TOTAL MEAN CURVATURE OF IMMERSED SURFACES IN $E^m$

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ABSTRACT. Total mean curvature and value-distribution of mean curvature for certain pseudo-umbilical surfaces are studied.

1. Introduction. In the classical theory of surfaces in a euclidean $m$-space $E^m$, the two most important curvatures are the so-called Gauss curvature $G$ and the mean curvature $\alpha$. It is well known that the Gauss curvature is intrinsic. The integral of Gauss curvature gives the beautiful Gauss-Bonnet formula, which holds for orientable compact surfaces as well as nonorientable ones,

$$\int_M G \, dV = 2\pi \chi(M),$$

where $dV$ and $\chi(M)$ denote the volume element and Euler-Poincaré characteristic of $M$. For the mean curvature of a compact surface $M$ in $E^m$ we have [2, I] (see also [5]),

$$\int_M \alpha^2 \, dV \geq 4\pi.$$

The equality holds when and only when $M$ is an ordinary 2-sphere in an affine 3-space. It is an interesting problem to improve inequality (1.2) for some special surfaces in $E^m$. In [2, III] the author obtains some results of this problem for surfaces in $E^4$. In this paper we shall study this problem for pseudo-umbilical surfaces in $E^m$ (for the definition of pseudo-umbilicity see §2). In particular, we shall prove the following:

THEOREM 1. Let $M$ be a compact pseudo-umbilical surface in $E^m$ with non-negative Gauss curvature. If we have

$$\int_M \alpha^2 \, dV \leq (2 + \pi)\pi,$$

then $M$ is homeomorphic to a 2-sphere.

THEOREM 2. Let $M$ be a compact flat pseudo-umbilical surface in $E^m$. 

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Then we have

\begin{equation}
\int_M \alpha^2 \, dV \geq 2\pi^2.
\end{equation}

The equality sign holds if and only if \( M \) is a Clifford torus, i.e., \( M \) is the product surface of two plane circles with the same radius.

2. Preliminaries. Let \( x : M \to \mathbb{E}^m \) be an isometric immersion of a surface \( M \) in an \( m \)-dimensional euclidean space \( \mathbb{E}^m \) and let \( \nabla \) and \( \nabla' \) be the covariant differentiations of \( M \) and \( \mathbb{E}^m \) respectively. Let \( X \) and \( Y \) be two tangent vector fields on \( M \). Then the second fundamental form \( h \) is given by

\begin{equation}
\nabla'_X Y = \nabla_X Y + h(X, Y).
\end{equation}

It is well known that \( h(X, Y) \) is a normal vector field on \( M \) and is symmetric on \( X \) and \( Y \). Let \( \xi \) be a normal vector field on \( M \); we write

\begin{equation}
\nabla'_X \xi = -A_\xi(X) + D_X \xi,
\end{equation}

where \(-A_\xi(X)\) and \( D_X \xi \) denote the tangential and normal components of \( \nabla'_X \xi \). Then we have

\begin{equation}
\langle A_\xi(X), Y \rangle = \langle h(X, Y), \xi \rangle
\end{equation}

where \( \langle \ , \ \rangle \) denotes the scalar product in \( \mathbb{E}^m \). A normal vector field \( \xi \) on \( M \) is said to be parallel (in the normal bundle) if \( D\xi = 0 \). The mean curvature vector \( H \) is defined by

\begin{equation}
H = \frac{1}{2} \text{trace } h.
\end{equation}

The length of \( H \), denoted by \( \alpha \), is called the mean curvature of \( M \). If the mean curvature vector \( H \) is nowhere zero and the second fundamental form \( h \) satisfies

\begin{equation}
\langle h(X, Y), H \rangle = \lambda \langle X, Y \rangle,
\end{equation}

for all tangent vectors \( X, Y \) on \( M \), then \( M \) is said to be pseudo-umbilical.

Let \( R \) and \( R^N \) be the curvature tensors associated with connections \( \nabla \) and \( D \), i.e., \( R \) and \( R^N \) are given respectively by

\[
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{\{X,Y\}}
\]

and

\[
R^N(X, Y) = [D_X, D_Y] - D_{\{X,Y\}}.
\]

For a surface \( M \) in \( \mathbb{E}^m \), if \( R \) vanishes identically, then \( M \) is said to be flat. If \( R^N \) vanishes identically, then \( M \) is said to have flat normal connection. Let \( e_1 \) and \( e_2 \) be orthonormal vector fields tangent to \( M \). Then the Gauss curvature \( G \) of \( M \) is a well-defined intrinsic function on \( M \) given by
The Gauss and Ricci equations are given respectively by
\[
(2.6) \quad \langle R(X, Y)Z, W \rangle = \langle h(X, W), h(Y, Z) \rangle - \langle h(X, Z), h(Y, W) \rangle,
\]
\[
(2.7) \quad \langle R^N(X, Y)\xi, \eta \rangle = \langle h(A_\xi(Y), X), \eta \rangle - \langle h(A_\xi(X), Y), \eta \rangle,
\]
where \(X, Y, Z, W\) are vector fields tangent to \(M\) and \(\xi, \eta\) are vector fields normal to \(M\). For the second fundamental form \(h\), we define the covariant derivative, denoted by \(\nabla_X\), to be
\[
(2.8) \quad (\nabla_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).
\]
The Codazzi equation is given by
\[
(2.9) \quad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z).
\]
Let \(e_1, e_2, \xi_3, \ldots, \xi_m\) be orthonormal vector fields, defined along \(M\), such that \(e_1, e_2\) are tangent to \(M\) and \(\xi_3, \ldots, \xi_m\) are normal to \(M\). For each \(r = 3, \ldots, m\), we simply denote \(A_{\xi_r}\) by \(A_r\). Let
\[
A_r = (h_{ij}^r)_{i,j=1,2}.
\]
With respect to the basis \(e_1, e_2\), we have \(h_{12}^r = h_{21}^r\). From (2.4) and (2.6) we find
\[
(2.10) \quad H = \frac{1}{2} \sum (h_{11}^r + h_{22}^r) \xi_r,
\]
\[
(2.11) \quad G = \sum (h_{11}^r h_{22}^r - h_{12}^r h_{12}^r).
\]

3. Mean curvature of pseudo-umbilical surfaces with \(R = R^N = 0\). In this section we shall prove the following results for later use.

**Theorem 3.** Let \(M\) be a flat pseudo-umbilical surface in \(E^m\) with flat normal connection. Then the mean curvature \(\alpha\) satisfies the following Laplace's equation,
\[
\Delta \ln \alpha = 0,
\]
where \(\Delta\) denotes the Laplacian on \(M\).

**Proof.** Since the normal connection of \(M\) is flat, the equation of Ricci implies that
\[
(3.1) \quad [A_r, A_s] = 0, \quad r, s = 3, \ldots, m.
\]
Now, let \(\xi_3, \ldots, \xi_m\) be chosen in the way that \(H = \alpha \xi_3\). Then by the pseudo-umbilicity of \(M\) in \(E^m\), we have
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(3.2) \[ A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h_{11}^r & h_{12}^r \\ h_{12}^r & -h_{11}^r \end{pmatrix}, \quad r = 4, \ldots, m. \]

By (3.1), we may choose orthonormal tangent vectors \( e_1, e_2 \) which diagonalize \( A_r \) simultaneously. With respect to such frame \( e_1, e_2, \xi_3, \ldots, \xi_m \), we have

(3.3) \[ A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_r = \begin{pmatrix} h_{11}^r & 0 \\ 0 & -h_{11}^r \end{pmatrix}, \quad r = 4, \ldots, m. \]

Now, by the flatness of \( M \) and (2.11), we have

(3.4) \[ \alpha^2 = \sum_{r=4}^{m} (h_{11}^r)^2. \]

For each \( p \in M \), let \( N_p \) be the vector space consisting of all normal vectors of \( M \) in \( E^m \) at \( p \) which are perpendicular to the mean curvature vector \( H \). On \( N_p \) we define a linear mapping into the set of all symmetric matrices of order 2 by \( \rho(\xi) = A_\xi \). Let \( O_p \) denote the kernel of \( \rho \). Then by (3.3) and (3.4) we see that \( \dim O_p = m - 4 \). Hence, we may choose a frame field \( e_1, e_2, \xi_3, \xi_4, \ldots, \xi_m \) such that, with respect to this frame, we have

(3.5) \[ A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_4 = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad A_s = 0, \quad s = 5, \ldots, m. \]

Since the normal connection is flat, there exist, at least locally, orthonormal normal vector fields \( \xi_3, \ldots, \xi_m \) such that \( \xi_3, \ldots, \xi_m \) are parallel (see [1, p. 99]), i.e.,

(3.6) \[ D\xi_3 = \cdots = D\xi_m = 0. \]

We put

\[ \xi_r = \sum_{s=3}^{m} a_{rs} \xi_s, \quad r = 3, \ldots, m. \]

Then \( (a_{rs}) \) is an orthogonal matrix of order \( m - 2 \).

Since \( M \) is two dimensional and local study of \( M \) is sufficient, we may assume that \( M \) is covered by an isothermal coordinate \((x, y)\) such that the metric on \( M \) has the form \( ds^2 = E(dx^2 + dy^2) \). In the following, we shall denote the coordinate vector fields \( \partial/\partial x \) and \( \partial/\partial y \) by \( X_1 \) and \( X_2 \) respectively. We put

\[ L = h(X_1, X_1), \quad M = h(X_1, X_2), \quad N = h(X_2, X_2) \]

and

\[ \nabla_{X_j} X_i = \sum_{k=1}^{2} \Gamma_{jk}^k X_k, \quad i, j = 1, 2. \]
Then we have
\[ \Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} = X_1 E/2E, \quad \Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11} = X_2 E/2E. \]

Therefore the Codazzi equation reduces to
\[ D_{X_2} L - D_{X_1} M = (X_2 E)H, \]
\[ D_{X_2} M - D_{X_1} N = -(X_1 E)H. \]

Since \( X_1 \) and \( X_2 \) are orthonormal, we may define a function \( \theta = \theta(x, y) \) by
\[ X_1 = \partial / \partial x = \cos \theta e_1 + \sin \theta e_2, \]
\[ X_2 = \partial / \partial y = -\sin \theta e_1 + \cos \theta e_2. \]

Then with respect to the frame field \( X_1, X_2, \xi_3, \ldots, \xi_m \), the second fundamental tensors are given by
\[ A_r = \begin{pmatrix} \alpha(a_{r_1} + a_{r_2} \cos 2\theta) & -\alpha a_{r_2} \sin 2\theta \\ -\alpha a_{r_2} \sin 2\theta & \alpha(a_{r_1} - a_{r_2} \cos 2\theta) \end{pmatrix}. \]

Since \( M \) is flat, we may assume that \( E = 1 \). Hence by (3.6) equations of Codazzi reduce to
\[ \frac{\partial}{\partial y} [\alpha(a_{r_1} + a_{r_2} \cos 2\theta)] = -\frac{\partial}{\partial x} [\alpha a_{r_2} \sin 2\theta], \]
\[ -\frac{\partial}{\partial y} [\alpha a_{r_2} \sin 2\theta] = \frac{\partial}{\partial x} [\alpha(a_{r_1} - a_{r_2} \cos 2\theta)]. \]

Multiplying \( a_{r_1} \) to (3.8) and summing over \( r \), then, by the fact that \( (a_{r_1}) \in O(m - 2) \), we find
\[ \frac{\partial}{\partial y} \ln \alpha = \sum_{r=3}^m \left\{ \left( \frac{\partial a_{r_1}}{\partial y} \right) a_{r_2} \cos 2\theta + \left( \frac{\partial a_{r_1}}{\partial x} \right) a_{r_2} \sin 2\theta \right\}. \]

Similarly, multiplying \( a_{r_1} \) to (3.9) and summing over \( r \), we have
\[ \frac{\partial}{\partial x} \ln \alpha = \sum_{r=3}^m \left\{ \left( \frac{\partial a_{r_1}}{\partial y} \right) a_{r_2} \sin 2\theta - \left( \frac{\partial a_{r_1}}{\partial x} \right) a_{r_2} \cos 2\theta \right\}. \]

Multiplying \( a_{r_2} \) to (3.8) and summing over \( r \), we find
\[ \sum a_{r_2} \left( \frac{\partial a_{r_1}}{\partial y} \right) + \left( \frac{\partial \ln \alpha}{\partial x} \right) \sin 2\theta + \left( \frac{\partial \ln \alpha}{\partial y} \right) \cos 2\theta = 2 \frac{\partial \theta}{\partial y} \sin 2\theta - 2 \frac{\partial \theta}{\partial x} \cos 2\theta. \]
Hence, by substituting (3.10) and (3.11) into this equation, we find

\[ \sum a_{r2} \frac{\partial a_{r1}}{\partial y} = \sin 2\theta \frac{\partial \theta}{\partial y} - \cos 2\theta \frac{\partial \theta}{\partial x}. \]

Similarly, by multiplying \( a_{r2} \) to (3.9), summing over \( r \), and by using (3.10) and (3.11), we get

\[ \sum a_{r2} \frac{\partial a_{r1}}{\partial x} = -\cos 2\theta \frac{\partial \theta}{\partial y} - \sin 2\theta \frac{\partial \theta}{\partial x}. \]

Substituting (3.12) and (3.13) into (3.10) and (3.11), we may find

\[ \frac{\partial \ln \alpha}{\partial x} = \frac{\partial \theta}{\partial y}, \quad \frac{\partial \ln \alpha}{\partial y} = -\frac{\partial \theta}{\partial x}. \]

From this we get \((\partial^2/\partial x^2)\ln \alpha + (\partial^2/\partial y^2)\ln \alpha = 0\). Since \( E = 1 \), this implies that \( \Delta \ln \alpha = 0 \). Q.E.D.

It shall be remarked that for a pseudo-umbilical surface in \( E^4 \), the normal connection is always flat.

As an application of Theorem 3, we have the following result concerning the value-distribution of mean curvature \( \alpha \).

**Theorem 4.** Let \( M \) be a complete flat pseudo-umbilical surface in \( E^m \) with flat normal connection. Then we have either

1. the mean curvature \( \alpha \) of \( M \) takes every value in \((0, \infty)\), or
2. the mean curvature \( \alpha \) of \( M \) takes only one value in \((0, \infty)\).

If case (2) holds, \( M \) is the product of two curves \( C_1 \) and \( C_2 \) where \( C_1 \) is a curve in \( E^n \) for some \( n; 1 < n < m \); and \( C_2 \) is a curve in \( E^{m-n} \) so that the first curvatures of \( C_1 \) and \( C_2 \) are equal.

**Proof.** Since \( M \) is flat and complete, \( M \) is parabolic in the sense that there exists no nonconstant negative subharmonic function on \( M \). Thus every subharmonic function on \( M \) which is bounded from above on \( M \) must be a constant function. By Theorem 3, \( \ln \alpha \) is a continuous harmonic function and so a subharmonic and also a superharmonic function on \( M \). Hence, if \( \alpha \) does not take every value in \((0, \infty)\), then \( \alpha \) must be constant. This proves the first part of the theorem.

Now, suppose that case (2) holds. Then \( \alpha \) is a nonzero constant. From (3.5) we see that with respect to the frame field \( e_1, e_2, \xi_3, \xi_4, \xi_5, \ldots, \xi_m \), we have

\[ A_3 = \begin{pmatrix} \sqrt{2\alpha} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_5 = \cdots = A_m, \]
where \( \tilde{e}_3 = \cos \theta \tilde{e}_3 + \sin \theta \tilde{e}_4, \quad \tilde{e}_4 = \sin \theta \tilde{e}_3 - \cos \theta \tilde{e}_4, \quad \tilde{e}_5 = \tilde{e}_5, \ldots, \quad \tilde{e}_m = \tilde{e}_m. \)

From (3.15) and the equations of structure we may easily find that both the distributions \( T_i = \{ae_i; a \in R\}, \quad i = 1, 2 \) are parallel. By the de Rham decomposition theorem we see that \( M = C_1 \times C_2 \) where \( C_1 \) (respectively, \( C_2 \)) is the maximal integral manifold of \( T_1 \) (resp. \( T_2 \)). Moreover, (3.15) implies that \( h(e_1, e_2) = 0. \) Hence, \( M \) is the product of \( C_1 \) and \( C_2 \) such that \( C_1 \) is a curve in an affine \( n \)-space \( E^m \) and \( C_2 \) is a curve in an affine \( (m-n) \)-space \( E^{m-n} \) in \( E^m \).

Let \( h' \) be the second fundamental form of \( C_i, \quad i = 1, 2. \) Then, (3.15) implies \( h'(e_i, e_j) = \sqrt{2} \alpha \tilde{e}_{i+2}, \quad i = 1, 2. \) Hence the first curvatures of \( C_1 \) and \( C_2 \) are equal. Q.E.D.

4. Proofs of Theorems 1 and 2. Let \( M \) be a compact pseudo-umbilical surface in \( E^m. \) Let \( e_1, e_2, \tilde{e}_3, \ldots, \tilde{e}_m \) be a local field of orthonormal frame defined along \( M \) such that \( e_1, e_2 \) are tangent to \( M \) and \( \tilde{e}_3, \ldots, \tilde{e}_m \) normal to \( M. \)

For a unit normal vector \( \tilde{e} \) at \( p \in M, \) the Lipschitz-Killing curvature \( K(p, \tilde{e}) \) is defined by \( K(p, \tilde{e}) = \det \Lambda_{\tilde{e}}. \) Let \( \tilde{e} = \sum_{r=3}^{m} \cos \theta_r \tilde{e}_r \) and \( \Lambda_r = (h'_{ij}). \) Then we have

\[
K(p, \tilde{e}) = \left( \sum \cos \theta_r h'_{11} \right) \left( \sum \cos \theta_r h'_{22} \right) - \left( \sum \cos \theta_r h'_{12} \right)^2
\]

The right-hand side of this equation is a quadratic form of \( \cos \theta_3, \ldots, \cos \theta_m. \) Hence, by choosing a suitable local frame field \( \tilde{e}_3, \ldots, \tilde{e}_m, \) we may write

\[
K(p, \tilde{e}) = \sum \lambda_{r-2}(p) \cos^2 \theta_r
\]

where \( \lambda_1, \ldots, \lambda_{m-2} \) are continuous functions defined on \( M \) and satisfy the following relations: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m-2}. \) From (2.11) we find

\[
G = \lambda_1 + \cdots + \lambda_{m-2}
\]

Since \( M \) is pseudo-umbilical, if we choose \( \tilde{e}_3 \) in the direction of the mean curvature vector, then (3.2) holds. From this we may easily see that \( K(p, \tilde{e}) \) takes its maximal value at \( \tilde{e}_3. \) Hence we have

\[
\lambda_1 = \alpha^2, \quad \lambda_2 \leq 0, \ldots, \lambda_{m-2} \leq 0.
\]

Now, let \( S_p \) be the unit \( (m-3) \)-sphere of all unit normal vectors at \( p \in M \) and \( d\sigma \) the volume element of \( S_p. \) Then the total absolute curvature \( K^*(p) \) at \( p \) is given by

\[
K^*(p) = \int_{S_p} |K(p, \tilde{e})| \, d\sigma
\]

and the total absolute curvature \( TA(M) \) of \( M \) in \( E^m \) (in the sense of Chern-Lashof [3]) is given by \( TA(M) = \int_M K^*(p) \, dV. \) Let \( H_i(M; F) \) be the \( i \)th homology group.
of $M$ over a field $F$ and $\beta(M; F)$ the dimension of $H_1(M; F)$. Then we have [3]

$$\tag{4.4} TA(M) \geq c_{m-1}\beta(M),$$

where $\beta(M) = \max \{\sum_{i=0}^2 \beta_i(M; F); F \text{ fields}\}$ and $c_{m-1}$ the area of unit $(m-1)$-sphere.

From (4.2) and (4.3) we find

$$|K(p, \xi)| = \left|G \cos^2 \theta_3 + \sum_{r=4}^m \lambda_{r-2}(\cos^2 \theta_r - \cos^2 \theta_3)\right|$$

$$\tag{4.5} \leq |G| \cos^2 \theta_3 - \sum_{r=4}^m \lambda_{r-2}|\cos^2 \theta_r - \cos^2 \theta_3|.$$

This implies that

$$K^*(p) \leq \frac{c_{m-1}}{2\pi} |G| - \sum_{r=4}^m \lambda_{r-2}\int_{S_p} |\cos^2 \theta_r - \cos^2 \theta_3| \, d\sigma.$$

On the other hand, since

$$\int_{S_p} |\cos^2 \theta_r - \cos^2 \theta_3| \, d\sigma = 2c_{m-1}/\pi^2,$$

(4.2), (4.3) and (4.5) imply

$$\alpha^2 \geq (\pi^2/2c_{m-1})K^*(p) + G(p) - (\pi/4)|G(p)|.$$

**Case (1).** $G = 0$. In this case, (4.4) and (4.6) imply

$$\int \alpha^2 \, dV \geq (\pi^2/2)\beta(M).$$

On the other hand, the flatness of $M$ implies that $M$ is either homeomorphic to a torus or a Klein bottle; in both cases, $\beta(M) = 4$. Hence (4.7) implies inequality (1.4). Now, if the equality of (1.4) holds, then we have

$$\tag{4.8} TA(M) = 4c_{m-1}.$$

Moreover, the inequality in (4.5) is actually an equality for all $(\theta_3, \ldots, \theta_m)$ satisfying $\cos^2 \theta_3 + \cos^2 \theta_4 + \cdots + \cos^2 \theta_m = 1$. Hence we have

$$\lambda_{m-2} = -\alpha^2, \quad \lambda_2 = \cdots = \lambda_{m-3} = 0.$$

From this we see that $[A_r, A_s] = 0$ for all $r, s = 3, \ldots, m$. By using equation (2.7) of Ricci, we see that the normal connection of $M$ in $E^m$ is flat. Thus, Theorem 4 implies that $M = C_1 \times C_2$, where $C_1$ and $C_2$ are two closed curves in $E^n$ and $E^{m-n}$, respectively, with the same first curvature for some $n, 1 < n < m$. On the other hand by a result of Kuiper [4], we have
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$$TA(M)/c_{m-1} = TA(C_1)TA(C_2)/c_{n-1}c_{m-n-1},$$

where $TA(C_i)$ is the total absolute curvature of $C_i$, $i = 1, 2$. Since $TA(C_1) \geq 2c_{n-1}$ and $TA(C_2) \geq 2c_{m-n-1}$, (4.8) implies $TA(C_1) = 2c_{n-1}$ and $TA(C_2) = 2c_{m-n-1}$. From these we know that both $C_1$ and $C_2$ are two plane circles with the same radius [3]. Thus $M$ is a Clifford torus. This proves Theorem 2.

Case (2). $G > 0$ and $G \neq 0$. In this case, $M$ is either homeomorphic to a sphere or a real projective plane. Now, suppose that $M$ is homeomorphic to a real projective plane. Then we have $\chi(M) = 1$ and $\beta(M) = 3$. Hence inequality (4.4) implies

$$\int_M \alpha^2 dV \geq \frac{3}{2} \pi^2 + \left(1 - \frac{\pi}{2}\right) \int_M G dV.$$

This, combining with Gauss-Bonnet's formula, gives

(4.10) $$\int_M \alpha^2 dV \geq (2 + \pi)\pi.$$

In the equality of (4.10) holds, then the inequality in (4.5) is actually an equality for all $(\theta_3, \ldots, \theta_m)$ satisfying $\cos^2 \theta_3 + \cdots + \cos^2 \theta_m = 1$. Hence, we have either $\lambda_2 = \cdots = \lambda_{m-2} = 0$ or $G = \lambda_3 = \cdots = \lambda_{m-2} = 0$ pointwise. Now, let $U = \{ p \in M : G(p) \neq 0 \}$. Then $U$ is a nonempty open subset of $M$. By the assumption of pseudo-umbilicity, $U$ is totally umbilical in $E^m$. Hence the Gauss curvature $G$ is positive constant on every component of $U$ (see [1, p. 49]). From this we know that $U$ is also a closed subset of $M$. Thus $U = M$ and $M$ is an ordinary 2-sphere in $E^m$. This is a contradiction. This proves Theorem 1. Q.E.D.

Remark 1. The real projective plane can be immersed in $E^5$ as a pseudo-umbilic surface with positive constant Gauss curvature and total mean curvature $6\pi$.

REFERENCES