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COMPLEX SPACE FORMS IMMERSED IN COMPLEX SPACE FORMS

BY

H. NAKAGAWA AND K. OGIUE(1)

ABSTRACT. We determine all the isometric immersions of complex space forms into complex space forms. Our result can be considered as the local version of a well-known result of Calabi.

A Kaehler manifold of constant holomorphic curvature is called a *complex space form*. By a *Kaehler submanifold* we mean a complex submanifold with the induced Kaehler metric. E. Calabi [1] gave a classification of Kaehler imbeddings of complete and simply connected complex space forms into complete and simply connected complex space forms. The local version of Calabi's result has been conjectured to be true by the second author [4] and he gave some partial solutions [2], [3].

The purpose of this paper is to prove the following two theorems which furnish the complete solutions to the conjecture. Throughout this paper we denote by $M_n(c)$ an n-dimensional complex space form of constant holomorphic curvature c.

THEOREM 1. *Let $M_n(c)$ be a Kaehler submanifold immersed in $M_{n+p}(\tilde{c})$. If $\tilde{c} > 0$ and the immersion is full, then $\tilde{c} = \nu c$ and $n + p = \binom{n+\nu}{\nu} - 1$ for some positive integer ν.*

THEOREM 2. *Let $M_n(c)$ be a Kaehler submanifold immersed in $M_{n+p}(\tilde{c})$. If $\tilde{c} \leqslant 0$, then $\tilde{c} = c$ (i.e., $M_n(c)$ is totally geodesic in $M_{n+p}(\tilde{c})$).*

1. **Kaehler submanifolds in $M_{n+p}(\tilde{c})$.** Let M be an n-dimensional Kaehler submanifold immersed in $M_{n+p}(\tilde{c})$. We choose a local field of unitary frames $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ in $M_{n+p}(\tilde{c})$ in such a way that, restricted to M,

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(2)Throughout this paper we use the following convention on the range of indices unless otherwise stated:

$A, B, C, \ldots = 1, \ldots, n, n+1, \ldots, n+p,$

$i, j, k, \ldots = 1, \ldots, n,$

$\alpha, \beta, \gamma, \ldots = n+1, \ldots, n+p.$

$e_1, \ldots, e_n$ are tangent to M. Let $\omega^1, \ldots, \omega^n, \omega^{n+1}, \ldots, \omega^{n+p}$ be the field of dual frames. Then the Kaehler metric $\tilde{g}$ of $M_{n+p}(\tilde{c})$ is given by $\tilde{g} = \Sigma_A \omega^A \overline{\omega}^A$ and the structure equations of $M_{n+p}(\tilde{c})$ are given by ([2])

$$d\omega^A + \sum_B \omega^A_B \wedge \omega^B = 0, \qquad \omega^A_B + \overline{\omega}^B_A = 0, \tag{1.1}$$

$$d\omega^A_B + \sum_C \omega^A_C \wedge \omega^C_B = \tilde{\Omega}^A_B, \qquad \tilde{\Omega}^A_B = \sum_{C,D} \tilde{R}^A_{BC\bar{D}}\omega^C \wedge \overline{\omega}^D. \tag{1.2}$$

Since $M_{n+p}(\tilde{c})$ is a complex space form of constant holomorphic curvature $\tilde{c}$, we have

$$\tilde{R}^A_{BC\bar{D}} = \frac{\tilde{c}}{4}(\delta^A_B\delta_{CD} + \delta^A_C\delta_{BD}). \tag{1.3}$$

Restricting these forms to M, we have

$$\omega^\alpha = 0, \tag{1.4}$$

and the Kaehler metric g of M is given by $g = \Sigma_i \omega^i \overline{\omega}^i$.

It follows from (1.1), (1.4) and Cartan's lemma that we may write

$$\omega^\alpha_i = \sum_j h^\alpha_{ij}\omega^j, \qquad h^\alpha_{ij} = h^\alpha_{ji}. \tag{1.5}$$

Moreover we obtain

$$d\omega^i + \sum_j \omega^i_j \wedge \omega^j = 0, \qquad \omega^i_j + \overline{\omega}^j_i = 0, \tag{1.6}$$

$$d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j = \Omega^i_j, \qquad \Omega^i_j = \sum_{k,l} R^i_{jk\bar{l}}\omega^k \wedge \overline{\omega}^l, \tag{1.7}$$

$$d\omega^\alpha_\beta + \sum_\gamma \omega^\alpha_\gamma \wedge \omega^\gamma_\beta = \Omega^\alpha_\beta, \qquad \Omega^\alpha_\beta = \sum_{k,l} R^\alpha_{\beta k\bar{l}}\omega^k \wedge \overline{\omega}^l. \tag{1.8}$$

From (1.2), (1.3), (1.4), (1.5), and (1.7), we have

$$\Omega^i_j = \sum_{k,l}\left\{\frac{\tilde{c}}{4}(\delta^i_j\delta_{kl} + \delta^i_k\delta_{jl}) - \sum_\alpha h^\alpha_{jk}\bar{h}^\alpha_{il}\right\}\omega^k \wedge \overline{\omega}^l. \tag{1.9}$$

Similarly from (1.2), (1.3), (1.4), (1.5), and (1.8), we have

$$\Omega^\alpha_\beta = \sum_{k,l}\left(\frac{\tilde{c}}{4}\delta^\alpha_\beta\delta_{kl} + \sum_j h^\alpha_{jk}\bar{h}^\beta_{jl}\right)\omega^k \wedge \overline{\omega}^l. \tag{1.10}$$

If we define h^α_{ijk} and $h^\alpha_{ij\bar{k}}$ by

$$\sum_k h^\alpha_{ijk}\omega^k + \sum_k h^\alpha_{ij\bar{k}}\overline{\omega}^k = dh^\alpha_{ij} - \sum_k h^\alpha_{kj}\omega^k_i - \sum_k h^\alpha_{ik}\omega^k_j + \sum_\beta h^\beta_{ij}\omega^\alpha_\beta, \tag{1.11}$$

then we can easily see

$$h^\alpha_{ijk} = h^\alpha_{ikj}, \qquad h^\alpha_{ij\bar{k}} = 0. \tag{1.12}$$

We may define inductively the successive derivatives of h^α_{ij} by

$$(1.13)\quad \begin{cases} \sum_l h^\alpha_{i_1\cdots i_k l}\omega^l + \sum_l h^\alpha_{i_1\cdots i_k \bar l}\bar\omega^l \\ \qquad = dh^\alpha_{i_1\cdots i_k} - \sum_{r=1}^k \sum_{j_r} h^\alpha_{i_1\cdots j_r\cdots i_k}\omega^{j_r}_{i_r} + \sum_\beta h^\beta_{i_1\cdots i_k}\omega^\alpha_\beta, \\ \sum_m h^\alpha_{i_1\cdots i_k \bar l m}\omega^m + \sum_m h^\alpha_{i_1\cdots i_k \bar l\bar m}\bar\omega^m \\ \qquad = dh^\alpha_{i_1\cdots i_k\bar l} - \sum_{r=1}^k \sum_{j_r} h^\alpha_{i_1\cdots j_r\cdots i_k \bar l}\omega^{j_r}_{i_r} \\ \qquad\quad - \sum_m h^\alpha_{i_1\cdots i_k \bar m}\bar\omega^m_l + \sum_\beta h^\beta_{i_1\cdots i_k\bar l}\omega^\alpha_\beta. \end{cases}$$

If $h^\alpha_{i_1\ldots i_k}$ is symmetric with respect to all indices, then taking the exterior derivative of the first equality of (1.13) and using (1.6), (1.7), and (1.8), we get

$$\sum_{l,m} h^\alpha_{i_1\cdots i_k lm}\omega^l\wedge\omega^m - \sum_{l,m}(h^\alpha_{i_1\cdots i_k l\bar m} - h^\alpha_{i_1\cdots i_k \bar m l})\omega^l\wedge\bar\omega^m$$
$$+ \sum_{l,m} h^\alpha_{i_1\cdots i_k \bar l\bar m}\bar\omega^l\wedge\bar\omega^m$$
$$= -\sum_{r=1}^k \sum_{j_r} h^\alpha_{i_1\cdots j_r\cdots i_k}\Omega^{j_r}_{i_r} + \sum_\beta h^\beta_{i_1\cdots i_k}\Omega^\alpha_\beta,$$

from which, together with (1.9) and (1.10), we obtain

$$(1.14)\qquad h^\alpha_{i_1\cdots i_k lm} = h^\alpha_{i_1\cdots i_k ml}, \qquad h^\alpha_{i_1\cdots i_k \bar l\bar m} = h^\alpha_{i_1\cdots i_k \bar m\bar l},$$

$$(1.15)\quad \begin{aligned} &h^\alpha_{i_1\cdots i_k l\bar m} - h^\alpha_{i_1\cdots i_k \bar m l} \\ &\quad = \frac{\tilde c}{4}\left\{(k-1)h^\alpha_{i_1\cdots i_k}\delta_{lm} + \sum_{r=1}^k h^\alpha_{i_1\cdots l\cdots i_k}\delta_{i_r m}\right\} \\ &\qquad - \sum_{r=1}^k \sum_{j_r,\beta} h^\alpha_{i_1\cdots j_r\cdots i_k}h^\beta_{i_r e}\bar h^\beta_{j_r m} - \sum_{j,\beta} h^\beta_{i_1\cdots i_k}h^\alpha_{jl}\bar h^\beta_{jm}. \end{aligned}$$

Since h^α_{ij} and h^α_{ijk} are symmetric with respect to all indices, (1.14) implies inductively that

$$(1.16)\qquad h^\alpha_{i_1\cdots i_k} \textit{ is symmetric with respect to all indices,}$$

and consequently (1.15) holds for all integers $k \geqslant 2$.

2. **Proof of theorems.** Since $M_n(c)$ is a complex space form of constant holomorphic curvature c, the curvature forms Ω_j^i are given by

$$\Omega_j^i = \sum_{k,l} R^i_{jk\bar{l}}\omega^k \wedge \overline{\omega}^l = \frac{c}{4}\sum_{k,l}(\delta_j^i\delta_{kl} + \delta_k^i\delta_{jl})\omega^k \wedge \overline{\omega}^l,$$

which, together with (1.9), implies

$$\sum_\alpha h^\alpha_{ik}\overline{h}^\alpha_{jl} = \frac{\widetilde{c}-c}{4}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}). \tag{2.1}$$

From (2.1) and the second property of (1.12), we obtain

$$\sum_\alpha h^\alpha_{i_1\cdots i_k}\overline{h}^\alpha_{jl} = 0 \quad \text{for } k \geqslant 3. \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.15), we have

$$h^\alpha_{i_1\cdots i_k l\bar{m}} = h^\alpha_{i_1\cdots i_k\bar{m}l} - \frac{\widetilde{c}-kc}{4}h^\alpha_{i_1\cdots i_k}\delta_{lm}$$

$$+ \frac{c}{4}\sum_{r=1}^k h^\alpha_{i_1\cdots l\cdots i_k}\delta_{i_r m} \quad \text{for } k \geqslant 3,$$

and, in particular, applying this relation to the first term of the right-hand side repeatedly and taking account of (1.12), we can obtain

$$h^\alpha_{i_1\cdots i_k\bar{l}} = -\frac{\widetilde{c}-(k-1)c}{4}\sum_{r=1}^k h^\alpha_{i_1\cdots\hat{i}_r\cdots i_k}\delta_{i_r l}, \tag{2.3}$$

where the notation ^ means the omission of the index i_r. Since h^α_{ijk} is symmetric, from (2.1), (2.2), and (2.3), we have

$$\begin{aligned} h^\alpha_{ijk}\overline{h}^\alpha_{lmn} = \frac{(\widetilde{c}-c)(\widetilde{c}-2c)}{4^2}(&\delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jn}\delta_{kl} \\ &+ \delta_{im}\delta_{jl}\delta_{kn} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}). \end{aligned} \tag{2.4}$$

First of all, as a generalization of (2.1) and (2.4), we shall prove the following.

LEMMA 1.

$$\sum_\alpha h^\alpha_{i_1\cdots i_k}\overline{h}^\alpha_{j_1\cdots j_l} = \begin{cases} 0 & \textit{for } k \neq l, \\ \dfrac{1}{4^{k-1}}\displaystyle\prod_{r=1}^{k-1}(\widetilde{c}-rc)\sum_\sigma \delta_{\sigma(i_1)j_1}\cdots\delta_{\sigma(i_k)j_k} & \\ & \textit{for } k = l, \end{cases} \tag{2.5}$$

where Σ_σ is the summation on all permutations with respect to indices $i_1, \ldots, i_k$.

PROOF. We shall prove the second equality by induction. The cases where $k = 2$, and $k = 3$ reduce to (2.1) and (2.4) respectively.

We suppose that the following equalities hold:

$$\sum_\alpha h^\alpha_{i_1\cdots i_l}\bar{h}^\alpha_{j_1\cdots j_l} = \frac{1}{4^{l-1}}\prod_{r=1}^{l-1}(\tilde{c} - rc)\sum_\sigma \delta_{\sigma(i_1)j_1}\cdots\delta_{\sigma(i_l)j_l} \tag{2.6}$$

for $l \leqslant k$.

Then it follows from (2.3) that

$$\sum_\alpha h^\alpha_{i_1\cdots i_k}\bar{h}^\alpha_{j_1\cdots j_k m} = -\sum_\alpha h^\alpha_{i_1\cdots i_k \bar{m}}\bar{h}^\alpha_{j_1\cdots j_k} \equiv 0$$
$$\left(\operatorname{mod}\sum_\alpha h^\alpha_{l_1\cdots l_{k-1}}\bar{h}^\alpha_{m_1\cdots m_k}\delta_{ij}\right).$$

Making use of (2.3) and the supposition (2.6) of the induction repeatedly, we see that

$$\sum_\alpha h^\alpha_{i_1\cdots i_k}\bar{h}^\alpha_{j_1\cdots j_k m} = 0, \quad \text{i.e.,} \quad \sum_\alpha h^\alpha_{i_1\cdots i_k m}\bar{h}^\alpha_{j_1\cdots j_k} = 0. \tag{2.7}$$

From (2.3) and the second equality of (2.7), we have

$$\sum_\alpha h^\alpha_{i_1\cdots i_{k+1}}\bar{h}^\alpha_{j_1\cdots j_{k+1}}$$
$$= \frac{1}{4^k}\prod_{r=1}^{k}(\tilde{c} - rc)\sum_\sigma \delta_{\sigma(i_1)j_1}\cdots\delta_{\sigma(i_{k+1})j_{k+1}}.$$

This shows that (2.5) holds for any integer k.

By the similar argument, we can prove the first equality of (2.5) noting that we may assume $k > l$ without loss of generality. Q.E.D.

From the second equality of (2.5) we have

$$\sum_\alpha\sum h^\alpha_{i_1\cdots i_{k+1}}\bar{h}^\alpha_{i_1\cdots i_{k+1}} \tag{2.8}$$
$$= \frac{1}{4^k}n(n+1)\cdots(n+k)(\tilde{c} - c)(\tilde{c} - 2c)\cdots(\tilde{c} - kc).$$

LEMMA 2. *If $c > 0$, then $\tilde{c} = \nu c$ for some positive integer ν.*

PROOF. Suppose that there exists no integer l such that $\tilde{c} = lc$. Then, since $c > 0$ so that $\tilde{c} > c > 0$, there exists an integer k satisfying $(k-1)c < \tilde{c} < kc$. For such k, the right-hand side of (2.8) is negative, which contradicts the fact that the left-hand side is nonnegative. Q.E.D.

Next we prove the following.

LEMMA 3. (1) *If* $\tilde{c} = \nu c$ *for a positive integer* ν, *then* $n + p \geqslant \binom{n+\nu}{\nu} - 1$.

(2) *If* $\tilde{c} > 0 \geqslant c$ *or* $0 \geqslant \tilde{c} > c$, *then* $n + p \geqslant \binom{n+\nu}{\nu} - 1$ *for any positive integer* ν.

PROOF. Put $m_l = \binom{n+l}{l} - (n+1)$ and suppose $p < m_\nu$. Let $H = (H^\lambda_\mu)$ be a matrix of order m_ν defined by

$$\begin{cases} H^{\alpha-n}_\mu = \begin{cases} h^\alpha_{i_1 i_2} & \text{for } \mu \leqslant m_2, \\ h^\alpha_{i_1 i_2 i_3} & \text{for } m_2 < \mu \leqslant m_3, \\ \cdots & \\ h^\alpha_{i_1 \cdots i_\nu} & \text{for } m_{\nu-1} < \mu \leqslant m_\nu, \end{cases} \\ H^\lambda_\mu = 0 \qquad\qquad\qquad \text{for } p < \lambda \leqslant m_\nu. \end{cases}$$

Then it follows from (2.5) that

$${}^t\bar{H}H = \begin{bmatrix} 2!\,\dfrac{\tilde{c}-c}{4} & & & \\ & \ddots & & \\ & & * & \\ & & & \ddots \\ & & & & \nu!\,\dfrac{\prod_{r=1}^{\nu-1}(\tilde{c}-rc)}{4^{\nu-1}} \end{bmatrix},$$

where

$$* = \frac{1}{4^{j-1}}(r_1! + r_2! + \cdots + r_l!)\prod_{s=1}^{j-1}(\tilde{c} - sc)$$

and $r_1 + \cdots + r_l = j$. The assumption on c and $\tilde{c}$ implies that the determinant of H is not zero, which contradicts the definition of the matrix H. Q.E.D.

Theorem 2 follows immediately from Lemma 3.

In order to complete the proof of Theorem 1, it suffices to show the following.

LEMMA 4. *If* $\tilde{c} = \nu c$ *for a positive integer* ν *and the immersion is full, then* $n + p = \binom{n+\nu}{\nu} - 1$.

PROOF. In consideration of Lemma 3(1), we suppose $n + p > \binom{n+\nu}{\nu} - 1$.

Let $A_2 = (A_{2\beta}^{\alpha})$ be a matrix of order p defined by $A_{2\beta}^{\alpha} = \Sigma_{k,l} h_{kl}^{\alpha} \bar{h}_{kl}^{\beta}$. It is easily seen that A_2 is a Hermitian matrix. Furthermore, we consider a $p \times \binom{n+1}{2}$ matrix $H_2 = (h_{ij}^{\alpha})$. Then these two matrices satisfy

$$A_2 H_2 = \frac{\tilde{c} - c}{2} H_2, \qquad H_2\, {}^t\bar{H}_2 = A_2,$$

and the matrix ${}^t\bar{H}_2 H_2$ is nonsingular, which implies that the rank of the matrix A_2 is equal to $\binom{n+1}{2}$. We denote by N_x the normal space to M at x, and define a mapping f_2 of $N_x \times N_x$ into a complex field C by

$$f_2(X, Y) = \sum_{\alpha,\beta} A_{2\beta}^{\alpha} \xi_\alpha \bar{\eta}_\beta,$$

where $X = \Sigma_\alpha \xi_\alpha e_\alpha$ and $Y = \Sigma_\beta \eta_\beta e_\beta$. Let H_p be the set of all Hermitian matrices of order p, which is considered as a complex vector space. The unitary group $U(p)$ operates on H_p as follows: For any Hermitian matrix $A \in H_p$ and any $U \in U(p)$, $U(A) = {}^t\bar{U} A U$.

Since the matrix A_2 is invariant under $U(p)$, the mapping f_2 is well defined and it is a positive semidefinite Hermitian form of rank $r = \binom{n+1}{2}$ so that it can be normalized as

$$f_2(X, X) = \lambda_1' \xi_{n+1}' \bar{\xi}_{n+1}' + \cdots + \lambda_r' \xi_{n+r}' \bar{\xi}_{n+r}'.$$

This means that we can choose a new unitary frame $(e_i, e_{\alpha_1}, e_\beta)$ at x such that

$$\text{(2.9)}\quad \omega_i^{\alpha_1} \neq 0, \qquad \omega_i^\beta = 0 \quad \text{for} \binom{n+1}{1} \leqslant \alpha_1 < \binom{n+2}{2}, \beta \geqslant \binom{n+2}{2}.$$

Similarly, we can choose a new unitary frame $(e_i, e_{\alpha_1}, e_{\alpha_2}, e_\beta)$ such that

$$\text{(2.10)}\quad \begin{aligned} &\omega_{\alpha_1}^{\alpha_2} \neq 0, \quad \omega_{\alpha_1}^{\beta} = 0 \\ &\qquad \text{for} \binom{n+1}{1} \leqslant \alpha_1 < \binom{n+2}{2}, \\ &\qquad\qquad \binom{n+2}{2} \leqslant \alpha_2 < \binom{n+3}{3}, \quad \beta \geqslant \binom{n+3}{3}. \end{aligned}$$

In fact, we take a unitary frame $(e_i, e_{\alpha_1}, e_\beta)$ satisfying property (2.9). Let $A_3 = (A_{3\beta}^{\alpha})$ be a matrix of order p defined by $A_{3\beta}^{\alpha} = \Sigma_{i,j,k} h_{ijk}^{\alpha} \bar{h}_{ijk}^{\beta}$. Then A_3 is also a Hermitian matrix of rank $\binom{n+2}{3}$ and it is invariant under $U(p)$. This implies that a mapping f_3 of $N_x \times N_x$ into C defined by

$$f_3(X, Y) = \sum_{\alpha,\beta} A_{3\beta}^{\alpha} \xi_\alpha \bar{\eta}_\beta,$$

where $X = \Sigma_\alpha \xi_\alpha e_\alpha$ and $Y = \Sigma_\beta \eta_\beta e_\beta$, is well defined and is also a positive semidefinite Hermitian form of rank $\binom{n+2}{3}$. Therefore we can choose a frame satisfying (2.10).

Since $h^\alpha_{i_1 \ldots i_l} = 0$ for $l \geqslant \nu + 1$, we can inductively choose a unitary frame $(e_i, e_{\alpha_1}, \ldots, e_{\alpha_{\nu-1}}, e_\beta)$ at x such that

$$
(2.11) \qquad \left\{ \begin{array}{lll}
\omega^\beta = 0 & & \text{for } \beta \geqslant \binom{n+1}{1}, \\
\omega_i^{\alpha_1} \neq 0, & \omega_i^\beta = 0 & \text{for } \beta \geqslant \binom{n+2}{2}, \\
\vdots & & \\
\omega_{\alpha_{r-1}}^{\alpha_r} \neq 0, & \omega^\beta_{\alpha_{r-1}} = 0 & \text{for } \beta \geqslant \binom{n+r+1}{r+1}, \\
\vdots & & \\
\omega_{\alpha_{\nu-2}}^{\alpha_{\nu-1}} \neq 0, & \omega^\beta_{\alpha_{\nu-2}} = 0 & \text{for } \beta \geqslant \binom{n+\nu}{\nu}, \\
\omega^\beta_{\alpha_{\nu-1}} = 0 & & \text{for } \beta \geqslant \binom{n+\nu}{\nu},
\end{array} \right.
$$

where $\binom{n+r}{r} \leqslant \alpha_r < \binom{n+r+1}{r+1}$. Now we consider a distribution $\mathfrak{M}$ on the frame bundle defined by

$$
\omega^\beta = 0, \quad \omega_i^\beta = 0, \quad \omega^\beta_{\alpha_1} = 0, \ldots, \omega^\beta_{\alpha_{\nu-1}} = 0 \quad \text{for } \beta \geqslant \binom{n+\nu}{\nu},
$$

where $\binom{n+r}{r} \leqslant \alpha_r < \binom{n+r+1}{r+1}$. Then it follows from the structure equations that

$$
\begin{aligned}
d\omega^\beta &= -\sum_{i=1}^n \omega_i^\beta \wedge \omega^i - \sum_{r=1}^{\nu-1} \sum_{\alpha_r} \omega^\beta_{\alpha_r} \wedge \omega^{\alpha_r} - \sum_\gamma \omega^\beta_\gamma \wedge \omega^\gamma \\
&\equiv 0 \quad (\mathrm{mod}\ \omega^\beta, \omega_i^\beta, \omega^\beta_{\alpha_1}, \ldots, \omega^\beta_{\alpha_{\nu-1}}), \\
d\omega_i^\beta &= -\sum_{j=1}^n \omega_j^\beta \wedge \omega_i^j - \sum_{r=1}^{\nu-1} \sum_{\alpha_r} \omega^\beta_{\alpha_r} \wedge \omega_i^{\alpha_r} - \sum_\gamma \omega^\beta_\gamma \wedge \omega_i^\gamma + \tilde{\Omega}_i^\beta \\
&\equiv 0 \quad (\mathrm{mod}\ \omega^\beta, \omega_i^\beta, \omega^\beta_{\alpha_1}, \ldots, \omega^\beta_{\alpha_{\nu-1}}), \\
d\omega^\beta_{\alpha_r} &= -\sum_{j=1}^n \omega_j^\beta \wedge \omega^j_{\alpha_r} - \sum_{s=1}^{\nu-1} \sum_{\alpha_s} \omega^\beta_{\alpha_s} \wedge \omega^{\alpha_s}_{\alpha_r} - \sum_\gamma \omega^\beta_\gamma \wedge \omega^\gamma_{\alpha_r} + \tilde{\Omega}^\beta_{\alpha_r} \\
&\equiv 0 \quad (\mathrm{mod}\ \omega^\beta, \omega_i^\beta, \omega^\beta_{\alpha_1}, \ldots, \omega^\beta_{\alpha_{\nu-1}}), \text{ where } \gamma \geqslant \binom{n+\nu}{\nu}.
\end{aligned}
$$

Therefore it is seen that the distribution $\mathfrak{M}$ is completely integrable.

For any point x in $M_n(c)$, let $M(x)$ be the maximal integral submanifold of $\mathfrak{M}$ through x. Then $M(x)$ is of $\{\binom{n+\nu}{\nu} - 1\}$-dimensional and by the construction it is totally geodesic in $M_{n+p}(\tilde{c})$. Thus there exists an $\{\binom{n+\nu}{\nu} - 1\}$-dimensional totally geodesic submanifold of $M_{n+p}(\tilde{c})$ in which the given submanifold $M_n(c)$ is immersed. This is a contradiction to the assumption that the immersion is full.

This completes the proof.

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DEPARTMENT OF MATHEMATICS, TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY, TOKYO, JAPAN (Current address of Hisao Nakagawa)

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823

Current address (K. Ogiue): Department of Mathematics, Tokyo Metropolitan University, Tokyo, Japan