ON THE STRUCTURE OF CERTAIN SUBALGEBRAS
OF A UNIVERSAL ENVELOPING ALGEBRA\(^{(1)}\)

BY

BERTRAM KOSTANT AND JUAN TIRAO

ABSTRACT. The representation theory of a semisimple group \(G\), from an algebraic point of view, reduces to determining the finite dimensional representation of the centralizer \(U^\mathfrak{t}\) of the maximal compact subgroup \(K\) of \(G\) in the universal enveloping algebra \(U\) of the Lie algebra \(\mathfrak{g}\) of \(G\). The theory of spherical representations has been determined in this way since by a result of Harish-Chandra \(U^\mathfrak{t}\) modulo a suitable ideal \(I\) is isomorphic to the ring of Weyl group \(W\) invariants \(U(a)^W\) in a suitable polynomial ring \(U(a)\). To deal with the general case one must determine the image of \(I/\mathfrak{t}\) in \(U(\mathfrak{t}) \otimes U(a)\), where \(\mathfrak{t}\) is the Lie algebra of \(K\). We prove that if \(W\) is replaced by the Kunze-Stein intertwining operators \(\tilde{W}\) then \(U^\mathfrak{t}\) suitably localized and completed is indeed isomorphic to \(U(\mathfrak{t}) \otimes U(a)^{\tilde{W}}\) suitably localized and completed.

1. Introduction. Let \(\mathfrak{g}\) be a real semisimple Lie algebra and let \(\mathfrak{g} = \mathfrak{t} + \mathfrak{p}\) be a Cartan decomposition of \(\mathfrak{g}\). If \(G\) is a Lie group, say with finite center, with Lie algebra \(\mathfrak{g}\), it is known that many of the fundamental questions concerning the infinite dimensional representation theory of \(G\) reduce to questions about the structure and finite dimensional representation theory of the algebra \(G^\mathfrak{t}\). Here \(G\) is the universal enveloping algebra, over \(\mathbb{C}\), of \(\mathfrak{g}\) and \(G^\mathfrak{t}\) is the centralizer of \(\mathfrak{t}\) in \(G\). Briefly, the reason for this is as follows (Theorem of Harish-Chandra): To any quasi-simple irreducible Banach space representation \(\pi\) of \(G\) there is associated an algebraically irreducible \(G\)-module \(V\) which is locally finite for \(\mathfrak{t}\) and which determines \(\pi\) up to infinitesimal equivalence. In fact one has a primary decomposition \(V = \bigoplus V_\delta\), where the sum is taken over the set \(\hat{\mathfrak{t}}\) of all equivalence classes \(\delta\) of finite dimensional irreducible \(\mathfrak{t}\)-modules, and the multiplicity of \(\delta\) is finite for any \(\delta \in \hat{\mathfrak{t}}\). Then, in particular, any \(V_\delta\) is finite dimensional and hence, a finite dimensional \(G^\mathfrak{t}\)-module. The point is that \(V\) itself as a \(G\)-module is completely determined by \(V_\delta\) as a \(G^\mathfrak{t}\)-module for any fixed \(\delta\) when \(V_\delta \neq 0\). See Lepowsky and McCollum [10] and Lepowsky [9] for a nice exposition of this. See also Dixmier [3].

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If $V_{s0} \neq 0$, where $s_0$ is the class of the trivial representation of $\mathfrak{f}$, then $\pi$ is called spherical. The approach above has been quite successful in dealing with spherical irreducible representations of $G$ (see e.g. Kostant [6]). Indeed, we may take $s = s_0$ and thus we have only to consider a quotient $G'/I$ instead of $G/\mathfrak{f}$. Here $I$ is the intersection of $G'$ with the left ideal in $G$ generated by $\mathfrak{f}$. Now by a theorem of Harish-Chandra, $G'/I$ is not only commutative but also isomorphic to a polynomial ring in $r$ variables where $r$ is the split rank of $G$. More precisely one has an algebra exact sequence

\begin{equation}
0 \rightarrow I \rightarrow G' \rightarrow \mathcal{A} \rightarrow 0
\end{equation}

where $a$ is a maximal abelian subalgebra of $\mathfrak{p}$, $\mathcal{A} \subset G$ is the universal enveloping algebra of $a$ (over $\mathbb{C}$) and $\mathcal{A}\tilde{W}$ is the ring of $\tilde{W}$-invariants in $\mathcal{A}$, $\tilde{W}$ being the translated Weyl group.

To investigate the general (not necessarily spherical) case along these lines one must look at $G'$ itself, not just $G'/I$. It is known (see e.g. Lepowsky [9]) that the map (1.1) may be replaced by an exact sequence (see Proposition 3.1)

\begin{equation}
0 \rightarrow G' \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow 0
\end{equation}

where $K$ is the universal enveloping algebra, over $\mathbb{C}$, of $\mathfrak{f}$, $M$ is the centralizer of $a$ in the analytic subgroup $K$ of $G$ with Lie algebra $\mathfrak{f}$, $K^M$ is the centralizer of $M$ in $K$ and $K^M \otimes A$ is given the tensor product algebra structure. Moreover $P$ is an antihomomorphism of algebras. In order to generalize (1.1) it is necessary to determine the image of $P$. Towards the end we introduce the subalgebra $B$ of all elements in $K^M \otimes A$ which commute with certain intertwining operators. Such operators are in 1:1 correspondence with the elements of the Weyl group $W$ and are rather closely related to the operators considered in [12] and also to those studied in [8] and [5]. To define $B$ we consider $K^M \otimes A$ as a subalgebra of a larger algebra. In fact the relation of $B$ to $K^M \otimes A$ may be taken as the generalization of the relation of $\mathcal{A}\tilde{W}$ to $\mathcal{A}$.

A result of Tirao shows that the image of $P$ lies in $B$ (Theorem 3.2). However, unlike (1.1), $P$ is not an anti-isomorphism of $G'$ onto $B$. But now we isolate an element $\gamma$ in the center of $G$ (hence in the center of $G'$). One notes the mapping $P$ extends to an exact sequence

\begin{equation}
0 \rightarrow G'_{\gamma} \rightarrow \mathcal{B}_{\gamma} \rightarrow 0
\end{equation}

where $G'_{\gamma}$ is the localization of the ring $G'$ with respect to $\gamma$ and $\mathcal{B}_{\gamma_0}$ is the localization of $\mathcal{B}$ with respect to $\gamma_0 = P(\gamma)$.

Now there are natural valuations (in the sense of ring theory) on $G'_{\gamma}$ and $\mathcal{B}_{\gamma_0}$ so that the extended map $P_{\gamma}$ is compatible with these valuations. Thus $P_{\gamma}$...
extends to a map $P_\Gamma$ of the respective completions $G^*_\Gamma$ and $B^{*\Gamma}_0$. Our main result is the following:

**Theorem.** The map $P_\Gamma: G^*_\Gamma \to B^{*\Gamma}_0$ is a surjective anti-isomorphism.

2. Let $G$ be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$; assume that $G$ has finite center. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{a}$ be a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. If $\alpha$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, we denote by $\mathfrak{g}_\alpha$ the corresponding root subspace. Fix a linear ordering on the dual of $\mathfrak{a}$ and set

$$\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^* = \sum_{\alpha > 0} \mathfrak{g}^{-\alpha}.$$  

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$.

Let $K$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{k}$, so $K$ is a maximal compact subgroup of $G$, and let $A, N, \overline{N}$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}, \mathfrak{n}$ and $\mathfrak{n}^*$ respectively. $G$ has the global Iwasawa decomposition $G = KAN$. For $x$ in $G$ we write $x = \kappa(x)(\exp H(x))n$ with $\kappa(x) \in K$, $H(x) \in \mathfrak{a}$, $n \in N$. Let $M$ (resp. $M'$) be the centralizer (resp. the normalizer) of $\mathfrak{a}$ in $K$; $W = M'/M$ is a finite group, the Weyl group.

Let $\mathfrak{a}_C^*$ be the complex dual of $\mathfrak{a}$. The Weyl group $W$ operates on $\mathfrak{a}_C^*$ by

$$\langle \omega(\lambda), H \rangle = \langle \lambda, \text{Ad}(w^{-1})H \rangle, \quad \lambda \in \mathfrak{a}_C^*, \ H \in \mathfrak{a},$$

where $\omega = wM$, $w \in M'$. Let $\rho(H) = \frac{1}{2} \text{tr(ad}(H))n$ for $H$ in $\mathfrak{a}$; in other words, $\rho$ is half the sum of the positive roots with multiplicities.

We shall consider a family $U^\lambda$ of continuous representations of $G$ parametrized by $\lambda \in \mathfrak{a}_C^*$ (which may be viewed as being induced from characters of $AN$) and realized on $L^2(K)$. Given $x$ in $G$, $U^\lambda(x)$ is defined by the prescription

$$(U^\lambda(x)f)(k) = e^{-\langle \lambda + \rho, H(x^{-1}k) \rangle} \cdot f(\kappa(x^{-1}k)), \quad f \in L^2(K)$$

(see Warner [14, p. 445]).

For $w \in M'$, define $\overline{N}_w = \overline{N} \cap w^{-1}Nw$. Clearly $\overline{N}_w$ depends only on the coset $\overline{w} = wM$. We introduce intertwining operators for the representations $U^\lambda$ by considering the formal integral (for the statement about convergence see Proposition 2.1 below).

$$(A(w, \lambda)f)(k) = \int_{\overline{N}_w} e^{-\langle \lambda + \rho, H(v) \rangle} f(k\kappa(v)) \, dv$$

where $\lambda \in \mathfrak{a}_C^*$, $f \in C^\infty(K)$ and the Haar measure $dv$ on $\overline{N}_w$ is normalized by (see Schiffmann [12, p. 35]),

$$\int_{\overline{N}_w} e^{-2\rho(H(v))} \, dv = 1.$$
If $\alpha$ is a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$, we denote by $H_\alpha$ the unique element in $\mathfrak{a}$ such that $\alpha(H) = B(H_\alpha, H)$ for $H \in \mathfrak{a}$ ($B$ is the Killing form of $\mathfrak{g}$).

We recall that every $\overline{w} \in W$ can be decomposed as the product of reflections with respect to the simple roots. The minimum integer $q$, such that there exist $q$ simple roots, not necessarily different, $\alpha_1, \ldots, \alpha_q$ with $\overline{w} = s_{\alpha_1} \cdots s_{\alpha_q}$ ($s_\alpha =$ reflection corresponding to $\alpha$) is by definition the length $l(\overline{w})$ of $\overline{w}$.

Let $C^\omega(K)$ denote the set of $C^\omega$ complex-valued functions on $K$ equipped with the usual Fréchet structure (Schwartz topology).

We come to Schiffmann’s results which can be found in [12], except that Schiffmann deals with the “induced picture”. We state them in the following proposition for future reference. One notes first that $C^\omega(K)$ is stable under the action of $U^\lambda(x), x \in G$.

**Proposition 2.1.** (i) The domain of convergence (absolute) of the intertwining integrals (2.1) is the set $S(\overline{w})$ of all $\lambda \in \mathfrak{a}_C^*$ such that $\text{Re}(\lambda(H_\alpha)) > 0$ for every positive root $\alpha$ such that $\overline{w}(\alpha) < 0$.

(ii) If $\lambda \in S(\overline{w})$, $A(w, \lambda)$ is a continuous endomorphism of $C^\omega(K)$, and $U^{\overline{w}(\lambda)}(x)A(w, \lambda)f = A(w, \lambda)U^\lambda(x)f$ for all $x \in G, f \in C^\omega(K)$.

(iii) Let $w_1, w_2 \in M'$ such that $l(\overline{w_1 \overline{w_2}}) = l(\overline{w_1}) + l(\overline{w_2})$. Then $S(\overline{w_1 \overline{w_2}}) = S(\overline{w_2}) \cap \overline{w_2}^{-1} S(\overline{w_1})$ and $A(w_1 w_2, \lambda) = A(w_1, \overline{w_2}(\lambda))A(w_2, \lambda)$ for $\lambda \in S(\overline{w_1 \overline{w_2}})$.

One uses this result to establish

**Proposition 2.2.** (i) Given $w \in M'$, for $\lambda \in S(\overline{w})$ the linear form

$$T(w, \lambda): f \mapsto \int_{\overline{w}^{-1}} e^{-(\lambda + \rho)H(V)} f(wK(v))dv, \quad f \in C^\omega(K),$$

defines a distribution on $K$.

(ii) Let $w_1 w_2 \in M'$ such that $l(\overline{w_1 \overline{w_2}}) = l(\overline{w_1}) + l(\overline{w_2})$. Then

$$T(w_1 w_2, \lambda) = T(w_1, \overline{w_2}(\lambda)) \ast T(w_2, \lambda) \quad \text{for } \lambda \in S(\overline{w_1 \overline{w_2}}).$$

**Proof.** (i) follows from Proposition 2.1(ii) by observing that $(T(w, \lambda), f) = (A(w, \lambda)f)(e)$. To prove (ii) we note that

$$A(w, \lambda)f = (T(w, \lambda) \ast \tilde{f})^\gamma, \quad f \in C^\omega(K),$$

where $\tilde{f}$ denotes the function $k \mapsto f(k^{-1})$. If $\lambda \in S(\overline{w_1 \overline{w_2}})$, Proposition 2.1(iii) gives

$$A(w_1 w_2, \lambda)f = A(w_1, \overline{w_2}(\lambda))A(w_2, \lambda)f, \quad f \in C^\omega(K),$$

which can be rewritten

$$(T(w_1 w_2, \lambda) \ast \tilde{f})^\gamma = (T(w_1, \overline{w_2}(\lambda)) \ast (A(w_2, \lambda)f)^\gamma)
= (T(w_1, \overline{w_2}(\lambda)) \ast T(w_2, \lambda) \ast \tilde{f})^\gamma.$$
Therefore,
\[ T(w_1w_2, \lambda) \ast \tilde{f} = T(w_1, \overline{w}_2(\lambda)) \ast T(w_2, \lambda) \ast \tilde{f}. \]

Now evaluating at the identity e and using the fact that \( \langle T, f \rangle = (T \ast \tilde{f})(e) \), we complete the proof of the proposition.

3. Let \( U \) be the universal enveloping algebra of the complexification \( u_C \) of the Lie algebra \( u \) of a Lie group \( U \). As is well known, we may regard \( U \) as the algebra of distributions on \( U \) whose support is the identity \( \{e\} \). One knows that \( D \in U \) defines a left invariant differential operator \( f \mapsto Df \) on \( G \) where \( Df = f \ast \tilde{D} \) and \( D \mapsto \tilde{D} \) is the usual antipode in \( U \). One also knows that \( \langle D, f \rangle = [Df](e), \ f \in C^\infty_c(U) \).

Let \( g_C, f_C, a_C, n_C \) be the complexifications of \( g, f, a, n \), respectively. We denote by \( G \) the universal enveloping algebra of \( g_C \), and by \( K, A \) and \( N \), the universal enveloping algebras of \( f_C, a_C \) and \( n_C \), respectively, regarded as canonically embedded in \( G \). We have
\[ G = KAN = KA(C1 + Nn_C) = KA + Gnc. \]

Let \( P: G \to fCA \) denote the corresponding projection map. We give \( KA \) an algebra structure by identifying it with the algebra \( K \otimes A \), and we also regard \( P \) as a map \( P: G \to K \otimes A \). Let \( G^K \) and \( K^M \) denote the centralizers of \( K \) in \( G \) and of \( M \) in \( K \), respectively. A proof of the following proposition can be found in Lepowsky [9].

**Proposition 3.1.** \( P \) defines an injective antihomomorphism of \( G^K \) into \( K^M \otimes A \).

The algebra \( A \) is just the symmetric algebra \( S(a_C) \); hence each linear mapping \( \lambda: a \to C \) extends uniquely to a homomorphism \( D \to D(\lambda) \) of \( A \) into \( C \) satisfying \( 1(\lambda) = 1 \). Now given \( \lambda \in a_C^* \) we can also consider the homomorphism \( K \otimes A \to K \) defined by \( E \otimes D \mapsto (E \otimes D)(\lambda) = D(\lambda)E \ (E \in K, D \in A) \).

We take the opportunity to prove the following unpublished result of Tirao.

**Theorem 3.2.** Given \( w \in M', \ \lambda \in S(w) \), we have
\[ T(w, \lambda) \ast P(D)(-\lambda - \rho) = P(D)(-\overline{w}(\lambda) - \rho) \ast T(w, \lambda) \ \text{for all} \ D \in G^K. \]

**Proof.** Consider the following identity
\[ e^{-(\overline{w}(\lambda) + \rho)H(x)} \int_{\overline{w}} e^{-(\lambda + \rho)H(v)}f(k(x)wK(v)) dv = \int_{\overline{w}} e^{-(\lambda + \rho)(H(v) + H(xwK(v)))}f(k(xwK(v))) dv, \ x \in G, \]
(3.1)
which is another way of writing $(U^\tilde{w}(\lambda)(x^{-1})A(w, \lambda)f)(e) = (A(w, \lambda)U^{\lambda}(x^{-1})f)(e)$ (cf. Proposition 2.1(ii)). Let $\varphi_1(x)$ and $\varphi_2(x)$ denote the left- and right-hand sides of (3.1), respectively. It is also convenient to introduce the following notation: given $\lambda \in \mathfrak{a}_C^*$ and $f \in C^\infty(K)$, let

$$F^\lambda_f(x) = e^{-\lambda \cdot \rho}H(x)f(\kappa(x)), \quad x \in G.$$ 

Then since $H(xn) = H(x)$ and $\kappa(xn) = \kappa(x)$ for $n \in N$ it follows that $DF^\lambda_f = 0$ for $D \in G_n$, i.e., $DF^\lambda_f = P(D)F^\lambda_f$, for $D \in G$. Since $H(x \exp H) = H(x) + H$ and $\kappa(x \exp H) = \kappa(x)$ ($H \in \mathfrak{n}$), we have $DF^\lambda_f = D(-\lambda - \rho)F^\lambda_f$, for $D \in A$.

Having in mind the decomposition $G = KA \oplus G_n$ it follows that $DF^\lambda_f = P(D)(-\lambda - \rho)F^\lambda_f$, for $D \in G$.

If $f$ is a continuous function on $K$ we shall write $f^R(k)$ for the composite function $f \circ R(k)$ where $R(k)$ is a right translation by $k \in K$.

Given $D \in G$, we have

$$[D\varphi_1](e) = \int_{N_u} e^{-\lambda \cdot \rho}H(u)[DF^\lambda_{f^R(wk(u))}](e) du$$

$$= \int_{N_u} e^{-\lambda \cdot \rho}H(u)[P(D)(-\bar{w}(\lambda) - \rho)F^\lambda_{f^R(wk(u))}](e) du$$

$$= \int_{N_u} e^{-\lambda \cdot \rho}H(u)[P(D)(-\bar{w}(\lambda) - \rho)F^\lambda_{f^R(wk(u))}](e) du$$

$$= \langle P(D)(-\bar{w}(\lambda) - \rho) \ast T(w, \lambda), f \rangle.$$

Now let $D \in G^K$ and differentiate $\varphi_2$ to obtain

$$[D\varphi_2](e) = \int_{N_u} e^{-\lambda \cdot \rho}H(u)[DF^\lambda_f(wk(u))](e) du$$

$$= \int_{N_u} e^{-\lambda \cdot \rho}H(u)[P(D)(-\lambda - \rho)f](wk(u)) du$$

$$= \int_{N_u} e^{-\lambda \cdot \rho}H(u)[P(D)(-\lambda - \rho)f](wk(u)) du$$

$$= \langle T(w, \lambda) \ast P(D)(-\lambda - \rho), f \rangle. \quad Q.E.D.$$ 

Given a finite dimensional irreducible representation $(V_\delta, \delta)$ of $K$ let us consider the maps

$$P_\delta = (\delta \otimes 1) \circ P : G^K \rightarrow \text{End}(V_\delta) \otimes A$$

and

$$p_\delta = (\text{tr} \otimes 1) \circ P_\delta : G^K \rightarrow A.$$ 

When $\delta$ is the trivial one-dimensional representation of $K$, $P_\delta$ (or $p_\delta$) gives
Harish-Chandra's famous homomorphism $\gamma: G^K \rightarrow A$. Theorem 3.2 generalizes that part of Harish-Chandra's theorem which asserts that the image of $\gamma$ is contained in the ring of $\tilde{W}$-invariants of $A$ ($\tilde{W}$ denotes the translated Weyl group). One also has the following result of Lepowsky (see [9])

$$p_\delta(D)(\lambda - \rho) = p_\delta(D)(\overline{w}(\lambda) - \rho)$$

for all $D \in G^K$, $\overline{w} \in \mathcal{W}$, $\lambda \in A^*_C$.

From Theorem 3.2 we get instead the more precise result

$$\delta(T(w, \lambda))p_\delta(D)(-\lambda - \rho) = p_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(T(w, \lambda))$$

for all $D \in G^K$, $w \in M'$, $\lambda \in S(\overline{w})$.

Note that $\delta(T(w, \lambda))$ is given by the integral

$$\delta(T(w, \lambda)) = \int_{\overline{w}} e^{-(\lambda + \rho)H(u)}\delta(w\kappa(u))dv, \quad \lambda \in S(\overline{w}).$$

Let $n = \dim N_w$. From Theorem 4.1, it will follow that there exists a non-zero complex number $t_w(\lambda)$ such that

$$\lim_{t \to +\infty} t^{n/2}\delta(T(w, t\lambda)) = t_w(\lambda)\delta(w), \quad w \in M',$$

uniformly on compact subsets of $S(\overline{w})$. Therefore, given any compact subset $\omega \subset S(\overline{w})$, for $t$ sufficiently large

$$\delta(T(w, t\lambda))$$

is invertible for all $\lambda \in \omega$.

Now it is clear that (3.3) implies (3.2).

Let $\mathcal{D}(K)$ denote the space of distributions on $K$ equipped with the topology of uniform convergence on bounded subsets of $C^\omega(K)$. Let $\mathcal{D}(K)^M$ be the centralizer of $M$ in $\mathcal{D}(K)$. We shall write $\delta_k$ for the Dirac measure at $k \in K$.

We can write

$$T(w, \lambda) = \delta_w \ast T'(\overline{w}, \lambda), \quad \lambda \in S(\overline{w}),$$

(cf. Proposition 2.2(i)) where $T'(\overline{w}, \lambda)$ is the distribution on $K$ defined by

$$\langle T'(\overline{w}, \lambda), f \rangle = \int_{\overline{w}} e^{-(\lambda + \rho)H(u)}f(\kappa(u))dv, \quad \lambda \in S(\overline{w}), \ f \in C^\omega(K).$$

Now

$$T'(\overline{w}, \lambda) \in \mathcal{D}(K)^M \text{ for } \lambda \in S(\overline{w}), \ \overline{w} \in \mathcal{W}.$$

In fact, for $\lambda \in S(\overline{w})$, $f \in C^\omega(K)$ and $m \in M$ we have

$$\langle \delta_m \ast T'(\overline{w}, \lambda) \ast \delta_{m^{-1}}, f \rangle = \int_{\overline{w}} e^{-(\lambda + \rho)H(u)}f(m\kappa(u)m^{-1})dv$$

$$\quad = \int_{\overline{w}} e^{-(\lambda + \rho)H(mum^{-1})}f(\kappa(mum^{-1}))dv$$
because $M$ normalizes $N$. But the Haar measure $du$ of $\overline{N_w}$ is invariant under $v \mapsto \alpha v \gamma^{-1}$; therefore $\delta_m \ast T'(\overline{w}, \lambda) \ast \delta_{m-1} = T'(\overline{w}, \lambda)$, which proves (3.6).

A consequence of Theorem 3.2 is the following

**Corollary 3.3.** Assume $\mathcal{D}(K)^M$ is abelian (which is precisely the case when $G$ is one of the following classical rank one groups: $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$). Then

$$\delta_w \ast P(D)(\lambda - \rho) = P(D)(\overline{w}(\lambda) - \rho) \ast \delta_w$$

for all $w \in M'$, $\lambda \in \mathfrak{a}_C^*$ and $D \in G^K$.

**Proof.** Let $\delta$ be any finite dimensional irreducible representation of $K$. From (3.3) and (3.5) we obtain

$$\delta(w)\delta(T'(\overline{w}, \lambda))P_\delta(D)(-\lambda - \rho) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w)\delta(T'(\overline{w}, \lambda))$$

for $w \in M'$, $\lambda \in S(\overline{w})$ and $D \in G^K$. But since $T'(\overline{w}, \lambda)$ and $P(D)(-\lambda - \rho)$ are in $\mathcal{D}(K)^M$ (cf. (3.6) and Proposition 3.1) we have

$$\delta(w)P_\delta(D)(-\lambda - \rho)\delta(T'(\overline{w}, \lambda)) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w)\delta(T'(\overline{w}, \lambda)).$$

Now because of (3.4) and (3.5) we obtain

$$\delta(w)P_\delta(D)(-\lambda - \rho) = P_\delta(D)(-\overline{w}(\lambda) - \rho)\delta(w), \quad \lambda \in \mathfrak{a}_C^*,$$

which in turn implies our assertion. Q.E.D.

4. Let $B$ be the set of all elements $B \in K^M \otimes A$ such that

$$(4.1) \quad T(w, \lambda) \ast B(-\lambda - \rho) = B(-\overline{w}(\lambda) - \rho) \ast T(w, \lambda)$$

for all $w \in M'$ and all $\lambda \in S(\overline{w})$. Clearly $B$ is a subalgebra of $K^M \otimes A$, and according to Theorem 3.2 it contains the image $P: G^K \rightarrow K^M \otimes A$. The principal objective now is to get information about the leading term of $B \in B$. The following theorem is needed and should be compared with results of Cohn [1].

**Theorem 4.1.** Given $w \in M'$ let $n = \dim \overline{N_w}$. For each $\lambda \in S(\overline{w})$ there exists a nonzero complex number $t_w(\lambda)$ such that

$$\lim_{t \rightarrow +\infty} t^{n/2}T(w, t\lambda) = t_w(\lambda)\delta_w$$

uniformly on compact subsets of $S(\overline{w})$.

**Proof.** We shall show that it is sufficient to consider the case when $\overline{w} = s_\alpha$ is the reflection corresponding to a simple root $\alpha$. In fact, given $w \in M'$ we can write $\overline{w} = s_{\alpha_1} \cdots s_{\alpha_q}$ where $q = l(\overline{w})$ and $\alpha_j$ ($j = 1, \ldots, q$) are simple roots. We can find elements $w_1, \ldots, w_q$ in $M'$ such that $\overline{w}_j = s_{\alpha_j}$ ($j = 1, \ldots, q$) and $w = w_1 \cdots w_q$. Now from Proposition 2.2(ii) it follows that
\[ T(w, \lambda) = T(w_1, \bar{w}_2 \cdots \bar{w}_q(\lambda)) \ast T(w_2, \bar{w}_3 \cdots \bar{w}_q(\lambda)) \ast \cdots \ast T(w_q, \lambda). \]

On the other hand if \( n_j = \dim N_{w_j} \) (\( j = 1, \ldots, q \)) we have \( n = n_1 + \cdots + n_q \) (cf. [12, Proposition 1.3, p. 12]). Therefore, if we assume the theorem when \( \bar{w} = s_\alpha \), \( \alpha \) a simple root, we get

\[
\lim_{t \to +\infty} t^{n/2} T(w, t\lambda) = \lim_{t \to +\infty} t^{n/2} T(w_1, w_2 \cdots w_q(t\lambda)) \ast \cdots \ast \lim_{t \to +\infty} t^{n/2} T(w_q, t\lambda) = t_{w_1}(\bar{w}_2 \cdots \bar{w}_q(\lambda)) \ast \cdots \ast t_{w_q}(\lambda) \delta_{w_1} \ast \cdots \ast \delta_{w_q} = t_w(\lambda) \delta_w,
\]

uniformly on compact subsets of \( S(\bar{w}) \). We have used the joint continuity of the convolution which is a consequence of the compactness of \( K \). Next we shall prove the case \( \bar{w} = s_\alpha \), \( \alpha \) a simple root, thus completing the proof of the theorem.

Let \( w \in M' \) be such that \( \bar{w} = s_\alpha \), where \( \alpha \) is a simple root. The Lie algebra \( \bar{n}_w \) of \( \bar{N}_w = \bar{N} \cap \bar{W}N_w \) is given by \( \bar{n}_w = g^{-\alpha} + g^{2\alpha} \). Let \( G_\alpha \) be the analytic subgroup whose Lie algebra is the smallest subalgebra of \( g \) containing \( g^{-\alpha}, g^{\alpha}, g^{\alpha}, \) and \( g^{2\alpha} \). Then \( G_\alpha \) is a semisimple Lie group with finite center. If we take \( K_\alpha = G_\alpha \cap K, A_\alpha = G_\alpha \cap A \) and \( N_\alpha = G_\alpha \cap N \) then \( G_\alpha = K_\alpha A_\alpha N_\alpha \) is an Iwasawa decomposition of \( G_\alpha \). The Lie algebra \( \mathfrak{a}_\alpha \) of \( A_\alpha \) is equal to \( RH_\alpha \), i.e. \( G_\alpha \) has real-rank one.

Let \( p = \dim g^{-\alpha} \) and \( q = \dim g^{-2\alpha} \), then \( \dim \bar{N}_w = p + q \). Since for \( \lambda \in S(\bar{w}) \), \( T(w, \lambda) = \delta_w \ast T'(w, \lambda) \) (cf. 3.5) it is enough to establish that

\[
\lim_{t \to +\infty} t^{(p+q)}/2 T'(w, t\lambda) = t_w(\lambda) \delta_e
\]

(\( 0 \neq t_w(\lambda) \in \mathbb{C} \)) uniformly on compact subsets of \( S(\bar{w}) \).

The distributions \( T'(w, \lambda) \) (\( \lambda \in S(\bar{w}) \)) on \( K \) come from the corresponding distributions on \( K_\alpha \) (by restriction from \( K \) to \( K_\alpha \)). This being a continuous map, the whole question reduces to the real-rank one group \( G_\alpha \).

If \( H_\alpha' = 2\alpha(H_\alpha)^{-1}H_\alpha \), then \( \rho(H_\alpha') = p + 2q \). Let \( z = (p + 2q)^{-1}\lambda(H_\alpha') \), \( \lambda \) is in \( S(\bar{w}) \) if and only if \( \text{Re } z > 0 \). We have to prove that given a compact subset \( \omega \) of the set of all \( z \in \mathbb{C} \) with \( \text{Re } z > 0 \)

\[
\lim_{t \to +\infty} t^{(p+q)/2} \int_{\bar{N}_w} e^{-(t \zeta + 1)\rho(H_\alpha')(\zeta)} f^{(\omega)}(\zeta) d\zeta = t_w(\lambda) f(e)
\]

uniformly for all \( z \in \omega \) and all \( f \) in each bounded subset of \( C^\omega(K_\alpha) \).

We drop the subscript \( \alpha \) and prove instead the following proposition which will complete the proof of Theorem 4.1.

Let \( S_{\Delta} \) be the sector in the complex plane of all \( z \in \mathbb{C} \) such that \( 0 < |z| < \infty, |\text{arg } z| < \pi/2 - \Delta \).
Proposition 4.2. Let $G$ be a connected semisimple Lie group with finite center and real-rank one. Let $n = \dim \mathbb{N}$. Then, there exists a positive constant $c$ such that

$$\lim_{z \to \infty} z^{n/2} \int_{\mathbb{N}} e^{-z^{p(H(u))}f(k(u))} dv = cf(e)$$

when $(z \to \infty, z \in S_{\Delta}, \Delta > 0)$ uniformly for all $f$ in each bounded subset of $C^\infty(K)$.

First we need a few lemmas.

Lemma 4.3. Let $e$ be a positive real number and $p$ a positive integer. Then

$$\int_0^e r^{p-1}(1 + r^2)^{-z} dr \sim \frac{\Gamma(p/2)}{z^{p/2}} (z \to \infty, z \in S_{\Delta}, \Delta > 0).$$

Proof. The asymptotic behavior of the above integral can be established, for example, by Laplace's method, after introducing the new variable $t = \log(1 + r^2)$ (see Erdélyi [4, p. 37]), or we can proceed more directly as follows. Write

$$\int_0^e r^{p-1}(1 + r^2)^{-z} dr = \int_0^\infty r^{p-1}(1 + r^2)^{-z} dr + g(z).$$

We have

$$\int_0^\infty r^{p-1}(1 + r^2)^{-z} dr = \Gamma(p/2)\Gamma(z-p/2)/2\Gamma(z) \quad (\Re z > p/2),$$

which is asymptotic to $\frac{\Gamma(p/2)}{z^{p/2}}$ (Stirling's formula; see Magnus [11, p. 12]).

On the other hand we can estimate $g(z)$ in the following way:

$$|g(z)| \leq \int_0^e r^{p-1}(1 + r^2)^{-\Re z} dr.$$

Given a positive real number $\delta$, there exists a positive number $A$ such that

$$r^{p+1} \leq \left(\frac{1 + r^2}{1 + \delta}\right)^{\Re z} \quad \text{for } r \geq A, \Re z \geq p + 1.$$

Now if we choose $\delta$ less than $e^2$, we can find another constant $B$ such that

$$r^{p+1} \leq B\left(\frac{1 + e^2}{1 + \delta}\right)^{\Re z} \quad \text{for } 0 < r < A, \Re z \geq p + 1.$$

Therefore, there exists $C$ such that

$$r^{p+1} \leq C\left(\frac{1 + r^2}{1 + \delta}\right)^{\Re z} \quad \text{for } r \geq \epsilon, \Re z \geq p + 1.$$

Hence $|g(z)| \leq Ce^{-1}(1 + \delta)^{-\Re z}$ for $\Re z \geq p + 1$, which implies that $g(z) = O(z^{-p/2})$ when $z \to \infty, z \in S_{\Delta}, \Delta > 0$. This proves the lemma. Q.E.D.
Let $B(e)$ $(e > 0)$ denote the set of all $(X, Y) \in \mathbb{R}^p \times \mathbb{R}^q$ such that $\|X\|$, $\|Y\| \leq e$.

**Lemma 4.4.** Let $p > 0$. There exists a positive constant $c_{p,q}$ such that

$$f(z) = \int_{B(e)} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-z} dX dY \sim c_{p,q} z^{-(p+q)/2}$$

when $z \to \infty$, $z \in S_{\Delta}$, $\Delta > 0$.

**Proof.** We have to consider two different cases: (a) $q = 0$ and (b) $q \neq 0$.

(a) Let $c_n$ be the Euclidean volume of the unit sphere in $\mathbb{R}^n$; in particular $c_1 = 2$.

The usual formula for integration in polar coordinates yields

$$\int_{B(e)} \sim \int_0^e r^{p-1}(1 + r^2)^{-2z} dr.$$

The assertion follows from Lemma 4.3 with $c_{p,0} = 2^{-(1+p/2)} \Gamma(p/2)c_p$.

(b) In this case

$$f(z) = c_p c_q \int_0^e \int_0^e r^{p-1}(1 + r^2)^{-2z} dr ds.$$

Letting for $0 < s \leq e$, $u = s(1 + r^2)^{-1}$ we find

$$f(z) = c_p c_q \int_0^e \int_0^e r^{p-1}(1 + r^2)^{-2z} dr ds \int_0^e u^{q-1}(1 + u^2)^{-z} du ds.$$

We can estimate $g(z)$ as follows:

$$|g(z)| \leq c_p c_q \int_0^e \int_0^e r^{p-1}(1 + r^2)^{-2z} dr ds \int_0^e u^{q-1}(1 + u^2)^{-z} du ds \leq c_p c_q |(1 + e^2)^{-1} - \int_0^e \int_0^e r^{p-1}(1 + r^2)^{-2z} dr ds.$$
\[ Q(X) = 4B(X, \theta(X))/B(H'_a, \theta(H'_a)), \]

\( \theta \) denotes the Cartan involution of \( g \). If \( v = \exp(X + Y), X \in g^{-\alpha}, Y \in g^{-2\alpha}, \) then (Helgason-Schiffmann, cf. [14, p. 38]) \( H(v) = (a/2)H_a' \) with \( \varepsilon^{2\alpha} = (1 + Q(X)/2)^2 + 2Q(Y). \) We make the identifications \( g^{-\alpha} \cong \mathbb{R}^p, g^{-2\alpha} \cong \mathbb{R}^q \) in such a way that \( \|X\|^2 = Q(X)/2, \|Y\|^2 = 2Q(Y) \) \( (X \in g^{-\alpha}, Y \in g^{-2\alpha}). \) Then the integral under study can be written

\[ I(z) = z^{(p+q)/2} \int_{\mathbb{R}^p \times \mathbb{R}^q} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-\frac{b}{2}} f(\kappa(\exp(X + Y))) \, dX \, dY, \]

where \( b = (p + 2q)/4. \)

Let \( B(\varepsilon) = \{(X, Y) \in \mathbb{R}^p \times \mathbb{R}^q : \|X\|, \|Y\| \leq \varepsilon\}. \) We can write \( I(z) \) as the sum of an integral over \( B(\varepsilon) \) and an integral over \( \mathbb{R}^p \times \mathbb{R}^q - B(\varepsilon). \) Call the two resulting integrals \( II(\varepsilon, z) \) and \( III(\varepsilon, z) \) respectively. First of all we shall prove that \( III(\varepsilon, z) \to 0 \) as \( z \to \infty, z \in S_\Delta, \Delta > 0, \) uniformly for all \( f \) in a bounded subset of \( C^\infty(K). \) There exists a constant \( C \) such that the integrand of \( III(\varepsilon, z) \) is bounded by

\[ C|z|^{(p+q)/2}(1 + \|X\|^2)^2 + \|Y\|^2)^{-b} \Re z. \]

Given \( d > 0, \) for \( z \in S_\Delta \) and \( |z| \) sufficiently large we have

\[ |z|^{(p+q)/2} \leq (1 + e^2)^b \Re z - d \leq ((1 + \|X\|^2)^2 + \|Y\|^2)^b \Re z - d \]

whenever \( (X, Y) \notin B(\varepsilon). \) Therefore the integrand of \( III(\varepsilon, z) \) is bounded by

\[ C((1 + \|X\|^2)^2 + \|Y\|^2)^{-d} \]

which is an integrable function for \( d > b \) (see Wallach [13, p. 262]). By the dominated convergence theorem we have \( \lim \) \( III(\varepsilon, z) = 0 \) when \( z \to \infty, z \in S_\Delta, \Delta > 0 \) uniformly on bounded subsets of \( C^\infty(K). \) In fact

\[ \lim|z|^{(p+q)/2}((1 + \|X\|^2)^2 + \|Y\|^2)^{-b} \Re z = 0 \quad (z \to \infty, z \in S_\Delta, \Delta > 0) \]

if \( (X, Y) \neq 0. \)

Now consider \( II(\varepsilon, z) \) and write

\[ II(\varepsilon, z) = f(\varepsilon)z^{(p+q)/2} \int_{B(\varepsilon)} ((1 + \|X\|^2)^2 + \|Y\|^2)^{-\frac{b}{2}} dX \, dY + II'(\varepsilon, z). \]

By Lemma 4.4 the first term tends to \( cf(\varepsilon) \) with \( c = c_{p, q}b^{-(p+q)/2} \) as \( z \to \infty, z \in S_\Delta, \Delta > 0. \) Therefore to complete the proof of the proposition it is enough to show that given a bounded subset \( B \) of \( C^\infty(K) \) and a positive \( \delta, \) there exists a positive \( \varepsilon \) such that \( |II'(\varepsilon, z)| < \delta \) for \( f \in B, z \in S_\Delta \) and \( |z| \) sufficiently large. Now
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\[ |\Pi'(e, z)| \leq |z|^{(p+q)/2} \int_{B(e)} (1 + ||X||^2 + ||Y||^2)^{-b \Re z} \]
\[ \cdot |f(\kappa(\exp(X + Y))) - f(e)| \, dX \, dY. \]

From Lemma 4.4 it also follows that

\[ |z|^{(p+q)/2} \int_{B(1)} ((1 + ||X||^2 + ||Y||^2)^{-b \Re z} \, dX \, dY \]

is bounded in \( S_\Delta \), say by a constant \( A \). Given \( \delta > 0 \) there exists \( \epsilon \) \((0 < \epsilon \leq 1)\) such that

\[ |f(\kappa(\exp(X + Y))) - f(e)| < \delta A^{-1} \quad \text{on } B(e) \]

for all \( f \in B \). Then \( |\Pi'(e, z)| < \delta \) for \( z \in S_\Delta \) and \( f \in B \), which completes the proof of Proposition 4.2. Q.E.D.

If \( B \in K^M \otimes A \) we can view \( B(\lambda) \) as a polynomial of degree \( d \) on \( a^*_C \) with coefficients in \( K^M \). Let \( B_d \in K^M \otimes A \) be the element such that \( B_d(\lambda) \) is the leading term (homogeneous of degree \( d \) in \( \lambda \)) of \( B(\lambda) \). The Weyl group \( W \) acts on \( K^M \) and on \( A \) via the adjoint representation, so we can define an action of \( W \) on \( K^M \otimes A \) by taking the tensor product action.

**Theorem 4.5.** If \( B \in B \) then the leading term \( B_d \) of \( B \) is \( W \)-invariant.

**Proof.** Given \( w \in M' \), let \( n = \dim N_w \). By hypothesis we have \( T(w, \lambda) * B(-\lambda - \rho) = B(-\bar{\lambda}(w) - \rho) * T(w, \lambda) \) for all \( \lambda \in S(\bar{w}) \). Now write

\[ t^{n/2} T(w, t\lambda) \cdot t^{-d} B(-t\lambda - \rho) = t^{-d} B(-t\bar{\lambda}(t) - \rho) * t^{n/2} T(w, t\lambda) \]

and let \( t \rightarrow +\infty \). From Theorem 4.1 we obtain

\[ t(w, \lambda) \delta_w \cdot B_d(-\lambda) = B_d(-\bar{\lambda}(w)) \cdot t(w, \lambda) \delta_w, \quad \lambda \in S(\bar{w}), \]

which proves our assertion. Q.E.D.

5. Let \( g \) be a real semisimple Lie algebra and let \( g = \mathfrak{g} + \mathfrak{p} \) be a Cartan decomposition. If subscript \( C \) denotes complexification one also has \( \mathfrak{g}_C = \mathfrak{f}_C + \mathfrak{p}_C \).

Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \) and let \( H \subseteq \text{Aut}(\mathfrak{p}_C) \) be the analytic subgroup corresponding to \( \text{ad}_{\mathfrak{p}_C} \mathfrak{f}_C \subset \text{End}(\mathfrak{p}_C) \). One knows that \( x \in \mathfrak{p}_C \) is semisimple if and only if \( x \in H \cdot \mathfrak{a}_C \) (see Kostant and Rallis [7, Theorem 1]).

Let \( r = \dim \mathfrak{a}_C \). If \( x \in \mathfrak{p}_C \) then one knows that \( \dim \mathfrak{p}_C^x \geq r \) where one puts \( \mathfrak{p}_C^x = (\text{Ker ad } x) \cap \mathfrak{p}_C \). An element \( x \in \mathfrak{p}_C \) is called regular if \( \dim \mathfrak{p}_C^x = r \). Similarly, let \( \mathfrak{f}_C^x = (\text{Ker ad } x) \cap \mathfrak{f}_C \). Let \( m \) be the centralizer of \( \mathfrak{a} \) in \( \mathfrak{f}_C \). One has [7, p. 770]

**Theorem 5.1.** For any \( x \in \mathfrak{p}_C \) one has \( \dim \mathfrak{f}_C^x - \dim \mathfrak{p}_C^x \) is independent of \( x \). In particular \( \dim \mathfrak{f}_C^x \geq \dim m \) and equality holds if and only if \( x \) is regular in \( \mathfrak{p}_C \).
Now for any \( x \in \mathfrak{p}_C \), \((\text{ad} \, x)^2\) leaves \( \mathfrak{p}_C \) stable. Let \( \alpha_x = (\text{ad} \, x)^2|_{\mathfrak{p}_C} \) and let \( r_x \) be the multiplicity of the zero eigenvalue of \( \alpha_x \).

**Proposition 5.2.** For all \( x \in \mathfrak{p}_C \), \( r_x \geq r \).

**Proof.** Clearly \( r_x \geq \dim(\ker \alpha_x) \geq \dim \mathfrak{p}_C \geq r \). Q.E.D.

Now we say that \( x \in \mathfrak{p}_C \) is s-regular if \( r_x = r \).

**Proposition 5.3.** An element \( x \) in \( \mathfrak{p}_C \) is s-regular if and only if \( x \) is regular and semisimple.

**Proof.** Assume \( x \in \mathfrak{p}_C \) is regular and semisimple. Then \( \ker \alpha_x = \mathfrak{p}_C \) because \( x \) is semisimple, and \( \dim \mathfrak{p}_C = r \) because \( x \) is regular. Hence \( x \) is s-regular.

Conversely, suppose \( x \) is s-regular. Now if \( y \in \mathfrak{p}_C \) let \( \mathfrak{g}_C(y) = \{ u \in \mathfrak{g}_C : (\text{ad} \, y)^n u = 0 \text{ for some } n \} \), and if \( q = \min(\dim(\mathfrak{g}_C(y) \cap \mathfrak{p}_C)) \) over all \( y \in \mathfrak{p}_C \), we let \( Q \) be the set of all \( y \in \mathfrak{p}_C \) such that \( q = \dim(\mathfrak{g}_C(y) \cap \mathfrak{p}_C) \). Since \( \dim(\mathfrak{g}_C(y) \cap \mathfrak{p}_C) \) is clearly the multiplicity of the zero eigenvalue of \( \alpha_y \) it follows that \( q = r \) and hence \( x \in Q \). One knows that \( Q = H \cdot (Q \cap a_C) \) [7, p. 765] so the elements in \( Q \) are semisimple. Hence \( x \) is semisimple, and since it is clearly regular, we have completed the proof of the proposition. Q.E.D.

The theorem we wish to prove is

**Theorem 5.4.** Let \( x \in \mathfrak{p}_C \) and let \( l_x \) be the multiplicity of the zero eigenvalue of \( \text{ad} \, x \) in \( \mathfrak{g}_C \). Then \( l_x \geq l = \dim a + \dim m \) where equality holds if and only if \( x \) is s-regular.

**Proof.** If \( x \) is s-regular then one has (cf. Theorem 5.1) \( \dim \mathfrak{g}_C = \dim \mathfrak{f}_C + \dim \mathfrak{p}_C = \dim m + \dim a \) where \( \mathfrak{g}_C = \ker \text{ad} \, x \). However since \( x \) is semisimple \( \dim \mathfrak{g}_C \) is the multiplicity of the zero eigenvalue of \( \text{ad} \, x \) in \( \mathfrak{g}_C \) establishing the theorem in one direction.

Conversely assume \( l_x = l \). For any \( y \in \mathfrak{p}_C \), \((\text{ad} \, y)^2\) leaves \( \mathfrak{f}_C \) stable. Let \( d_y \) be the multiplicity of the zero eigenvalue of \((\text{ad} \, y)^2\) in \( \mathfrak{f}_C \). We have \( l_y = d_y + r_y \geq \dim \mathfrak{f}_C \geq \dim m \) and \( r_y \geq r \). Also \( l_y \) is equal to the multiplicity of the zero eigenvalue of \((\text{ad} \, y)^2\) in \( \mathfrak{g}_C \) as well as \( \text{ad} \, y \). Therefore \( l_x = l \) implies \( r_x = r \) which concludes the proof of the theorem. Q.E.D.

For any vector space \( V \), let \( S(V) \) denote the symmetric algebra over \( V \). For every nonnegative integer \( i \), let \( S^i(V) \) denote the homogeneous subspace of \( S(V) \) of degree \( i \).

Let \( n = \dim \mathfrak{g}_C \) and let \( \mathfrak{g}_C' \) be the dual of \( \mathfrak{g}_C \). Now for any \( x \in \mathfrak{g}_C \) let \( \det(r - \text{ad} \, x) = \Sigma a_i(x) t^i \) be the characteristic polynomial of \( \text{ad} \, x \). One has \( a_i \in (S^{n-i}(\mathfrak{g}_C'))^G_C \) is an invariant polynomial where \( G_C \) denotes the adjoint group of \( \mathfrak{g}_C \). Consider \( a_i \). We then have
Corollary 5.5. Let \( x \in \mathfrak{p}_C \). Then \( a_i(x) = 0 \) for all \( i < l \) and \( a_j(x) = 0 \) if and only if \( x \in \mathfrak{p}_C \) is not \( s \)-regular.

Let \( \mathfrak{p}_C^* \subset \mathfrak{p}_C \) denote the set of all \( s \)-regular elements in \( \mathfrak{p}_C \). Let \( a = a_l \).

Now let \( b = a|_\mathfrak{p} \), so that \( b \in \mathfrak{S}^{n-l}(\mathfrak{p}_C') \), where \( \mathfrak{p}_C' \) denotes the dual of \( \mathfrak{p}_C \). We note that \( b \neq 0 \) and in fact \( \mathfrak{p}_C^* = \{ x \in \mathfrak{p}_C : b(x) \neq 0 \} \). More explicitly if \( \Delta \) is the set of roots, counting multiplicities of \( (a_C, \mathfrak{g}_C) \), then card \( \Delta = n - l \) and \( b|_{\mathfrak{a}_C} = \prod_{\alpha \in \Delta} \alpha \).

6. Now we regard \( \mathfrak{S}(\mathfrak{p}_C') \) as a subalgebra of \( \mathfrak{S}(\mathfrak{g}_C') \) where if \( f \in \mathfrak{S}(\mathfrak{p}_C') \) then \( f \) is also regarded as a function on \( \mathfrak{g}_C \) such that if \( z \in \mathfrak{g}_C, z = x + y, x \in \mathfrak{f}_C, y \in \mathfrak{p}_C \) then \( f(x + y) = f(y) \).

It follows that if \( \mathfrak{g}_C^* = \mathfrak{f}_C + \mathfrak{p}_C^* \) then \( b \in \mathfrak{S}^{n-l}(\mathfrak{g}_C^*) \) and \( \mathfrak{g}_C^* = \{ z \in \mathfrak{g}_C : b(z) \neq 0 \} \). That is, \( \mathfrak{g}_C^* \) is an open affine subvariety of \( \mathfrak{g}_C \) and the affine algebra of \( \mathfrak{g}_C^* \) is the localization \( \mathfrak{S}(\mathfrak{g}_C')_{b} \) of \( \mathfrak{S}(\mathfrak{g}_C') \) by \( b \), so that \( \mathfrak{S}(\mathfrak{g}_C')_{b} \) is the ring of all rational functions on \( \mathfrak{g}_C \) of the form \( f/b^k \) where \( f \in \mathfrak{S}(\mathfrak{g}_C') \) and \( k \in \mathbb{Z} \).

Now let \( a_C^* = \{ x \in a_C : \alpha(x) \neq 0 \) for all \( \alpha \in \Delta \} \); then \( \mathfrak{f}_C + a_C^* = \{ z \in \mathfrak{f}_C + a_C : b_0(z) \neq 0 \} \) where \( b_0 = b|_{\mathfrak{f}_C + a_C} \). Thus \( \mathfrak{f}_C + a_C^* \) is an affine variety whose affine algebra is the localization \( \mathfrak{S}((\mathfrak{f}_C + a_C'))_{b_0} \) of \( \mathfrak{S}((\mathfrak{f}_C + a_C')) \) by \( b_0 \). By now the injection map \( \mathfrak{f}_C + a_C^* \hookrightarrow \mathfrak{f}_C + \mathfrak{p}_C^* = \mathfrak{g}_C^* \) of affine varieties induces contravariantly the restriction homomorphism

\[
\mathfrak{S}(\mathfrak{g}_C')_{b} \rightarrow \mathfrak{S}((\mathfrak{f}_C + a_C'))_{b_0}
\]

of affine algebras.

Now let \( K_C \) be the subgroup of \( G_C \) corresponding to \( ad \mathfrak{f}_C \). Then the affine variety \( \mathfrak{g}_C^* \) is clearly stable under the action of the reductive algebraic group \( K_C \) and hence the ring of \( K_C \)-invariants \( A = \mathfrak{S}(\mathfrak{g}_C')_{K_C} \) is an affine ring (finitely generated). Also if \( M_C' \) is the normalizer of \( a_C \) in \( K_C \) then \( M_C' \) is a reductive algebraic group operating on the affine variety \( \mathfrak{f}_C + a_C^* \) and hence \( A_0 = \mathfrak{S}((\mathfrak{f}_C + a_C'))_{b_0} \) is also an affine ring. Since \( M_C' \subseteq K_C \) the homomorphism (6.1) restricted to \( A \) induces a homomorphism

\[
\pi: A \rightarrow A_0.
\]

We will prove the following theorem of Kostant.

Theorem 6.1. The homomorphism \( \pi: A \rightarrow A_0 \) is an isomorphism of algebras.

We first establish some lemmas. Let \( O \) be the set of all \( K_C \) orbits in \( \mathfrak{g}_C^* \) and let \( O_0 \) be the set of \( M_C' \) orbits in \( \mathfrak{f}_C + a_C^* \).

Lemma 6.2. If \( O \in O \) then \( O \cap (\mathfrak{f}_C + a_C^*) = O_0 \) is an \( M_C' \) orbit and the correspondence \( O \mapsto O_0 \) defines a bijection \( O \rightarrow O_0 \).
The only thing we really have to prove is that given \( x, y \in O \cap (t_C + a_C^*) \) there exists \( k \in M'_C \) such that \( y = k \cdot x \). Write \( x = x_1 + x_2 \), \( y = y_1 + y_2 \) where \( x_1, y_1 \in t_C \) and \( x_2, y_2 \in a_C^* \). We know that there is \( k \in K_C \) such that \( y = k \cdot x \) and therefore \( y_2 = k \cdot x_2 \). Now we use the fact (see [7]) that if two elements in \( a_C^* \) are \( K_C \)-conjugate then they are \( M'_C \)-conjugate. Hence there exist \( m_1 \in M'_C \) such that \( y_2 = m_1 \cdot x_2 \). Then \( m_1^{-1}k \cdot x_2 = x_2 \). Since \( x_2 \) is \( s \)-regular \( m_1^{-1}k = m \) centralizes \( a_C^* \) (cf. [7, Lemma 20]). Thus \( k = m_1m \in M'_C \) and the lemma is proved.

**Lemma 6.3.** With respect to the bijection \( O \leftrightarrow O_0 \) of the previous lemma one has: \( O \) is closed if and only if \( O_0 \) is closed.

**Proof.** Assume \( O_0 \) is closed and \( x_n \rightarrow x, x_n \in O, x \in g_C^* \). Then by applying an element in \( K_C \) we may assume \( x \in t_C + a_C^* \). Then we may find \( k_n \in K_C, k_n \rightarrow e \) such that \( k_n \cdot x_n \in t_C + a_C^* \) so that \( k_n \cdot x_n \rightarrow x \). But \( k_n \cdot x_n \in O_0 \) therefore \( x \in O_0 \). Hence \( O \) is closed.

**Proof of Theorem 6.1.** We first observe that \( \pi: A \rightarrow A_0 \) is injective. Indeed if \( 0 \neq f \in A \) we must show \( f|t_C + a_C^* \neq 0 \). But if \( f|t_C + a_C^* = 0 \) then \( 0 = f|K_C \cdot (t_C + a_C^*) \). Thus \( f = 0 \) since \( K_C \cdot (t_C + a_C^*) = a_C^* \). Thus we may regard \( A \subset A_0 \). But now we assert: (1) \( A \) is integrally closed in its quotient field \( Q \); (2) if \( \tilde{A}_0 \) (resp. \( \tilde{A} \)) denotes the set of all homomorphisms \( \chi: A_0 \rightarrow C \) (resp. \( \chi: A \rightarrow C \)) then the map \( \tilde{A}_0 \rightarrow \tilde{A}, \chi \rightarrow \chi \circ \pi \) is a bijection.

To establish (1) we note that if \( f \in Q \) satisfies a monic polynomial equation with coefficients in \( A \subset S(a_C^*b) \) then \( f \in S(a_C^*b) \), a localization of a polynomial ring, is integrally closed. Because \( f \in Q \), \( f = a_1/a_2, a_1, a_2 \in A \) and \( a_2 \neq 0 \). Hence \( fa_2 = a_1 \); applying \( k \in K_C \) we get \( f^k a_2 = a_1 = fa_2 \), therefore \( f^k = f \), i.e. \( f \in A \).

Now (2) follows from Lemma 6.3 since one knows that the natural map \( O \rightarrow \tilde{A} \) and \( O_0 \rightarrow \tilde{A}_0 \) give a bijection between the set of all closed orbits in \( O \) and \( \tilde{A} \), and the set of all closed orbits in \( O_0 \) and \( \tilde{A}_0 \), respectively. (See e.g. Dieudonné [2].) Now (2) implies that \( \tilde{A}_0 \rightarrow \tilde{A} \) is a bijective, birational map of affine varieties. But (1) implies that \( \tilde{A} \) is normal. Hence by Zariski’s Main Theorem (see e.g. [15, p. 413]) the map \( \tilde{A}_0 \rightarrow \tilde{A} \) is an isomorphism and hence \( A_0 = A \). Q.E.D.

7. A valuation on a ring \( R \) is a map \( \nu: R \rightarrow Z \cup \{ -\infty \} \) such that: (1) \( \nu(r) = -\infty \) if and only if \( r = 0 \), (2) \( \nu(r + s) \leq \max(\nu(r), \nu(s)) \), (3) \( \nu(rs) = \nu(r) + \nu(s) \). If \( R_n = \{ r \in R : \nu(r) \leq n \} \) then \( R_n \subset R_{n+1}, \bigcap_{-\infty < n < \infty} R_n = \{ 0 \} \) and \( R_n \) \( (n \in Z) \) defines a system of neighborhoods of \( 0 \) and hence a topology on \( R \). The valuation also defines a uniform structure on \( R \) so that we may complete \( R \) obtaining a ring \( \overline{R} \). To each \( \bar{r} \in \overline{R} \) there is a Cauchy sequence \( r_n \in R \) such that \( r_n \rightarrow \bar{r} \). If \( r_n \rightarrow \bar{r} \) and \( s_n \rightarrow \bar{s} \) then \( r_ns_n \rightarrow \bar{r}\bar{s}, r_n + s_n \rightarrow \bar{r} + \bar{s} \).
Now if $R$ is an integral domain and it satisfies the Ore condition (i.e.: $Ra \cap Rb \neq \{0\}$ for all $a, b \neq 0$) then $\nu$ extends to $Q(R)$, the left quotient division ring of $R$, by setting $\nu(a^{-1}b) = \nu(b) - \nu(a)$.

**Example.** Let $\mathfrak{h}$ be a Lie algebra and $\mathfrak{j} \subset \mathfrak{h}$ any subalgebra in $\mathfrak{h}$. Let $J \subset H$ be the corresponding universal enveloping algebras. Let $H(n)$ be the usual filtration of $H$. Thus $H(n)$ is spanned by $x_1 \cdots x_j$, $x_i \in \mathfrak{h}$, $j \leq n$, and the identity. We claim that $JH(n) = H(n)J$.

To prove, for example, that $JH(n) \subset H(n)J$ one notices that if $x_1, \ldots, x_j \in \mathfrak{h}$, $y \in \mathfrak{j}$, then by induction

$$yx_1 \cdots x_j = x_1 \cdots x_jy + \sum_{i=1}^j x_1 \cdots [y, x_i] \cdots x_j \in H(n)J$$

if $j \leq n$. Thus we get a new filtration of $H$ by putting $H_n = JH(n)$ since now $H_nH_m \subset H_{n+m}$.

**Theorem 7.1.** If $0 \neq a \in H$ let $\nu(a) = \min n$ such that $a \in H_n$ and let $\nu(0) = -\infty$. Then $\nu$ is a valuation on $H$.

**Proof.** Let $\mathfrak{q}$ be a linear complement of $\mathfrak{j}$ in $\mathfrak{h}$ so that $\mathfrak{h} = \mathfrak{j} + \mathfrak{q}$ (direct sum). Let $y_1, \ldots, y_k$ be a basis of $\mathfrak{q}$. Then

$$(7.1) \quad H = \bigoplus_{(m_1, \ldots, m_k)} Jy_1^{m_1}y_2^{m_2} \cdots y_k^{m_k}$$

by the Birkhoff-Witt theorem. In fact, if $u \in H$, $u = \sum u_{m_1, \ldots, m_k}$ where $u_{m_1, \ldots, m_k} \in Jy_1^{m_1} \cdots y_k^{m_k}$, then $\nu(u) = \max u_{m_1, \ldots, m_k} |m|$ where $m = (m_1, \ldots, m_k)$, $|m| = \sum_{i=1}^k m_i$.

The only thing to be proved is that $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in H$.

Since clearly $\nu(ab) \leq \nu(a) + \nu(b)$ to prove the equality we may assume that $a = \sum_{|m| = \nu(a)} a_m$, $b = \sum_{|n| = \nu(b)} b_n$ where $a_m = c_my^m$, $b_n = d_n y^n$, $y_m = y_1^{m_1} \cdots y_k^{m_k}$; $c_m, d_n \in J$. Now let $u \rightarrow \rho(u)$ be the usual valuation of an element $u \in H$ (the case where $\mathfrak{j} = 0$). Then one has $\rho(a) = \nu(a) + \alpha(a)$ and $\rho(b) = \nu(b) + \alpha(b)$ where for any $u \in H$ one puts $\alpha(u) = \max \rho(e_r)$ where $e_r \in J$ is such that $u = \Sigma e_r y^r$. But now if $v = \Sigma c_m d_n y^{m+n}$ where the sum is over all pairs $(m, n)$ such that $|m| = \nu(a)$, $\rho(c_m) = \alpha(a)$, $|n| = \nu(b)$, $\rho(d_n) = \alpha(b)$ then clearly

$$(7.2) \quad ab - v \in H(\rho(a) + \rho(b) - 1).$$

On the other hand, since $\rho(ab) = \rho(a) + \rho(b)$ it follows that $v \notin H(\rho(a) + \rho(b) - 1)$, so that $v$ can be written $v = \Sigma_{|s| \leq \nu(a) + \nu(b)} e_r y^s$ where $\alpha(v) = \alpha(a) + \alpha(b)$. On the other hand by (7.2) one has, for some $f_s \in J$, $ab - v = \Sigma |s| \leq \nu(a) + \nu(b) f_s y^s$ and $\rho(f_s) \leq \alpha(a) + \alpha(b)$ for $|s| = \nu(a) + \nu(b)$. Thus one cannot have $e_r + f_r \neq 0$ for all $r$ where $|r| = \nu(a) + \nu(b)$. This implies $\nu(ab) = \nu(a) + \nu(b)$. Q.E.D.
We can identify the universal enveloping algebra of the direct sum \( \mathfrak{g}_C + \mathfrak{a}_C \) with \( K \otimes A \). Since \( \mathfrak{g}_C \) is a subalgebra in \( \mathfrak{g}_C \oplus \mathfrak{a}_C \) it defines a valuation \( \nu_0 \) on \( K \otimes A \). Now \( A \) is graded \( A = \bigoplus A_i \). If \( \nu_0(u) = d \), \( u \in K \otimes A \) then there exists a unique \( u_d \in K \otimes A_d \) such that
\[
\nu_0(u - u_d) < d. \tag{7.3}
\]

Since \( K \otimes A \) satisfies the Ore condition and is an integral domain, \( \nu_0 \) extends to the quotient division ring \( Q(K \otimes A) \).

We let \( \nu \) be the valuation on \( G \) and on \( Q(G) \) its quotient division ring, defined also by \( \mathfrak{g}_C \).

A proof of the following proposition can be found in Lepowsky \[9\]. We first recall that the map \( P: G \rightarrow K \otimes A \) was the projection defined by the decomposition \( G = K \otimes A \oplus G_{nc} \). Let \( \lambda: S(\mathfrak{g}_C) \rightarrow G \) denote the symmetrization mapping. We note that \( \lambda \) is defined on \( S(\mathfrak{p}_C) \) by regarding \( S(\mathfrak{p}_C) \subset S(\mathfrak{g}_C) \). Let \( q \) be the orthogonal complement of \( \mathfrak{a} \) in \( \mathfrak{p} \) with respect to the Killing form of \( g \), and let \( q_C \subset \mathfrak{p}_C \) be the complexification of \( q \). Then \( S(\mathfrak{p}_C) = S(\mathfrak{a}_C) \oplus q_C S(\mathfrak{p}_C) \), so that
\[
G = (K \otimes A) \oplus (K \otimes \lambda(q_C S(\mathfrak{p}_C))).
\]
Let \( F: G \rightarrow K \otimes A \) denote the projection onto the first summand in this decomposition.

**Proposition 7.2.** (i) If \( u \in G^K \) then \( \nu(u) = \nu_0(F(u)) \). (ii) If \( 0 \neq u \in G \) then \( \nu_0(P(u) - F(u)) < \nu(u) \).

**Corollary 7.3.** If \( u \in G^K \) then \( \nu(u) = \nu_0(P(u)) \).

**Proof.** If \( u \in G^K \) we have \( \nu_0(P(u) - F(u)) < \nu_0(F(u)) \); hence the leading term of \( F(u) \) is equal to the leading term of \( P(u) \), and therefore \( \nu_0(P(u)) = \nu_0(F(u)) = \nu(u) \).

Now let \( \delta: \mathfrak{g}_C \rightarrow \mathfrak{g}_C' \) be the isomorphism defined by the Killing form of \( \mathfrak{g}_C \). We may extend \( \delta \) to an algebra, \( G_C \)-isomorphism of symmetric algebras \( \delta: S(\mathfrak{g}_C) \rightarrow S(\mathfrak{g}_C') \), \( G_C \) being the adjoint group of \( \mathfrak{g}_C \).

Let \( a \in (S^{n-1}(\mathfrak{g}_C'))^{G_C} \) be as at the end of §5 and let \( a = \delta^{-1}(a) \) so that \( a \in (S^{n-1}(\mathfrak{g}_C'))^{G_C} \). Finally \( \lambda: S(\mathfrak{g}_C) \rightarrow G \) is a \( G_C \)-linear isomorphism and we put \( \gamma = \lambda(\alpha) \) so that \( \gamma \in \text{Cent } G \subset G^K \). Now \( \gamma_0 = P(\gamma) \in B \subset K^M \otimes A \) (cf. §4) and hence \( P \) induces an antihomomorphism \( P_\gamma: G^K_\gamma \rightarrow B_{\gamma_0} \). Note that \( \gamma_0 \in \text{Center } K^M \otimes A \) since one easily has \( \gamma_0 \in M \otimes A \) where \( M \) is the enveloping algebra of the Lie algebra of \( M \). Clearly \( P \) is compatible with valuations (see Corollary 7.3). Therefore \( P_\gamma \) extends to a map \( P_\Gamma \) of the respective completions \( G^K_\Gamma \) and \( B_{\Gamma_0} \). We have

**Theorem 7.4.** The map \( P_\Gamma: G^K_\Gamma \rightarrow B_{\Gamma_0} \) is a surjective anti-isomorphism.
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Proof. To prove the theorem it is sufficient to show that given \( u_0 \in B_{\gamma_0} \) there exists \( u \in \mathcal{O}_{\gamma}^K \) such that

(a) \( \nu_0(u_0) = \nu(u) \), and
(b) \( \nu_0(u_0 - P_{\gamma}(u)) < \nu_0(u_0) \).

In fact it suffices only to prove (b) since (b) \( \Rightarrow \) (a). This is clear since, by writing \( P_{\gamma}(u) = u_0 - (u_0 - P_{\gamma}(u)) \) and \( u_0 = (u_0 - P_{\gamma}(u)) + P_{\gamma}(u) \), (2) implies that \( \nu_0(u_0) = \nu_0(P_{\gamma}(u)) \). But \( \nu_0(P_{\gamma}(u)) = \nu(u) \).

Note that we may assume that \( u_0 \in B \). Indeed assume the theorem is true in this case. Write \( u_0 = f_0/\gamma_0^t \) where \( f_0 \in B \). But then there exists \( f \in G_{\gamma}^K \) such that \( \nu_0(f_0 - P_{\gamma}(f)) < \nu_0(f_0) \). Hence

\[
\nu_0(u_0 - P_{\gamma}(f/\gamma_0^t)) = \nu_0(f_0 - P_{\gamma}(f)) - \nu_0(\gamma_0^t) < \nu_0(f_0) - \nu_0(\gamma_0^t) = \nu_0(u_0).
\]

Thus we assume \( u_0 = f_0 \in B \).

Let \( G_{(n)} \) be the usual filtration of \( G \) and \( B_{(n)} \) be the usual filtration for \( B \). (See beginning of this section.) Now let \( \sigma^n: G_{(n)} \to S^n(\mathfrak{g}_C') \) be the linear map defined by composing \( \lambda^{-1}: G_{(n)} \to S_{(n)}(\mathfrak{g}_C) = \Sigma_{j=0}^n S^j(\mathfrak{g}_C) \) with \( \delta: S_{(n)}(\mathfrak{g}_C) \to S_{(n)}(\mathfrak{g}_C') \) and then with the projection \( S_{(n)}(\mathfrak{g}_C') \to S^n(\mathfrak{g}_C') \). It is clear that \( \sigma^n \) is a \( G_C \)-linear map. Let \( \sigma_0^n: (K \otimes A)_{(n)} \to S^n((t_C \oplus a_C)') \) be defined similarly so that \( \sigma_0^n \) is a \( K_C \)-linear map. It then follows easily from the definition of the map \( F: G \to K \otimes A \) and the Birkhoff-Witt theorem that one has a commutative diagram

\[
\begin{array}{ccc}
G_{(n)} & \xrightarrow{\sigma^n} & S^n(\mathfrak{g}_C') \\
| F \downarrow & & \downarrow \pi \\
(K \otimes A)_{(n)} & \xrightarrow{\sigma_0^n} & S^n((t_C \oplus a_C)')
\end{array}
\]

where, recall, \( \pi: S(\mathfrak{g}_C') \to S((t_C \oplus a_C)') \) is the restriction map \( \varphi \to \varphi(t_C \oplus a_C) \)

for \( \varphi \in S(\mathfrak{g}_C') \).

Now let \( \rho \) be the usual valuation on \( G \). Thus if \( u \in G \) then \( \rho(u) = -\infty \) if \( u = 0 \), otherwise \( \rho(u) \) is the least \( n \in \mathbb{Z}_+ \) such that \( u \in G_{(n)} \). Note that if \( u \in G_{(n)} \) then \( \lambda^n(u) \neq 0 \) if and only if \( \rho(u) = n \). One defines the valuation \( \rho_0 \) on \( K \otimes A \) similarly. Now since \( \pi \) is injective on \( S(\mathfrak{g}_C')^{KC} \) it then follows from (7.4) that, for any \( u \in G^K \), \( \rho(u) = \rho_0(Fu) \).

The proof of Theorem 7.4 will follow easily from

**Lemma 7.5.** For any \( 0 \neq f_0 \in B \) there exists \( j \in \mathbb{Z}_+ \) and \( w \in G^K \) such that \( \nu_0(f_0 \gamma_0^t - P(w)) < \nu_0(f_0 \gamma_0^t) \). (Note that this implies \( \nu_0(f_0 \gamma_0^t) = \nu_0(P(w)) \) and since (Corollary 7.3) \( \nu_0(P(w)) = \nu(w) \) this also implies \( \nu_0(f_0 \gamma_0^t) = \nu(w) \).

**Proof of Lemma 7.5.** For any \( 0 \neq x \in K \otimes A \) let \( \tilde{x} \in K \otimes A \) be the unique element defined so that if \( \nu_0(x) = d \) then \( \tilde{x} \in K \otimes A_d \) and \( \nu_0(x - \tilde{x}) < d \).
Clearly \( \widetilde{x} \neq 0 \) and \( \rho_0(\widetilde{x}) \gg \nu_0(\widetilde{x}) = \nu_0(x) = d \). Put \( \beta(x) = \rho_0(\widetilde{x}) - \nu_0(\widetilde{x}) \).

We will prove the lemma by induction on \( \beta(f_0) \). Let \( \tau: S(\mathfrak{g}_C') \to G \) be the \( G_C \)-linear map defined by putting \( \tau = \lambda \circ \delta^{-1} \). Thus \( \tau(S(\mathfrak{n}(\mathfrak{g}_C'))) = G(n) \) and \( \sigma^\tau \circ \tau \) is the identity on \( S^n(\mathfrak{g}_C') \).

Now assume \( \beta(f_0) = 0 \). Thus \( \tilde{f}_0 \in A_d \) where \( d = \nu_0(f_0) \). But then, by Theorem 4.5, \( \tilde{f}_0 \) and hence \( \sigma_0^d(\tilde{f}_0) \in S^d(\mathfrak{a}_c') \) is Weyl group invariant. Thus there exists (see e.g. [16, Theorem 6.10]) \( \xi \in S^d(\mathfrak{g}_c')^K \subset S^d(\mathfrak{g}_c')^G \) such that \( \pi(\xi) = \sigma_0^d(\tilde{f}_0) \). But then if \( w = \tau \xi \) one has \( w \in G^K \). But by (7.4) one has \( \sigma_0^d(F(w)) = \sigma_0^d(\tilde{f}_0) \). Thus \( \rho_0(\tilde{f}_0 - F(w)) < d \). But \( \nu_0(\tilde{f}_0 - F(w)) < \rho_0(f_0) = d \). Thus \( \nu_0(\tilde{f}_0 - F(w)) < d \). But then \( \nu_0(F(w) - F(w')) < d \) by Proposition 7.2. Thus \( \nu_0(f_0 - F(w)) < d \) proving the lemma for \( \beta(f_0) = 0 \).

Now assume \( \beta(f_0) > 0 \) and assume the lemma is true for smaller values. Again let \( d = \nu_0(\tilde{f}_0) = \nu_0(f_0) \).

Now put \( m = \rho_0(\tilde{f}_0) \) so that \( m - d = \beta(f_0) \). But by Theorem 4.5 \( 0 \neq \sigma^m_0(\tilde{f}_0) \in S^m((\mathfrak{t}_c \oplus \mathfrak{a}_c')) \) is Weyl group invariant. But then by Theorem 6.1 there exist \( i \in \mathbb{Z}_+ \) and \( \psi \in S^r(\mathfrak{g}_c')^K \) where \( r = m + i(n - 1) \) such that \( \psi \|_{\mathfrak{t}_c \oplus \mathfrak{a}_c} = \sigma^m_0(\tilde{f}_0)b_0^i \). Furthermore since \( \sigma^m_0(\tilde{f}_0)b_0^i \in S(\mathfrak{t}_c') \otimes S^p(\mathfrak{a}_c') \) where \( p = d + i(n - 1) \) it follows from the injectivity of \( \pi|S(\mathfrak{g}_c')^K \) that \( \psi \in S(\mathfrak{t}_c') \otimes S^p(\mathfrak{a}_c') \). It follows therefore, if we put \( u = \tau(\psi) \in G^K \), that \( \nu(u) \leq p \). On the other hand by (7.4) one has

\[
\sigma_0^d(F(u)) = \sigma_0^m(\tilde{f}_0)b_0^i \neq 0.
\]

But since \( 0 \neq \sigma_0^d(F(u)) \in S(\mathfrak{t}_c') \otimes S^p(\mathfrak{a}_c') \) it follows that \( \nu_0(F(u)) \geq p \). Thus

\[
\nu_0(F(u)) = \nu(u) = p
\]

since \( \nu(u) = \nu_0(F(u)) \).

Now by definition \( \gamma_0 = P(\gamma) \) and \( \gamma = \tau(a) \) where \( a \in S^{n-1}(\mathfrak{g}_c') \) is defined as in \$5. \) Obviously, then \( \rho(\gamma) = n - l \) so that \( \rho_0(\gamma) \leq n - l \).

Now Proposition 7.2 clearly implies that for any \( v \in G^K \) one has

\[
\widetilde{P(v)} = \widetilde{F(v)}.
\]

Now we assert that \( \rho_0(\widetilde{\gamma}_0) = n - l \) and in fact

\[
\sigma_0^{n-l}(\widetilde{\gamma}_0) = b_0 \in S^{n-l}(\mathfrak{a}_c').
\]

Indeed by definition \( a_\alpha'c = b_0 \). But then if \( a_0 = \pi(a) \) one has that \( a_0 = b_0 + a_1 \) where \( a_1 \in S(\mathfrak{t}_c') \otimes S_{n-l-1}(\mathfrak{a}_c') \). But by (7.4) \( \sigma_0^{n-l}(F(\gamma)) = a_0 \) and hence \( \sigma_0^{n-l}(\tilde{F}(\gamma)) = b_0 \). Then by (7.7) \( \sigma_0^{n-l}(\widetilde{\gamma}_0) = b_0 \) establishing (7.8) and hence also that \( \rho_0(\widetilde{\gamma}_0) = n - l \). This implies \( \rho_0(\gamma_0) = n - l \) since \( n - l = \rho(\gamma) \geq \)
\[ \rho_0(\gamma_0) \geq \rho_0(\tilde{\gamma}_0) = n - l. \] Note that (7.8) also implies that \( \nu_0(\gamma_0) = \nu_0(\tilde{\gamma}_0) = n - l. \)

Now for any \( w, v \in K \otimes A \) note that \( \overline{wv} = \overline{vw} \). Hence \( f_0\gamma_0 = \tilde{f}_0\gamma_0 \). Thus since \( \nu_0(x) = \nu_0(\hat{x}) \) for \( 0 \neq x \in K \otimes A \) this implies that

\[ (7.9) \quad \nu_0(f_0\gamma_0) = p. \]

On the other hand by (7.6) and Corollary 7.3, \( \nu_0(Pu) = p \). If \( \nu_0(f_0\gamma_0 - Pu) < p \) we are done. Assume therefore, that \( \nu_0(f_0\gamma_0 - Pu) = p \). Thus \( f_0\gamma_0 \) and \( Pu \) are distinct elements of \( K \otimes A_p \) and hence if \( x = f_0\gamma_0 - Pu \) one has \( x \in B \) and \( \tilde{x} = f_0\gamma_0 - P(u) \in K \otimes A_p \). But \( \rho_0(f_0\gamma_0) = \rho_0(\tilde{f}_0) \rho_0(\gamma_0) = r. \) However by (7.5) \( r = \rho_0(u) \geq \rho_0(P(u)) \). Thus \( r \geq \rho_0(\tilde{x}) \). On the other hand one has \( \sigma_0(f_0\gamma_0) = \sigma_0(\tilde{f}_0\gamma_0) \). But then

\[ (7.10) \quad \sigma_0(f_0\gamma_0) = \sigma_0(\tilde{f}_0\gamma_0) \]

by (7.8) since if \( y \in (K \otimes A_{(y)}) \) and \( z \in (K \otimes A_{(z)}) \) then

\[ \sigma_0^+(yz) = \sigma_0^+(yz) \sigma_0^+(yz). \]

Now \( \sigma_0(F(u)) = \sigma_0^+(F(\hat{u})) \) by (7.5). We assert that \( \sigma_0(F(u)) = \sigma_0(F(\hat{u})). \) Indeed \( \sigma_0(F(\hat{u})) \in S(t^*_C) \otimes S^p(q^*_C) \) and \( \sigma_0(F(\hat{u})) \in S(t^*_C) \otimes S^p(\alpha^*_C) \) since \( \nu_0(F(\hat{u})) = p \) by (7.6). However one necessarily has \( \sigma_0(F(u)) - F(\hat{u}) \in S(t^*_C) \otimes S(q^*_C) \) since \( \nu_0(F(\hat{u})) < p \). Thus \( \sigma_0(F(\hat{u})) = \sigma_0(F(u)) = \sigma_0(f_0\gamma_0) \). But \( \sigma_0(P(u)) = \sigma_0(F(\hat{u})) \) by (7.7). Thus recalling (7.10), one has \( \sigma_0(\tilde{x}) = 0 \) so that \( \rho_0(\tilde{x}) < r. \) But then \( \beta(x) = \rho_0(\tilde{x}) - p < r - p = m - d. \) The induction assumption then applies to \( x \) so that for some \( k \in Z_+ \) there exists \( v \in G^k \) such that \( \nu_0(\gamma_0^k - Pu) = \nu_0(x\gamma_0^k) \). But \( x\gamma_0^k = (f_0\gamma_0 - Pu) \gamma_0^k = f_0\gamma_0^k - Pu \) where \( j = k + f. \) Then \( \nu_0(f_0\gamma_0^k - Pu) = \nu_0(x\gamma_0) = p + k(n - l) = d + j(n - l) = \nu_0(f_0\gamma_0). \) Q.E.D.

To finish the proof of the theorem let \( 0 \neq f_0 \in B \) and let \( f \in Z_+ \) and \( w \in G^k \) be given by Lemma 7.5. Now put \( f = w/\gamma^l. \) Then

\[ \nu_0(f_0 - P(f)) = \nu_0(f_0\gamma_0^l - P(f)\gamma_0^l) - j(n - l) \]

\[ = \nu_0(f_0\gamma_0^l - P(w)) - j(n - l) < \nu_0(f_0\gamma_0^l) - j(n - l) = \nu_0(f_0). \] Q.E.D.

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139