INFINITE-DIMENSIONAL WHITEHEAD AND VIETORIS THEOREMS IN SHAPE AND PRO-HOMOTOPY

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ABSTRACT. In Theorem 3.3 and Remark 3.4 conditions are given under which an infinite-dimensional Whitehead theorem holds in pro-homotopy. Applications to shape theory are given in Theorems 1.1, 1.2, 4.1 and 4.2.

1. Introduction. Whitehead theorems in the shape theory of finite-dimensional spaces have been proved by Moszyńska [26] and by Mardešić [22], while in [7] we proved a Whitehead theorem in pro-homotopy theory for inverse systems of complexes whose dimensions are bounded. On first sight, the prospects for removing the hypotheses of finite dimension looked bleak, because of the counterexamples in [13], [11], [1, p. 35], [5] and [4]. However, by restricting ourselves just enough to avoid these counterexamples we have been able to prove reasonable theorems. We were led to them by reading the papers of Borsuk [31] and of Kozlowski and Segal [17]. For compact metric spaces (compacta) their theorem reads: a movable compactum whose shape groups are trivial is shape equivalent to a point. Our generalizations of this are Theorems 4.1 and 4.2 below. Confining ourselves in this introduction to the compact metric case, our theorem becomes:

Theorem 1.1. Let \( \varphi: X \to Y \) be a pointed shape morphism between pointed connected compacta, which induces isomorphisms on the (inverse limit) shape groups. If \( X \) is movable and \( Y \) is an FANR (in the pointed sense) then \( \varphi \) is a pointed shape equivalence.

This is proved by combining Theorem 4.2, below, with Theorem 5.1 of our paper [7].

Theorem 1.1 has a geometrical consequence of some interest. A map \( f: X \to Y \) between compacta is called a CE map (or cell-like map or Vietoris map) if for each \( y \in Y \), \( f^{-1}(y) \) is shape equivalent to a point. If \( X \) and \( Y \) are ANR's then \( f \) must be a homotopy equivalence (see [12], [15] and the references...
If $X$ and $Y$ are finite-dimensional compacta, $f$ must be a shape equivalence (see [28] and [15]): Anderson (unpublished) was able to remove the requirement that $X$ be finite dimensional. But if one also removes the requirement that $Y$ be finite dimensional, counterexamples exist: Taylor [29] constructed a CE map from a nonmovable compactum onto the Hilbert Cube, while Keesling [14] modified this example to get a CE map from the Hilbert Cube onto a nonmovable compactum: clearly these cannot be shape equivalences. Kozlowski and Segal have gone further, by constructing [32] a CE map between movable compacta of different shapes. The theorems in this paper imply the following "infinite-dimensional Vietoris theorem" (which is proved by combining Theorem 1.1, above, with Theorem 2.3 of K. Kuperberg's paper [18]).

**Theorem 1.2.** Let $f: (X, x) \to (Y, y)$ be a CE map between pointed connected compacta. If $(X, x)$ is movable and $(Y, y)$ is an FANR (in the pointed sense), then $f$ is a pointed shape equivalence.

Our principal tool is a Whitehead theorem in pro-homotopy, Theorem 3.3. Roughly, it says that a weak equivalence in pro-homotopy from an inverse system $\{X_\alpha\}$ of finite-dimensional complexes to a finite-dimensional complex $Y$ is an equivalence provided $\{X_\alpha\}$ is movable. The point is that the dimensions of the complexes $X_\alpha$ need not be bounded.

In Remarks 3.4 and 4.4 we indicate how the hypotheses on $X$ and $Y$ of Theorem 1.1 can be replaced by the hypothesis that $\varphi$ be a "movable morphism."

**Note added May 1, 1975.** J. Dydak [39] has extended our shape theoretic results. It is not clear whether his methods can be adapted to improve our pro-homotopy results.

**2. Notation, terminology and a lemma.** We follow the notational conventions set out in §§2 and 3 of [7]. The principal items are listed below. Shape terminology is introduced in §4.

If $C$ is a category, pro-$C$ denotes a certain category of inverse systems in $C$ indexed by directed sets: for a description of the morphisms and other properties of pro-$C$ see [5] or [22]; for the original more general version see the Appendix of [1]. $C_{\text{maps}}$ denotes the category whose objects are the morphisms of $C$ and whose morphisms from an object $f$ to an object $g$ are the commutative square diagrams

\[
\begin{array}{ccc}
\, & \, & \, \\
\downarrow & f & \downarrow \\
\, & \, & \, \\
\downarrow & g & \downarrow \\
\, & \, & \,
\end{array}
\]

in $C$. There is an obvious functor pro-$\left(C_{\text{maps}}\right) \to \text{pro-}\left(C_{\text{maps}}\right)$ and we say that the object $\{X_\alpha \to Y_\alpha\}$ of pro-$\left(C_{\text{maps}}\right)$ "induces" its image $\{X_\alpha\} \to \{Y_\alpha\}$ under this functor: see §3 of [7].
We suppress bonding morphisms and the indexing directed set, denoting an object of pro-$C$ by $\{X_\alpha\}$. $\{X_\alpha\}$ is movable if for each $\alpha$ there exists $\beta \geq \alpha$ such that for all $\gamma \geq \beta$ the bond $p_{\alpha \beta}: X_\beta \to X_\alpha$ factors as $p_{\alpha \beta} = p_{\alpha \gamma} \circ r_{\beta \gamma}$ where $r_{\beta \gamma}: X_\beta \to X_\gamma$ is a morphism of $C$. $\{X_\alpha\}$ is uniformly movable if for each $\alpha$ there exists $\beta \geq \alpha$ such that the bond $p_{\alpha \beta}$ factors as $p_{\alpha \beta} = p_\alpha \circ r_\beta$ where $r_\beta: X_\beta \to \{X_\alpha\}$ is a morphism of pro-$C$ and $p_\alpha: \{X_\alpha\} \to X_\alpha$ is the projection morphism of pro-$C$. ($C$ is, of course, embedded as a full subcategory of pro-$C$.)

A directed set is closure finite if each element has only finitely many predecessors.

Categories used include: Groups (groups and homorphisms); $T_0$ (pointed connected topological spaces and pointed continuous functions); $HT_0$ (the homotopy category corresponding to $T_0$); $CW_0$ (pointed connected CW complexes and pointed continuous functions); $H_0$ (the homotopy category corresponding to $CW_0$); $HT_0$-pairs (pointed pairs of connected spaces and pointed homotopy classes of maps); $H_0$,pairs (pointed pairs of connected CW complexes and pointed homotopy classes of maps).

We call an object of pro-Groups a pro-group. We always suppress base points of spaces.

The definition of uniform movability becomes simpler in the case of pro-groups. A pro-group $G = \{G_\alpha\}$ is (clearly) uniformly movable if and only if for each $\alpha$ there exists $\beta \geq \alpha$ such that the bond $p_{\alpha \beta}: G_\beta \to G_\alpha$ factors as $p_{\alpha \beta} = p_\alpha \circ r_\beta$ where $r_\beta: G_\beta \to \lim G$ is a homomorphism and $p_\alpha: \lim G \to G_\alpha$ is projection on the $\alpha$ factor.

**Lemma 2.1.** Let $G = \{G_\alpha\}$ be a uniformly movable pro-group. Let $H$ be a group, let $f: G \to H$ be a morphism of pro-Groups and let $p: \lim G \to G$ be the projection morphism. If $f \equiv f \circ p$ is an isomorphism (of groups) then $f$ is an isomorphism (of pro-groups).

**Proof.** The required inverse to $f$ is $p \circ \widetilde{f}^{-1}$. It is trivial that $f \circ (p \circ \widetilde{f}^{-1})$ is the identity of $G$. To show that $(p \circ \widetilde{f}^{-1}) \circ f$ is the identity of $\{G_\alpha\}$ it is enough to show that given $\alpha$ there exists $\gamma \geq \alpha$ such that $p_\alpha \circ \widetilde{f}^{-1} \circ f_\beta \circ p_{\beta \gamma} = p_{\alpha \gamma}$. Since $G$ is uniformly movable the above remark implies that there exist $\beta \geq \alpha$ and $r_\beta: G_\beta \to \lim G$ such that $p_{\alpha \beta} = p_\alpha \circ r_\beta$. Let $f$ be represented by homomorphisms $f_\alpha: G_\alpha \to H$. Choose $\gamma \geq \beta$ such that $f_\beta \circ p_{\beta \gamma} = f_\alpha \circ p_{\alpha \beta} \circ p_{\beta \gamma}$. Then

$$f_\beta \circ p_{\beta \gamma} = f_\alpha \circ p_\alpha \circ r_\beta \circ p_{\beta \gamma} = \widetilde{f} \circ r_\beta \circ p_{\beta \gamma}.$$  

So

$$p_\alpha \circ \widetilde{f}^{-1} \circ f_\beta \circ p_{\beta \gamma} = p_\alpha \circ r_\beta \circ p_{\beta \gamma} = p_{\alpha \beta} \circ p_{\beta \gamma} = p_{\alpha \gamma}. \qed$$
3. A Whitehead theorem in pro-homotopy. The principal result here is Theorem 3.3. Lemma 3.1 and Proposition 3.2 are the modifications needed to obtain an infinite-dimensional Whitehead theorem from [22] and [7].

If \( f: X \to Y \) is a morphism of \( T_0 \), \( M(f) \) denotes the reduced mapping cylinder of \( f \), and \( X \) is regarded as a subset of \( M(f) \) in the usual manner. Thus \((M(f), X)\) is an object of \( T_{0,\text{pairs}} \). If \( f = \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\} \) is an object of \( (T_{0,\text{maps}}) \) then \( M(f) = \{(M(f_\alpha), X_\alpha)\} \) is a well-defined object of \( \text{pro-}(T_{0,\text{pairs}}) \) and so induces an object of \( \text{pro-}(HT_{0,\text{pairs}}) \); see §3 of [7].

**Lemma 3.1.** Let \( f = \{X_\alpha \xrightarrow{f_\alpha} Y_\alpha\} \) be an object of \( \text{pro-}(CW_{0,\text{maps}}) \) whose domain \( \{X_\alpha\} \) is movable in \( \text{pro-}H_0 \) and whose range \( \{Y_\alpha\} \) is such that every \( Y_\alpha \) is the same (pointed) complex \( Y \), and every bond \( Y_\beta \to Y_\alpha \) is the identity map. Then \( \{(M(f_\alpha), X_\alpha)\} \) is movable in \( \text{pro-}(HT_{0,\text{pairs}}) \).

**Proof.** Let \( p_{\alpha \gamma}: X_\beta \to X_\alpha \) denote the appropriate bond of \( \{X_\alpha\} \). Given \( \alpha \), there exists \( \beta \geq \alpha \) such that for all \( \gamma \geq \beta \) there is a pointed map \( r^{\beta \gamma}: X_\beta \to X_\gamma \) with the property that \( p_{\alpha \beta} \) is pointedly homotopic to \( p_{\alpha \gamma} \circ r^{\beta \gamma} \). By Theorem 2.8.9 of [37], \( p_{\alpha \gamma} \) may be replaced by a fibration: to be precise, there exist a pointed fibration \( p'_{\alpha \gamma}: X_\gamma' \to X_\alpha \), and an inclusion \( i: X_\gamma \to X_\gamma' \) as a pointed strong deformation retract, such that \( p'_{\alpha \gamma} \circ i = p_{\alpha \gamma} \). Since \( p'_{\alpha \gamma} \) is a fibration, there is a pointed map \( s^{\beta \gamma}: X_\beta \to X_\gamma' \) such that \( p_{\alpha \gamma} \circ s^{\beta \gamma} = p_{\alpha \beta} \). Letting \( f'_\gamma = f_\alpha \circ p'_{\alpha \gamma} \) we have a commutative diagram in \( CW_0 \):

\[
\begin{array}{ccc}
X_\alpha & \xrightarrow{f_\alpha} & Y \\
\uparrow{p_{\alpha \gamma}} & & \downarrow{f'_{\gamma}} \\
X_\gamma & \xrightarrow{f'_\gamma} & X_\gamma' \end{array}
\]

From it, we obtain a commutative diagram in \( T_{0,\text{pairs}} \):

\[
\begin{array}{ccc}
(M(f_\alpha), X_\alpha) & \xrightarrow{q_{\alpha \beta}} & (M(f_\beta), X_\beta) \\
\downarrow{q_{\alpha \gamma}} & & \downarrow{t^{\beta \gamma}} \\
(M(f_\gamma), X_\gamma) & \xrightarrow{j} & (M(f'_\gamma), X'_\gamma) \end{array}
\]
$j$ induces pointed homotopy equivalences $M(f_\gamma) \to M(f'_\gamma)$ and $X_\gamma \to X'_\gamma$; by Lemma 1 of [34], we could deduce that $j$ induces an isomorphism in $HT_{0,\text{pairs}}$ if we knew that $(M(f_\gamma), X_\gamma)$ and $(M(f'_\gamma), X'_\gamma)$ were isomorphic in $HT_{0,\text{pairs}}$ to pointed CW pairs. It is not hard to show that $(M(f_\gamma), X_\gamma)$ has this property ($f_\gamma$ is homotopic to a cellular map; use Lemma 3.9 of [7]). But it is not clear that the same is true of $(M(f'_\gamma), X'_\gamma)$. To avoid the problem, we apply the composite functor “geometric realization of the singular complex” [33], $| \cdot | \circ S : T_0 \to CW_0$, to the diagram $(\ast)$, and thus obtain the following commutative diagram in $CW_{0,\text{pairs}}$, analogous to $(\ast)$:

\[
\begin{array}{cccc}
(M(|Sf_\alpha|), |SX_\alpha|) & \xrightarrow{\bar{q}_{\alpha\beta}} & (M(|Sf_\beta|), |SX_\beta|) \\
\downarrow \bar{q}_{\alpha\gamma} & & \downarrow \bar{t}_{\beta\gamma} \\
(M(|Sf'_\gamma|), |SX'_\gamma|) & \xrightarrow{\bar{f}} & (M(|Sf'_\beta|), |SX'_\beta|)
\end{array}
\]

where the maps are obtained from those of $(\ast)$ in the obvious way. Now, Lemma 1 of [34] implies that $\bar{f}$ induces an isomorphism in $HT_{0,\text{pairs}}$. It follows that $\bar{q}_{\alpha\beta}$ can be lifted in $HT_{0,\text{pairs}}$ through $\bar{q}_{\alpha\gamma}$, so that $\{(M(|Sf_\alpha|), |SX_\alpha|)\}$ is movable in $HT_{0,\text{pairs}}$ (where the bonds are induced by the maps $\bar{q}_{\alpha\beta}$). The argument is completed by observing that $\{(M(|Sf_\alpha|), |SX_\alpha|)\}$ is isomorphic in pro-$HT_{0,\text{pairs}}$ to $\{(M(f_\alpha), X_\alpha)\}$. To see this, observe that there is a commutative diagram in $T_{0,\text{pairs}}$

\[
\begin{array}{cccc}
(M(f_\alpha), X_\alpha) & \xrightarrow{q_{\alpha\beta}} & (M(|Sf_\alpha|), |SX_\alpha|) \\
\downarrow q_{\alpha\beta} & & \downarrow \bar{q}_{\alpha\beta} \\
(M(f_\beta), X_\beta) & \xrightarrow{\bar{q}_{\alpha\beta}} & (M(|Sf_\beta|), |SX_\beta|)
\end{array}
\]

whose horizontal morphisms are induced by the canonical maps $|SX_\alpha| \to X_\alpha$, $|SY| \to Y$, etc. As explained above $(M(f_\alpha), X_\alpha)$ is isomorphic in $HT_{0,\text{pairs}}$ to a pointed CW pair; and $(M(|Sf_\alpha|), |SX_\alpha|)$ is itself a pointed CW pair. So, by Lemma 1 of [34], the horizontal morphisms are isomorphisms in $HT_{0,\text{pairs}}$. Since $\{(M(f_\alpha))\}$ is isomorphic to a movable object, it is itself movable. 

\[\square\]

**Proposition 3.2.** Let $\{(P_{\alpha}, P'_\alpha)\}$ be a movable object of pro-$HT_{0,\text{pairs}}$ indexed by a closure finite directed set. Assume that each $P_{\alpha}$ is a finite-dimensional simplicial complex and that $P'_\alpha$ is a subcomplex of $P_{\alpha}$. If $\{\pi_k(P_{\alpha}, P'_\alpha)\}$ is trivial for all $k$, then the “inclusion” $\{P'_\alpha\} \to \{P_{\alpha}\}$ is an isomorphism in pro-$H_0$. 

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Proof. The proof is almost identical to that of Theorem 2 of [22]. Movability makes unnecessary the hypothesis in [22] that the dimensions of the complexes $P_{\alpha}$ be bounded. For each $\alpha$ choose $\beta(\alpha) \geq \alpha$ such that for every $\gamma \geq \beta(\alpha)$ there exists a morphism of $H_0$-pairs, $s^{\beta \gamma}: (P_{\beta}, P_{\beta}) \to (P_{\gamma}, P_{\gamma})$, such that $q_{\alpha \gamma} \circ s^{\beta \gamma} = q_{\alpha \beta}$ where $q_{\lambda \mu}: (P_{\lambda}, P_{\mu}) \to (P_\mu, P_\mu)$ denotes the appropriate bonding morphism. Following 2.3 of [22], assume $\beta(\alpha) \leq \beta(\tilde{\alpha})$ whenever $\alpha \leq \tilde{\alpha}$.

Claim 1. For each $\alpha$, each pointed pair of finite-dimensional simplicial complexes $(K, K')$ and each pointed map $\varphi: (K, K') \to (P_{\beta(\alpha)}, P'_{\beta(\alpha)})$ there exists a pointed map $\psi: K \to P'_{\beta(\alpha)}$ such that (inclusion) $\circ \psi$ is pointedly homotopic to (bond) $\circ \varphi$ in $P_{\alpha}$ and $\psi|K'$ is pointedly homotopic to (bond) $\circ \varphi|K'$ in $P'_{\alpha}$.

Proof of Claim 1. By Lemma 1 (§6.2) of [22] with $n + 1 = \dim K$ and $\alpha \ast \geq \beta(\alpha)$; movability implies that $\varphi$ can be lifted to $(P_{\alpha \ast}, P'_{\alpha \ast})$, hence $\psi$ exists.

Claim 2. Given $\alpha$ and a pointed finite-dimensional complex $L$, let $\varphi_0, \varphi_1: L \to P'_{\beta(\alpha)}$ be pointed maps such that (inclusion) $\circ \varphi_0$ and (inclusion) $\circ \varphi_1$ are pointedly homotopic in $P_{\beta(\alpha)}$. Then (bond) $\circ \varphi_0$ and (bond) $\circ \varphi_1$ are pointedly homotopic in $P'_{\alpha}$.

Proof of Claim 2. By Lemma 2 (§6.3) of [22] with $n = \dim L$: movability implies that $\varphi_0$ and $\varphi_1$ can be lifted to $P'_{\alpha \ast}$, and the claim follows.

The remainder of the proof is similar to the corresponding proof in §6.4 of [22]. Claims 1 and 2 are used in place of Lemmas 1 and 2 of [22].

Theorem 3.3. Let $Y$ be a pointed complex, $\{X_{\alpha}\}$ an object of pro-CW$_0$ and $g: X \to Y$ a morphism of pro-H$_0$. Assume $Y$ and each $X_{\alpha}$ are finite dimensional, and that the object of pro-H$_0$ induced by $\{X_{\alpha}\}$ is movable. If $g \# : \{\pi_k(X_{\alpha})\} \to \pi_k(Y)$ is an isomorphism (in the category pro-Groups) for every $k$, then $g$ induces an isomorphism of pro-H$_0$.

Proof. The proof is similar to that of Theorem 3.1 of [7]. Since $Y$ is a complex, we may represent $g$ by a morphism of pro-CW$_0$ and hence replace it by an object $f \equiv \{X'_\gamma \xrightarrow{f_\gamma} Y'_\gamma\}$ of pro-(CW$_0$,maps) indexed by a closure finite directed set such that: $\{X'_\gamma\}$ is movable, each $X'_\gamma$ is finite dimensional, each $Y'_\gamma$ is $Y$, and each bond of $\{Y'_\gamma\}$ is the identity map; see §3 of [7]. $f_\#: \{\pi_k(X'_\gamma)\} \to \{\pi_k(Y'_\gamma)\}$ is an isomorphism of pro-groups for each $k$. By Lemma 3.8 of [7], $\{\pi_k(M(f_\gamma), X'_\gamma)\}$ is trivial, where $\{M(f_\gamma)\}$ is the reduced mapping cylinder object of pro-CW$_0$ corresponding to $f$ (see §3 of [7]). Each $M(f_\gamma)$ is finite dimensional.

By Lemma 3.1, above, $\{(M(f_\gamma), X'_\gamma)\}$ is movable in pro-(HT$_0$,pairs). The rest of the proof is as in [7], except that Proposition 3.12 of [7] is replaced by the above Proposition 3.2.

Remark 3.4. There is a variation on Theorem 3.3. Following [9], define $\tilde{H}$-CW$_0$,maps to be the category whose objects are those of CW$_0$,maps and whose
morphisms are homotopy classes of morphisms of $CW_{0, maps}$, where two morphisms $(a_1, a_2)$ and $(b_1, b_2)$ from $f: X \rightarrow Y$ to $f': X' \rightarrow Y'$ are defined to be **homotopic** if there is a morphism $(\theta_1, \theta_2)$ from $f \times 1: X \times I \rightarrow Y \times I$ to $f': X' \rightarrow Y'$ such that $\theta_i$ is a homotopy between $a_i$ and $b_i$, $i = 1, 2$. Call an object $(X_\gamma, f_\gamma: Y_\gamma)$ of pro-$CW_{0, maps}$ $H$-movable if it induces a movable object of pro-$H_{CW_{0, maps}}$. Call a morphism $g: \{X_\alpha\} \rightarrow \{Y_\beta\}$ of pro-$CW_0$ movable if $g$ is isomorphic in (pro-$H_0$) maps to the object of (pro-$H_0$) maps induced by such an $H$-movable $(f_\gamma)$. If each $X_\alpha$ and each $Y_\beta$ is finite dimensional, if $g$ is movable, and if $g$ induces isomorphisms of homotopy pro-groups, then $g$ induces an isomorphism in pro-$H_0$. The proof is similar to that of Theorem 3.3. The hypotheses make it possible to by-pass Lemma 3.1: clearly $\{(M(f_\gamma), X_\gamma)\}$ is movable in pro-$HT_{0, pairs}$.

4. Whitehead theorems in shape. All spaces mentioned will be paracompact Hausdorff, so our shape theory may be understood either in the sense of [21] or [27], since these two theories agree on such spaces [19], [25]. For compact Hausdorff spaces these theories agree with that of [23], and for compact metric spaces they agree with that of [3] (see [24]).

We refer the reader to [25] or to §3 of [22] for an account of how the shape theory of spaces is fully and faithfully reflected in pro-homotopy theory. In particular, if $X$ and $Y$ are pointed connected spaces, there is a functorial bijection between the (pointed) shape morphisms from $X$ to $Y$ and the morphisms of pro-$H_0$ from $\{X_\alpha\}$ to $\{Y_\beta\}$, where $\{X_\alpha\}$ and $\{Y_\beta\}$ are objects of pro-$H_0$ (unique up to isomorphism) which are “associated” with $X$ and $Y$ respectively. A shape morphism $\varphi: X \rightarrow Y$ is a weak shape equivalence if the corresponding $f: \{X_\alpha\} \rightarrow \{Y_\beta\}$ induces isomorphisms $f_\#_\kappa: \{\pi_k(X_\alpha)\} \rightarrow \{\pi_k(Y_\beta)\}$ in pro-Groups for each $k \geqslant 1$. $\varphi$ is a very weak shape equivalence if $f_\#_\kappa: \varprojlim \{\pi_k(X_\alpha)\} \rightarrow \varprojlim \{\pi_k(Y_\beta)\}$ is an isomorphism in Groups for each $k \geqslant 1$. $X$ is movable [resp. uniformly movable] if $\{X_\alpha\}$ is movable [resp. uniformly movable] in pro-$H_0$.

Every object of pro-$CW_0$ gives rise to an object of pro-$H_0$, but (apart from the case of countably indexed systems) it is unknown whether every object of pro-$H_0$ “comes from” an object of pro-$CW_0$. The Vietoris functor [27] allows one to associate objects “coming from” pro-$CW_0$ with spaces, but the complexes involved are infinite dimensional. It is for these reasons that we confine ourselves to compact Hausdorff spaces in the theorems which follow.

**Theorem 4.1.** Let $X$ be a movable pointed connected compact Hausdorff space, let $Y$ be pointed shape equivalent to a pointed connected CW complex
and let \( \varphi: X \to Y \) be a pointed shape morphism. If \( \varphi \) is a weak shape equivalence, it is a pointed shape equivalence.

**Proof.** Assume \( Y \) is a CW complex. First assume \( Y \) is a finite-dimensional complex. As we shall see, no generality is lost by this.

Let \( \{X_\alpha\} \) be an object of pro-CW\(_0\) whose inverse limit is homeomorphic to \( X \). Then \( \{X_\alpha\} \) is associated with \( X \) in the sense of [25]. Let \( g: \{X_\alpha\} \to Y \) be a morphism of pro-H\(_0\) associated with \( \varphi \) in the sense of [25]. By Theorem 3.3, \( g \) induces an isomorphism in pro-H\(_0\). Hence, by [25], \( \varphi \) is a shape equivalence.

If \( Y \) is not finite dimensional we show that it must be (pointed) homotopy equivalent to a finite-dimensional complex. Since \( X \) is compact, \( g \) may be represented by a continuous map \( g_{\alpha_0}: X_{\alpha_0} \to Y \) for some \( \alpha_0 \), and hence \( g \) factors through a finite subcomplex \( K \) of \( Y \). So \( \tilde{g}: \{X_\alpha\} \to \tilde{Y} \) factors through \( \tilde{K} \) (where we have applied the pointed universal cover functor \( \tilde{\cdot} \)). Since \( g \) is a weak equivalence in pro-H\(_0\), so also is \( \tilde{g} \). Hence \( g \) and \( \tilde{g} \) are \( \tilde{\cdot} \)-isomorphisms [1, §4]; therefore, they induce isomorphisms on homology pro-groups and cohomology groups with every possible coefficient bundle (see 4.4 of [1]). Since \( K \) and \( \tilde{K} \) are finite dimensional, the homology of \( \tilde{Y} \) and the cohomology of \( Y \) vanish above the dimension of \( K \). By Theorem E of [30], \( Y \) is homotopy equivalent (hence pointed homotopy equivalent) to a finite-dimensional complex. \( \square \)

**Theorem 4.2.** Let \( X \) be a uniformly movable pointed connected compact Hausdorff space, let \( Y \) be pointed shape equivalent to a pointed CW complex, and let \( \varphi: X \to Y \) be a morphism in pointed shape. If \( \varphi \) is a very weak shape equivalence, it is a pointed shape equivalence. Furthermore, if \( X \) is metrizable it is only necessary to assume that \( X \) is movable.

**Proof.** By Lemma 2.1, \( \varphi \) is a weak shape equivalence, so the conclusion follows from Theorem 4.1. For metric compacta the concepts of “movable” and “uniformly movable” coincide, by [38] (see also Theorem 4.7 of [16] and Remark 6.7 of [35]) so the last statement is justified.

**Remark 4.3.** Various criteria are available for deciding if a given space \( Y \) is shape equivalent to a CW complex (as required in Theorems 4.1 and 4.2). See [10], [6], [7], [8].

**Remark 4.4.** Following Remark 3.4, one may define the notion of “movable shape morphism”: the special case of “movable map” is discussed in [9]. One may then prove that if \( \varphi: X \to Y \) is a movable pointed shape morphism between metric compacta and if \( \varphi \) is a very weak shape equivalence, then \( \varphi \) is a shape equivalence. A remark on p. 4 of [2] (incorrect as stated, but correct in the countable case) is used instead of Lemma 2.1 to show that \( \varphi \) is a weak shape equivalence.
equivalence. Then Remark 3.4 is used instead of Theorem 3.3 to complete the proof. Compare with [36].

**Remark 4.5.** If one interchanges the properties of $X$ and $Y$ in Theorems 4.1 and 4.2, making $Y$ movable (or uniformly movable) and $X$ shape equivalent to a complex, the resulting "theorems" are false. Counterexamples are given in [5]. However, if one also requires $X$ to be compact metric (or, equivalently, to be an FANR: see [6]) we do not know a counterexample. Added in proof: there is none; see [39].

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