is an \( \alpha \) and a regular set isomorphism \( T \) so that (24) and (25) are satisfied for \( U \); hence (26) and (28) must hold. Since \( A^2 = \lambda^2 I \), we have from (26) that

\[ \lambda^2 f(t) = (A^2 f)(t) = \lambda^2 \alpha(t) \alpha \circ T(t)f \circ T \circ T(t) \quad \text{a.e.} \]

for each \( f \in L^p \). If \( E \) is any subset of finite measure, we may take \( f = \chi_E \) so that

\[ \chi_E(t) = \alpha(t) \alpha \circ T(t) \chi_{TE} \circ T(E) \quad \text{a.e.} \]
It follows from this that $T \circ T(E) = E$ modulo a set of measure zero and $\alpha(t)\alpha \circ T(t) = 1$ a.e. on sets of finite measure. The extension to sets of infinite measure follows readily from the $\sigma$-finiteness of $\Omega$, and (27), (29) are established.

Next suppose (26), (27), (28), and (29) are satisfied by $A$, $\alpha$, $T$, $\lambda$. If $E \in \Sigma$, we have

$$\int_\Omega \chi_E \circ T = \int_\Omega \chi_{TE} = \mu(TE)$$

$$= \int_{T \circ T(E)} |\alpha|^P \text{ by (28)}$$

$$= \int_E |\alpha|^P \text{ by (29)}$$

$$= \int_\Omega |\alpha|^P \chi_E.$$  

In the same way, it can be shown that if $f$ is a simple function with support of finite measure then

$$\int_\Omega f \circ T = \int_\Omega |\alpha|^P f$$

and finally, if $|\alpha|^P f$ is integrable, then $f \circ T$ is integrable and

$$\text{(31)} \quad \int_\Omega f \circ T = \int_\Omega |\alpha|^P f \quad \text{for all } f \in L^P$$

since any measurable $f$ is the a.e. limit of a sequence of simple functions with finite support [9, p. 224].

Now suppose $f, g \in L^P$. Then using the given conditions along with (31) and the fact that $T$ distributes across products, we obtain

$$[A f, g] = \lambda \|g\| \int \alpha(f \circ T) \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{ sgn } g$$

$$= \lambda \|g\| \int \alpha f \circ T |\alpha|^P |\alpha \circ T|^p |\alpha | \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{ sgn } \text{ sgn}(\alpha \circ T) \text{ sgn } g$$

$$= \lambda \|g\| \int |\alpha|^P(f \circ T) |\alpha \circ T|^p |\alpha | \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{ sgn } (\alpha \circ T) \text{ sgn } g$$

$$= \|A g\| \int (f \circ T) |\alpha \circ T|^p |\alpha | \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{ sgn } (\alpha \circ T) \text{ sgn } g$$

$$= \|A g\| \int f \left( \frac{|\alpha|^P |\alpha|^P |\alpha \circ T|^p |\alpha | \left( \frac{|g|}{\|g\|} \right)^{p-1} \text{ sgn } (\alpha \circ T) \text{ sgn } g$$

$$= \|A g\| \int f \left( \frac{|A g|}{\|A g\|} \right) \frac{\lambda}{|\lambda|} \text{ sgn } (\alpha \circ T) \text{ sgn } g$$

and $A$ is adjoint abelian.
A characterization of adjoint abelian operators on $l^p$ which is included in Theorem 5, has been obtained previously in [1] and [4].

The results above are related to some recent work of Byrne and Sullivan [2] on contractive projections on $l^p$. A projection $P$ on $l^p$ is called contractive if $\|P\| = 1$. An isometry $U$ with the property that $U^2 = I$ is called a reflection. In [2], it is proved that $P$ and $I - P$ are contractive if and only if $P = (I + U)/2$ for some reflection $U$. The next two corollaries are immediate from Theorems 4 and 5 and the work in [2].

**Corollary 2.** A nonzero operator $A$ on $l^p$ is adjoint abelian if and only if $A$ is a real multiple of a reflection.

**Corollary 3.** Both $P$ and $I - P$ are contractive projections on $l^p$ if and only if there is a real number $\lambda$ and an adjoint abelian operator $A$ such that $P = (I + \lambda A)/2$.

Stampfli [12] has proved that an operator $B$ on a weakly complete Banach space has a proper invariant subspace if it commutes with an adjoint abelian operator $A$ where $A \neq \lambda I$.

**Corollary 4.** Let $B$ be a bounded operator $L^p(\Omega, \Sigma, \mu)$ where $1 < p < \infty$, $p \neq 2$. If there exists a regular, measure preserving set isomorphism $T$ such that $T$ is not the identity, $T \circ T(E) = E$ modulo sets of measure zero for all $E \in \Sigma$, and

$$B(f \circ T) = Bf \circ T \text{ for all } f \in L^p,$$

then $B$ has a proper invariant subspace.

**Proof.** If we define $A$ on $L^p$ by $Af = f \circ T$, then (26), (27), (28), and (29) are satisfied by taking $\alpha = 1$. Hence $A$ is adjoint abelian and (32) is simply the condition that $A$ commutes with $B$.

4. Adjoint abelian operators and isometries. We have shown that adjoint abelian operators on $C(K)$ and $L^p(\Omega, \Sigma, \mu)$ are multiples of isometries. This is also the case for adjoint abelian operators on certain spaces of class $S$ discussed in [4]. The next theorem characterizes the types of isometries which can give rise to adjoint abelian operators in this manner.

**Theorem 6.** Let $U$ be an isometry on the Banach space $X$ and $\lambda$ a real scalar. The operator $A = \lambda U$ is adjoint abelian if and only if $U^2 = I$.

**Proof.** Let $U$ be an isometry with $U^2 = I$. By a theorem of Koehler and Rosenthal [5] there exists a s.i.p $[\cdot, \cdot]$ compatible with the norm such that

$$[Ux, Uy] = [x, y] \text{ for all } x, y \in X.$$
Hence,
\[
[\lambda Ux, y] = [\lambda Ux, U^2y] \\
= [\lambda x, Uy] \quad \text{from (33)} \\
= [x, \lambda Uy].
\]

Next suppose \( A = \lambda U \) is adjoint abelian and \( U \) is an isometry. Then for every \( x \in X \)
\[
(34) \quad [(\lambda U)^2 x, x] = [\lambda Ux, \lambda Ux] = \|\lambda Ux\|^2 = \lambda^2 \|x\|^2.
\]
It follows from (34) that
\[
(35) \quad [(U^2 - I)x, x] = 0 \quad \text{for every } x.
\]
By Theorem 5 of [7] we conclude that \( U^2 = I \).

One could give slightly different proofs of the "sufficiency" parts of Theorems 2 and 5 by using Theorem 6, showing that the given conditions characterize reflections.

From Theorem 6 above and Theorem 1 of [11] the next corollary is immediate.

**COROLLARY 5.** Every adjoint abelian operator which is a multiple of an isometry is necessarily a scalar operator.

In particular, as mentioned in the introduction, every adjoint abelian operator on \( C(K) \) or \( L^p \) is scalar.

In conclusion, we raise the following question: On what Banach spaces is every adjoint abelian operator a real multiple of an isometry?

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**BIBLIOGRAPHY**


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