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# ON THE STRUCTURE OF CERTAIN SUBALGEBRAS OF A UNIVERSAL ENVELOPING ALGEBRA(1)

BY

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ABSTRACT. The representation theory of a semisimple group $G$, from an algebraic point of view, reduces to determining the finite dimensional representation of the centralizer $U^{\mathfrak{k}}$ of the maximal compact subgroup $K$ of $G$ in the universal enveloping algebra $U$ of the Lie algebra $\mathfrak{g}$ of $G$. The theory of spherical representations has been determined in this way since by a result of Harish-Chandra $U^{\mathfrak{k}}$ modulo a suitable ideal $I$ is isomorphic to the ring of Weyl group $W$ invariants $U(\mathfrak{a})^W$ in a suitable polynomial ring $U(\mathfrak{a})$. To deal with the general case one must determine the image of $U^{\mathfrak{k}}$ in $U(\mathfrak{k}) \otimes U(\mathfrak{a})$, where $\mathfrak{k}$ is the Lie algebra of $K$. We prove that if $W$ is replaced by the Kunze-Stein intertwining operators $\widetilde{W}$ then $U^{\mathfrak{k}}$ suitably localized and completed is indeed isomorphic to $U(\mathfrak{k}) \otimes U(\mathfrak{a})^{\widetilde{W}}$ suitably localized and completed.

**1. Introduction.** Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. If $G$ is a Lie group, say with finite center, with Lie algebra $\mathfrak{g}$, it is known that many of the fundamental questions concerning the infinite dimensional representation theory of $G$ reduce to questions about the structure and finite dimensional representation theory of the algebra $\mathcal{G}^{\mathfrak{k}}$. Here $\mathcal{G}$ is the universal enveloping algebra, over $\mathbf{C}$, of $\mathfrak{g}$ and $\mathcal{G}^{\mathfrak{k}}$ is the centralizer of $\mathfrak{k}$ in $\mathcal{G}$. Briefly, the reason for this is as follows (Theorem of Harish-Chandra): To any quasi-simple irreducible Banach space representation $\pi$ of $G$ there is associated an algebraically irreducible $\mathcal{G}$-module $V$ which is locally finite for $\mathfrak{k}$ and which determines $\pi$ up to infinitesimal equivalence. In fact one has a primary decomposition $V = \bigoplus V_\delta$, where the sum is taken over the set $\hat{\mathfrak{k}}$ of all equivalence classes $\delta$ of finite dimensional irreducible $\mathfrak{k}$-modules, and the multiplicity of $\delta$ is finite for any $\delta \in \hat{\mathfrak{k}}$. Then, in particular, any $V_\delta$ is finite dimensional and hence, a finite dimensional $\mathcal{G}^{\mathfrak{k}}$-module. The point is that $V$ itself as a $\mathcal{G}$-module is completely determined by $V_\delta$ as a $\mathcal{G}^{\mathfrak{k}}$-module for any fixed $\delta$ when $V_\delta \neq 0$. See Lepowsky and McCollum [10] and Lepowsky [9] for a nice exposition of this. See also Dixmier [3].

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If $V_{\delta_0} \neq 0$, where $\delta_0$ is the class of the trivial representation of $\mathfrak{k}$, then $\pi$ is called spherical. The approach above has been quite successful in dealing with spherical irreducible representations of $G$ (see e.g. Kostant [6]). Indeed, we may take $\delta = \delta_0$ and thus we have only to consider a quotient $G^{\mathfrak{k}}/I$ instead of $G^{\mathfrak{k}}$. Here $I$ is the intersection of $G^{\mathfrak{k}}$ with the left ideal in $G$ generated by $\mathfrak{k}$. Now by a theorem of Harish-Chandra, $G^{\mathfrak{k}}/I$ is not only commutative but also isomorphic to a polynomial ring in $r$ variables where $r$ is the split rank of $G$. More precisely one has an algebra exact sequence

$$(1.1) \qquad 0 \longrightarrow I \longrightarrow G^{\mathfrak{k}} \xrightarrow{\gamma} A^{\widetilde{W}} \longrightarrow 0$$

where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$, $A \subset G$ is the universal enveloping algebra of $\mathfrak{a}$ (over C) and $A^{\widetilde{W}}$ is the ring of $\widetilde{W}$-invariants in $A$, $\widetilde{W}$ being the translated Weyl group.

To investigate the general (not necessarily spherical) case along these lines one must look at $G^{\mathfrak{k}}$ itself, not just $G^{\mathfrak{k}}/I$. It is known (see e.g. Lepowsky [9]) that the map (1.1) may be replaced by an exact sequence (see Proposition 3.1)

$$0 \longrightarrow G^{\mathfrak{k}} \xrightarrow{P} K^M \longrightarrow A$$

where $K$ is the universal enveloping algebra, over C, of $\mathfrak{k}$, $M$ is the centralizer of $\mathfrak{a}$ in the analytic subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$, $K^M$ is the centralizer of $M$ in $K$ and $K^M \otimes A$ is given the tensor product algebra structure. Moreover $P$ is an antihomomorphism of algebras. In order to generalize (1.1) it is necessary to determine the image of $P$. Towards the end we introduce the subalgebra $B$ of all elements in $K^M \otimes A$ which commute with certain intertwining operators. Such operators are in 1 : 1 correspondence with the elements of the Weyl group $W$ and are rather closely related to the operators considered in [12] and also to those studied in [8] and [5]. To define $B$ we consider $K^M \otimes A$ as a subalgebra of a larger algebra. In fact the relation of $B$ to $K^M \otimes A$ may be taken as the generalization of the relation of $A^{\widetilde{W}}$ to $A$.

A result of Tirao shows that the image of $P$ lies in $B$ (Theorem 3.2). However, unlike (1.1), $P$ is not an anti-isomorphism of $G^{\mathfrak{k}}$ onto $B$. But now we isolate an element $\gamma$ in the center of $G$ (hence in the center of $G^{\mathfrak{k}}$). One notes the mapping $P$ extends to an exact sequence

$$0 \longrightarrow G^{\mathfrak{k}}_\gamma \xrightarrow{P_\gamma} B_{\gamma_0}$$

where $G^{\mathfrak{k}}_\gamma$ is the localization of the ring $G^{\mathfrak{k}}$ with respect to $\gamma$ and $B_{\gamma_0}$ is the localization of $B$ with respect to $\gamma_0 = P(\gamma)$.

Now there are natural valuations (in the sense of ring theory) on $G^{\mathfrak{k}}_\gamma$ and $B_{\gamma_0}$ so that the extended map $P_\gamma$ is compatible with these valuations. Thus $P_\gamma$

extends to a map $P_\Gamma$ of the respective completions $G_\Gamma^{\mathfrak{k}}$ and $\mathcal{B}_{\Gamma_0}$. Our main result is the following:

THEOREM. *The map* $P_\Gamma$: $G_\Gamma^{\mathfrak{k}} \longrightarrow \mathcal{B}_{\Gamma_0}$ *is a surjective anti-isomorphism.*

2. Let $G$ be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$; assume that $G$ has finite center. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. If $\alpha$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, we denote by $\mathfrak{g}^\alpha$ the corresponding root subspace. Fix a linear ordering on the dual of $\mathfrak{a}$ and set

$$\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}^\alpha \quad \text{and} \quad \bar{\mathfrak{n}} = \sum_{\alpha>0} \mathfrak{g}^{-\alpha}.$$

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$.

Let $K$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{k}$, so $K$ is a maximal compact subgroup of $G$, and let $A$, $N$, $\bar{N}$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}, \mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. $G$ has the global Iwasawa decomposition $G = KAN$. For $x$ in $G$ we write $x = \kappa(x)(\exp H(x))n$ with $\kappa(x) \in K$, $H(x) \in \mathfrak{a}$, $n \in N$. Let $M$ (resp. $M'$) be the centralizer (resp. the normalizer) of $\mathfrak{a}$ in $K$; $W = M'/M$ is a finite group, the Weyl group.

Let $\mathfrak{a}_{\mathbf{C}}^*$ be the complex dual of $\mathfrak{a}$. The Weyl group $W$ operates on $\mathfrak{a}_{\mathbf{C}}^*$ by

$$\langle \bar{w}(\lambda), H\rangle = \langle \lambda, \operatorname{Ad}(w^{-1})H\rangle, \qquad \lambda \in \mathfrak{a}_{\mathbf{C}}^*,\ H \in \mathfrak{a},$$

where $\bar{w} = wM$, $w \in M'$. Let $\rho(H) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}(H)|\mathfrak{n})$ for $H$ in $\mathfrak{a}$; in other words, $\rho$ is half the sum of the positive roots with multiplicities.

We shall consider a family $U^\lambda$ of continuous representations of $G$ parametrized by $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ (which may be viewed as being induced from characters of $AN$) and realized on $L^2(K)$. Given $x$ in $G$, $U^\lambda(x)$ is defined by the prescription

$$(U^\lambda(x)f)(k) = e^{-(\lambda+\rho)H(x^{-1}k)} \cdot f(\kappa(x^{-1}k)), \qquad f \in L^2(K)$$

(see Warner [14, p. 445]).

For $w \in M'$, define $\bar{N}_w = \bar{N} \cap w^{-1}Nw$. Clearly $\bar{N}_w$ depends only on the coset $\bar{w} = wM$. We introduce intertwining operators for the representations $U^\lambda$ by considering the formal integral (for the statement about convergence see Proposition 2.1 below).

$$(2.1) \qquad (A(w, \lambda)f)(k) = \int_{\bar{N}_w} e^{-(\lambda+\rho)H(v)} f(kw\kappa(v))\, dv$$

where $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, $f \in C^\infty(K)$ and the Haar measure $dv$ on $\bar{N}_w$ is normalized by (see Schiffmann [12, p. 35]),

$$\int_{\bar{N}_w} e^{-2\rho(H(v))} dv = 1.$$

If $\alpha$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, we denote by $H_\alpha$ the unique element in $\mathfrak{a}$ such that $\alpha(H) = B(H_\alpha, H)$ for $H \in \mathfrak{a}$ ($B$ is the Killing form of $\mathfrak{g}$).

We recall that every $\overline{w} \in W$ can be decomposed as the product of reflections with respect to the simple roots. The minimum integer $q$, such that there exist $q$ simple roots, not necessarily different, $\alpha_1, \ldots, \alpha_q$ with $\overline{w} = s_{\alpha_1} \cdots s_{\alpha_q}$ ($s_\alpha$ = reflection corresponding to $\alpha$) is by definition the length $l(\overline{w})$ of $\overline{w}$.

Let $C^\infty(K)$ denote the set of $C^\infty$ complex-valued functions on $K$ equipped with the usual Fréchet structure (Schwartz topology).

We come to Schiffmann's results which can be found in [12], except that Schiffmann deals with the "induced picture". We state them in the following proposition for future reference. One notes first that $C^\infty(K)$ is stable under the action of $U^\lambda(x)$, $x \in G$.

PROPOSITION 2.1. (i) *The domain of convergence* (*absolute*) *of the intertwining integrals* (2.1) *is the set* $S(\overline{w})$ *of all* $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ *such that* $\mathrm{Re}(\lambda(H_\alpha)) > 0$ *for every positive root* $\alpha$ *such that* $\overline{w}(\alpha) < 0$.

(ii) *If* $\lambda \in S(\overline{w})$, $A(w, \lambda)$ *is a continuous endomorphism of* $C^\infty(K)$, *and* $U^{\overline{w}(\lambda)}(x)A(w, \lambda)f = A(w, \lambda)U^\lambda(x)f$ *for all* $x \in G$, $f \in C^\infty(K)$.

(iii) *Let* $w_1, w_2 \in M'$ *such that* $l(\overline{w}_1\overline{w}_2) = l(\overline{w}_1) + l(\overline{w}_2)$. *Then* $S(\overline{w}_1\overline{w}_2) = S(\overline{w}_2) \cap \overline{w}_2^{-1}S(\overline{w}_1)$ *and* $A(w_1w_2, \lambda) = A(w_1, \overline{w}_2(\lambda))A(w_2, \lambda)$ *for* $\lambda \in S(\overline{w}_1\overline{w}_2)$.

One uses this result to establish

PROPOSITION 2.2. (i) *Given* $w \in M'$, *for* $\lambda \in S(\overline{w})$ *the linear form*

$$T(w, \lambda): f \longmapsto \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} f(w\kappa(v))\,dv, \qquad f \in C^\infty(K),$$

*defines a distribution on* $K$.

(ii) *Let* $w_1 w_2 \in M'$ *such that* $l(\overline{w}_1\overline{w}_2) = l(\overline{w}_1) + l(\overline{w}_2)$. *Then*

$$T(w_1w_2, \lambda) = T(w_1, \overline{w}_2(\lambda)) * T(w_2, \lambda) \quad \text{for } \lambda \in S(\overline{w}_1\overline{w}_2).$$

PROOF. (i) follows from Proposition 2.1(ii) by observing that $\langle T(w, \lambda), f\rangle = (A(w, \lambda)f)(e)$. To prove (ii) we note that

$$A(w, \lambda)f = (T(w, \lambda) * \check{f})^{\vee}, \qquad f \in C^\infty(K),$$

where $\check{f}$ denotes the function $k \longmapsto f(k^{-1})$. If $\lambda \in S(\overline{w}_1\overline{w}_2)$, Proposition 2.1(iii) gives

$$A(w_1w_2, \lambda)f = A(w_1, \overline{w}_2(\lambda))A(w_2, \lambda)f, \qquad f \in C^\infty(K),$$

which can be rewritten

$$\begin{aligned}(T(w_1w_2, \lambda) * \check{f})^{\vee} &= (T(w_1, \overline{w}_2(\lambda)) * (A(w_2, \lambda)f)^{\vee})^{\vee} \\ &= (T(w_1, \overline{w}_2(\lambda)) * T(w_2, \lambda) * \check{f})^{\vee}.\end{aligned}$$

Therefore,

$$T(w_1 w_2, \lambda) * \check{f} = T(w_1, \overline{w}_2(\lambda)) * T(w_2, \lambda) * \check{f}.$$

Now evaluating at the identity $e$ and using the fact that $\langle T, f\rangle = (T * \check{f})(e)$, we complete the proof of the proposition.

3. Let $\mathfrak{U}$ be the universal enveloping algebra of the complexification $\mathfrak{u}_{\mathbf{C}}$ of the Lie algebra $\mathfrak{u}$ of a Lie group $U$. As is well known, we may regard $\mathfrak{U}$ as the algebra of distributions on $U$ whose support is the identity $\{e\}$. One knows that $D \in \mathfrak{U}$ defines a left invariant differential operator $f \longmapsto Df$ on $G$ where $Df = f * \check{D}$ and $D \longmapsto \check{D}$ is the usual antipode in $\mathfrak{U}$. One also knows that $\langle D, f\rangle = [Df](e)$, $f \in C_c^\infty(U)$.

Let $\mathfrak{g}_{\mathbf{C}}, \mathfrak{k}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}}, \mathfrak{n}_{\mathbf{C}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, respectively. We denote by $\mathfrak{G}$ the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$, and by $\mathfrak{K}$, $\mathfrak{A}$ and $\mathfrak{N}$, the universal enveloping algebras of $\mathfrak{k}_{\mathbf{C}}, \mathfrak{a}_{\mathbf{C}}$ and $\mathfrak{n}_{\mathbf{C}}$, respectively, regarded as canonically embedded in $\mathfrak{G}$. We have

$$\mathfrak{G} = \mathfrak{K}\mathfrak{A}\mathfrak{N} = \mathfrak{K}\mathfrak{A}(\mathbf{C}1 + \mathfrak{N}\mathfrak{n}_{\mathbf{C}}) = \mathfrak{K}\mathfrak{A} + \mathfrak{G}\mathfrak{n}_{\mathbf{C}}.$$

Let $P$: $\mathfrak{G} \longrightarrow \mathfrak{K}\mathfrak{A}$ denote the corresponding projection map. We give $\mathfrak{K}\mathfrak{A}$ an algebra structure by identifying it with the algebra $\mathfrak{K} \otimes \mathfrak{A}$, and we also regard $P$ as a map $P$: $\mathfrak{G} \longrightarrow \mathfrak{K} \otimes \mathfrak{A}$. Let $\mathfrak{G}^K$ and $\mathfrak{K}^M$ denote the centralizers of $K$ in $\mathfrak{G}$ and of $M$ in $\mathfrak{K}$, respectively. A proof of the following proposition can be found in Lepowsky [9].

PROPOSITION 3.1. *$P$ defines an injective antihomomorphism of $\mathfrak{G}^K$ into $\mathfrak{K}^M \otimes \mathfrak{A}$.*

The algebra $\mathfrak{A}$ is just the symmetric algebra $S(\mathfrak{a}_{\mathbf{C}})$; hence each linear mapping $\lambda$: $\mathfrak{a} \longrightarrow \mathbf{C}$ extends uniquely to a homomorphism $D \longrightarrow D(\lambda)$ of $\mathfrak{A}$ into $\mathbf{C}$ satisfying $1(\lambda) = 1$. Now given $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ we can also consider the homomorphism $\mathfrak{K} \otimes \mathfrak{A} \longrightarrow \mathfrak{K}$ defined by $E \otimes D \longmapsto (E \otimes D)(\lambda) = D(\lambda)E$ $(E \in \mathfrak{K}, D \in \mathfrak{A})$.

We take the opportunity to prove the following unpublished result of Tirao.

THEOREM 3.2. *Given $w \in M'$, $\lambda \in S(\overline{w})$, we have*

$$T(w, \lambda) * P(D)(-\lambda - \rho) = P(D)(-\overline{w}(\lambda) - \rho) * T(w, \lambda) \quad \textit{for all } D \in \mathfrak{G}^K.$$

PROOF. Consider the following identity

$$\begin{aligned} &e^{-(\overline{w}(\lambda)+\rho)H(x)} \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} f(\kappa(x)w\kappa(v))\, dv \\ &= \int_{\overline{N}_w} e^{-(\lambda+\rho)(H(v)+H(xw\kappa(v)))} f(\kappa(xw\kappa(v)))\, dv, \qquad x \in G, \end{aligned} \tag{3.1}$$

which is another way of writing $(U^{\overline{w}(\lambda)}(x^{-1})A(w,\lambda)f)(e) = (A(w,\lambda)U^{\lambda}(x^{-1})f)(e)$ (cf. Proposition 2.1(ii)). Let $\varphi_1(x)$ and $\varphi_2(x)$ denote the left- and right-hand sides of (3.1), respectively. It is also convenient to introduce the following notation: given $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ and $f \in C^\infty(K)$, let

$$F_f^\lambda(x) = e^{-(\lambda+\rho)H(x)} f(\kappa(x)), \qquad x \in G.$$

Then since $H(xn) = H(x)$ and $\kappa(xn) = \kappa(x)$ for $n \in N$ it follows that $DF_f^\lambda = 0$ for $D \in \mathcal{G}\mathfrak{n}_{\mathbf{C}}$, i.e., $DF_f^\lambda = P(D)F_f^\lambda$, for $D \in \mathcal{G}$. Since $H(x \exp H) = H(x) + H$ and $\kappa(x \exp H) = \kappa(x)$ $(H \in \mathfrak{a})$, we have $DF_f^\lambda = D(-\lambda-\rho)F_f^\lambda$, for $D \in \mathcal{A}$. Having in mind the decomposition $\mathcal{G} = \mathcal{K}\mathcal{A} \oplus \mathcal{G}\mathfrak{n}_{\mathbf{C}}$ it follows that $DF_f^\lambda = P(D)(-\lambda-\rho)F_f^\lambda$, for $D \in \mathcal{G}$.

If $f$ is a continuous function on $K$ we shall write $f^{R(k)}$ for the composite function $f \circ R(k)$ where $R(k)$ is a right translation by $k \in K$.

Given $D \in \mathcal{G}$, we have

$$\begin{aligned}
[D\varphi_1](e) &= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [DF^{\overline{w}(\lambda)}_{f^{R(w\kappa(v))}}](e)\,dv \\
&= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [P(D)(-\overline{w}(\lambda)-\rho)F^{\overline{w}(\lambda)}_{f^{R(w\kappa(v))}}](e)\,dv \\
&= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [P(D)(-\overline{w}(\lambda)-\rho)\, f^{R(w\kappa(v))}](e)\,dv \\
&= \langle P(D)(-\overline{w}(\lambda)-\rho) * T(w,\lambda), f\rangle.
\end{aligned}$$

Now let $D \in \mathcal{G}^K$ and differentiate $\varphi_2$ to obtain

$$\begin{aligned}
[D\varphi_2](e) &= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [DF_f^\lambda](w\kappa(v))\,dv \\
&= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [P(D)(-\lambda-\rho)F_f^\lambda](w\kappa(v))\,dv \\
&= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)} [P(D)(-\lambda-\rho)f](w\kappa(v))\,dv \\
&= \langle T(w,\lambda) * P(D)(-\lambda-\rho), f\rangle. \quad \text{Q.E.D.}
\end{aligned}$$

Given a finite dimensional irreducible representation $(V_\delta, \delta)$ of $K$ let us consider the maps

$$P_\delta = (\delta \otimes 1) \circ P\colon \mathcal{G}^K \longrightarrow \operatorname{End}(V_\delta) \otimes \mathcal{A}$$

and

$$p_\delta = (\operatorname{tr} \otimes 1) \circ P_\delta\colon \mathcal{G}^K \longrightarrow \mathcal{A}.$$

When $\delta$ is the trivial one-dimensional representation of $K$, $P_\delta$ (or $p_\delta$) gives

Harish-Chandra's famous homomorphism $\gamma$: $\mathcal{G}^K \longrightarrow \mathcal{A}$. Theorem 3.2 generalizes that part of Harish-Chandra's theorem which asserts that the image of $\gamma$ is contained in the ring of $\widetilde{W}$-invariants of $\mathcal{A}$ ($\widetilde{W}$ denotes the translated Weyl group). One also has the following result of Lepowsky (see [9])

$$p_\delta(D)(\lambda - \rho) = p_\delta(D)(\overline{w}(\lambda) - \rho) \tag{3.2}$$

for all $D \in \mathcal{G}^K$, $\overline{w} \in W$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$.

From Theorem 3.2 we get instead the more precise result

$$\delta(T(w, \lambda))P_\delta(D)(-\lambda - \rho) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(T(w, \lambda)) \tag{3.3}$$

for all $D \in \mathcal{G}^K$, $w \in M'$, $\lambda \in S(\overline{w})$.

Note that $\delta(T(w, \lambda))$ is given by the integral

$$\delta(T(w, \lambda)) = \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)}\delta(w\kappa(v))\,dv, \qquad \lambda \in S(\overline{w}).$$

Let $n = \dim \overline{N}_w$. From Theorem 4.1, it will follow that there exists a nonzero complex number $t_w(\lambda)$ such that

$$\lim_{t\to+\infty} t^{n/2}\delta(T(w, t\lambda)) = t_w(\lambda)\delta(w), \qquad w \in M',$$

uniformly on compact subsets of $S(\overline{w})$. Therefore, given any compact subset $\omega \subset S(\overline{w})$, for $t$ sufficiently large

$$\delta(T(w, t\lambda)) \text{ is invertible for all } \lambda \in \omega. \tag{3.4}$$

Now it is clear that (3.3) implies (3.2).

Let $\mathcal{D}(K)$ denote the space of distributions on $K$ equipped with the topology of uniform convergence on bounded subsets of $C^\infty(K)$. Let $\mathcal{D}(K)^M$ be the centralizer of $M$ in $\mathcal{D}(K)$. We shall write $\delta_k$ for the Dirac measure at $k \in K$.

We can write

$$T(w, \lambda) = \delta_w * T'(\overline{w}, \lambda), \qquad \lambda \in S(\overline{w}), \tag{3.5}$$

(cf. Proposition 2.2(i)) where $T'(\overline{w}, \lambda)$ is the distribution on $K$ defined by

$$\langle T'(\overline{w}, \lambda), f\rangle = \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)}f(\kappa(v))\,dv, \qquad \lambda \in S(\overline{w}),\ f \in C^\infty(K).$$

Now

$$T'(\overline{w}, \lambda) \in \mathcal{D}(K)^M \quad \text{for } \lambda \in S(\overline{w}),\ \overline{w} \in W. \tag{3.6}$$

In fact, for $\lambda \in S(\overline{w})$, $f \in C^\infty(K)$ and $m \in M$ we have

$$\begin{aligned}\langle \delta_m * T'(\overline{w}, \lambda) * \delta_{m^{-1}}, f\rangle &= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(v)}f(m\kappa(v)m^{-1})\,dv \\ &= \int_{\overline{N}_w} e^{-(\lambda+\rho)H(mvm^{-1})}f(\kappa(mvm^{-1}))\,dv\end{aligned}$$

because $M$ normalizes $N$. But the Haar measure $dv$ of $\overline{N}_w$ is invariant under $v \longmapsto mvm^{-1}$; therefore $\delta_m * T'(\overline{w}, \lambda) * \delta_{m^{-1}} = T'(\overline{w}, \lambda)$, which proves (3.6).

A consequence of Theorem 3.2 is the following

COROLLARY 3.3. *Assume $\mathcal{D}(K)^M$ is abelian (which is precisely the case when $G$ is one of the following classical rank one groups:* SO$(n, 1)$ *or* SU$(n, 1)$). *Then*

$$\delta_w * P(D)(\lambda - \rho) = P(D)(\overline{w}(\lambda) - \rho) * \delta_w$$

*for all $w \in M'$, $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$ and $D \in \mathcal{G}^K$.*

PROOF. Let $\delta$ be any finite dimensional irreducible representation of $K$. From (3.3) and (3.5) we obtain

$$\delta(w)\delta(T'(\overline{w}, \lambda))P_\delta(D)(-\lambda - \rho) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w)\delta(T'(\overline{w}, \lambda))$$

for $w \in M'$, $\lambda \in S(\overline{w})$ and $D \in \mathcal{G}^K$. But since $T'(\overline{w}, \lambda)$ and $P(D)(-\lambda - \rho)$ are in $\mathcal{D}(K)^M$ (cf. (3.6) and Proposition 3.1) we have

$$\delta(w)P_\delta(D)(-\lambda - \rho)\delta(T'(\overline{w}, \lambda)) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w)\delta(T'(\overline{w}, \lambda)).$$

Now because of (3.4) and (3.5) we obtain

$$\delta(w)P_\delta(D)(-\lambda - \rho) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w), \qquad \lambda \in \mathfrak{a}_{\mathbf{C}}^*,$$

which in turn implies our assertion. Q.E.D.

4. Let $\mathcal{B}$ be the set of all elements $B \in \mathcal{K}^M \otimes \mathcal{A}$ such that

$$T(w, \lambda) * B(-\lambda - \rho) = B(-\overline{w}(\lambda) - \rho) * T(w, \lambda) \tag{4.1}$$

for all $w \in M'$ and all $\lambda \in S(\overline{w})$. Clearly $\mathcal{B}$ is a subalgebra of $\mathcal{K}^M \otimes \mathcal{A}$, and according to Theorem 3.2 it contains the image $P$: $\mathcal{G}^K \longrightarrow \mathcal{K}^M \otimes \mathcal{A}$. The principal objective now is to get information about the leading term of $B \in \mathcal{B}$. The following theorem is needed and should be compared with results of Cohn [1].

THEOREM 4.1. *Given $w \in M'$ let $n = \dim \overline{N}_w$. For each $\lambda \in S(\overline{w})$ there exists a nonzero complex number $t_w(\lambda)$ such that*

$$\lim_{t \to +\infty} t^{n/2} T(w, t\lambda) = t_w(\lambda)\delta_w$$

*uniformly on compact subsets of $S(\overline{w})$.*

PROOF. We shall show that it is sufficient to consider the case when $\overline{w} = s_\alpha$ is the reflection corresponding to a simple root $\alpha$. In fact, given $w \in M'$ we can write $\overline{w} = s_{\alpha_1} \cdots s_{\alpha_q}$ where $q = l(\overline{w})$ and $\alpha_j$ $(j = 1, \ldots, q)$ are simple roots. We can find elements $w_1, \ldots, w_q$ in $M'$ such that $\overline{w}_j = s_{\alpha_j}$ $(j = 1, \ldots, q)$ and $w = w_1 \cdots w_q$. Now from Proposition 2.2(ii) it follows that

$$T(w, \lambda) = T(w_1, \overline{w}_2 \cdots \overline{w}_q(\lambda)) * T(w_2, \overline{w}_3 \cdots w_q(\lambda)) * \cdots * T(w_q, \lambda).$$

On the other hand if $n_j = \dim \overline{N}_{w_j}$ $(j = 1, \ldots, q)$ we have $n = n_1 + \cdots + n_q$ (cf. [12, Proposition 1.3, p. 12]). Therefore, if we assume the theorem when $\overline{w} = s_\alpha$, $\alpha$ a simple root, we get

$$\begin{aligned}\lim_{t\to+\infty} t^{n/2}T(w, t\lambda) &= \lim_{t\to+\infty} t^{n_1/2}T(w_1, w_2 \cdots w_q(t\lambda)) * \cdots * \lim_{t\to+\infty} t^{n_q/2}T(w_q, t\lambda)\\ &= t_{w_1}(\overline{w}_2 \cdots \overline{w}_q(\lambda)) \cdots t_{w_q}(\lambda)\delta_{w_1} * \cdots * \delta_{w_q} = t_w(\lambda)\delta_w,\end{aligned}$$

uniformly on compact subsets of $S(\overline{w})$. We have used the joint continuity of the convolution which is a consequence of the compactness of $K$. Next we shall prove the case $\overline{w} = s_\alpha$, $\alpha$ is a simple root, thus completing the proof of the theorem.

Let $w \in M'$ be such that $\overline{w} = s_\alpha$, where $\alpha$ is a simple root. The Lie algebra $\overline{\mathfrak{n}}_w$ of $\overline{N}_w = \overline{N} \cap \overline{w}'Nw$ is given by $\overline{\mathfrak{n}}_w = \mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha}$. Let $G_\alpha$ be the analytic subgroup whose Lie algebra is the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{g}^{-2\alpha}, \mathfrak{g}^{-\alpha}, \mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{2\alpha}$. Then $G_\alpha$ is a semisimple Lie group with finite center. If we take $K_\alpha = G_\alpha \cap K$, $A_\alpha = G_\alpha \cap A$ and $N_\alpha = G_\alpha \cap N$ then $G_\alpha = K_\alpha A_\alpha N_\alpha$ is an Iwasawa decomposition of $G_\alpha$. The Lie algebra $\mathfrak{a}_\alpha$ of $A_\alpha$ is equal to $RH_\alpha$, i.e. $G_\alpha$ has real-rank one.

Let $p = \dim \mathfrak{g}^{-\alpha}$ and $q = \dim \mathfrak{g}^{-2\alpha}$, then $\dim \overline{N}_w = p + q$. Since for $\lambda \in S(\overline{w})$, $T(w, \lambda) = \delta_w * T'(w, \lambda)$ (cf. 3.5) it is enough to establish that

$$\lim_{t\to+\infty} t^{(p+q)/2}T'(\overline{w}, t\lambda) = t_w(\lambda)\delta_e$$

$(0 \neq t_w(\lambda) \in \mathbf{C})$ uniformly on compact subsets of $S(\overline{w})$.

The distributions $T'(w, \lambda)$ $(\lambda \in S(\overline{w}))$ on $K$ come from the corresponding distributions on $K_\alpha$ (by restriction from $K$ to $K_\alpha$). This being a continuous map, the whole question reduces to the real-rank one group $G_\alpha$.

If $H'_\alpha = 2\alpha(H_\alpha)^{-1}H_\alpha$, then $\rho(H'_\alpha) = p + 2q$. Let $z = (p + 2q)^{-1}\lambda(H'_\alpha)$; $\lambda$ is in $S(\overline{w})$ if and only if $\operatorname{Re} z > 0$. We have to prove that given a compact subset $\omega$ of the set of all $z \in \mathbf{C}$ with $\operatorname{Re} z > 0$

$$\lim_{t\to+\infty} t^{(p+q)/2}\int_{\overline{N}_w} e^{-(tz+1)\rho(H(v))}f(\kappa(v))\,dv = t_w(\lambda)f(e)$$

uniformly for all $z \in \omega$ and all $f$ in each bounded subset of $C^\infty(K_\alpha)$.

We drop the subscript $\alpha$ and prove instead the following proposition which will complete the proof of Theorem 4.1.

Let $S_\Delta$ be the sector in the complex plane of all $z \in \mathbf{C}$ such that $0 < |z| < \infty$, $|\arg z| < \pi/2 - \Delta$.

PROPOSITION 4.2. *Let $G$ be a connected semisimple Lie group with finite center and real-rank one. Let $n = \dim \bar{N}$. Then, there exists a positive constant $c$ such that*

$$\lim z^{n/2} \int_{\bar{N}} e^{-z\rho(H(v))} f(\kappa(v))\, dv = cf(e)$$

*when* ($z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$) *uniformly for all $f$ in each bounded subset of* $C^\infty(K)$.

First we need a few lemmas.

LEMMA 4.3. *Let $\epsilon$ be a positive real number and $p$ a positive integer. Then*

$$\int_0^\epsilon r^{p-1}(1 + r^2)^{-z} dr \sim \tfrac{1}{2}\Gamma(p/2) z^{-p/2} \qquad (z \longrightarrow \infty,\ z \in S_\Delta,\ \Delta > 0).$$

PROOF. The asymptotic behavior of the above integral can be established, for example, by Laplace's method, after introducing the new variable $t = \log(1 + r)^2$ (see Erdélyi [4, p. 37]), or we can proceed more directly as follows. Write

$$\int_0^\epsilon r^{p-1}(1 + r^2)^{-z} dr = \int_0^\infty r^{p-1}(1 + r^2)^{-z} dr + g(z).$$

We have

$$\int_0^\infty r^{p-1}(1 + r^2)^{-z} dr = \Gamma(p/2)\Gamma(z - p/2)/2\Gamma(z) \qquad (\operatorname{Re} z > p/2),$$

which is asymptotic to $\tfrac{1}{2}\Gamma(p/2)z^{-p/2}$ ($z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$) (Stirling's formula; see Magnus [11, p. 12]).

On the other hand we can estimate $g(z)$ in the following way:

$$|g(z)| \leqslant \int_\epsilon^\infty r^{p-1}(1 + r^2)^{-\operatorname{Re} z} dr.$$

Given a positive real number $\delta$, there exists a positive number $A$ such that

$$r^{p+1} \leqslant \left(\frac{1 + r^2}{1 + \delta}\right)^{\operatorname{Re} z} \qquad \text{for } r \geqslant A,\ \operatorname{Re} z \geqslant p + 1.$$

Now if we choose $\delta$ less than $\epsilon^2$, we can find another constant $B$ such that

$$r^{p+1} \leqslant B\left(\frac{1 + \epsilon^2}{1 + \delta}\right)^{\operatorname{Re} z} \qquad \text{for } 0 \leqslant r \leqslant A,\ \operatorname{Re} z \geqslant p + 1.$$

Therefore, there exists $C$ such that

$$r^{p+1} \leqslant C\left(\frac{1 + r^2}{1 + \delta}\right)^{\operatorname{Re} z} \qquad \text{for } r \geqslant \epsilon,\ \operatorname{Re} z \geqslant p + 1.$$

Hence $|g(z)| \leqslant C\epsilon^{-1}(1 + \delta)^{-\operatorname{Re} z}$ for $\operatorname{Re} z \geqslant p + 1$, which implies that $g(z) = O(z^{-p/2})$ when $z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$. This proves the lemma. Q.E.D.

Let $B(\epsilon)$ $(\epsilon > 0)$ denote the set of all $(X, Y) \in \mathbf{R}^p \times \mathbf{R}^q$ such that $\|X\|$, $\|Y\| \leqslant \epsilon$.

LEMMA 4.4. *Let $p > 0$. There exists a positive constant $c_{p,q}$ such that*

$$f(z) = \int_{B(\epsilon)} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-z}\, dX\, dY \sim c_{p,q} z^{-(p+q)/2}$$

*when $z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$.*

PROOF. We have to consider two different cases: (a) $q = 0$ and (b) $q \neq 0$. Let $c_n$ be the Euclidean volume of the unit sphere in $\mathbf{R}^n$; in particular $c_1 = 2$.

(a) The usual formula for integration in polar coordinates yields

$$f(z) = c_p \int_0^\epsilon r^{p-1}(1 + r^2)^{-2z}\, dr.$$

The assertion follows from Lemma 4.3 with $c_{p,0} = 2^{-(1+p/2)}\Gamma(p/2)c_p$.

(b) In this case

$$f(z) = c_p c_q \int_0^\epsilon \int_0^\epsilon r^{p-1} s^{q-1}((1 + r^2)^2 + s^2)^{-z}\, dr\, ds.$$

Letting for $0 \leqslant s \leqslant \epsilon$, $u = s(1 + r^2)^{-1}$ we find

$$\begin{aligned} f(z) &= c_p c_q \int_0^\epsilon r^{p-1}(1 + r^2)^{q-2z} \int_0^{\epsilon(1+r^2)^{-1}} u^{q-1}(1 + u^2)^{-z}\, du\, dr \\ &= c_p c_q \int_0^\epsilon r^{p-1}(1 + r^2)^{q-2z}\, dr \int_0^\epsilon u^{q-1}(1 + u^2)^{-z}\, du - g(z) \end{aligned}$$

where

$$g(z) = c_p c_q \int_0^\epsilon r^{p-1}(1 + r^2)^{q-2z} \int_{\epsilon(1+r^2)^{-1}}^\epsilon u^{q-1}(1 + u^2)^{-z}\, du\, dr.$$

We can estimate $g(z)$ as follows:

$$\begin{aligned} |g(z)| &\leqslant c_p c_q \int_0^\epsilon r^{p-1}(1 + r^2)^{q-2\operatorname{Re} z} \int_{\epsilon(1+r^2)^{-1}}^\epsilon u^{q-1}(1 + u^2)^{-\operatorname{Re} z}\, du\, dr \\ &\leqslant c_p c_q \epsilon^{q+1} \delta (1 + \delta^2)^{-\operatorname{Re} z} \int_0^\epsilon r^{p-1}(1 + r^2)^{q-2\operatorname{Re} z}\, dr \end{aligned}$$

where $\delta = \epsilon(1 + \epsilon^2)^{-1}$. Therefore

$$g(z) = O((\operatorname{Re} z)^{-p/2}(1 + \delta)^{-\operatorname{Re} z}) = O((1 + \delta)^{-z}) \quad (z \longrightarrow \infty, z \in S_\Delta, \Delta > 0).$$

Hence $f(z) \sim c_{p,q} z^{-(p+q)/2}$ $(z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0)$ where

$$c_{p,q} = 2^{-(2+p/2)}\Gamma(p/2)\Gamma(q/2)c_p c_q. \quad \text{Q.E.D.}$$

PROOF OF PROPOSITION 4.2. Let $\alpha$ be the simple root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then $\bar{\mathfrak{n}} = \mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha}$. Let $p = \dim \mathfrak{g}^{-\alpha}$ and $q = \dim \mathfrak{g}^{-2\alpha}$. Let $Q$ be the quadratic form on $\mathfrak{g}$ defined by

$$Q(X) = 4B(X, \theta(X))/B(H'_\alpha, \theta(H'_\alpha)),$$

$\theta$ denotes the Cartan involution of $\mathfrak{g}$. If $v = \exp(X + Y)$, $X \in \mathfrak{g}^{-\alpha}$, $Y \in \mathfrak{g}^{-2\alpha}$, then (Helgason-Schiffmann, cf. [14, p. 38]) $H(v) = (a/2)H'_\alpha$ with $e^{2a} = (1 + Q(X)/2)^2 + 2Q(Y)$. We make the identifications $\mathfrak{g}^{-\alpha} \simeq \mathbf{R}^p$, $\mathfrak{g}^{-2\alpha} \simeq \mathbf{R}^q$ in such a way that $\|X\|^2 = Q(X)/2$, $\|Y\|^2 = 2Q(Y)$ $(X \in \mathfrak{g}^{-\alpha}, Y \in \mathfrak{g}^{-2\alpha})$. Then the integral under study can be written

$$\mathrm{I}(z) = z^{(p+q)/2} \iint_{\mathbf{R}^p \times \mathbf{R}^q} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-zb} f(\kappa(\exp(X + Y)))\, dX\, dY,$$

where $b = (p + 2q)/4$.

Let $B(\epsilon) = \{(X, Y) \in \mathbf{R}^p \times \mathbf{R}^q : \|X\|, \|Y\| \leqslant \epsilon\}$. We can write $\mathrm{I}(z)$ as the sum of an integral over $B(\epsilon)$ and an integral over $\mathbf{R}^p \times \mathbf{R}^q - B(\epsilon)$. Call the two resulting integrals $\mathrm{II}(\epsilon, z)$ and $\mathrm{III}(\epsilon, z)$ respectively. First of all we shall prove that $\mathrm{III}(\epsilon, z) \longrightarrow 0$ as $z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$, uniformly for all $f$ in a bounded subset of $C^\infty(K)$. There exists a constant $C$ such that the integrand of $\mathrm{III}(\epsilon, z)$ is bounded by

$$C|z|^{(p+q)/2}((1 + \|X\|^2)^2 + \|Y\|^2)^{-b\,\mathrm{Re}\,z}.$$

Given $d > 0$, for $z \in S_\Delta$ and $|z|$ sufficiently large we have

$$|z|^{(p+q)/2} \leqslant (1 + \epsilon^2)^{b\,\mathrm{Re}\,z - d} \leqslant ((1 + \|X\|^2)^2 + \|Y\|^2)^{b\,\mathrm{Re}\,z - d}$$

whenever $(X, Y) \notin B(\epsilon)$. Therefore the integrand of $\mathrm{III}(\epsilon, z)$ is bounded by

$$C((1 + \|X\|^2)^2 + \|Y\|^2)^{-d}$$

which is an integrable function for $d > b$ (see Wallach [13, p. 262]). By the dominated convergence theorem we have $\lim \mathrm{III}(\epsilon, z) = 0$ when $z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$ uniformly on bounded subsets of $C^\infty(K)$. In fact

$$\lim |z|^{(p+q)/2}((1 + \|X\|^2)^2 + \|Y\|^2)^{-b\,\mathrm{Re}\,z} = 0 \qquad (z \longrightarrow \infty, z \in S_\Delta, \Delta > 0)$$

if $(X, Y) \neq 0$.

Now consider $\mathrm{II}(\epsilon, z)$ and write

$$\mathrm{II}(\epsilon, z) = f(e) z^{(p+q)/2} \int_{B(\epsilon)} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-zb}\, dX\, dY + \mathrm{II}'(\epsilon, z).$$

By Lemma 4.4 the first term tends to $cf(e)$ with $c = c_{p,q} b^{-(p+q)/2}$ as $z \longrightarrow \infty$, $z \in S_\Delta$, $\Delta > 0$. Therefore to complete the proof of the proposition it is enough to show that given a bounded subset $B$ of $C^\infty(K)$ and a positive $\delta$, there exists a positive $\epsilon$ such that $|\mathrm{II}'(\epsilon, z)| < \delta$ for $f \in B$, $z \in S_\Delta$ and $|z|$ sufficiently large. Now

$$|\mathrm{II}'(\epsilon, z)| \leq |z|^{(p+q)/2} \int_{B(\epsilon)} (1 + \|X\|^2 + \|Y\|^2)^{-b \operatorname{Re} z}$$
$$\cdot\, |f(\kappa(\exp(X+Y))) - f(e)|\, dX\, dY.$$

From Lemma 4.4 it also follows that

$$|z|^{(p+q)/2} \int_{B(1)} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-b \operatorname{Re} z}\, dX\, dY$$

is bounded in $S_\Delta$, say by a constant $A$. Given $\delta > 0$ there exists $\epsilon$ $(0 < \epsilon \leq 1)$ such that

$$|f(\kappa(\exp(X+Y))) - f(e)| < \delta A^{-1} \quad \text{on } B(\epsilon)$$

for all $f \in B$. Then $|\mathrm{II}'(\epsilon, z)| < \delta$ for $z \in S_\Delta$ and $f \in B$, which completes the proof of Proposition 4.2. Q.E.D.

If $B \in \mathcal{K}^M \otimes \mathcal{A}$ we can view $B(\lambda)$ as a polynomial of degree $d$ on $\mathfrak{a}_{\mathbf{C}}^*$ with coefficients in $\mathcal{K}^M$. Let $B_d \in \mathcal{K}^M \otimes \mathcal{A}$ be the element such that $B_d(\lambda)$ is the leading term (homogeneous of degree $d$ in $\lambda$) of $B(\lambda)$. The Weyl group $W$ acts on $\mathcal{K}^M$ and on $\mathcal{A}$ via the adjoint representation, so we can define an action of $W$ on $\mathcal{K}^M \otimes \mathcal{A}$ by taking the tensor product action.

THEOREM 4.5. *If $B \in \mathcal{B}$ then the leading term $B_d$ of $B$ is $W$-invariant.*

PROOF. Given $w \in M'$, let $n = \dim \overline{N}_w$. By hypothesis we have $T(w, \lambda) * B(-\lambda - \rho) = B(-\overline{w}(\lambda) - \rho) * T(w, \lambda)$ for all $\lambda \in S(\overline{w})$. Now write

$$t^{n/2} T(w, t\lambda) * t^{-d} B(-t\lambda - \rho) = t^{-d} B(-\overline{w}(t\lambda) - \rho) * t^{n/2} T(w, t\lambda)$$

and let $t \longrightarrow +\infty$. From Theorem 4.1 we obtain

$$t(w, \lambda)\delta_w * B_d(-\lambda) = B_d(-\overline{w}(\lambda)) * t(w, \lambda)\delta_w, \qquad \lambda \in S(\overline{w}),$$

which proves our assertion. Q.E.D.

5. Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition. If subscript $\mathbf{C}$ denotes complexification one also has $\mathfrak{g}_{\mathbf{C}} = \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_{\mathbf{C}}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $H \subseteq \operatorname{Aut}(\mathfrak{p}_{\mathbf{C}})$ be the analytic subgroup corresponding to $\operatorname{ad}_{\mathfrak{p}_{\mathbf{C}}} \mathfrak{k}_{\mathbf{C}} \subset \operatorname{End}(\mathfrak{p}_{\mathbf{C}})$. One knows that $x \in \mathfrak{p}_{\mathbf{C}}$ is semisimple if and only if $x \in H \cdot \mathfrak{a}_{\mathbf{C}}$ (see Kostant and Rallis [7, Theorem 1]).

Let $r = \dim \mathfrak{a}_{\mathbf{C}}$. If $x \in \mathfrak{p}_{\mathbf{C}}$ then one knows that $\dim \mathfrak{p}_{\mathbf{C}}^x \geq r$ where one puts $\mathfrak{p}_{\mathbf{C}}^x = (\operatorname{Ker} \operatorname{ad} x) \cap \mathfrak{p}_{\mathbf{C}}$. An element $x \in \mathfrak{p}_{\mathbf{C}}$ is called regular if $\dim \mathfrak{p}_{\mathbf{C}}^x = r$. Similarly, let $\mathfrak{k}_{\mathbf{C}}^x = (\operatorname{Ker} \operatorname{ad} x) \cap \mathfrak{k}_{\mathbf{C}}$. Let $\mathfrak{m}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. One has [7, p. 770]

THEOREM 5.1. *For any $x \in \mathfrak{p}_{\mathbf{C}}$ one has* $\dim \mathfrak{k}_{\mathbf{C}}^x - \dim \mathfrak{p}_{\mathbf{C}}^x$ *is independent of $x$. In particular* $\dim \mathfrak{k}_{\mathbf{C}}^x \geq \dim \mathfrak{m}$ *and equality holds if and only if $x$ is regular in $\mathfrak{p}_{\mathbf{C}}$.*

Now for any $x \in \mathfrak{p}_{\mathbf{C}}$, $(\mathrm{ad}\, x)^2$ leaves $\mathfrak{p}_{\mathbf{C}}$ stable. Let $\alpha_x = (\mathrm{ad}\, x)^2|_{\mathfrak{p}_{\mathbf{C}}}$ and let $r_x$ be the multiplicity of the zero eigenvalue of $\alpha_x$.

PROPOSITION 5.2. *For all* $x \in \mathfrak{p}_{\mathbf{C}}$, $r_x \geqslant r$.

PROOF. Clearly $r_x \geqslant \dim(\ker \alpha_x) \geqslant \dim \mathfrak{p}_{\mathbf{C}}^x \geqslant r$. Q.E.D.

Now we say that $x \in \mathfrak{p}_{\mathbf{C}}$ is *s-regular* if $r_x = r$.

PROPOSITION 5.3. *An element $x$ in $\mathfrak{p}_{\mathbf{C}}$ is s-regular if and only if $x$ is regular and semisimple.*

PROOF. Assume $x \in \mathfrak{p}_{\mathbf{C}}$ is regular and semisimple. Then $\ker \alpha_x = \mathfrak{p}_{\mathbf{C}}^x$ because $x$ is semisimple, and $\dim \mathfrak{p}_{\mathbf{C}}^x = r$ because $x$ is regular. Hence $x$ is $s$-regular.

Conversely, suppose $x$ is $s$-regular. Now if $y \in \mathfrak{p}_{\mathbf{C}}$ let $\mathfrak{g}_{\mathbf{C}}^0(y) = \{u \in \mathfrak{g}_{\mathbf{C}}: (\mathrm{ad}\, y)^n u = 0 \text{ for some } n\}$, and if $q = \min(\dim(\mathfrak{g}_{\mathbf{C}}^0(y) \cap \mathfrak{p}_{\mathbf{C}}))$ over all $y \in \mathfrak{p}_{\mathbf{C}}$, we let $Q$ be the set of all $y \in \mathfrak{p}_{\mathbf{C}}$ such that $q = \dim(\mathfrak{g}_{\mathbf{C}}^0(y) \cap \mathfrak{p}_{\mathbf{C}})$. Since $\dim(\mathfrak{g}_{\mathbf{C}}^0(y) \cap \mathfrak{p}_{\mathbf{C}})$ is clearly the multiplicity of the zero eigenvalue of $\alpha_y$ it follows that $q = r$ and hence $x \in Q$. One knows that $Q = H \cdot (Q \cap \mathfrak{a}_{\mathbf{C}})$ [7, p. 765] so the elements in $Q$ are semisimple. Hence $x$ is semisimple, and since it is clearly regular, we have completed the proof of the proposition. Q.E.D.

The theorem we wish to prove is

THEOREM 5.4. *Let $x \in \mathfrak{p}_{\mathbf{C}}$ and let $l_x$ be the multiplicity of the zero eigenvalue of* $\mathrm{ad}\, x$ *in* $\mathfrak{g}_{\mathbf{C}}$. *Then* $l_x \geqslant l = \dim \mathfrak{a} + \dim \mathfrak{m}$ *where equality holds if and only if $x$ is s-regular.*

PROOF. If $x$ is $s$-regular then one has (cf. Theorem 5.1) $\dim \mathfrak{g}_{\mathbf{C}}^x = \dim \mathfrak{k}_{\mathbf{C}}^x + \dim \mathfrak{p}_{\mathbf{C}}^x = \dim \mathfrak{m} + \dim \mathfrak{a}$ where $\mathfrak{g}_{\mathbf{C}}^x = \ker \mathrm{ad}\, x$. However since $x$ is semisimple $\dim \mathfrak{g}_{\mathbf{C}}^x$ is the multiplicity of the zero eigenvalue of $\mathrm{ad}\, x$ in $\mathfrak{g}_{\mathbf{C}}$ establishing the theorem in one direction.

Conversely assume $l_x = l$. For any $y \in \mathfrak{p}_{\mathbf{C}}$, $(\mathrm{ad}\, y)^2$ leaves $\mathfrak{k}_{\mathbf{C}}$ stable. Let $d_y$ be the multiplicity of the zero eigenvalue of $(\mathrm{ad}\, y)^2$ in $\mathfrak{k}_{\mathbf{C}}$. We have $l_y = d_y + r_y$, $d_y \geqslant \dim \mathfrak{k}_{\mathbf{C}}^y \geqslant \dim \mathfrak{m}$ and $r_y \geqslant r$. Also $l_y$ is equal to the multiplicity of the zero eigenvalue of $(\mathrm{ad}\, y)^2$ in $\mathfrak{g}_{\mathbf{C}}$ as well as $\mathrm{ad}\, y$. Therefore $l_x = l$ implies $r_x = r$ which concludes the proof of the theorem. Q.E.D.

For any vector space $V$, let $S(V)$ denote the symmetric algebra over $V$. For every nonnegative integer $i$, let $S^i(V)$ denote the homogeneous subspace of $S(V)$ of degree $i$.

Let $n = \dim \mathfrak{g}_{\mathbf{C}}$ and let $\mathfrak{g}_{\mathbf{C}}'$ be the dual of $\mathfrak{g}_{\mathbf{C}}$. Now for any $x \in \mathfrak{g}_{\mathbf{C}}$ let $\det(t - \mathrm{ad}\, x) = \Sigma a_i(x) t^i$ be the characteristic polynomial of $\mathrm{ad}\, x$. One has $a_i \in (S^{n-i}(\mathfrak{g}_{\mathbf{C}}'))^{G_{\mathbf{C}}}$ is an invariant polynomial where $G_{\mathbf{C}}$ denotes the adjoint group of $\mathfrak{g}_{\mathbf{C}}$. Consider $a_l$. We then have

COROLLARY 5.5. *Let* $x \in \mathfrak{p}_{\mathbf{C}}$. *Then* $a_i(x) = 0$ *for all* $i < l$ *and* $a_l(x) = 0$ *if and only if* $x \in \mathfrak{p}_{\mathbf{C}}$ *is not s-regular.*

Let $\mathfrak{p}_{\mathbf{C}}^* \subset \mathfrak{p}_{\mathbf{C}}$ denote the set of all $s$-regular elements in $\mathfrak{p}_{\mathbf{C}}$. Let $a = a_l$. Now let $b = a|_{\mathfrak{p}_{\mathbf{C}}}$ so that $b \in S^{n-l}(\mathfrak{p}'_{\mathbf{C}})$, where $\mathfrak{p}'_{\mathbf{C}}$ denotes the dual of $\mathfrak{p}_{\mathbf{C}}$. We note that $b \neq 0$ and in fact $\mathfrak{p}_{\mathbf{C}}^* = \{x \in \mathfrak{p}_{\mathbf{C}}\colon b(x) \neq 0\}$. More explicitly if $\Delta$ is the set of roots, counting multiplicities of $(\mathfrak{a}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$, then card $\Delta = n - l$ and $b|_{\mathfrak{a}_{\mathbf{C}}} = \Pi_{\alpha\in\Delta}\alpha$.

6. Now we regard $S(\mathfrak{p}'_{\mathbf{C}})$ as a subalgebra of $S(\mathfrak{g}'_{\mathbf{C}})$ where if $f \in S(\mathfrak{p}'_{\mathbf{C}})$ then $f$ is also regarded as a function on $\mathfrak{g}_{\mathbf{C}}$ such that if $z \in \mathfrak{g}_{\mathbf{C}}$, $z = x + y$, $x \in \mathfrak{k}_{\mathbf{C}}$, $y \in \mathfrak{p}_{\mathbf{C}}$ then $f(x + y) = f(y)$.

It follows that if $\mathfrak{g}_{\mathbf{C}}^* = \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_{\mathbf{C}}^*$ then $b \in S^{n-l}(\mathfrak{g}'_{\mathbf{C}})$ and $\mathfrak{g}_{\mathbf{C}}^* = \{z \in \mathfrak{g}_{\mathbf{C}}\colon b(z) \neq 0\}$. That is, $\mathfrak{g}_{\mathbf{C}}^*$ is an open affine subvariety of $\mathfrak{g}_{\mathbf{C}}$ and the affine algebra of $\mathfrak{g}_{\mathbf{C}}^*$ is the localization $S(\mathfrak{g}'_{\mathbf{C}})_b$ of $S(\mathfrak{g}'_{\mathbf{C}})$ by $b$, so that $S(\mathfrak{g}'_{\mathbf{C}})_b$ is the ring of all rational functions on $\mathfrak{g}_{\mathbf{C}}$ of the form $f/b^k$ where $f \in S(\mathfrak{g}'_{\mathbf{C}})$ and $k \in \mathbf{Z}$.

Now let $\mathfrak{a}_{\mathbf{C}}^* = \{x \in \mathfrak{a}_{\mathbf{C}}\colon \alpha(x) \neq 0 \text{ for all } \alpha \in \Delta\}$; then $\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^* = \{z \in \mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}\colon b_0(z) \neq 0\}$ where $b_0 = b|\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}$. Thus $\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*$ is an affine variety whose affine algebra is the localization $S((\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})')_{b_0}$ of $S((\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})')$ by $b_0$. By now the injection map $\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^* \longrightarrow \mathfrak{k}_{\mathbf{C}} + \mathfrak{p}_{\mathbf{C}}^* = \mathfrak{g}_{\mathbf{C}}^*$ of affine varieties induces contravariantly the restriction homomorphism

$$S(\mathfrak{g}'_{\mathbf{C}})_b \longrightarrow S((\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})')_{b_0} \tag{6.1}$$

of affine algebras.

Now let $K_{\mathbf{C}}$ be the subgroup of $G_{\mathbf{C}}$ corresponding to ad $\mathfrak{k}_{\mathbf{C}}$. Then the affine variety $\mathfrak{g}_{\mathbf{C}}^*$ is clearly stable under the action of the reductive algebraic group $K_{\mathbf{C}}$ and hence the ring of $K_{\mathbf{C}}$-invariants $A = S(\mathfrak{g}'_{\mathbf{C}})_b^{K_{\mathbf{C}}}$ is an affine ring (finitely generated). Also if $M'_{\mathbf{C}}$ is the normalizer of $\mathfrak{a}_{\mathbf{C}}$ in $K_{\mathbf{C}}$ then $M'_{\mathbf{C}}$ is a reductive algebraic group operating on the affine variety $\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*$ and hence $A_0 = S((\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}})')_{b_0}^{M'_{\mathbf{C}}}$ is also an affine ring. Since $M'_{\mathbf{C}} \subset K_{\mathbf{C}}$ the homomorphism (6.1) restricted to $A$ induces a homomorphism

$$\pi\colon A \longrightarrow A_0. \tag{6.2}$$

We will prove the following theorem of Kostant.

THEOREM 6.1. *The homomorphism* $\pi\colon A \longrightarrow A_0$ *is an isomorphism of algebras.*

We first establish some lemmas. Let $\mathcal{O}$ be the set of all $K_{\mathbf{C}}$ orbits in $\mathfrak{g}_{\mathbf{C}}^*$ and let $\mathcal{O}_0$ be the set of $M'_{\mathbf{C}}$ orbits in $\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*$.

LEMMA 6.2. *If* $O \in \mathcal{O}$ *then* $O \cap (\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*) = O_0$ *is an* $M'_{\mathbf{C}}$ *orbit and the correspondence* $O \longmapsto O_0$ *defines a bijection* $\mathcal{O} \longrightarrow \mathcal{O}_0$.

PROOF. The only thing we really have to prove is that given $x, y \in O \cap (\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*)$ there exists $k \in M'_{\mathbf{C}}$ such that $y = k \cdot x$. Write $x = x_1 + x_2$, $y = y_1 + y_2$ where $x_1, y_1 \in \mathfrak{k}_{\mathbf{C}}$ and $x_2, y_2 \in \mathfrak{a}_{\mathbf{C}}^*$. We know that there is $k \in K_{\mathbf{C}}$ such that $y = k \cdot x$ and therefore $y_2 = k \cdot x_2$. Now we use the fact (see [7]) that if two elements in $\mathfrak{a}_{\mathbf{C}}$ are $K_{\mathbf{C}}$-conjugate then they are $M'_{\mathbf{C}}$-conjugate. Hence there exists $m_1 \in M'_{\mathbf{C}}$ such that $y_2 = m_1 \cdot x_2$. Then $m_1^{-1}k \cdot x_2 = x_2$. Since $x_2$ is $s$-regular $m_1^{-1}k = m$ centralizes $\mathfrak{a}_{\mathbf{C}}$ (cf. [7, Lemma 20]). Thus $k = m_1 m \in M'_{\mathbf{C}}$ and the lemma is proved.

LEMMA 6.3. *With respect to the bijection $O \mapsto O_0$ of the previous lemma one has: $O$ is closed if and only if $O_0$ is closed.*

PROOF. Assume $O_0$ is closed and $x_n \longrightarrow x$, $x_n \in O$, $x \in \mathfrak{g}_{\mathbf{C}}^*$. Then by applying an element in $K_{\mathbf{C}}$ we may assume $x \in \mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*$. Then we may find $k_n \in K_{\mathbf{C}}$, $k_n \longrightarrow e$ such that $k_n \cdot x_n \in \mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*$ so that $k_n \cdot x_n \longrightarrow x$. But $k_n \cdot x_n \in O_0$ therefore $x \in O_0$. Hence $O$ is closed.

PROOF OF THEOREM 6.1. We first observe that $\pi: A \longrightarrow A_0$ is injective. Indeed if $0 \neq f \in A$ we must show $f|\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^* \neq 0$. But if $f|\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^* = 0$ then $0 = f|K_{\mathbf{C}} \cdot (\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*)$. Thus $f = 0$ since $K_{\mathbf{C}} \cdot (\mathfrak{k}_{\mathbf{C}} + \mathfrak{a}_{\mathbf{C}}^*) = \mathfrak{g}_{\mathbf{C}}^*$. Thus we may regard $A \subset A_0$. But now we assert: (1) $A$ is integrally closed in its quotient field $Q$; (2) if $\hat{A}_0$ (resp. $\hat{A}$) denotes the set of all homomorphisms $\chi: A_0 \longrightarrow \mathbf{C}$ (resp. $\chi: A \longrightarrow \mathbf{C}$) then the map $\hat{A}_0 \longrightarrow \hat{A}$, $\chi \longrightarrow \chi \circ \pi$ is a bijection.

To establish (1) we note that if $f \in Q$ satisfies a monic polynomial equation with coefficients in $A \subset S(\mathfrak{g}'_{\mathbf{C}})_b$ then $f \in S(\mathfrak{g}'_{\mathbf{C}})_b$ since $S(\mathfrak{g}'_{\mathbf{C}})_b$, a localization of a polynomial ring, is integrally closed. Because $f \in Q$, $f = a_1/a_2$, $a_1, a_2 \in A$ and $a_2 \neq 0$. Hence $fa_2 = a_1$; applying $k \in K_{\mathbf{C}}$ we get $f^k a_2 = a_1 = fa_2$, therefore $f^k = f$, i.e. $f \in A$.

Now (2) follows from Lemma 6.3 since one knows that the natural map $\mathcal{O} \longrightarrow \hat{A}$ and $\mathcal{O}_0 \longrightarrow \hat{A}_0$ give a bijection between the set of all closed orbits in $\mathcal{O}$ and $\hat{A}$, and the set of all closed orbits in $\mathcal{O}_0$ and $\hat{A}_0$, respectively. (See e.g. Dieudonné [2].) Now (2) implies that $\hat{A}_0 \longrightarrow \hat{A}$ is a bijective, birational map of affine varieties. But (1) implies that $\hat{A}$ is normal. Hence by Zariski's Main Theorem (see e.g. [15, p. 413]) the map $\hat{A}_0 \longrightarrow \hat{A}$ is an isomorphism and hence $A_0 = A$. Q.E.D.

7. A *valuation* on a ring $R$ is a map $\nu: R \longrightarrow Z \cup \{-\infty\}$ such that: (1) $\nu(r) = -\infty$ if and only if $r = 0$, (2) $\nu(r + s) \leqslant \max(\nu(r), \nu(s))$, (3) $\nu(rs) = \nu(r) + \nu(s)$. If $R_n = \{r \in R.\ \nu(r) \leqslant n\}$ then $R_n \subset R_{n+1}$, $\bigcap_{-\infty < n < \infty} R_n = \{0\}$ and $R_n$ $(n \in \mathbf{Z})$ defines a system of neighborhoods of 0 and hence a topology on $R$. The valuation also defines a uniform structure on $R$ so that we may complete $R$ obtaining a ring $\overline{R}$. To each $\overline{r} \in \overline{R}$ there is a Cauchy sequence $r_n \in R$ such that $r_n \longrightarrow \overline{r}$. If $r_n \longrightarrow \overline{r}$ and $s_n \longrightarrow \overline{s}$ then $r_n s_n \longrightarrow \overline{r}\overline{s}$, $r_n + s_n \longrightarrow \overline{r} + \overline{s}$.

Now if $R$ is an integral domain and it satisfies the Ore condition (i.e.: $Ra \cap Rb \neq \{0\}$ for all $a, b \neq 0$) then $\nu$ extends to $Q(R)$, the left quotient division ring of $R$, by setting $\nu(a^{-1}b) = \nu(b) - \nu(a)$.

EXAMPLE. Let $\mathfrak{h}$ be a Lie algebra and $\mathfrak{j} \subset \mathfrak{h}$ any subalgebra in $\mathfrak{h}$. Let $J \subset H$ be the corresponding universal enveloping algebras. Let $H_{(n)}$ be the usual filtration of $H$. Thus $H_{(n)}$ is spanned by $x_1 \cdots x_j$, $x_i \in \mathfrak{h}, j \leqslant n$, and the identity. We claim that $JH_{(n)} = H_{(n)}J$.

To prove, for example, that $JH_{(n)} \subset H_{(n)}J$ one notices that if $x_1, \ldots, x_j \in \mathfrak{h}$, $y \in \mathfrak{j}$, then by induction

$$yx_1 \cdots x_j = x_1 \cdots x_j y + \sum_{i=1}^{j} x_1 \cdots [y, x_i] \cdots x_j \in H_{(n)}J$$

if $j \leqslant n$. Thus we get a new filtration of $H$ by putting $H_n = JH_{(n)}$ since now $H_n H_m \subset H_{n+m}$.

THEOREM 7.1. *If* $0 \neq a \in H$ *let* $\nu(a) = \min n$ *such that* $a \in H_n$ *and let* $\nu(0) = -\infty$. *Then* $\nu$ *is a valuation on* $H$.

PROOF. Let $\mathfrak{q}$ be a linear complement of $\mathfrak{j}$ in $\mathfrak{h}$ so that $\mathfrak{h} = \mathfrak{j} + \mathfrak{q}$ (direct sum). Let $y_1, \ldots, y_k$ be a basis of $\mathfrak{q}$. Then

$$H = \bigoplus_{(m_1,\ldots,m_k)} J y_1^{m_1} y_2^{m_2} \cdots y_k^{m_k} \tag{7.1}$$

by the Birkhoff-Witt theorem. In fact, if $u \in H$, $u = \Sigma u_{m_1,\ldots,m_k}$ where $u_{m_1,\ldots,m_k} \in J y_1^{m_1} \cdots y_k^{m_k}$, then $\nu(u) = \max_{u_m \neq 0} |m|$ where $m = (m_1, \ldots, m_k)$, $|m| = \Sigma_{i=1}^k m_i$.

The only thing to be proved is that $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in H$. Since clearly $\nu(ab) \leqslant \nu(a) + \nu(b)$ to prove the equality we may assume that $a = \Sigma_{|m|=\nu(a)} a_m$, $b = \Sigma_{|n|=\nu(b)} b_n$ where $a_m = c_m y^m$, $b_n = d_n y^n$, $y^m = y_1^{m_1} \cdots y_k^{m_k}$; $c_m, d_n \in J$. Now let $u \to \rho(u)$ be the usual valuation of an element $u \in H$ (the case where $\mathfrak{j} = 0$). Then one has $\rho(a) = \nu(a) + \alpha(a)$ and $\rho(b) = \nu(b) + \alpha(b)$ where for any $u \in H$ one puts $\alpha(u) = \max_r \rho(e_r)$ where $e_r \in J$ is such that $u = \Sigma e_r y^r$. But now if $v = \Sigma c_m d_n y^{m+n}$ where the sum is over all pairs $(m, n)$ such that $|m| = \nu(a)$, $\rho(c_m) = \alpha(a)$, $|n| = \nu(b)$, $\rho(d_n) = \alpha(b)$ then clearly

$$ab - v \in H_{(\rho(a)+\rho(b)-1)}. \tag{7.2}$$

On the other hand, since $\rho(ab) = \rho(a) + \rho(b)$ it follows that $v \notin H_{\rho(a)+\rho(b)-1}$, so that $v$ can be written $v = \Sigma_{|r|=\nu(a)+\nu(b)} e_r y^r$ where $\alpha(v) = \alpha(a) + \alpha(b)$. On the other hand by (7.2) one has, for some $f_s \in J$, $ab - v = \Sigma_{|s| \leqslant \nu(a)+\nu(b)} f_s y^s$ and $\rho(f_s) < \alpha(a) + \alpha(b)$ for $|s| = \nu(a) + \nu(b)$. Thus one cannot have $e_r + f_r = 0$ for all $r$ where $|r| = \nu(a) + \nu(b)$. This implies $\nu(ab) = \nu(a) + \nu(b)$. Q.E.D.

We can identify the universal enveloping algebra of the direct sum $\mathfrak{k}_C + \mathfrak{a}_C$ with $K \otimes A$. Since $\mathfrak{k}_C$ is a subalgebra in $\mathfrak{k}_C \oplus \mathfrak{a}_C$ it defines a valuation $\nu_0$ on $K \otimes A$. Now $A$ is graded $A = \bigoplus A_i$. If $\nu_0(u) = d$, $u \in K \otimes A$ then there exists a unique $u_d \in K \otimes A_d$ such that

$$\nu_0(u - u_d) < d. \tag{7.3}$$

Since $K \otimes A$ satisfies the Ore condition and is an integral domain, $\nu_0$ extends to the quotient division ring $Q(K \otimes A)$.

We let $\nu$ be the valuation on $G$ and on $Q(G)$ its quotient division ring, defined also by $\mathfrak{k}_C$.

A proof of the following proposition can be found in Lepowsky [9]. We first recall that the map $P$: $G \longrightarrow K \otimes A$ was the projection defined by the decomposition $G = K \otimes A \oplus G\mathfrak{n}_C$. Let $\lambda$: $S(\mathfrak{g}_C) \longrightarrow G$ denote the symmetrization mapping. We note that $\lambda$ is defined on $S(\mathfrak{p}_C)$ by regarding $S(\mathfrak{p}_C) \subset S(\mathfrak{g}_C)$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ with respect to the Killing form of $\mathfrak{g}$, and let $\mathfrak{q}_C \subset \mathfrak{p}_C$ be the complexification of $\mathfrak{q}$. Then $S(\mathfrak{p}_C) = S(\mathfrak{a}_C) \oplus \mathfrak{q}_C S(\mathfrak{p}_C)$, so that

$$G = (K \otimes A) \oplus (K \otimes \lambda(\mathfrak{q}_C S(\mathfrak{p}_C))).$$

Let $F$: $G \longrightarrow K \otimes A$ denote the projection onto the first summand in this decomposition.

PROPOSITION 7.2. (i) *If* $u \in G^K$ *then* $\nu(u) = \nu_0(F(u))$. (ii) *If* $0 \neq u \in G$ *then* $\nu_0(P(u) - F(u)) < \nu(u)$.

COROLLARY 7.3. *If* $u \in G^K$ *then* $\nu(u) = \nu_0(P(u))$.

PROOF. If $u \in G^K$ we have $\nu_0(P(u) - F(u)) < \nu_0(F(u))$; hence the leading term of $F(u)$ is equal to the leading term of $P(u)$, and therefore $\nu_0(P(u)) = \nu_0(F(u)) = \nu(u)$.

Now let $\delta$: $\mathfrak{g}_C \longrightarrow \mathfrak{g}'_C$ be the isomorphism defined by the Killing form of $\mathfrak{g}_C$. We may extend $\delta$ to an algebra, $G_C$-isomorphism of symmetric algebras $\delta$: $S(\mathfrak{g}_C) \longrightarrow S(\mathfrak{g}'_C)$, $G_C$ being the adjoint group of $\mathfrak{g}_C$.

Let $a \in (S^{n-l}(\mathfrak{g}'_C))^{G_C}$ be as at the end of §5 and let $\alpha = \delta^{-1}(a)$ so that $\alpha \in (S^{n-l}(\mathfrak{g}_C))^{G_C}$. Finally $\lambda$: $S(\mathfrak{g}_C) \longrightarrow G$ is a $G_C$-linear isomorphism and we put $\gamma = \lambda(\alpha)$ so that $\gamma \in \text{Cent } G \subset G^K$. Now $\gamma_0 = P(\gamma) \in B \subset K^M \otimes A$ (cf. §4) and hence $P$ induces an antihomomorphism $P_\gamma$: $G^K_\gamma \longrightarrow B_{\gamma_0}$. Note that $\gamma_0 \in \text{Center } K^M \otimes A$ since one easily has $\gamma_0 \in M \otimes A$ where $M$ is the enveloping algebra of the Lie algebra of $M$. Clearly $P$ is compatible with valuations (see Corollary 7.3). Therefore $P_\gamma$ extends to a map $P_\Gamma$ of the respective completions $G^K_\Gamma$ and $B_{\Gamma_0}$. We have

THEOREM 7.4. *The map* $P_\Gamma$: $G^K_\Gamma \longrightarrow B_{\Gamma_0}$ *is a surjective anti-isomorphism.*

PROOF. To prove the theorem it is sufficient to show that given $u_0 \in \mathcal{B}_{\gamma_0}$ there exists $u \in G_\gamma^K$ such that

(a) $\nu_0(u_0) = \nu(u)$, and

(b) $\nu_0(u_0 - P_\gamma(u)) < \nu_0(u_0)$.

In fact it suffices only to prove (b) since (b) $\Rightarrow$ (a). This is clear since, by writing $P_\gamma(u) = u_0 - (u_0 - P_\gamma(u))$ and $u_0 = (u_0 - P_\gamma(u)) + P_\gamma(u)$, (2) implies that $\nu_0(u_0) = \nu_0(P_\gamma(u))$. But $\nu_0(P_\gamma u) = \nu(u)$.

Next note that we may assume that $u_0 \in \mathcal{B}$. Indeed assume the theorem is true in this case. Write $u_0 = f_0/\gamma_0^i$ where $f_0 \in \mathcal{B}$. But then there exists $f \in G_\gamma^K$ such that $\nu_0(f_0 - P_\gamma f) < \nu_0(f_0)$. Hence

$$\nu_0(u_0 - P_\gamma(f/\gamma^i)) = \nu_0(f_0 - P_\gamma(f)) - \nu_0(\gamma_0^i) < \nu_0(f_0) - \nu_0(\gamma_0^i) = \nu_0(u_0).$$

Thus we assume $u_0 = f_0 \in \mathcal{B}$.

Let $G_{(n)}$ be the usual filtration of $G$ and $\mathcal{B}_{(n)}$ be the usual filtration for $\mathcal{B}$. (See beginning of this section.) Now let $\sigma^n$: $G_{(n)} \longrightarrow S^n(\mathfrak{g}_C')$ be the linear map defined by composing $\lambda^{-1}$: $G_{(n)} \longrightarrow S_{(n)}(\mathfrak{g}_C) = \Sigma_{j=0}^n S^j(\mathfrak{g}_C)$ with $\delta$: $S_{(n)}(\mathfrak{g}_C) \longrightarrow S_{(n)}(\mathfrak{g}_C')$ and then with the projection $S_{(n)}(\mathfrak{g}_C') \longrightarrow S^n(\mathfrak{g}_C')$. It is clear that $\sigma^n$ is a $G_C$-linear map. Let $\sigma_0^n$: $(K \otimes A)_{(n)} \longrightarrow S^n((\mathfrak{k}_C \oplus \mathfrak{a}_C)')$ be defined similarly so that $\sigma_0^n$ is a $K_C$-linear map. It then follows easily from the definition of the map $F$: $G \longrightarrow K \otimes A$ and the Birkhoff-Witt theorem that one has a commutative diagram

$$\begin{array}{ccc} G_{(n)} & \xrightarrow{\sigma^n} & S^n(\mathfrak{g}_C') \\ \downarrow F & & \downarrow \pi \\ (K \otimes A)_{(n)} & \xrightarrow{\sigma_0^n} & S^n((\mathfrak{k}_C \oplus \mathfrak{a}_C)') \end{array} \tag{7.4}$$

where, recall, $\pi$: $S(\mathfrak{g}_C') \longrightarrow S((\mathfrak{k}_C \oplus \mathfrak{a}_C)')$ is the restriction map $\varphi \longrightarrow \varphi|(\mathfrak{k}_C \oplus \mathfrak{a}_C)$ for $\varphi \in S(\mathfrak{g}_C')$.

Now let $\rho$ be the usual valuation on $G$. Thus if $u \in G$ then $\rho(u) = -\infty$ if $u = 0$, otherwise $\rho(u)$ is the least $n \in \mathbf{Z}_+$ such that $u \in G_{(n)}$. Note that if $u \in G_{(n)}$ then $\lambda^n(u) \neq 0$ if and only if $\rho(u) = n$. One defines the valuation $\rho_0$ on $K \otimes A$ similarly. Now since $\pi$ is injective on $S(\mathfrak{g}_C')^{K_C}$ it then follows from (7.4) that, for any $u \in G^K$, $\rho(u) = \rho_0(Fu)$.

The proof of Theorem 7.4 will follow easily from

LEMMA 7.5. *For any* $0 \neq f_0 \in \mathcal{B}$ *there exists* $j \in \mathbf{Z}_+$ *and* $w \in G^K$ *such that* $\nu_0(f_0\gamma_0^j - P(w)) < \nu_0(f_0\gamma_0^j)$. *(Note that this implies* $\nu_0(f_0\gamma_0^j) = \nu_0(P(w))$ *and since (Corollary 7.3)* $\nu_0(P(w)) = \nu(w)$ *this also implies* $\nu_0(f_0\gamma_0^j) = \nu(w)$.*)*

PROOF OF LEMMA 7.5. For any $0 \neq x \in K \otimes A$ let $\widetilde{x} \in K \otimes A$ be the unique element defined so that if $\nu_0(x) = d$ then $\widetilde{x} \in K \otimes A_d$ and $\nu_0(x - \widetilde{x}) < d$.

Clearly $\widetilde{x} \neq 0$ and $\rho_0(\widetilde{x}) \geq \nu_0(\widetilde{x}) = \nu_0(x) = d$. Put $\beta(x) = \rho_0(\widetilde{x}) - \nu_0(x)$.

We will prove the lemma by induction on $\beta(f_0)$. Let $\tau: S(\mathfrak{g}'_C) \longrightarrow G$ be the $G_C$-linear map defined by putting $\tau = \lambda \circ \delta^{-1}$. Thus $\tau(S_{(n)}(\mathfrak{g}'_C)) = G_{(n)}$ and $\sigma^n \circ \tau$ is the identity on $S^n(\mathfrak{g}'_C)$.

Now assume $\beta(f_0) = 0$. Thus $\widetilde{f}_0 \in A_d$ where $d = \nu_0(f_0)$. But then, by Theorem 4.5, $\widetilde{f}_0$ and hence $\sigma_0^d(\widetilde{f}_0) \in S^d(\mathfrak{a}'_C)$ is Weyl group invariant. Thus there exists (see e.g. [16, Theorem 6.10]) $\xi \in S^d(\mathfrak{p}'_C)^{K_C} \subseteq S^d(\mathfrak{g}'_C)^{K_C}$ such that $\pi(\xi) = \sigma_0^d(\widetilde{f}_0)$. But then if $w = \tau\xi$ one has $w \in G^K$. But by (7.4) one has $\sigma_0^d(F(w)) = \sigma_0^d(\widetilde{f}_0)$. Thus $\rho_0(\widetilde{f}_0 - F(w)) < d$. But $\nu_0(\widetilde{f}_0 - F(w)) \leq \rho_0(\widetilde{f}_0 - F(w))$ and $\rho_0(\widetilde{f}_0) = \nu_0(\widetilde{f}_0) = d$. Thus $\nu_0(\widetilde{f}_0 - F(w)) < \nu_0(\widetilde{f_0}) = d$. But then $\nu_0(f_0 - F(w)) < d = \nu_0(f_0)$. Hence $\nu_0(F(w)) = d$. But then $\nu_0(F(w) - P(w)) < d$ by Proposition 7.2. Thus $\nu_0(f_0 - P(w)) < d$ proving the lemma for $\beta(f_0) = 0$.

Now assume $\beta(f_0) > 0$ and assume the lemma is true for smaller values. Again let $d = \nu_0(\widetilde{f}_0) = \nu_0(f_0)$.

Now put $m = \rho_0(\widetilde{f}_0)$ so that $m - d = \beta(f_0)$. But by Theorem 4.5 $0 \neq \sigma_0^m(\widetilde{f}_0) \in S^m((\mathfrak{k}_C \oplus \mathfrak{a}_C)')$ is Weyl group invariant. But then by Theorem 6.1 there exist $i \in \mathbf{Z}_+$ and $\psi \in S^r(\mathfrak{g}'_C)^{K_C}$ where $r = m + i(n - l)$ such that $\psi|\mathfrak{k}_C \oplus \mathfrak{a}_C = \sigma_0^m(\widetilde{f}_0)b_0^i$. Furthermore since $\sigma_0^m(\widetilde{f}_0)b_0^i \in S(\mathfrak{k}'_C) \otimes S^p(\mathfrak{a}'_C)$ where $p = d + i(n - l)$ it follows from the injectivity of $\pi|S(\mathfrak{g}'_C)^{K_C}$ that $\psi \in S(\mathfrak{k}'_C) \otimes S^p(\mathfrak{p}'_C)$. It follows therefore, if we put $u = \tau(\psi) \in G^K$, that $\nu(u) \leq p$. On the other hand by (7.4) one has

$$\sigma_0^r(F(u)) = \sigma_0^m(f_0)b_0^i \neq 0. \tag{7.5}$$

But since $0 \neq \sigma_0^r(F(u)) \in S(\mathfrak{k}_C) \otimes S^p(\mathfrak{a}'_C)$ it follows that $\nu_0(F(u)) \geq p$. Thus

$$\nu_0(F(u)) = \nu(u) = p \tag{7.6}$$

since $\nu(u) = \nu_0(F(u))$.

Now by definition $\gamma_0 = P(\gamma)$ and $\gamma = \tau(a)$ where $a \in S^{n-l}(\mathfrak{g}'_C)$ is defined as in §5. Obviously, then $\rho(\gamma) = n - l$ so that $\rho_0(\gamma_0) \leq n - l$.

Now Proposition 7.2 clearly implies that for any $v \in G^K$ one has

$$\widetilde{P(v)} = \widetilde{F(v)}. \tag{7.7}$$

Now we assert that $\rho_0(\widetilde{\gamma}_0) = n - l$ and in fact

$$\sigma_0^{n-l}(\widetilde{\gamma}_0) = b_0 \in S^{n-l}(\mathfrak{a}'_C). \tag{7.8}$$

Indeed by definition $a|\mathfrak{a}'_C = b_0$. But then if $a_0 = \pi(a)$ one has that $a_0 = b_0 + a_1$ where $a_1 \in S(\mathfrak{k}'_C) \otimes S_{(n-l-1)}(\mathfrak{a}'_C)$. But by (7.4) $\sigma_0^{n-l}(F(\gamma)) = a_0$ and hence $\sigma_0^{n-l}(\widetilde{F(\gamma)}) = b_0$. Then by (7.7) $\sigma_0^{n-l}(\widetilde{\gamma}_0) = b_0$ establishing (7.8) and hence also that $\rho_0(\widetilde{\gamma}_0) = n - l$. This implies $\rho_0(\gamma_0) = n - l$ since $n - l = \rho(\gamma) \geq$

$\rho_0(\gamma_0) \geqslant \rho_0(\widetilde{\gamma}_0) = n - l$. Note that (7.8) also implies that $\nu_0(\gamma_0) = \nu_0(\widetilde{\gamma}_0) = n - l$.

Now for any $w, v \in K \otimes A$ note that $\widetilde{wv} = \widetilde{w}\widetilde{v}$. Hence $\widetilde{f_0\gamma_0^i} = \widetilde{f}_0\widetilde{\gamma}_0^i$. Thus since $\nu_0(x) = \nu_0(\widetilde{x})$ for $0 \neq x \in K \otimes A$ this implies that

$$\nu_0(f_0\gamma_0^i) = p. \tag{7.9}$$

On the other hand by (7.6) and Corollary 7.3, $\nu_0(Pu) = p$. If $\nu_0(f_0\gamma_0^i - Pu) < p$ we are done. Assume therefore, that $\nu_0(f_0\gamma_0^i - Pu) = p$. Thus $\widetilde{f_0\gamma_0^i}$ and $\widetilde{Pu}$ are distinct elements of $K \otimes A_p$ and hence if $x = f_0\gamma_0^i - P(u)$ one has $x \in B$ and $\widetilde{x} = \widetilde{f_0\gamma_0^i} - \widetilde{P(u)} \in K \otimes A_p$. But $\rho_0(\widetilde{f_0\gamma_0^i}) = \rho_0(\widetilde{f}_0)\rho_0(\widetilde{\gamma}_0)^i = r$. However by (7.5) $r = \rho(u) \geqslant \rho_0(\widetilde{P(u)})$. Thus $r \geqslant \rho_0(\widetilde{x})$. On the other hand one has $\sigma_0^r(\widetilde{f_0\gamma_0^i}) = \sigma_0^r(\widetilde{f}_0\widetilde{\gamma}_0^i)$. But then

$$\sigma_0^r(\widetilde{f_0\gamma_0^i}) = \sigma_0^m(\widetilde{f}_0)b_0^i \tag{7.10}$$

by (7.8) since if $y \in (K \otimes A)_{(s)}$ and $z \in (K \otimes A)_{(t)}$ then

$$\sigma_0^{s+t}(yz) = \sigma_0^s(y)\sigma_0^t(z).$$

Now $\sigma_0^r(F(u)) = \sigma_0^m(\widetilde{f}_0)b_0^i$ by (7.5). We assert that $\sigma_0^r(F(u)) = \sigma_0^r(F(\widetilde{u}))$. Indeed $\sigma_0^r(\widetilde{F(u)}) \in S(\mathfrak{k}_C') \otimes S^p(\mathfrak{g}_C')$ and $\sigma_0^r(\widetilde{F(u)}) \in S(\mathfrak{k}_C') \otimes S^p(\mathfrak{a}_C')$ since $\nu_0(F(u)) = p$ by (7.6). However one necessarily has $\sigma_0^r(F(u)) - \widetilde{F}(u) \in S(\mathfrak{k}_C') \otimes S_{(p-1)}(\mathfrak{a}_C')$ since $\nu_0(F(u) - \widetilde{F}(u)) < p$. Thus $\sigma_0^r(\widetilde{F}(u)) = \sigma_0^r(F(u)) = \sigma_0^m(\widetilde{f}_0)b_0^i$. But $\sigma_0^r(\widetilde{P(u)}) = \sigma_0^r(\widetilde{F(u)})$ by (7.7). Thus recalling (7.10), one has $\sigma_0^r(\widetilde{x}) = 0$ so that $\rho_0(\widetilde{x}) < r$. But then $\beta(x) = \rho_0(\widetilde{x}) - p < r - p = m - d$. The induction assumption then applies to $x$ so that for some $k \in \mathbf{Z}_+$ there exists $v \in G^K$ such that $\nu_0(x\gamma_0^k - P(v)) < \nu_0(x\gamma_0^k)$. But $x\gamma_0^k = (f_0\gamma_0^i - P(u))\gamma_0^k = f_0\gamma_0^j - P(\gamma^k u)$ where $j = k + i$. Now put $w = v + \gamma^k u \in G^K$. Then $\nu_0(f_0\gamma_0^j - P(w)) < \nu_0(x\gamma_0^k) = p + k(n-l) = d + j(n-l) = \nu_0(f_0\gamma_0^j)$. Q.E.D.

To finish the proof of the theorem let $0 \neq f_0 \in B$ and let $j \in \mathbf{Z}_+$ and $w \in G^K$ be given by Lemma 7.5. Now put $f = w/\gamma^j$. Then

$$\nu_0(f_0 - P_\gamma(f)) = \nu_0(f_0\gamma_0^j - P_\gamma(f)\gamma_0^j) - j(n-l)$$

$$= \nu_0(f_0\gamma_0^j - P(w)) - j(n-l) < \nu_0(f_0\gamma_0^j) - j(n-l) = \nu_0(f_0). \quad \text{Q.E.D.}$$

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