CONTINUOUS COHOMOLOGY FOR COMPACTLY SUPPORTED VECTORFIELDS ON \( \mathbb{R}^n \)

BY

STEVEN SHNIDER

ABSTRACT. In this paper we study the Gelfand-Fuks cohomology of the Lie algebra of compactly supported vectorfields on \( \mathbb{R}^n \) and establish the degeneracy of a certain spectral sequence at the \( E_1 \) level. We apply this result to the study of another spectral sequence introduced by Resetnikov for the cohomology of the algebra of vectorfields on \( \mathbb{S}^m \).

Let \( L \) be the Lie algebra of compactly supported smooth vectorfields on a manifold \( M \). For \( U \) a precompact open subset of \( M \) let \( L_U \) be the set of vectorfields supported in \( U \) with the \( C^\infty \) topology, then \( L = \bigcup_{U \subset M} L_U \) and we give \( L \) the topology of a strict inductive limit. Let \( C^q(L) \) be the vector space of all continuous skewsymmetric \( \mathbb{R} \)-multilinear functions from \( L \times \cdots \times L \) (\( q \) times) into \( \mathbb{R} \). Define

\[
d^q : C^q(L) \to C^{q+1}(L),
\]

\[
(d^q \lambda)(\xi_1, \ldots, \xi_{q+1}) = \sum (-1)^{i+j} \lambda([\xi_i, \xi_j], \ldots, \xi_i, \ldots, \xi_j, \ldots, \xi_{q+1})
\]

where \([ , , ]\) denotes the Lie bracket of vectorfields and \( ^\wedge \) indicates omission. Then \( d^{q+1} \circ d^q = 0 \) and \( C^*(L) = \bigoplus_{q=0}^{\infty} C^q(L) \) is a differential complex with differential \( d = \bigoplus d^q \). The cohomology of \( (C^*(L), d) \) is known as the Gelfand-Fuks cohomology of \( L \) with coefficients in \( \mathbb{R} \).

Let \( pr_i : M^q \to M \) be the projection on the \( i \)th factor of the \( q \)-fold cartesian product of \( M \) and let \( pr_i^*T \) be the pull-back of the tangent bundle to \( M \) along \( pr_i \). Define \( T^q = pr_1^*T \otimes \cdots \otimes pr_q^*T \) as a bundle over \( M^q \). A vectorfield \( \xi \) on \( M \) defines a section \( pr_i^*T \) in a natural way and a \( q \)-tuple \((\xi_1, \ldots, \xi_q)\) of vectorfields defines a section \( pr_1^*\xi_1 \otimes \cdots \otimes pr_q^*\xi_q \) of \( T^q \) over \( M^q \). Linear combinations of sections of this type are dense in the space of compactly supported sections of \( T^q \), denoted \( [T^q]_C \), with the inductive limit topology defined similarly to that on \( L = [T]_C \). Thus an element \( \lambda \in C^q(L) \) defines a continuous function...
\( \tilde{\lambda} : [T^q]_C \rightarrow \mathbb{R} \). If \( \text{Hom}_R([T^q]_C, \mathbb{R}) \) denotes the continuous \( \mathbb{R} \) multilinear functions, then we have a map \( C^q(L) \rightarrow \text{Hom}_R([T^q]_C, \mathbb{R}) \). If we let \( B^q(L) \) denote the set of not necessarily skewsymmetric continuous \( \mathbb{R} \)-multilinear functions \( L \times \cdots \times L \rightarrow \mathbb{R} \), then we have an isomorphism:

\[
(1) \quad B^q(L) \cong \text{Hom}_R([T^q]_C, \mathbb{R}).
\]

Let \( \Sigma_q \) be the permutation group on \( q \)-letters and corresponding to \( \sigma \in \Sigma_q \) and \( \lambda \in B^q(L) \) let \( \sigma \circ \lambda \in B^q(L) \) be defined by

\[
(\sigma \circ \lambda)(\xi_1, \ldots, \xi_q) = e_{\sigma} \lambda(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(q)})
\]

where \( e_{\sigma} \) is the sign of \( \sigma \) as a permutation. With these definitions \( C^q(L) \) is the subspace of \( \Sigma_q \) invariants in \( B^q(L) \).

\[
(2) \quad B^q(L)^{\Sigma_q} = C^q(L).
\]

Let \( \mathcal{D}(M^q) \) be the space of distributions on \( M^q \),

\[
\mathcal{D}(M^q) = \text{Hom}_R(C^\infty_0(M^q), \mathbb{R}) = \text{Hom}_R([1]_C, \mathbb{R}).
\]

Consider \( C^\infty_0(M^q) \) as a left \( C^\infty(M^q) \) module making \( \mathcal{D}(M^q) \) a right \( C^\infty(M^q) \) module. Then

\[
\text{Hom}_R([T^q]_C, \mathbb{R}) = \text{Hom}_R([T^q] \otimes C^\infty_0(M^q) [1]_C, \mathbb{R})
\]

\[
= \text{Hom}_C^\infty(M^q)([T^q], \text{Hom}([1]_C, \mathbb{R}))
\]

\[
= \text{Hom}_C^\infty(M^q)([T^q], \mathcal{D}(M^q)) \cong \mathcal{D}(M^q) \otimes C^\infty_0(M^q) [T^q^*].
\]

Let \( \Sigma_q \) act on \( M^q \) by permuting factors \( \sigma(x_1, \ldots, x_q) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(q)}) \). This induces an action on \( C^\infty_0(M^q) \) and by duality on \( \mathcal{D}(M^q) \). Let \( \Sigma_q \) act on \( T^q^* \) by permuting factors and multiplying by \( e_{\sigma} \), then for \( \omega_1 \otimes \cdots \otimes \omega_q \in [T^q^*], \xi_1 \otimes \cdots \otimes \xi_q \in [T^q]_C \) and \( u \in \mathcal{D}(M^q) \),

\[
\sigma(u \otimes \omega_1 \otimes \cdots \otimes \omega_q)[\xi_1 \otimes \cdots \otimes \xi_q] = e_{\sigma}(\sigma \circ u \otimes \omega_{\sigma^{-1}(1)} \otimes \cdots \otimes \omega_{\sigma^{-1}(q)})[\xi_1 \otimes \cdots \otimes \xi_q]
\]

\[
= e_{\sigma}(\sigma \circ u)[(\omega_{\sigma^{-1}(1)}, \xi_1)_{x_1} \cdots (\omega_{\sigma^{-1}(q)}, \xi_q)_{x_q}]
\]

\[
= e_{\sigma}(\sigma \circ u)[(\omega_{\sigma^{-1}(1)}, \xi_{\sigma(1)})_{x_1} \cdots (\omega_{\sigma^{-1}(q)}, \xi_{\sigma(q)})_{x_q}]
\]

\[
= e_{\sigma}(\sigma \circ u)(\omega_1, \xi_{\sigma(1)})_{x_1} \cdots (\omega_q, \xi_{\sigma(q)})_{x_q}
\]

\[
= e_{\sigma}(\sigma \circ u) \omega_1 \otimes \cdots \otimes \omega_q [\xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(q)}].
\]

Therefore

\[
(3) \quad (\mathcal{D}(M^q) \otimes C^\infty_0(M^q) [T^q^*])^{\Sigma_q} \cong C^q(L).
\]
To compute the cohomology of $C^q(L)$ we use the spectral sequence defined as follows. Let $\mathcal{D}(M^q)|_{M^q}$ be the distributions with support on the subset $M^q_k = \{(x_1, \ldots, x_q)\}$ at most $k$ of the points $x_i \in M$. Set

$$C^q_k(L) = (\mathcal{D}(M^q)|_{M^q_k} \otimes \mathcal{C}^\omega(M^q) [T^*])^\subset,$$

then $C^q_k(L) \subset C^q_{k+1}(L)$ and $d^q C^q_k(L) \subset C^q_{k+1}(L)$. If we define $F^{-k} C^q = C^q_k$ we have a decreasing filtration preserved by the differential and thus a cohomology spectral sequence.

Note that $M^q_k$ is a union of submanifolds. In fact if $S$ is a partition of $q$ elements into $k$ sets, let $M^o_S$ be the set of points in $M^q$ consisting of $(x_1, \ldots, x_q)$ such that if $i, j$ are in the same subset of the partition then $x_i = x_j$. There is an obvious diffeomorphism of $M^k$ and $M^o_S$, and $M^o_S = \bigcup S$ a partition of $M^q$. Any element of $\mathcal{D}(M^q)|_{M^q_S}$ can be written as a sum of normal derivatives of distributions on $M^o_S$, see Schwartz [4]. P. Trauber in his Princeton thesis [6] has used the isomorphism (4) and this fact to give a nice description of the $E_0$ term of the spectral sequence and then applied the methods of relative homological algebra to compute $E_1$. We summarize his results below, making the obvious extension to the case of compactly supported vectorfields. Let $D(M)$ be the differential operators on $M$, not necessarily of finite order, topologized as follows. For $U$ a precompact open subset of $M$, let $D^k(U)$ be the differential operators of at most order $k$ on smooth functions with support in $U$. As sections of a vector bundle $D^k(U)$ has a nuclear locally convex topology and so the inductive limit $D(U) = \lim_k D^k(U)$ does also. For $U \subset V$ there is a restriction map $D(V) \to D(U)$ and the precompact open subsets of $M$ together with these restriction maps form a directed system. Let $D(M) = \lim_{U \subset M} D(U)$, as a projective limit of nuclear spaces it is a nuclear space. If we use the cofinal family $U^q = U \times \cdots \times U$ ($q$ times) of precompact open sets on $M^q$ to define the topology on $D(M^q)$, then because

$$D^k(U^q) \cong D^k(U) \hat{\otimes} \cdots \hat{\otimes} D^k(U)$$

and $\hat{\otimes}$ is an exact functor we have $D(U^q) \cong D(U) \hat{\otimes} \cdots \hat{\otimes} D(U)$ and $D(M^q) \cong D(M) \hat{\otimes} \cdots \hat{\otimes} D(M)$. Similarly $[T^q]^* \cong [T]^* \hat{\otimes} \cdots \hat{\otimes} [T]^*$. Let $D(M^q)|_{M^q_S}$ be the differential operators $C^0_0(M^q) \to C^0_0(M^q_S)$. Composition on the left defines a left $D(M^q_S)$ module structure on $D(M^q)|_{M^q_S}$ and $C^\omega(M^q_S) \subset D(M^q_S)$.

Relative to these structures we have the following

**Proposition (Trauber [6]).**

(a) $\mathcal{D}(M^q)|_{M^q_S} \cong \mathcal{D}(M^q)|_{M^q} \otimes_{D(M^q)} D(M^q)|_{M^q_S}$,

(b) $D(M^q)|_{M^q_S} \cong C^\omega(M^q_S) \otimes_{C^\omega(M^q)} D(M^q)$,

where the $C^\omega(M^q)$ module structure on $C^\omega(M^q_S)$ is restriction followed by multiplication. Using these isomorphisms we have
$U(M^q) \otimes_{C^\infty(M^q)} [T^q]$ 

$\cong U'(M_S^q) \otimes_{D(M_S^q)} C^\infty(M_S^q) \otimes_{C^\infty(M^q)} D(M^q) \otimes_{C^\infty(M^q)} [T^q]$ 

$\cong U'(M_S^q) \otimes_{D(M_S^q)} C^\infty(M_S^q) \otimes_{C^\infty(M^q)} (D(M) \otimes \cdots \otimes D(M))$ 

$\otimes_{C^\infty(M)} \cdots \otimes_{C^\infty(M)} ([T^*] \otimes \cdots \otimes [T^*])$ 

$\cong U'(M_S^q) \otimes_{D(M_S^q)} C^\infty(M_S^q) \otimes_{C^\infty(M^q)} D(M) \otimes_{C^\infty(M^q)} [T^*]$ 

$\otimes \cdots \otimes D(M) \otimes_{C^\infty(M)} [T^*]$.

Let $D \otimes T^* = D(M) \otimes_{C^\infty(M)} [T^*]$ and let $X$ be the elements of positive degree in the exterior algebra over $C^\infty(M)$ of $D \otimes T^*$ let $X^k = X \otimes \cdots \otimes X$ ($k$ times) and let $X^k(q)$ be the subspace of $X^k$ consisting of elements with $q$ factors of $T^*$. Trauber proves the following

**Theorem (Trauber [6]).**

(a) $C^k(L) \cong (U(M^q))_{M^q_S} \otimes_{C^\infty(M^q)} [T^q^*] \cong (U'(M^k) \otimes_{D(M^k)} X^k(q))^\Sigma_k$,

(b) $\frac{F_{-k} C^*(L)}{F_{-k+1} C^*(L)} \cong \left( U'(M^k) \otimes_{D(M^k)} X^k \right)^\Sigma_k$.

He also points out the following interpretation of the isomorphism (a).

Let $J^k(T)$ be the bundle of $k$-jets on $M$, for $U$ a precompact open set let $[J^k(T)]_U$ be the sections with support in $U$, this is a Frechet nuclear space. Define $[J^\infty(T)]_C = \lim_{\searrow} [J^k(T)]_U$. This is a nuclear l.c.s. such that

$$D \otimes T^* = \text{Hom}_{C^\infty(M)}([J^\infty(T)]_C, C^\infty(M)).$$

There is a continuous function $j^\infty: [T]_C \mapsto [J^\infty(T)]_C$ which associates to any compactly supported vectorfield its infinite jet at each point. The bundle $J^\infty(T)$ has a canonical connection $\triangledown: [J^\infty(T)]_C \mapsto [T^* \otimes J^\infty(T)]_C$ introduced by Spencer, see [2]. If $\xi \in [J^\infty(T)]_C$ then $\tilde{\xi} = j^\infty(\xi)$ for some $\xi \in [T]_C$ if and only if $\triangledown \tilde{\xi} = 0$ in $[T^* \otimes J^\infty(T)]_C$. The connection $\triangledown$ has 0 curvature and thus gives a representation of $D(M)$ on $[J^\infty(T)]_C$. The image of $j^\infty$ is the subspace of $D(M)$ invariants in $[J^\infty(T)]_C$. Using the isomorphism $D(M^q) \cong D(M) \otimes \cdots \otimes D(M)$ we get a representation of $D(M^q)$ on $[J^\infty(T)]_C$.

(2) For any vector bundle $E$ with connection $\triangledown: E \mapsto T^* \otimes E$ we write $\triangledown_X$ for the germ of a differential operator $(\triangledown_X)(\xi) = (\triangledown_X(\varphi)(X)) \in E_\varphi$ where $X \in T_\varphi$ and $\varphi \in E_\varphi$. If $\triangledown_X \triangledown_Y - \triangledown_Y \triangledown_X - \triangledown_{[X,Y]} = 0$ we say the connection has curvature zero and we get a Lie algebra representation of $[T] \mapsto [\text{Diff } E]$ differential operators on $E$. This extends to a representation $D(M) \mapsto [\text{Diff } E]$. 

---

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
which we will also denote by \( V \) also. For \( \eta_1 \otimes \cdots \otimes \eta_q \in [U^\infty(T)]_C \otimes \cdots \o [U^\infty(T)]_C \) and \( \xi_1 \otimes \cdots \otimes \xi_q \in D(M) \otimes \cdots \otimes D(M) \),

\[
\nabla_{\eta_1 \otimes \cdots \otimes \eta_q} \xi_1 \otimes \cdots \otimes \xi_q = \nabla_{\eta_1} \xi_1 \otimes \cdots \otimes \nabla_{\eta_q} \xi_q.
\]

Now \( L_C \xrightarrow{j^\infty} [U^\infty(T)]_C \) is a Lie algebra map; therefore there is a cochain map \( C^q([U^\infty(T)]_C) \xrightarrow{j^\infty} C^q(L) \) which is the same as

\[
\mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [U^\infty(T)]_C^* \otimes \cdots \otimes [U^\infty(T)]_C^* \xrightarrow{(j^\infty)^*} \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^*.
\]

or equivalently

\[
\mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} D \otimes T^* \otimes \cdots \otimes D \otimes T^*
\]

\[
\xrightarrow{(j^\infty)^*} \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^*.
\]

Since the image of \( j^\infty \) is the subspace of \( D(M) \) invariants it is not hard to see that \( (j^\infty)^* \) factors through the tensor product over \( D(M^q) \) to give an isomorphism

\[
\mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} D \otimes T^* \otimes \cdots \otimes D \otimes T^* \rightarrow \mathcal{U}'(M^q) \otimes_{C^\infty(M^q)} [T^q]^*.
\]

This allows us to identify the differential on the complex \( X \) appearing in the previous theorem: \( X \) is the exterior algebra on \([U^\infty(T)]_C^*\) and the differential \( d_X \) on \( X \) is the usual coboundary operator in the cochain complex on the dual of a Lie algebra. We can restate the previous theorem

\[
\mathcal{U}'(M^k) \otimes_{D(M^k)} D^+ \otimes [U^\infty(T)]_C^* \otimes \cdots \otimes [U^\infty(T)]_C^* \xrightarrow{(\Sigma)^*} F^{-k}C^* \rightarrow F^{-k-1}C^* (L)
\]

as cochain complexes with the isomorphism induced by \((j^\infty)^*\).

To compute \( H^*(F^{-k}/F^{-k+1}C^*) \) we note that \( X^k \) is flat as a \( D(M^k) \) module since \( X = \Lambda^+ D \otimes T^* \) is flat as a \( D \) module in each degree of the exterior power. Therefore the higher derived functors of \( \otimes_{D(M^k)} X^k \) in the category of differential complexes vanish.

\[
\text{Tor}_p^{D(M^k)}(A, X^k) = 0, \quad p > 0,
\]

\[
\text{Tor}_0^{D(M^k)}(A, X^k) = H^*(A \otimes_{D(M^k)} X^k, d_{X^k}).
\]

However we can also compute the differential derived functor by resolving \( X^k \).

Let \( Y_p = D(M^k) \otimes \Lambda^p [T(M^k)] \) define \( \partial_p: Y_p \rightarrow Y_{p-1} \) by

\[
\partial_p(u \otimes \xi_1 \wedge \cdots \wedge \xi_p) = \sum (-1)^{i-j} u \xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p
\]

\[
\cdot \sum (-1)^{1+j} u \otimes [\xi_j, \xi_i] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_p,
\]

\[
\wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p.
\]
Then $Y = \bigoplus Y_p$ gives a resolution of $C^\infty(M^k)$ as a left $D(M^k)$ module and tensoring on the right over $C^\infty(M^k)$ with $X^k$ we get a resolution:

$$D(M^k) \otimes_{C^\infty(M^k)} \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k$$

(9)

$$\varepsilon_0$$

$$X^k$$

Let $A$ be a right $D(M^k)$ module then tensoring on the left over $D(M^k)$ with $A$

$$A \otimes_{C^\infty(M^k)} \Lambda^*[T(M^k)] \otimes_{C^\infty(M^k)} X^k$$

(10)

$$\text{id} \otimes \varepsilon_0$$

$$A \otimes_{D(M^k)} X^k$$

as an augmented complex with homology (making $X^k$ a chain complex using negative indexing) equal to

$$\text{Tor}^1_{D(M^k)}(A, X^k) = H_q(A \otimes_{D(M^k)} X^k).$$

Computing the $\partial$ spectral sequence of the double complex we have

$$E^1_{p, -q} \cong A \otimes \Lambda^p[T(M^k)] \otimes_{C^\infty(M^k)} H^{-q}(X^k).$$

Here we need an additional fact. Let $L$ be the algebra of formal power series vectorfields, i.e., the fiber of $J^\infty(T)$ over a point of $M$, $L = \lim_k J^k(T)_x$. Let $L^* = \lim_k J^k(T)^*_x$, then $H(X) \cong C^\infty(M) \otimes_R H(\Lambda^+ L^*)$ and the $D(M)$ module structure on $H(X)$ is trivial, see [5] or [1a, pp. 205–206]. Therefore, we have

$$H(X^k) \cong C^\infty(M^k) \otimes H(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*$$

with $D(M^k)$ acting trivially. Hence

$$E^2_{p, -q} \cong H(A \otimes C^\infty(M^k) \Lambda[T(M^k)]) \otimes_R H(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*),$$

(11)

$$E^\infty_{p, q} \cong \text{Gr}_q H(A \otimes_{D(M^k)} X^k).$$

Let $\mathcal{D}(M^k - M^k_{k-1})$ be the distributions on $M^k - M^k_{k-1}$ which extend to distributions on $M^k$. The inclusion $i: C^\infty_0(M^k - M^k_{k-1}) \to C^\infty_0(M^k)$ induces an isomorphism

$$\mathcal{D}'(M^k)/\mathcal{D}'(M^k)|_{M^k_{k-1}} \cong \mathcal{D}'(M^k - M^k_{k-1}).$$

Since $\mathcal{D}'(M^k - M^k_{k-1})$ is dense in $\mathcal{D}'(M^k - M^k_{k-1})$ and $\mathcal{D}'(M^k - M^k_{k+1}) \otimes_{C^\infty(M^k)}$
Let \( F^{-k}C^*(L)/F^{-k+1}C^*(L) \) be considered as a chain complex using negative indexing; then there is a homology spectral sequence with

\[
E^2_{p,-q} \cong (H^p_{C}(M^k - M^k_{k-1}) \otimes \Lambda^q(T(M^k))) \otimes \Lambda^p_{R}(C^*(M^k)) \otimes \Lambda^q_{R}(C^*(M^k)) \\
E^{\infty}_{p,-q} = \text{Gr}_{p}(H^q_{R}(F^{-k}/F^{-k+1})).
\]

In the special case when \( M = R^n \) we have \( X \cong C'^{\infty}(M) \otimes R \Lambda^+L^* \) and \( X^k \cong C'^{\infty}(M^k) \otimes R \Lambda^+L^* \). This gives the following isomorphism

\[
\mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C'^{\infty}(M^k)} \Lambda[T(M^k)] \otimes_{C'^{\infty}(M^k)} X^k \\
\cong \mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C'^{\infty}(M^k)} \Lambda[T(M^k)] \otimes_{C'^{\infty}(M^k)} C'^{\infty}(M^k) \\
\otimes_{R} \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^* \\
\cong (\mathcal{D}'(M^k - M^k_{k-1}) \otimes_{C'^{\infty}(M^k)} \Lambda[T(M^k)]) \otimes_{R} (\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*). 
\]

One can apply the Kunneth theorem to the latter complex, therefore its homology is

\[
H(\mathcal{D}'(M^k - M^k_{k-1}) \otimes \Lambda[T(M^k)]) \otimes_{R} H^*(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*) \\
and we conclude that \( E^2 = E^{\infty} \).
\]

**Theorem 2.** If \( L \) is the Lie algebra of compactly supported vector fields on \( R^n \), then with respect to the filtration defined earlier there is a spectral sequence with

\[
E^{-k, l+k}_{1} = H^l \frac{F^{-k}C^*(L)}{F^{-k+1}C^*(L)} \\
\cong \bigoplus_{q-p=1} [H^p_{C}(R^n)^k - (R^n)^{k-1}_C]^* \otimes_{R} H^q(\Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*)] \Sigma^k.
\]
We will give an explicit expression for this isomorphism and show that the spectral sequence collapses at $E_1$.

When $M = \mathbb{R}^n$ we can find a global basis $[T(M^k)]$ as a $C^\infty(M^k)$ module which consists of commuting vectorfields; then

$$[T(M^k)] \cong C^\infty(\mathbb{R}^{nk}) \otimes \mathbb{R}^{nk}, \quad \Lambda[T(M^k)] \cong C^\infty(\mathbb{R}^{nk}) \otimes_{\mathbb{R}} \Lambda\mathbb{R}^{nk}.$$  

Let $\widetilde{\mathcal{X}}^k = C^\infty(\mathbb{R}^{nk}) \otimes \Lambda(L \oplus \cdots \oplus L)^*,$ i.e., the full exterior algebra. It is clear that $X^k$ is a direct summand of $\widetilde{\mathcal{X}}^k$ as a $D(M^k)$ module. Let $j$ be the inclusion and $\pi$ the projection $X^k \xrightarrow{j} \widetilde{\mathcal{X}}^k \xrightarrow{\pi} X^k$. Both $i$ and $\pi$ are cochain maps. Since $L \cong \mathbb{R}^n \oplus L^0$ we have $L \oplus \cdots \oplus L \cong \mathbb{R}^{nk} \oplus L^0 \oplus \cdots \oplus L^0$ and there is an obvious interior product $\Lambda\mathbb{R}^{nk} \otimes_{\mathbb{R}} \widetilde{\mathcal{X}}^k \rightarrow \widetilde{\mathcal{X}}^k$. Using the isomorphisms given above we get a map

$$\tilde{\iota}: \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} \mathbb{R}^{nk} \rightarrow \widetilde{\mathcal{X}}^k.$$  

Composing on the right with $\text{id} \otimes j$ and on the left with $\pi$ we get

$$\iota: \Lambda[T(M^k)] \otimes_{C^\infty(M^k)} X^k \rightarrow X^k$$  

which we will denote

$$\iota: \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \mapsto \xi_1 \wedge \cdots \wedge \xi_p \downarrow \alpha.$$  

Tensoring on the left over $C^\infty(M^k)$ with $D(M^k)$

$$\text{id} \otimes \iota: D(M^k) \otimes_{C^\infty(M^k)} [T(M^k)] \otimes_{C^\infty(M^k)} X^k \rightarrow D(M^k) \otimes_{C^\infty(M^k)} X^k.$$  

Composition with the left module structure on $X^k$ with $D(M^k) \otimes X^k \rightarrow X^k$ gives

$$\psi: D(M^k) \otimes_{C^\infty(M^k)} [T(M^k)] \otimes_{C^\infty(M^k)} X^k \rightarrow X^k,$$

$$u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha \mapsto u(\xi_1 \wedge \cdots \wedge \xi_p \downarrow \alpha).$$  

We will show that $\psi$ is a cochain map. Passing to $\Sigma_k$ invariants we get an explicit isomorphism for the $E^1$ term of the spectral sequence given in the previous theorem.

The map $\psi$ is defined with respect to a fixed parallelisation of $T(M^k)$, with respect to which we have

$$D(M^k) \otimes_{C^\infty(M^k)} [T(M^k)] \otimes_{C^\infty(M^k)} X^k$$

$$\cong D(\mathbb{R}^{nk}) \otimes_{\mathbb{R}} \Lambda\mathbb{R}^{nk} \otimes_{\mathbb{R}} \Lambda^+L^* \otimes \cdots \otimes \Lambda^+L^*.$$  

The differential is given by

$$d(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) = \sum (-1)^{i-1} u \xi_i \otimes \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_p \otimes \alpha$$

$$+ (-1)^{i} u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes d_L \alpha$$  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
where \( d_L \) is the differential in \( \Lambda L^* \otimes \cdots \otimes \Lambda L^* \),

\[
d\psi(u \otimes \xi_1 \wedge \cdots \wedge \xi_p \otimes \alpha) = d(u(\xi_1 \wedge \cdots \wedge \xi_p \perp \alpha)) = ud(\xi_1 \wedge \cdots \wedge \xi_p \perp \alpha) = u \left( \sum (-1)^{p-1}(\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p \perp \alpha \partial \xi_i) + (-1)^p(\xi_1 \wedge \cdots \wedge \xi_p \perp d L \alpha) \right).
\]

By definition \( ad \) is the adjoint representation of \( L \oplus \cdots \oplus L \) on \( \Lambda(L \oplus \cdots \oplus L)^* \) dual to the adjoint representation of \( L \oplus \cdots \oplus L \) on \( \Lambda(L \oplus \cdots \oplus L) \). For \( \alpha \in \Lambda(L \oplus \cdots \oplus L)^* \) and \( \xi_1, \ldots, \xi_p \in \mathbb{R}^{nk} \) we have \( ad \xi_i \alpha = \xi_i \cdot \alpha \) where \( \cdot \) indicates the module structure and \( \xi_i \) are considered as constant coefficient differential operators. Furthermore \( (\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_p \perp \alpha \partial \xi_i) = \) \( ad \xi_i(\xi_1 \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_p \perp \alpha) \), thus \( \psi \) is a cochain map.

We can represent the induced map on cohomology

\[
[H^p_c(R^{nk} - (\mathbb{R}^n)^k_{k-1}) \otimes_R H^q(\Lambda^* + L^* \otimes \cdots \otimes \Lambda^* + L^*)]_{\Sigma_k} \to H^{q-p}(F^{-k}/F^{-k+1})
\]

more conveniently as follows. For \( \eta \in L, j^\infty(\eta) \in C_0^\infty(M) \otimes L \) so if \( \alpha \in \Lambda L^* \) we can form \( j^\infty(\eta) \perp \alpha \in C_0^\infty(M) \otimes \Lambda L^* \). For \( \alpha = \sum \alpha_1 \otimes \cdots \otimes \alpha_k \in \Lambda^* + L^* \otimes \cdots \otimes \Lambda^* + L^* \) and for \( S \) a partition \( \{a_1, \ldots, a_q\}(a_1, \ldots, b_{s_2}) \cdots (c_1, \ldots, c_{s_k}) \) of \( q \) into \( k \) sets it makes sense to partition a set of \( q \) vectorfield \( \eta_1, \ldots, \eta_q \) into \( \eta_{a_1}, \ldots, \eta_{a_{s_1}}, \ldots, \eta_{b_1}, \ldots, \eta_{b_{s_2}}, \ldots, \eta_{c_1}, \ldots, \eta_{c_{s_k}} \) and form

\[
\sum_i (f^\infty(\eta_{a_1}) \wedge \cdots \wedge f^\infty(\eta_{a_{s_1}}) \perp \alpha_1) \wedge (f^\infty(\eta_{b_1}) \wedge \cdots \wedge f^\infty(\eta_{b_{s_2}}) \perp \alpha_2) \wedge \cdots \wedge (f^\infty(\eta_{c_1}) \wedge \cdots \wedge f^\infty(\eta_{c_{s_k}}) \perp \alpha_k).
\]

We will write \( f^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_q) \perp \alpha \) to mean the interior product just defined. Let \( i: \mathbb{R}^{nk} \to L \oplus \cdots \oplus L \) be the injection defined earlier and \( \Lambda(L \oplus \cdots \oplus L)^* \to \Lambda R^{nk}^* \) the extension of the dual map to exterior algebras. Let \( \phi \) be the isomorphism

\[
C_0^\infty(R^{nk}) \otimes \Lambda R^{nk}* \xrightarrow{\phi} \Omega_c(R^{nk})
\]

given by the choice of a parallelism. Finally for \( S \), the partition above, let \( \varepsilon_S \) be the sign of the permutation

\[
\begin{pmatrix}
1 & \cdots & S_1 & \cdots & k - S_k & + 1 & \cdots & k \\
\end{pmatrix}
\]

Then for \( \lambda \in \mathcal{D}(R^{nk} - (\mathbb{R}^n)^k_{k-1}) \otimes \Lambda^p [T(M^{k})] \alpha \in (\Lambda^* + L^* \otimes \cdots \otimes \Lambda^* + L^*)^q \) we have
\[
\psi(\lambda \otimes \alpha)(\eta_1, \ldots, \eta_{q-p})
\]
(14)
\[
= \sum \epsilon_s \psi(\phi^* f^\infty(\eta_1) \land \cdots \land f^\infty(\eta_{q-p}) \land s \lambda)
\]
and
\[
\psi(d \lambda \otimes \alpha) + (-1)^{q-p} \psi(\lambda \otimes d_L \alpha) = d_0(\psi(\lambda \otimes \alpha))
\]
(15)
where \(d\) is the differential in \(\mathcal{P}(M^k - M_{k-1}^k) \otimes \Lambda[T(M^k)]\), \(d_L\) is the differential in \(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*\) and \(d_0\) is the differential in \(F^k C^*(L)/F^{-k+1} C^*(L)\).

Let \(v_i \in \mathbb{R}^n\) and \((v_1, \ldots, v_k) \in \mathbb{R}^{nk}\) and let \(R^{nk}_{i,j} = \{v_i \mid v_i = v_j\}\) then \((\mathbb{R}^n)^k = \bigcup_{i<j} R^{nk}_{i,j}\). Let \(R^{nk} \cup \{\infty\} = S^{nk}\) and \(R^{nk}_{i,j} \cup \{\infty\} = S^{nk}_{i,j}\), then
\[
H^p_c(\mathbb{R}^{nk} - (\mathbb{R}^n)_{k-1})^* = H^p_c \left( \mathbb{R}^{nk} - \bigcup_{i<j} R^{nk}_{i,j} \right)
\]
\[
= H^p_c \left( S^{nk} - \bigcup_{i<j} S^{nk}_{i,j} \right)
\]
\[
\cong H^p \left( S^{nk}, \bigcup_{i<j} S^{nk}_{i,j} \right).
\]

Hence
\[
H^p_c(\mathbb{R}^{nk} - (\mathbb{R}^n)_{k-1})^* \cong H^p \left( S^{nk}, \bigcup_{i<j} S^{nk}_{i,j} \right)
\]
and composing these isomorphisms with \(\psi\) we have
\[
\Phi: \left( H^p \left( S^{nk}, \bigcup_{i<j} S^{nk}_{i,j} \right) \otimes H^q(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*) \right)^{\Sigma k}
\]
(16)
\[
\rightarrow E_1^{-k,q-p+k}.
\]

For \(\sum_{i=1}^m [\sigma_i] \otimes [\alpha_i]\) an element of the left-hand side if we choose representative cycles \(\sigma_i\) and representative cocycles \(\alpha_i\) we get a representative element of \(\Phi(\sum_{i=1}^m [\sigma_i] \otimes [\alpha_i]).\)

\[
(\eta_1, \ldots, \eta_{q-p})
\]
(17)
\[
\rightarrow \sum_{j=1}^m \int \sum \epsilon_s \phi^* f^\infty(\eta_1) \land \cdots \land f^\infty(\eta_{q-p}) \land s \alpha_i).
\]

If we pull back \(d_1: E_1^{-k,h+k} \rightarrow E_1^{-k+1,h+k}\) by the isomorphism \(\Phi\) we get a mapping for \(q - p = h\),
It is computed as follows. For \( \eta_1, \eta_2, \ldots, \eta_{n+1} \in L, \)

\[
\Phi\left( \sum_{i=1}^{m} [\alpha_i] \otimes [\beta_i] \right)(\eta_1, \ldots, \eta_{n+1}) = \sum_{i<j<h+1} (-1)^{i+j}\Phi\left( \sum_{i=1}^{m} [\alpha_i] \otimes [\beta_i] \right)
\]

\[
= \sum_{i<j<h+1} \sum_{l=1}^{s} \int_{\alpha_i} e_S\phi^*(j^\infty([\eta_i, \eta_j]) \wedge j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_{h+1}) - S\alpha_i)
\]

\[
= \sum_{i<j<h+1} \sum_{l=1}^{m} \sum_{s} \int_{\alpha_i} e_S\phi^*([j^\infty(\eta_i), j^\infty(\eta_j)] \wedge j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_{h+1}) - S\alpha_i)
\]

Now \( \alpha_i \) is a tensor product of \( k \) cycles \( \alpha_{i,j} \in Z(\Lambda^+L^*) \). To compute the last term we see what is happening to each \( \alpha_{i,j} \). For \( \alpha \in Z^f(\Lambda L^*) \) and \( \eta_1, \ldots, \eta_S \in L \)

\[
\sum_{i<j<s}^{i+j} \phi^*([j^\infty(\eta_i), j^\infty(\eta_j)] \wedge j^\infty(\eta_1) \wedge \cdots \wedge j^\infty(\eta_{s+1}) - \alpha)
\]

\[
= \sum_{i<j<s} \sum_{i_1<i_2<\cdots<i_{t-s}<n} (-1)^{i+j}\alpha([j^\infty(\eta_i), j^\infty(\eta_j)], j^\infty(\eta_1) \cdots j^\infty(\eta_{t-s})
\]

\[
\cdots j^\infty(\eta_t) \cdots j^\infty(\eta_s), e_{i_1} \cdots e_{i_{t-s}}
\]

\[
dx_{i_1} \wedge \cdots \wedge dx_{t-s}
\]
COMPACTLY SUPPORTED VECTORFIELDS ON $\mathbb{R}^n$

\[ + \sum_{r,j} \sum_{i_1 < i_2 < \ldots < i_{t-S}} (-1)^{r+S+j} \alpha([e_{i_r}, f^{\alpha}(\eta_j)], f^{\alpha}(\eta_1) \cdots f^{\alpha}(\eta_t)) \]

\[ \cdots f^{\alpha}(\eta_t), e_{i_1} \cdots e_{i_{t-S}} \]

\[ \frac{\partial}{\partial x^r} \alpha(f^{\alpha}(\eta_1), \ldots, f^{\alpha}(\eta_S), e_{i_1} \cdots e_{i_{t-S}}) \]

\[ dx^{i_1} \wedge \cdots \wedge dx^{i_{t-S}} \]

\[ = d\phi^*(f^{\alpha}(\eta_1) \wedge \cdots \wedge f^{\alpha}(\eta_S)) \wedge \alpha. \]

This shows what happens to each factor of $\alpha_i$; hence the end product is

\[ \Phi(\bar{\alpha} \sum [\alpha_1] \otimes [\alpha_t])(n_1, \ldots, n_{h+1}) \]

\[ = \sum \sum \int_{\sigma^1} e_{\sigma^1} d\phi^*(f^{\alpha}(\eta_1) \wedge \cdots \wedge f^{\alpha}(\eta_{n+1}) \wedge \sigma^0) \]

\[ = \sum \sum \int_{\sigma^1} e_{\sigma^1} \phi^*(f^{\alpha}(\eta_1) \wedge \cdots \wedge f^{\alpha}(\eta_{n+1}) \wedge \sigma^0) \]

where $S'$ ranges over partitions of $h + 1$ elements into $k$ sets. We can decompose $\partial \sigma_1$ into a sum of $\partial_{(i,j)} \sigma_1$ where $|\partial_{(i,j)} \sigma_1| \subseteq S^{n-k-n}$. When $\phi^*(f^{\alpha}(\eta_1) \wedge \cdots \wedge f^{\alpha}(\eta_{n+1}) \wedge \sigma^0)$ is integrated over $S^{n-k-n}$, the $i$th and $j$th factors are identified by restricting to the diagonal in the product of the $i$th and $j$th factors. This gives a mapping

\[ H(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*) \cong H(\Lambda^+ L^* \otimes \cdots \otimes H(\Lambda^+ L^*)) \]

by multiplying the $i$th and $j$th factors, just as restriction to the diagonal induces the cup product in singular cohomology. Therefore the $\partial \alpha$ operator involves multiplication in the cohomology algebra of the formal Lie algebra. It is known that this multiplication is trivial [5], [7], so $\partial \alpha = 0$. In a similar way one can see that all the higher differentials involve multiplication in the formal algebra so we have

**Theorem 3.** There is a spectral sequence for the continuous cohomology of the algebra of compactly supported vectorfields on $\mathbb{R}^n$ which collapses at the $E_1$ level.

\[ E^{-k, l+k} \cong \bigoplus_{q-p=l} H_p \left( S^{n-k}_{i,j} \cup \bigcup_{i<j} S^{n-k-n}_{i,j} \right) \otimes \bigotimes_{k} H^+(L) \]

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let $L$ be the algebra of vector fields on the $n$ sphere $S^n$, let $p \in S^n$ and let $\widetilde{L}$ be the ideal of vector fields flat at $p$ in some, hence any, coordinate system. Let $C^*(L)$ be the Gelfand-Fuks complex for the continuous cohomology of $L$, and define a filtration

$$F^kC^q(L) = \{ \lambda \in C^q(L) \mid \lambda(\xi_1, \ldots, \xi_k) = 0 \text{ if } q - k + 1 \text{ of } \xi_i \text{ are in } \widetilde{L} \},$$

then $F^k \supset F^{k+1}$ and $dF^k \subset F^k$. This is the filtration defining the Hochschild-Serre spectral sequence for $H(L)$ with respect to the ideal $\widetilde{L}$.

$$E^p_{p,q} \simeq H^p(L/\widetilde{L}, H^q(\widetilde{L})), \quad E^p_{m,q} \simeq \text{Gr}_p(H^{p+q}(L)).$$

There is an exact sequence of Lie algebras

$$0 \rightarrow \widetilde{L} \rightarrow L \rightarrow L \rightarrow 0.$$ 

Thus $E^p_{p,q} \simeq H^p(L, H^q(\widetilde{L}))$. The action of $L$ on $H^q(\widetilde{L})$ is defined as follows: for $\eta \in L$ let $\widetilde{\eta} \in L$ be a vector field such that $j^\omega(\widetilde{\eta})_p = \eta$ then Lie derivation with respect to $\widetilde{\eta}$ is a map $D_{\eta}: \widetilde{L} \rightarrow \widetilde{L}$ which in turn defines a cochain map $D_{\eta}: C^*(\widetilde{L}) \rightarrow C^*(\widetilde{L})$ and therefore a map $D_\eta: H^*(\widetilde{L}) \rightarrow H^*(\widetilde{L})$. If $j^\omega(\widetilde{\eta}_1)_p = j^\omega(\widetilde{\eta}_2)_p$ then $\widetilde{\eta}_1 - \widetilde{\eta}_2 \in \widetilde{L}$ and as is well known $D_{\eta_1 - \eta_2}$ induces the trivial map in cohomology, so $D_{\eta_1} = D_{\eta_2}$. Reshetnikov [3] has stated the following theorem for arbitrary $M$ but it is not clear to us that his proof is correct.

**Theorem.** Since $L$ acts trivially on $H^*(\widetilde{L})$, the $E_2$ term of the previous spectral sequence is $E^p_{p,q} \simeq H^p(L) \otimes H^q(\widetilde{L})$. Furthermore if $L_C$ is the algebra of compactly supported vector fields on $\mathbb{R}^n$, then $H^q(\widetilde{L}) \simeq H^q(L_C)$.

**Proof.** Let $\{U_i\}$ be a decreasing sequence of open sets which form a neighborhood basis at $p$. Let $K_i = S^n - U_i$; then $K_i$ is compact, and if we define $\phi: S^n - \{p\} \rightarrow \mathbb{R}^n$ by stereographic projection with $p$ as north pole then the $\phi(K_i)$ form a compact exhaustion of $\mathbb{R}^n$. Let $L_i$ be the algebra of vector fields on $S^n$ with support in $K_i$, there are inclusions $\iota^i_j: L_j \rightarrow L_i$; therefore, we can define $L_\infty = \lim L_i$. Clearly $L_\infty \cong L_C$, compactly supported vector fields. Let $\psi^i: L_i \rightarrow \widetilde{L}$ be the inclusion; then $\psi^i \circ \iota^i_j = \psi^j$ so we can define $\psi: L_\infty \rightarrow \widetilde{L}$. This induces $\psi^*: H^i(\widetilde{L}) \rightarrow H^i(L_\infty)$. For $\eta \in L$ let $\overline{\eta} \in L$ be a vector field such that $j^\omega(\overline{\eta})_p = \eta$ and supp $\overline{\eta} \subset U_i$; then for $\lambda \in H^*(\widetilde{L})$ we have $\eta \cdot [\lambda] = [D_\eta^\omega \lambda]$ for any $i$. Clearly $\psi^* [D_\eta^\omega \lambda] = 0$ and from the fact that $\psi^* \eta \cdot [\lambda] = 0$ if and only if $\psi^* \eta \cdot [\lambda] = 0$ for all $i$ we conclude $\psi^* \eta \cdot [\lambda] = 0$. To conclude the proof it is sufficient to show that $\psi^*$ is injective. In fact, $\psi^*$ is an isomorphism. To see this, look at the spectral sequences defined at the beginning of the paper.

Since $\widetilde{L}$ can be thought of as rapidly decreasing vector fields on $\mathbb{R}^n$, the space that arises in defining $C^*(\widetilde{L})$ is $S'(\mathbb{R}^n)$. From this observation we see that the spectral sequence converging to $H^*(F^{-k}C^*(\widetilde{L}))/F^{-k+1}C^*(\widetilde{L}))$, which is $E_1$ of
another spectral sequence, has \( E^2_{p,-q} \):

\[
[H_p(S'(R^n_k / S'(R^n_k)_{(R^n_k)_{k-1}}) \otimes \Lambda[T(R^n_k)]) \otimes H^q(\Lambda^+ L^* \otimes \cdots \otimes \Lambda^+ L^*)]^\Sigma k.
\]

We can identify the factor on the left from the following exact sequences (see Schwartz [4]). If \( p \) is the north pole of \( S^{n_k} \),

\[
0 \to E'(S^{n_k})_p \to E'(S^{n_k}) \to S'(R^n_k) \to 0
\]

\[
0 \to E'(S^{n_k})_p \to E'(S^{n_k}) \bigcup_{i<j<k} S^{n_k-n}_{(i,j)} \to S'(R^n_k)_{(R^n_k)_{k-1}} \to 0.
\]

Thus

\[
H_p(S'(R^n_k / S'(R^n_k)_{(R^n_k)_{k-1}}) \otimes \Lambda[T(R^n_k)]) \cong H_p(S^{n_k}, \bigcup S^{n_k-n}_{(i,j)})
\]

and \( E^2_{p,-q}(\tilde{L}) \cong E^2_{p,-q}(L_\infty) \).

BIBLIOGRAPHY


5. S. Shnider, Notes on Gel’fand-Fuks cohomology, Available from Math. Dept, Princeton University, Princeton, N. J.


DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON,
NEW JERSEY 08540