BOUNDARY VALUE PROBLEMS FOR
SECOND ORDER DIFFERENTIAL EQUATIONS
IN CONVEX SUBSETS OF A BANACH SPACE

BY

KLAUS SCHMITT(1) AND PETER VOLKMANN

ABSTRACT. Let $E$ be a real Banach space, $C$ a closed, convex subset of $E$ and $f: [0, 1] \times E \times E \to E$ be continuous. Let $u_0, u_1 \in C$ and consider the boundary value problem

$$(*) \quad u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1.$$

We establish sufficient conditions in order that $(*)$ have a solution $u: [0, 1] \to C$.

Introduction. Let $C$ be a closed, convex subset of the real Banach space $E$ and let $f: [0, 1] \times C \times E \to E$ be a function with the property

$$\varphi \in E^* (2), \quad x \in C, \quad \varphi(x) = \max_{q \in C} \langle q, \varphi(q) \rangle \Rightarrow \varphi(f(t, x, y)) \geq 0. \quad y \in E, \quad \varphi(y) = 0, \quad 0 \leq t \leq 1.$$

In this paper we show that under some additional (sometimes rather restrictive) assumptions the boundary value problem (BVP)

$$(2) \quad u'' = f(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1,$$

$(u_0, u_1 \in C)$ has a solution $u: [0, 1] \to C$. We note that (1) describes the behavior of $f$ on the boundary $\partial C$ of $C$, for if $\varphi \neq 0$, then condition (1) implies $x \in \partial C$. In case $E = \mathbb{R}^n$, $n$-dimensional Euclidean space, and $C$ is bounded with int $C (3) \neq \emptyset$, various results of this type exist in the literature (see e.g. [5] for a survey of such results). In this finite dimensional situation the general case may easily be obtained by projection methods. On the other hand, if $E$ is infinite dimensional, certain additional assumptions, either on $E$ or on $f$ seem to be needed to pass from the case int $C \neq \emptyset$ to the general case.

The paper is divided into two parts. In the first part we assume $f(t, x, y)$

Received by the editors August 23, 1974.
Key words and phrases. Boundary value problems, second order differential equations in Banach spaces.

(1) Research was performed while the first named author was a Visiting Professor at Universität Karlsruhe. His research was supported in part by U. S. Army research grant OAH-C-04-74-G-0208.
(2) $E^*$ denotes the space of all bounded linear functionals on $E$.
(3) “int” denotes the interior of a set.

Copyright © 1976, American Mathematical Society 397
to be completely continuous and satisfy a Nagumo type growth condition with respect to $y$. Then it is known [6] that if $C$ is bounded and $\text{int } C \neq \emptyset$, the BVP (2) has a solution $u: [0, 1] \rightarrow C$. In Theorem 1 we show that the same conclusion holds in case $C$ is a closed, bounded, convex subset of a uniformly convex space $E$, or in case $C$ is a compact convex subset. (The existence of a solution $u: [0, 1] \rightarrow C$ of (2) for certain compact convex $C$ in $l^p$, $1 < p < \infty$, has already been treated by Thompson [7]; his methods, however, are quite different from ours.) In the second part we assume $f$ in (2) to be independent of $u'$, continuous on $[0, 1] \times E$ and satisfy a Lipschitz condition

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|, \quad x, y \in E,$$

where $L < \pi^2$. Under these assumptions the existence of a unique solution $u: [0, 1] \rightarrow E$ of (2) follows easily by means of the contraction mapping principle, see e.g. [1] where the one dimensional case is treated, so one only needs to show that $u: [0, 1] \rightarrow C$. This is done (Theorem 2) by using results and techniques formerly used by Redheffer and Walter [4] and in [8], [9], [10] in the study of invariance properties of sets relative to initial value problems for first order equations. A final result (Theorem 3) shows that it suffices to assume $f$ to be defined on $[0, 1] \times C$, provided the continuity of $f$ relative to $t$ is uniform with respect to $x \in C$.

1. Completely continuous right-hand sides. Throughout this section we assume that $f: [0, 1] \times C \times E \rightarrow E$ is completely continuous.

**Theorem 1.** Let $C$ be a closed, bounded, convex subset of $E$ and assume there exists a continuous projection $P: E \rightarrow C$ assigning to each $x \in E$ a nearest point $P_x \in C$ (i.e., $\|x - P_x\| = \text{dist}(C, x) \equiv \inf_{q \in C} \|q - x\|$; such $P$ always exists if the Banach space $E$ is uniformly convex in the sense of Clarkson [2]), or assume $C$ is compact. Let $u_0, u_1 \in C$ and let $f$ satisfy (1) and the growth condition

$$\|f(t, x, y)\| \leq \omega(\|y\|) \quad (0 \leq t \leq 1, x \in C, y \in E),$$

where $\omega: [0, \infty) \rightarrow (0, \infty)$ is a continuous nondecreasing function with $\lim_{s \rightarrow \infty} s^2/\omega(s) = \infty$. Then the BVP (2) has a solution $u: [0, 1] \rightarrow C$.

**Proof.** 1. If $C$ is closed, bounded, convex and $\text{int } C \neq \emptyset$, the above result holds without further assumptions on $C$, [6, Theorem 4.1].

2. A further result [6, Lemma 2.1] which is needed in what is to follow and which makes use of the properties of $\omega$ is the following: For each $R > 0$ there exists $M$ (depending only on $R$ and $\omega$) such that: if $u: [0, 1] \rightarrow E$ is twice continuously differentiable and
\[ ||u(t)|| \leq R, \quad ||u''(t)|| \leq \omega(||u'(t)||), \quad 0 \leq t \leq 1, \]
then \[ ||u'(t)|| \leq M, 0 \leq t \leq 1. \]

3. Let \( C \) be such that there exists a continuous projection \( P: E \to C \) as in the statement of Theorem 1. Define \( \tilde{f}: [0, 1] \times E \times E \to E \) by
\[ \tilde{f}(t, x, y) = f(t, Px, y). \]

For each \( \epsilon > 0 \) the set \( C_\epsilon \) defined by
\[ C_\epsilon = \{ x \in E : \text{dist}(C, x) \leq \epsilon \} \]
is a closed, bounded, convex subset of \( E \) with \( \text{int} C_\epsilon \neq \emptyset \). We shall show next that the result of [6] stated in 1. above may be applied to \( \tilde{f} \) and \( C_\epsilon \).

Obviously \( \tilde{f} \) is completely continuous and verifies the estimate
\[ \|\tilde{f}(t, x, y)\| \leq \omega(\|y\|) \quad (0 \leq t \leq 1, x, y \in E). \]

Let us show (1) with \( C \) and \( f \) replaced by \( C_\epsilon \) and \( \tilde{f} \), respectively, i.e.
\[ \varphi \in E^*, x \in C_\epsilon, \varphi(x) = \max_{q \in C_\epsilon} \varphi(q) \]
\[ y \in E, \varphi(y) = 0, 0 \leq t \leq 1 \]

Let \( x \in C_\epsilon \), then \( \|x - Px\| \leq \epsilon \). Thus, if \( q \in C \), we have that \( q + x - Px \in C_\epsilon \).

The hypotheses of (6) consequently imply
\[ \varphi(q) \geq \varphi(q + x - Px) = \varphi(q) + \varphi(x) - \varphi(Px), \]
and since \( q \in C \) was arbitrary, it follows that
\[ \varphi(Px) = \max_{q \in C} \varphi(q). \]

Using (1), we therefore obtain
\[ \varphi(\tilde{f}(t, x, y)) = \varphi(f(t, Px, y)) \geq 0, \]
proving (6).

Using Theorem 4.1 of [6] we conclude the existence of a solution \( u_\epsilon: [0, 1] \to C_\epsilon \) of the BVP
\[ u_\epsilon'' = f(t, u_\epsilon, u_\epsilon'), \quad u_\epsilon(0) = u_0, \quad u_\epsilon(1) = u_1. \]

4. We now employ a limiting process (letting \( \epsilon \to 0 \)) to obtain the desired conclusion.

Let \( \{ \epsilon_n \} \) be a monotone decreasing sequence of real numbers with \( \lim_{n \to \infty} \epsilon_n = 0 \). Denote by \( u_n = u_{\epsilon_n} \), where \( u_{\epsilon_n}: [0, 1] \to C_{\epsilon_n} \) is a solution of (7), with \( \epsilon \) replaced by \( \epsilon_n \). Choose \( R > 0 \) such that \( ||u_n(t)|| \leq R, 0 \leq t \leq 1, n = 1, 2, \ldots \). Using (5) and 2, we obtain the existence of a constant \( M > 0 \)

such that \( ||u_n(t)|| \leq M, 0 \leq t \leq 1, n = 1, 2, \ldots \).

Let \( G \) denote the Green's function
\[ G(t, s) = \begin{cases} -s(1 - t), & 0 \leq s \leq t \leq 1, \\ -t(1 - s), & 0 \leq t \leq s \leq 1; \end{cases} \]

then

\[ u_n(t) = \int_0^1 G(t, s) \tilde{f}(s, u_n(s), u'_n(s)) \, ds + (1 - t)u_0 + tu_1 \]

and

\[ u'_n(t) = \int_0^1 \frac{d}{dt} G(t, s) \tilde{f}(s, u_n(s), u'_n(s)) \, ds + u_1 - u_0. \]

Using the complete continuity of \( \tilde{f} \), the uniform boundedness of \( \{u_n\} \), \( \{u'_n\} \) and (8), (9) we conclude that \( \{u_n\} \) and \( \{u'_n\} \) are equicontinuous sequences and that there exists a compact set \( K \subseteq E \) such that \( u_n(t), u'_n(t) \in K, 0 \leq t \leq 1, n = 1, 2, \ldots \).

We may thus employ the theorem of Ascoli-Arzelà to obtain a subsequence of \( \{u_n\} \) which converges to a solution \( u \) of

\[ u'' = \tilde{f}(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1. \]

Since, further, \( \text{dist}(C, u_n(t)) \leq \varepsilon_n \) and \( \lim_{n \to \infty} \varepsilon_n = 0 \), we obtain \( \text{dist}(C, u(t)) = 0 \), from which follows that \( u: [0, 1] \to C \) and \( \tilde{f}(t, u, u') = f(t, u, u') \), proving that \( u \) is a solution of (2).

5. We next consider the case where \( C \) is a compact convex subset of \( E \) (here no additional assumptions on \( E \) are needed). Choose \( R > 0 \) such that: \( x \in C \Rightarrow \|x\| \leq R \). Determine \( M = M(R, \omega) \) according to 2. above. Define \( Q: E \to E \) by

\[ Qy = \begin{cases} y, & \|y\| \leq M, \\ M\|y\|/\|y\|, & \|y\| > M, \end{cases} \]

and put

\[ \tilde{f}(t, x, y) = f(t, x, Qy) \quad (0 \leq t \leq 1, x \in C, y \in E). \]

The complete continuity of \( f \) implies that of \( \tilde{f} \). Hence the range of \( \tilde{f} \) is contained in some compact set \( K \subseteq E \), and (1) and (4) are satisfied by \( \tilde{f} \).

Let \( E_1 \) denote the closed linear span of \( C, K \) and restrict \( \tilde{f} \) to \( \tilde{f}: [0, 1] \times C \times E_1 \to E_1 \). Since \( C \) and \( K \) are compact, \( E_1 \) is a separable Banach space. Using a result of Clarkson [2] we may equip \( E_1 \) with a new norm \( \| \cdot \|_1 \), equivalent to \( \| \cdot \| \), such that \( E_1 \) becomes strictly convex. Hence to each \( x \in E_1 \) there corresponds a unique nearest point (with respect to \( \| \cdot \|_1 \)) \( Px \) in \( C \). Since (1) holds with \( E, f \) replaced by \( E_1, \tilde{f} \) (\( \varphi \in E_1^* \) with \( \varphi(x) = \max_{q \in C} \varphi(q) \) is extendable to a \( \Phi \in E^* \) with the same property) and since \( \tilde{f} \) is bounded and the projection \( P \), just defined, is continuous, we may apply the arguments of 3. and 4. to obtain a
solution \( u: [0, 1] \rightarrow C \) of
\[
(10) \quad u'' = \tilde{f}(t, u, u'), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1.
\]
Returning to the original norm we have that \( \|u(t)\| \leq R, 0 \leq t \leq 1 \), and by the monotonicity of \( \omega \) we find \( \|u''(t)\| \leq \omega(\|u'(t)\|) \), implying \( \|u'(t)\| \leq M, 0 \leq t \leq 1 \). Hence the definition of \( \tilde{f} \) shows that \( u \) is a solution of (2).

2. Right-hand sides satisfying a Lipschitz condition. Throughout this section we shall assume that \( f \) is independent of \( u' \) and satisfies a Lipschitz condition
\[
(11) \quad \|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad (0 \leq t \leq 1; x, y \in E).
\]

Theorem 2. Let \( C \) be a closed, convex subset of \( E \) and let \( u_0, u_1 \in C \). Assume that \( f: [0, 1] \times E \rightarrow E \) is continuous and satisfies the Lipschitz condition (11) with \( L < \pi^2 \). Further assume
\[
(12) \quad 0 < t < 1, \quad x \in C, \quad \varphi(x) = \max_{q \in C} \varphi(q) \Rightarrow \varphi(f(t, x)) \geq 0.
\]

Then the BVP
\[
(13) \quad u'' = f(t, u), \quad u(0) = u_0, \quad u(1) = u_1, \quad 0 \leq t \leq 1,
\]
has a unique solution \( u: [0, 1] \rightarrow C \).

Proof. 1. For our proof we need a formula first established for closed, convex cones by Redheffer and Walter [4] equivalent to (12):
\[
(14) \quad \lim_{h \to 0^+} \frac{1}{h} \text{dist}(C, x - hf(t, x)) = 0 \quad (0 \leq t \leq 1, x \in C)
\]
(see [8]). Letting (for \( \xi > 0 \))
\[
C_\xi = \{ x \in E: \text{dist}(C, x) \leq \xi \}
\]
\( (C_0 = C) \), using (11) and (12) and a result from [10] we obtain
\[
(15) \quad \lim_{h \to 0^+} \frac{1}{h} \text{dist}(C_\xi, x - hf(t, x)) \leq L\xi \quad (0 \leq t \leq 1, x \in C_\xi).
\]
(In [10] this formula is written with \( \lim \sup \) in place of \( \lim \), however, since \( C \) is convex the limit exists.)

2. Let \( \tilde{E} = E \oplus R \) normed by \( \|(x, \xi)\| = \max(\|x\|, \|\xi\|) \). With \( p = (\theta, 1) \) (\( \theta = \text{zero element of } E \)) we may write
\[
\tilde{E} = E \oplus \mathbb{R} = \{ x + \xi p: x \in E, \xi \in \mathbb{R} \}.
\]
Via the natural embedding, we consider \( E \) as a subspace of \( \tilde{E} \). Let
\[
\tilde{C} = \{ x + \xi p: \text{dist}(C, x) \leq \xi \},
\]
then \( \tilde{C} \) is a closed, convex subset of \( \tilde{E} \) with nonempty interior. Define \( \tilde{f}: [0, 1] \rightarrow \tilde{E} \)
\[ f(t, x + \xi p) = f(t, x) - L \xi p \quad (0 \leq t \leq 1, x + \xi p \in \tilde{C}). \]

Then \( \tilde{f} \) is continuous and satisfies a Lipschitz condition with Lipschitz constant \( L \) with respect to its second argument:

\[ \| \tilde{f}(t, \tilde{x}) - \tilde{f}(t, \tilde{y}) \| \leq L \| \tilde{x} - \tilde{y} \| \quad (0 \leq t \leq 1, \tilde{x}, \tilde{y} \in \tilde{C}). \]

Our method of proof requires a condition analogous to (12) for \( f \) and \( \tilde{C} \), namely:

\[ \left( 0 < f < 1, x \in \tilde{C}, \tilde{\varphi}(\tilde{x}) = \max_{q \in \tilde{C}} \tilde{\varphi}(q) \right) \Rightarrow \tilde{\varphi}(\tilde{f}(t, \tilde{x})) > 0. \]

That (18) follows from (12) has already been sketched in [9] for the case where \( C \) is a closed, convex cone; our proof to follow is patterned after the one in [9]. (For general closed, convex \( C \) (18) has been established in [8] for \( f \) defined by \( \tilde{f}(t, x + \xi p) = f(t, x) - 4L \xi p \). That result, however, is not sufficient for our purposes.)

3. To prove (18) we use the equivalence of (12) and (14) (applied to \( \tilde{C} \) and \( \tilde{f} \)) and verify

\[ \lim_{n \to 0} \frac{1}{h} \text{dist}(\tilde{C}, \tilde{x} - h\tilde{f}(t, \tilde{x})) = 0 \quad (0 < t < 1, \tilde{x} \in \tilde{C}). \]

Let \( t \in [0, 1] \) and \( \tilde{x} = x + \xi p \in \tilde{C} \), i.e., \( x \in C_\xi \). Then (15) implies that for \( \epsilon > 0 \) there exists \( h_0(\epsilon) \) such that

\[ h^{-1} \, \text{dist}(C_\xi, x - hf(t, x)) < L \xi + \epsilon \quad (0 < h \leq h_0(\epsilon)). \]

Thus there exists \( y_h \in C_\xi \) (i.e., \( y_h + \xi p \in \tilde{C} \)) such that

\[ \| x - hf(t, x) - y_h \| < hL \xi + h \epsilon, \]

implying

\[ x - hf(t, x) - y_h + h(L \xi + \epsilon)p \in \tilde{K} = \{ y + \eta p : y \in E, \| y \| \leq \eta \}. \]

Now \( \tilde{C} + \tilde{K} \subseteq \tilde{C} \) and \( y_h + \xi p \in \tilde{C} \), yielding

\[ x + \xi p - h[f(t, x) - L \xi p] + h \epsilon p \in \tilde{C}, \]

from which, in turn, it follows that

\[ h^{-1} \, \text{dist}(\tilde{C}, \tilde{x} - h\tilde{f}(t, \tilde{x})) < \epsilon \quad (0 < h \leq h_0(\epsilon)), \]

implying (19).

4. Define \( P: \tilde{E} \to \tilde{C} \) by

\[ P(x + \xi p) = \begin{cases} x + \xi p, & \text{dist}(C, x) \leq \xi, \\ x + \text{dist}(C, x)p, & \text{dist}(C, x) > \xi. \end{cases} \]

Then it is easily seen that
Extending \( \tilde{f} \) to \([0, 1] \times \tilde{E} \) by setting
\[
\tilde{f}(t, \tilde{x}) = \tilde{f}(t, P\tilde{x}) \quad (0 \leq t \leq 1, \tilde{x} \in \tilde{E}),
\]
we see by (21) that (17) remains valid for the extended function (with the same Lipschitz constant).

Letting
\[
\mathcal{C}_\eta = \mathcal{C} - \eta p = \{\tilde{x} - \eta p: \tilde{x} \in \mathcal{C}\} \quad (\eta \geq 0; \mathcal{C}_0 = \mathcal{C})
\]
we see that (18) holds with \( \mathcal{C} \) replaced by \( \mathcal{C}_\eta \), i.e.,
\[
(23) \begin{cases}
0 \leq t \leq 1, \tilde{\varphi} \in \tilde{E}^*, \tilde{x} \in \mathcal{C}_\eta, \\
\tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \mathcal{C}_\eta} \tilde{\varphi}(\tilde{q}),
\end{cases}
\]
for if \( \tilde{x} = x + \xi p \) and \( \tilde{\varphi} \neq 0 \) satisfy the hypotheses of (23), then \( \tilde{x} \in \partial \mathcal{C}_\eta \) and therefore \( x + (\xi + \eta) p = \tilde{x} + \eta p \in \partial \mathcal{C} \). Thus \( \text{dist}(C, x) = \xi + \eta \), which combined with (20) yields \( P\tilde{x} = x + (\xi + \eta)p = \tilde{x} + \eta p \). Therefore \( \tilde{\varphi}(P\tilde{x}) = \max_{\tilde{q} \in \tilde{E}} \tilde{\varphi}(\tilde{q}) \). Using (18) we obtain \( \tilde{\varphi}(\tilde{f}(t, P\tilde{x})) > 0 \), which by (22) implies (23).

5. The function \( \sigma: \tilde{E} \to \mathbb{R} \), defined by
\[
(24) \quad \sigma(x + \xi p) = \begin{cases}
0, & \text{dist}(C, x) \leq \xi, \\
\text{dist}(C, x) - \xi, & \text{dist}(C, x) > \xi,
\end{cases}
\]
satisfies a Lipschitz condition with Lipschitz constant 2. Choose \( e > 0 \) such that
\( L_1 = L + 2e < \pi^2 \). Then
\[
\hat{f}(t, \tilde{x}) = \tilde{f}(t, \tilde{x}) - e\sigma(\tilde{x})p
\]
satisfies
\[
\|\hat{f}(t, \tilde{x}) - \hat{f}(t, \tilde{y})\| \leq L_1 \|\tilde{x} - \tilde{y}\| \quad (0 \leq t \leq 1, \tilde{x}, \tilde{y} \in \tilde{E});
\]
further it follows from (23) and (24) that
\[
(25) \begin{cases}
0 \leq t \leq 1, \eta > 0, \tilde{\varphi} \in \tilde{E}^*, \tilde{\varphi} \neq 0, \tilde{x} \in \mathcal{C}_\eta, \\
\tilde{\varphi}(\tilde{x}) = \max_{\tilde{q} \in \mathcal{C}_\eta} \tilde{\varphi}(\tilde{q}),
\end{cases}
\]
\[
\Rightarrow \tilde{\varphi}(\hat{f}(t, \tilde{x})) > 0.
\]
Because \( L_1 < \pi^2 \), the BVP
\[
(26) \quad \tilde{u}'' = \hat{f}(t, \tilde{u}), \quad \tilde{u}(0) = u_0, \quad \tilde{u}(1) = u_1,
\]
has a unique solution \( \tilde{u}: [0, 1] \to \tilde{E} \) (this fact has already been mentioned in the introduction). It is the purpose of the next paragraphs to show that \( \tilde{u} \) is a solution of (13) with values in \( C \).
6. There exists a smallest $\eta > 0$ such that $\tilde{u} : [0, 1] \to \mathcal{C}_\eta$ ($\tilde{u}$ is the solution of (26)). Suppose $\eta > 0$. Then there exists $t_0 \in (0, 1)$ such that $\tilde{u}(t_0) \in \partial \mathcal{C}_\eta$ ($\tilde{u}(0), \tilde{u}(1) \in \text{int } \mathcal{C}_\eta$). We may thus choose $\tilde{\varphi} \in \mathcal{E}^*$, $\tilde{\varphi} \neq 0$, such that $\tilde{\varphi}(\tilde{u}(t_0)) = \max_{\tilde{\varphi} \in \mathcal{C}_\eta} \tilde{\varphi}(\tilde{q})$. By (25)

$$\tilde{\varphi}(\tilde{f}(t_0, \tilde{u}(t_0))) > 0.$$  

On the other hand, the scalar function $\rho(t) = \tilde{\varphi}(\tilde{u}(t))$, $0 \leq t \leq 1$, attains its maximum at $t_0$, hence $\rho''(t_0) \leq 0$. But

$$\rho''(t_0) = \tilde{\varphi}(\tilde{u}''(t_0)) = \tilde{\varphi}(\tilde{f}(t_0, \tilde{u}(t_0))),$$

contradicting (27). Thus $\tilde{u} : [0, 1] \to \mathcal{C}_0 = \mathbb{C}$.

7. It now follows from the definition of $\tilde{f}$ that $\tilde{f}(t, \tilde{u}(t)) = \tilde{f}(t, \tilde{u}(t))$.

Thus $\tilde{u}$ is the solution of the BVP

$$\tilde{u}'' = \tilde{f}(t, \tilde{u}), \quad \tilde{u}(0) = u_0, \quad \tilde{u}(1) = u_1.$$  

Using the notation $\tilde{u}(t) = u(t) + \eta(t)p$ ($\eta(t) \in \mathbb{E}$, $\eta(t) \in \mathbb{R}$, $0 < t < 1$), we may decompose (28) into

$$u'' = f(t, u), \quad u(0) = u_0, \quad u(1) = u_1,$$  

with the further constraint

$$0 < t < 1,$$

$$\eta'' = -L \eta, \quad \eta(0) = 0, \quad \eta(1) = 0,$$

$$\text{dist}(C, u(t)) < \eta(t).$$

Since, however, $L < \pi^2$, it follows that $\eta(t) \equiv 0$, and thus $\text{dist}(C, u(t)) = 0$, i.e., $u : [0, 1] \to C$. This completes the proof of Theorem 2.

**Theorem 3.** Theorem 2 remains valid if $f(t, x)$ is only defined on $[0, 1] \times C$, but is uniformly continuous in $t$ with respect to $x$, i.e.,

$$\sup_{x \in C} \|f(t_n, x) - f(t, x)\| \to 0 \quad \text{as } t_n \to t.$$  

**Proof.** We embed $E$ via an isometric isomorphism in some Banach space $B(S)$ of bounded functions on some set $S$ (e.g. $S = \{\varphi \in E^* : \|\varphi\| < 1\}$). Then (12) remains valid with $B(S)^*$ in place of $E^*$. Thus we may consider the problem in $B(S)$ instead of $E$; in particular we may consider $f : [0, 1] \times C \to B(S)$, where $C \subseteq B(S)$. By adopting the coordinate conventions and writing the elements $z \in B(S)$ as $z = (z_\sigma)_{\sigma \in S}$ ($z_\sigma \in \mathbb{R}$, $\|z\| = \sup_{\sigma \in S} |z_\sigma|$), we define $f_\phi : [0, 1] \times C \to \mathbb{R}$ ($\phi \in \mathbb{S}$) by

$$f_\phi(t, x) = f(t, x) \phi \quad (0 \leq t \leq 1, x \in C, \phi \in \mathbb{S}).$$
The Lipschitz continuity of \( f \) implies that of \( f_0 \), i.e.,
\[
|f_0(t, x) - f_0(t, y)| \leq L\|x - y\| \quad (0 \leq t \leq 1, x, y \in C, \sigma \in S).
\]

A result of McShane [3] implies that the function
\[
\tilde{f}_0(t, x) = \sup_{q \in C} (f_0(t, q) - L\|q - x\|) \quad (x \in B(S))
\]
is an extension of \( f_0 \) to \([0, 1] \times B(S)\), such that
\[
|\tilde{f}_0(t, x) - \tilde{f}_0(t, y)| \leq L\|x - y\| \quad (0 \leq t \leq 1, x, y \in B(S), \sigma \in S).
\]

Define \( \tilde{f} : [0, 1] \times B(S) \to B(S) \) by
\[
\tilde{f}(t, x) = \tilde{f}_0(t, x) \quad (0 \leq t \leq 1, x \in B(S), \sigma \in S).
\]

Then \( \tilde{f} \) is an extension of \( f \) to \([0, 1] \times B(S)\) and satisfies (11). By (32) \( \tilde{f}(t, x) \)
is also continuous with respect to \( t \). We may therefore use Theorem 2 to conclude that the BVP
\[
u'' = \tilde{f}(t, u), \quad u(0) = u_0, \quad u(1) = u_1 \quad (u_0, u_1 \in C)
\]
has a solution \( u : [0, 1] \to C \). Since \( \tilde{f} \) is an extension of \( f \), \( u \) is a solution of the original problem.

REFERENCES


7. R. C. Thompson, Differential inequalities for infinite second order systems and an application to the method of lines, J. Differential Equations 17 (1975), 421-434.


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112

MATHEMATISCHES INSTITUT I, UNIVERSITÄT KARLSRUHE (TH), 75 KARLSRUHE 1, FEDERAL REPUBLIC OF GERMANY