

ISOTROPIC TRANSPORT PROCESS ON A RIEMANNIAN MANIFOLD

BY

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ABSTRACT. We construct a canonical Markov process on the tangent bundle of a complete Riemannian manifold, which generalizes the isotropic scattering transport process on Euclidean space. By inserting a small parameter it is proved that the transition semigroup converges to the Brownian motion semigroup provided that the latter preserves the class C_0 . The special case of a manifold of negative curvature is considered as an illustration.

1. Introduction. In order to construct a diffusion process on a differentiable manifold it is necessary, in general, to solve stochastic differential equations in coordinate patches and then piece together the resulting local diffusions [4]. In the case of the sphere S^{n-1} , Stroock [10] has shown that the Brownian motion may be obtained by solving a single stochastic differential equation on R^n , whose solution stays on the sphere of its own accord.

The purpose of this paper is to show that, on a wide class of Riemannian manifolds, the Brownian motion can be approximated in law by a globally defined stochastic process—the isotropic transport process. The paths of this process are piecewise geodesic. The joint position-velocity motion is a Markov process on the tangent bundle of the manifold. In the case of R^n , it coincides with the usual transport process [11]. Using a theorem of Kurtz [8], we show that the Brownian motion on the manifold can be approximated, in the sense of weak convergence, by a sequence of isotropic transport processes. As a by-product we obtain a formula for the Laplace-Beltrami operator as the spherical average of the second covariant derivative along the geodesic flow field in the tangent bundle.

2. Geodesic flow field. Let M be a complete C^∞ Riemannian manifold of dimension n . $T_x(M)$ denotes the tangent space at $x \in M$; $T(M)$ denotes the tangent bundle over M . For $\tilde{x} = (x, \xi) \in T(M)$, we consider the geodesic flow $X = \{X^{(x, \xi)}(t), t \geq 0\}$. This satisfies the properties that $X^{(x, \xi)}(0) = x$, $\dot{X}^{(x, \xi)}(0) = \xi$ and $\dot{X}^{(x, \xi)}(t)$ is a parallel vector field along $\{X^{(x, \xi)}(t), t \geq 0\}$.

Received by the editors May 11, 1975.

AMS (MOS) subject classifications (1970). Primary 60J25, 60J65, 53C20; Secondary 58G99, 47D05.

Key words and phrases. Isotropic transport process, geodesic flow, Brownian motion, contraction semigroup.

⁽¹⁾ Supported by the National Science Foundation MPS71-02838 A04.

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We consider the lifting of X to a curve \tilde{X} in $T(M)$, defined by $\tilde{X}^{(x,\xi)}(t) = (X^{(x,\xi)}(t), \dot{X}^{(x,\xi)}(t))$. Let Z be the tangent vector to this curve at $t = 0$. Z is a vector field on $T(M)$ and is called the *geodesic flow field* [6, 132]. We let D_Z denote the operation of (covariant) differentiation defined by Z .

Let $C_0(T(M))$ be the Banach space of continuous functions on $T(M)$ with $\lim_{d(x,x_0) \rightarrow \infty} f(x) = 0$. [Here d is a metric consistent with the manifold topology and x_0 is a fixed point of M .] $C_0^k(T(M)) = \{f \in C_0(T(M)): f \text{ has } k \text{ continuous partial derivatives}\}$. $\|f\| = \sup_{\tilde{x} \in T(M)} |f(\tilde{x})|$. Finally, set

$$(T_t^0 f)(\tilde{x}) = f(\tilde{X}^{\tilde{x}}(t)), \quad (t > 0)$$

$$(R_\lambda^0 f)(\tilde{x}) = \int_0^\infty e^{-\lambda t} (T_t^0 f)(\tilde{x}) dt \quad (\lambda > 0).$$

PROPOSITION 2.1. R_λ^0 maps $C_0^1(T(M))$ into $C_0^1(T(M))$ and satisfies

$$(2.2) \quad (\lambda - D_Z) R_\lambda^0 f = f \quad (f \in C_0^1(T(M))).$$

PROOF. Let $f \in C_0^1(T(M))$. By the smooth dependence on initial conditions, $(x, \xi) \rightarrow \tilde{X}^{(x,\xi)}(t)$ is a C^∞ mapping for each $t > 0$. Therefore $T_t^0 f \in C^1(T(M))$. The dominated convergence theorem implies that $R_\lambda^0 f \in C^1(T(M))$. To show that $R_\lambda^0 f$ vanishes at ∞ , let $\epsilon > 0$. Choose $T > 0$ such that $e^{-\lambda T} \|f\|/\lambda \leq \epsilon/2$. Since $f \in C_0(T(M))$, we can find $R > 0$ such that $|f(\tilde{x})| \leq \epsilon\lambda/2$ if $d(\tilde{x}, \tilde{x}_0) \geq R$. Now let $d(\tilde{x}, \tilde{x}_0) \geq R + T$; then $d(\tilde{X}_t, \tilde{x}_0) \geq R$ for $t \leq T$. Hence

$$\begin{aligned} |R_\lambda^0 f(x)| &\leq \left| \int_0^T e^{-\lambda t} f(\tilde{X}(t)) dt \right| + \left| \int_T^\infty e^{-\lambda t} f(\tilde{X}(t)) dt \right| \\ &\leq \frac{\epsilon\lambda}{2} \left(\frac{1 - e^{-\lambda T}}{\lambda} \right) + \frac{\epsilon}{2} \leq \epsilon, \end{aligned}$$

for $d(\tilde{x}, \tilde{x}_0) \geq R + T$. Thus $R_\lambda^0 f \in C_0(T(M))$.

To verify (2.2) write, for $h > 0$,

$$\begin{aligned} T_h^0 R_\lambda^0 f &= \int_0^\infty e^{-\lambda t} T_{t+h}^0 f dt = e^{\lambda h} \int_h^\infty e^{-\lambda u} T_u^0 f du \\ &= e^{\lambda h} \left[R_\lambda^0 f - \int_0^h e^{-\lambda u} (T_u^0 f) du \right]. \end{aligned}$$

But $(T_u^0 g)(\tilde{x}) = g(\tilde{X}^{\tilde{x}}(u))$. Therefore

$$\begin{aligned} D_Z(R_\lambda^0 f) &\equiv \lim_{h \rightarrow 0} \frac{T_h^0(R_\lambda^0 f) - R_\lambda^0 f}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{e^{\lambda h} - 1}{h} R_\lambda^0 f - \frac{1}{h} \int_0^h e^{-\lambda u} (T_u^0 f) du \right] = \lambda R_\lambda^0 f - f. \end{aligned}$$

3. Isotropic transport process. Let (Ω, \mathcal{B}, P) be a probability space. On Ω we assume given a sequence $\{e_n\}_{n \geq 1}$ of independent random variables with the common distribution

$$P\{e_n > t\} = e^{-t} \quad (t > 0, n = 1, 2, \dots).$$

We set $\tau_0 = 0$, $\tau_n = e_1 + \dots + e_n$ ($n \geq 1$). Let $\mu_x(d\eta)$ be the unique rotationally invariant probability measure on the unit sphere in $T_x(M)$. For $(x, \xi) \in T(M)$, we define a sequence of random variables $\{Y_1, \xi_1, Y_2, \xi_2, \dots\}$ as follows:

$$Y_0 = x, \quad \xi_0 = \xi.$$

Assuming that the joint distributions of $\{Y_1, \xi_1, \dots, Y_{n-1}, \xi_{n-1}\}$ have been defined, we set

$$Y_n = X^{(Y_{n-1}, \xi_{n-1})}(e_n) \quad (n = 1, 2, 3, \dots),$$

$$P\{\xi_n \in d\eta\} = \mu_{Y_n}(d\eta) \quad (n = 1, 2, 3, \dots).$$

It is readily verified that the probability law of (Y_1, ξ_1, \dots) is well defined by these conditional distributions. We now define a stochastic process $\{Y(t), t \geq 0\}$ by

$$Y^{(x, \xi)}(t) = X^{(Y_n, \xi_n)}(t - \tau_n) \quad (\tau_n \leq t \leq \tau_{n+1}).$$

The curve $t \rightarrow Y(t) \in M$ is piecewise differentiable and therefore the tangent vector $\dot{Y}(t)$ is defined for $t \neq \tau_n$ ($n \geq 1$). We normalize $\dot{Y}(t)$ so that $\dot{Y}(\tau_n) = \dot{Y}(\tau_n + 0)$. The pair $\widetilde{Y}(t) = (Y(t), \dot{Y}(t))$ will be called the *isotropic transport process* on $T(M)$.

Let

$$T_t f(\tilde{x}) = E\{f(Y^{(x, \xi)}(t), \dot{Y}^{(x, \xi)}(t))\},$$

$$R_\lambda f(\tilde{x}) = \int_0^\infty e^{-\lambda t} T_t f(\tilde{x}) dt.$$

THEOREM 3.1. $\{T_t, t \geq 0\}$ is a strongly continuous contraction semigroup on $C_0(T(M))$. For $f \in C_0^1(T(M))$, the function $u = T_t f$ satisfies the differential equation

$$\frac{\partial u}{\partial t}(t, x, \xi) = (D_Z u)(t, x, \xi) + \int_{T_x(M)} [u(t, x, \eta) - u(t, x, \xi)] \mu_x(d\eta).$$

This result will be established by working with the operators $\{R_\lambda, \lambda > 0\}$. Of course the semigroup property follows from the Markov property which can be established along the lines of [2, pp. 65–68]. But since we make no explicit use of the Markov property below, this proof is omitted.

PROPOSITION 3.2. R_λ maps $C_0(T(M))$ into $C_0(T(M))$ and satisfies

$$(3.2a) \quad R_\lambda f = R_{1+\lambda}^0 f + R_{1+\lambda}^0 P R_\lambda f \quad (f \in C_0(T(M)))$$

where $Pf(x, \xi) = \int_{T_x(M)} f(x, \eta) \mu_x(d\eta)$.

PROOF. Following a similar argument above, we have

$$(R_\lambda f)(\tilde{x}) = \left\{ \int_0^T + \int_T^\infty \right\} e^{-\lambda t} E f(\tilde{Y}(t)) dt.$$

Given $\epsilon > 0$, choose T such that $e^{-\lambda T} \|f\|/T < \epsilon/2$. Choose R such that $|f(\tilde{x})| \leq \epsilon$ for $d(x, x_0) \geq R$. Then for $d(x, x_0) \geq R + T$, the first integral is $\leq \epsilon/2$. Thus $|R_\lambda f(\tilde{x})| \leq \epsilon$ for $d(x, x_0) \geq R + T$. To prove (3.2a), we write

$$\begin{aligned} (R_\lambda f)(\tilde{x}) &= E \left\{ \int_0^{\tau_1} + \int_{\tau_1}^\infty \right\} e^{-\lambda t} f(\tilde{Y}(t)) dt \\ &= E \int_0^\infty I_{(t < \tau_1)} e^{-\lambda t} f(\tilde{X}(t)) dt \\ &\quad + E \left\{ e^{-\lambda \tau_1} \int_0^\infty e^{-\lambda s} f(\tilde{X}(\tau_1 + s)) ds \right\}. \end{aligned}$$

The first integral $= \int_0^\infty e^{-t} e^{-\lambda t} f(\tilde{X}(t)) dt = (R_{1+\lambda}^0 f)(\tilde{x})$. For the second, note that $\tilde{X}(\tau_1 + s) = (X^{(Y_1, \xi_1)}(s), X^{(Y_1, \xi_1)}(s))$. Taking the conditional expectation with respect to (Y_1, ξ_1) and noting that $Y_1 = X^{(x, \xi)}(\tau_1)$, we have

$$\begin{aligned} E \left\{ e^{-\lambda \tau_1} \int_0^\infty e^{-\lambda s} f(\tilde{X}(\tau_1 + s)) ds \right\} &= E[e^{-\lambda \tau_1} (R_\lambda f)(Y_1, \xi_1)] \\ &= E[e^{-\lambda \tau_1} (PR_\lambda f)(Y_1)] = \int_0^\infty e^{-\lambda s} (PR_\lambda f)(X^{(x, \xi)}(s)) e^{-s} ds \\ &= (R_{1+\lambda}^0 PR_\lambda f)(x, \xi). \end{aligned}$$

PROPOSITION 3.3. *Let $f \in C_0^1(T(M))$. Then $R_\lambda f \in C_0^1(T(M))$ and*

$$(3.3a) \quad (\lambda - D_Z - P + I)(R_\lambda f) = f.$$

PROOF. Using Proposition 3.2, we iterate (3.2a), obtaining

$$R_\lambda f = R_{1+\lambda}^0 f + \sum_{n=1}^\infty (R_{1+\lambda}^0 P)^n (R_{1+\lambda}^0 f).$$

The series converges uniformly due to the estimation $\|R_{1+\lambda}^0 P\| \leq 1/(1 + \lambda)$. Thus $R_\lambda f \in C_0(T(M))$. For $f \in C_0^1(T(M))$, this series may be differentiated term-by-term and the differentiated series also converges uniformly. Hence $R_\lambda f \in C_0^1(T(M))$ which was to be shown.

To prove (3.3a), we apply the operator $(1 + \lambda - D_Z)$ to both sides of (3.2a) and use (2.2). Thus

$$(1 + \lambda - D_Z)R_\lambda f = f + PR_\lambda f,$$

which was to be proved.

PROPOSITION 3.4. *Let $u \in C_0^1(T(M))$, $(\lambda - D_Z - P + I)u = 0$ for some $\lambda > 0$. Then $u \equiv 0$.*

PROOF. Assume that $\sup_{\tilde{x} \in T(M)} u(x) > 0$. Then this sup is assumed at some $\tilde{x}_0 \in M$, for otherwise $\exists \tilde{x}_n \rightarrow \infty$ such that $u(\tilde{x}_n) \rightarrow \sup u(x) > 0$ which contradicts $f \in C_0(T(M))$. Now at \tilde{x}_0 , $D_Z u(\tilde{x}_0) = 0$ and $Pu(\tilde{x}_0) - u(\tilde{x}_0) \leq 0$. But $Pu(\tilde{x}_0) - u(\tilde{x}_0) = \lambda u(\tilde{x}_0) > 0$, a contradiction. Therefore $u(\tilde{x}) \leq 0$ on $T(M)$. Applying the argument to $-u$, we see that $u(\tilde{x}) \geq 0$ on $T(M)$. Thus $u \equiv 0$.

To complete the proof of Theorem 3.1, we appeal to the Hille-Yosida theorem. $u = R_\lambda f$ is the unique solution of the equation $(\lambda - D_Z - P + I)u = f$ and satisfies $\|u\| \leq \|f\|/\lambda$. Hence there exists a strongly continuous contraction semigroup $\{\bar{T}_t, t \geq 0\}$ whose resolvent operators are given by $\{R_\lambda, \lambda > 0\}$. By the uniqueness of Laplace transform, we conclude that $\bar{T}_t = T_t$.

To identify the domain of T_t , we recall that $\mathcal{D} = R_\lambda(C_0(T(M)))$ which is independent of λ . Clearly $C_0^1(T(M)) \subseteq \mathcal{D}$. Now if $u \in \mathcal{D}$, $u = R_\lambda f$ for some $f \in C_0(T(M))$. By 3.2a, $u = R_{1+\lambda}^0 g$, $g = f + PR_\lambda f$. But for any $g \in C_0(T(M))$ the proof of Proposition 2.1 shows that $R_\lambda^0 g$ is differentiable in the Z direction, and that $D_Z(R_\lambda^0 g) = \lambda(R_\lambda^0 g) - g$. Hence $D_Z u$ exists and is an element of $C_0(T(M))$. Thus $\mathcal{D} = \{u \in C_0(M): D_Z u \in C_0(M)\}$. The proof is now complete.

4. Convergence to Brownian motion. We introduce a parameter $\epsilon > 0$ and consider a one-parameter family of isotropic transport processes corresponding to the backward equation

$$(4.1) \quad \partial u / \partial t = \epsilon D_Z u + (Pu - u).$$

This process can be constructed by replacing $\{\xi_n\}_{n \geq 1}$ by $\{\epsilon \xi_n\}_{n \geq 1}$ in the definition of $\tilde{Y}(t)$. The solution of equation (4.1) with $u(0, x, \xi) = f(x, \xi)$ will be denoted by $T_t^{(\epsilon)} f$.

The Laplace-Beltrami operator Δ defines a Markov process on M , the Brownian notion $\{B^x(t), t < \zeta\}$ where ζ is the lifetime. We introduce the semigroup

$$U_t f(x) = E[f(B^x(t)), t < \zeta]$$

whose infinitesimal generator is an extension of Δ . We shall assume the following:

$$(4.2) \quad U_t \text{ maps } C_0(T(M)) \text{ into } C_0(T(M)).$$

THEOREM 4.3. *Assume (4.2). Then for $f \in C_0(M)$,*

$$\lim_{\epsilon \rightarrow 0} T_{t/\epsilon^2}^{(\epsilon)} f = U_{t/n} f$$

uniformly on M .

If (4.2) is satisfied, then $P(\zeta = \infty) = 1$, but not conversely. (4.2) is satisfied if (a) M is compact or (b) M has bounded negative curvature (see below).

To prepare the proof, we set $B = P - I$, $A = D_Z$. Then

$$(4.4) \quad PAPf = 0, \quad f \in C_0^1(T(M)).$$

$$(4.5) \quad \{f: Bf = 0\} = C_0(M),$$

$$(4.6) \quad \lim_{t \rightarrow \infty} e^{tB}f = Pf, \quad f \in C_0(T(M)).$$

The following proposition defines B^{-1} .

PROPOSITION 4.7. *Let $g \in C_0(T(M))$ with $\int_{T_x(M)} g(x, \xi) \mu_x(d\xi) = 0$. The unique solution of the equations*

$$Bf = g, \quad \int_{T_x(M)} f(x, \xi) \mu_x(d\xi) = 0$$

is given by $f(x, \xi) = -g(x, \xi)$.

The proof is omitted.

We now define the operator

$$(4.7a) \quad C = PAB^{-1}AP.$$

PROPOSITION 4.8. *Let $f \in C_0^2(M)$. Then $Cf = n^{-1}\Delta f$, where Δ is the Laplace-Beltrami operator on M .*

PROOF. We first verify that the components of any covariant vector (ξ_1) satisfy

$$(4.9) \quad \int_{T_x(M)} \xi_i \xi_j \mu_x(d\xi) = g_{ij}(x)/n \quad (1 \leq i, j \leq n).$$

Indeed, by the rotational invariance of μ_x , we must have $\langle \xi, \mu \rangle \langle \xi, v \rangle \mu_x(d\xi) = \langle \mu, v \rangle / n$ for any two vectors $\mu, v \in T_x(M)$. The left-hand member of this equation is $\mu_k v_s g^{ik}(x) g^{js}(x) \int \xi_i \xi_j \mu_x(d\xi)$ and the right-hand side is $g^{ij}(x) \mu_i v_j$. Clearly the only possible choice is (4.9). Now we write the second covariant derivative

$$D_Z D_Z f = g^{il} \xi_l g^{js} \xi_s \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right).$$

Using (4.7a), we have

$$\begin{aligned} nPD_Z D_Z f &= g_{ls}(x) g^{il}(x) g^{js}(x) \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) \\ &= g^{ij}(x) \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) \\ &= g^{ij}(x) D_i D_j f, \end{aligned}$$

which is the usual expression for the Laplace-Beltrami operator [5].

PROOF OF THEOREM 4.3. We apply the method of T. G. Kurtz [8, Theorem 2.2]. $u = T_{t/\epsilon^2}^{(\epsilon)} f$ satisfies the equation

$$du/dt = A/\epsilon + Bu/\epsilon^2$$

where

$$PAP = 0, \quad \lim_{\lambda \rightarrow \infty} \int_0^\infty \lambda e^{-\lambda t} e^{tB} f dt = Pf.$$

We must show that $C_0^1(M) \subseteq$ the closure of the range of $\lambda - \Delta$, $\lambda > 0$. But (4.2) implies [1] that the resolvent operator maps $C_0(M) \rightarrow C_0(M)$. Therefore, the equation $(\lambda - \Delta)f = g \in C_0^1(M)$ is solved by $f(x) = E\{\int_0^\infty e^{-\lambda t} g(B_t^x) dt\} \in C_0(M)$. Hence by the theorem of Kurtz, we have for $f \in C_0^1(M)$,

$$\lim_{\epsilon \rightarrow 0} e^{t(A/\epsilon + B/\epsilon^2)} f = e^{tPAB^{-1}AP} Pf = e^{t\Delta/n} Pf$$

in the norm of $C_0(T(M))$, which was to be shown.

5. Manifolds of negative curvature. In order to verify (4.2) in some non-compact cases, we assume in addition that M is an analytic simply connected manifold of negative curvature. In this case, the problem was considered by Azencott [1], who used a method based on Hasminskii's test. We will show below that by using a simple observation of Itô [7], (4.2) may be proved by a direct examination of the stochastic equations.

Indeed, in this case M is homeomorphic to its tangent space at some $x_0 \in M$. Taking geodesic polar coordinates at x , we have [3]

$$r^2(B_t) = \int_0^t 2r(B_s) dW_s + \int_0^t \left[2 + 2\frac{\Theta'}{\Theta}(B_s)r(B_s) \right] ds$$

where Θ is the volume element in these coordinates $\Theta'/\Theta \leq (n-1)b \coth br$, where $K \geq -b^2$. Now we apply Itô's formula to $f = \log(1 + r^2)$:

$$\begin{aligned} \log[1 + r^2(B_t)] &= \int_0^t \frac{2r(B_s) dW_s}{1 + r^2(B_s)} + \int_0^t \frac{2 + 2(\Theta'/\Theta)(B_s)r(B_s)}{1 + r^2(B_s)} ds \\ &\quad - \int_0^t \frac{4r^2(B_s)}{[1 + r^2(B_s)]^2} ds. \end{aligned}$$

To estimate the expectation, note that the first term above is an L^2 -martingale and hence has mean 0. The third term is negative. For the second term, note that $|x \coth x| \leq 2 + 2x$. Therefore

$$\begin{aligned} E[\log[1 + r^2(B_t)]] &\leq \int_0^t \frac{2 + 4(n-1)[1 + br(B_s)]}{1 + r^2(B_s)} ds \\ &\leq [4n - 2 + 2b(n-1)]t \end{aligned}$$

where we have used the inequality $r/(1 + r^2) \leq \frac{1}{2}$.

PROPOSITION 5.1. *Let M be an analytic simply connected Riemann mani-*

fold with $K \geq -b^2$. Then the diffusion semigroup U_t maps $C_0(M)$ into $C_0(M)$.

PROOF. Let $f \in C_0(M)$: given $\epsilon > 0$, let K_R be a geodesic ball such that $|f(x)| \leq \epsilon$ if $x \notin K_R$. Now

$$U_t f(x) = Ef(B_t^x) = \left\{ \int_{K_R^c} + \int_{K_R} \right\} f(y)P_t^x(dy).$$

The first integral is bounded by ϵ , by definition of K_R . To estimate the second integral, notice that if $B_t^x \in K_R$, then $d(B_t^x, x) \geq d(x, x_0) - R$. Therefore if $d(x, x_0) > R$,

$$\begin{aligned} \int_{K_R^c} P_t^x(dy) &\leq P\{d(B_t^x, x) \geq d(x, x_0) - R\} \\ &\leq P\{\log[1 + d^2(B_t^x, x)] \geq \log[1 + (d(x, x_0) - R)^2]\} \\ &\leq t(4n - 2 + 2b(n - 1))/\log(1 + (d(x, x_0) - R)^2). \end{aligned}$$

Now let $d(x, x_0) \rightarrow \infty$. Thus $\overline{\lim} |T_t f(x)| \leq \epsilon$ for each $\epsilon > 0$. Hence $U_t f \in C_0(M)$ which was to be proved.

ADDED IN PROOF. We have just learned of similar approximations of Brownian motion by geodesics in the works of Jorgensen [12] and Malliavin [13]. The latter construction generalizes the classical balayage method of Poincaré.

REFERENCES

1. R. Azencott, *Behavior of diffusion semigroups at infinity*, Bull. Soc. Math. France **102** (1974), 193–240.
2. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Pure and Appl. Math., vol. 29, Academic Press, New York and London, 1968. MR 41 #9348.
3. A. Debiard, B. Gaveau and E. Mazet, *Temps d'arrêt des diffusions riemanniennes*, C. R. Acad. Sci. Paris Sér. A-B **278** (1974), A723–A725. MR 49 #6381a.
4. R. Gangolli, *On the construction of certain diffusions on a differentiable manifold*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 406–419. MR 29 #2870.
5. R. Goldberg, *Curvature and homology*, Academic Press, New York, 1968.
6. N. J. Hicks, *Notes on differential geometry*, Van Nostrand Math. Studies, no. 3, Van Nostrand, Princeton, N. J., 1965. MR 31 #3936.
7. K. Itô, *Stability of stochastic dynamical systems*, Lecture Notes in Math., vol. 294, Springer-Verlag, New York, 1973, pp. 1–7.
8. T. G. Kurtz, *A limit theorem for perturbed operator semigroups with applications to random evolutions*, J. Functional Analysis **12** (1973), 55–67.
9. ———, *Semigroups of conditioned shifts and approximation of Markov processes*, Ann. Probability **3** (1975), 618–642.
10. D. W. Stroock, *On the growth of stochastic integrals*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **18** (1971), 340–344. MR 44 #4825.
11. S. Watanabe and T. Watanabe, *Convergence of isotropic scattering transport process to Brownian motion*, Nagoya Math. J. **40** (1970), 161–171. MR 43 #5606.
12. E. Jorgensen, *The central limit problem for geodesic random walk*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **32** (1975), 1–64.
13. P. Malliavin, *Diffusions et géométrie différentielle globale*, Lecture Notes, August 1975, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, Paris 5.