YONEDA PRODUCTS IN THE CARTAN-EILENBERG
CHANGE OF RINGS SPECTRAL SEQUENCE WITH
APPLICATIONS TO $BP_*(BO(n))$

BY
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ABSTRACT. Yoneda product structure is defined on a Cartan-Eilenberg
change of rings spectral sequence. Application is made to a factorization theo-
rem for the $E_2$-term of the Adams spectral sequence for Brown-Peterson homol-
ogy of the classifying spaces $BO(n)$.

This paper gives an algebraic decomposition of the $E_2$-term of the Adams
spectral sequence of the reduced mod 2 Brown-Peterson homol-
ogy [3], [4] of
the classifying space $BO(n)$.

The first section gives algebraic preliminaries and the statement of the main
result. In §2 Yoneda products are introduced in a Cartan-Eilenberg change of
rings spectral sequence [5] used to compute the required Ext module. The proof
of the main theorem is given in §3.

The main results of §§2 and 3 are contained in the author's doctoral dis-
sertation at the University of Chicago under Arunas Liulevicius, to whom grateful
acknowledgement is made for his time and helpful suggestions.

1. Preliminaries; statement of results. This section outlines the algebraic
constructions needed to construct the spectral sequence of §2 and to introduce
the main theorem.

Let $F$ be a field. An algebra $A$ will be a positively graded, augmented, as-
sociative $F$-algebra. Let $\bar{A}$ denote the augmentation ideal of $A$. Let $B_s(A, A) =
A \otimes \bar{A}^s \otimes A$, where $\bar{A}$ is the $s$-fold tensor product of $\bar{A}$. Form the 2-sided bar
construction [10] $B(A, A) = \sum_{s \geq 0} B_s(A, A)$ and let $\delta$ denote the standard bound-
ary map. In all that follows the degree of an element refers to its total degree.

Let $\overline{B}(A) = F \otimes_A B(A, A) \otimes_A F$ with induced boundary $\overline{\delta}$ and let $\overline{C}(A) =
\overline{B}(A)^\ast$ with coboundary $\overline{\delta} = (\overline{\delta})^\ast$. Recall that $\overline{C}(A) = (\bar{A}^\ast)^\ast$ and that $\overline{C}(A)$ is a
differential algebra under the cup-product

$$[\alpha_1, \ldots, \alpha_k] [\beta_1, \ldots, \beta_l] = [\alpha_1, \ldots, \alpha_k \beta_1, \ldots, \beta_l];$$

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(1.2) \( \bar{\delta}(\alpha \beta) = \bar{\delta}(\alpha)\beta + (-1)^{\text{deg} \alpha} \alpha \bar{\delta}(\beta) \).

If \( M \) is a positively graded left \( A \)-module, let \( B(F, M) = F \otimes_A B(A, A) \otimes_A M \) have induced boundary \( \bar{\delta}_M \) and let \( C(F, M) = B(F, M)^* \) have coboundary \( \delta_M = \bar{\delta}_M^* \). Then \( C(F, M) \) is a differential \( \overline{C}(A) \)-module under the cup-product, that is

\[
\delta_M(\alpha \cdot \beta \otimes \lambda) = \bar{\delta}(\alpha) \cdot (\beta \otimes \lambda) + (-1)^{\text{deg} \alpha} \alpha \cdot \bar{\delta}_M(\beta \otimes \lambda)
\]

where \( \alpha, \beta \in \overline{C}(A), \lambda \in M^* \).

The cup-product on \( \overline{C}(A) \) induces a product map on \( H^{**}(A) = \text{Ext}^**_A(F, F) = H_{**}(\overline{C}(A)) \) and a structure map

\[
H^{s+t}(A) \otimes \text{Ext}^{s'+r'}_A(M, F) \rightarrow \text{Ext}^{s+r', s+r'}_A(F, F);
\]

both structures are called Yoneda products.

Hereafter \( A \) denotes the mod 2 Steenrod algebra.

To compute \( \widetilde{BP}_*(BO(n)) \), we have the Adams spectral sequence

\[ E_2 = \text{Ext}_A(H^*(BP; Z_2) \otimes \tilde{H}^*(BO(n); Z_2), Z_2) \Rightarrow BP_*(BO(n)) \otimes I_2, \]

where \( I_2 \) denotes the 2-adic integers. We have [4]

\[ H^*(BP; Z_2) = A/A(Q_0, Q_1, \ldots) \]

where the \( Q_i \in A \) are defined by Milnor [11]; recall that \( Q_0 = Sq^1 \) and \( Q_i = [Q_{i-1}, Sq^{2i}] \). Let \( E = \wedge(Q_0, Q_1, \ldots) \), where \( \wedge \) denotes exterior algebra over \( Z_2 \), then \( H^*(BP; Z_2) = A \otimes_{E} Z_2 \), so by a standard change of rings theorem [7]

\[ E_2 \cong \text{Ext}_E(\tilde{H}^*(BO(n); Z_2), Z_2) = \text{Ext}_{E_*}(Z_2, \tilde{H}_*(BO(n); Z_2)). \]

Here we write \( E_* \) for \( \text{Hom}(E, Z_2) \) following Milnor’s convention since \( E_* \) occurs in the context of homology. The second Ext of (1.6) is one of \( E_* \)-comodules; see Adams [3].

Since \( E \) is a Hopf algebra, so is \( E_* \), which is an exterior algebra over \( Z_2 \) on generators \( \beta_1, \beta_2, \ldots \) which form a dual basis to \( Q_0, Q_1, \ldots \) respectively.

We have [8]

\[
\tilde{H}_*(BO(n); Z_2) = \tilde{H}_*(MO(n); Z_2) \otimes \tilde{H}_*(BO(n-1); Z_2)
\]

As \( A_* \)-comodules. The first summand may be described as follows: let \( MO \) denote the Thom spectrum for the orthogonal groups, then

\[
H_*(MO; Z_2) = Z_2[b_1, b_2, \ldots]
\]

where \( b_i \in H_i(MO; Z_2) \) is the image of \( x_{i+1} \in H_{i+1}(RP^{\infty}; Z_2) \) under the composite.
The subgroup $\widetilde{H}^*(MO(n); Z_2)$ is the span of monomials in the $b_i$ of degree $\leq n$.

The coaction map \[ (1.7) \] given on generators by

\[ (1.8) \]

is a ring homomorphism.

The cohomology of $E$, $H^{**}(E)$, is the polynomial algebra $Z_2[q_0, q_1, \ldots]$ where $q_i \in H^1, 2^{i+1}-1(E)$.

Let $E(m) = \bigwedge(Q_0, \ldots, Q_m)$. Let $M = M(n)$ denote either $\widetilde{H}^*(BO(n); Z_2)$ or $\widetilde{H}^*(MO(n); Z_2)$. The main result states:

**Theorem 1.9.** Under the Yoneda product, $\text{Ext}_E(M, Z_2)$ is a free $Z_2[q_n, q_{n+1}, \ldots]$-module on $\text{Ext}^*_{E(n-1)}(M, Z_2)$. Hence the $E_2$-term in the mod 2 Adams spectral sequence of $BP^{*}(BO(n))$ is given by:

\[ E_2 \approx Z_2[q_n, q_{n+1}, \ldots] \otimes \text{Ext}^*_{E(n-1)}(\widetilde{H}^*(BO(n); Z_2), Z_2). \]

2. A change of rings spectral sequence with products. The program for the proof of Theorem 1.9 is to show

\[ (2.1) \]

for $r \geq n$. For this purpose we use a spectral sequence of Cartan and Eilenberg [5] to which we have added the structure of Yoneda products.

Let $\varphi: S \rightarrow A$ be a homomorphism of algebras in the sense of §1, that is $\varphi$ has degree zero and commutes with the augmentations. The map $\varphi$ is called (left) normal if $\varphi(S)A$ is a left ideal of $A$. If $\varphi$ is normal, $A/\varphi(S)A \cong F \otimes_S A$ is an algebra.

**Theorem 2.2.** Let $\varphi: S \rightarrow A$ be a left normal homomorphism of algebras such that $A$ is projective as a left $S$-module. Let $T = F \otimes_S A$. Let $M$ be a left $A$-module and $C$ a left $T$-module. Then there is a spectral sequence

\[ (2.2) \]

Here $A$ and $M$ are $S$-modules through $\varphi$, and $C$ is an $A$-module through the projection $\pi: A \rightarrow A/\varphi(S)A \cong T$. The left $T$-operations on $\text{Tor}_q^S(F, M)$ are induced by left multiplication in $T$ through the isomorphism

\[ (2.3) \]

The outer Ext may be computed as one of $T^*$-comodules, so that

\[ (2.4) \]

$E_2 \cong \text{Ext}^*_{T^*}(C^*, \text{Ext}_q^S(M, F))$. 

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Note that $E_2^{0,m}$ may be regarded as the submodule of $T^*$-primitives of $\text{Ext}^m_\mathcal{A}(M, F)$, and it will follow from the construction that the edge homomorphism

$$\text{Ext}^m_\mathcal{A}(M, F) \to E_2^{0,m} \subseteq E_2^{0,m} \subseteq \text{Ext}^m_\mathcal{A}(M, F)$$

coincides with the induced map $\text{Ext}^m_\varphi(M, F)$.

To form the products, consider the case $C = F$. Note that $H^*(A)$ acts on $\text{Ext}^m_\mathcal{A}(M, F)$ through $H^*(\varphi)$, and so induces a structure map

$$H^s(A) \otimes E_r^{p,q} \xrightarrow{\mu_1} E_r^{p,q+s}. $$

On the other hand, by (2.4) the Yoneda product gives a structure map

$$H^s(A) \otimes E_r^{p,q} \xrightarrow{\mu_2} E_{r+s}^{p,q}. $$

In the theorems below, let $F_p$ denote the $p$th filtration of $\text{Ext}^m_\mathcal{A}(M, F)$ in the spectral sequence and let

$$\rho_p: F_p \to F_p/F_p+1 = E_\infty^p$$

be the projection.

**Theorem 2.5.** The spectral sequence of Theorem 2.2 with $C = F$ admits structure maps

$$H^s(A) \otimes E_r^{p,q} \xrightarrow{\lambda_r} E_r^{p,q+s}, \quad 1 \leq r \leq \infty,$$

such that:

1. $d_r: E_r^{p,q} \to E_r^{p+r,q-r+1}$ is a left (graded) $H^*(A)$-module homomorphism;
2. $\lambda_{r+1}$ is induced from $\lambda_r$ by passing to subquotients;
3. $\lambda_2 = \mu_1$;
4. the following diagram commutes, where $Y$ denotes the restriction of the Yoneda product map to $H^*(A) \otimes F_p$:

$$\begin{array}{ccc}
H^s(A) \otimes F_p & \xrightarrow{Y} & F_p \\
\downarrow \rho_p & & \downarrow \rho_p \\
H^s(A) \otimes E_\infty^p & \xrightarrow{\lambda_\infty} & E_\infty^p
\end{array}$$

**Theorem 2.7.** The spectral sequence of Theorem 2.2 with $C = F$ admits structure maps

$$H^s(T) \otimes E_r^{p,q} \xrightarrow{\theta_r} E_r^{p+r,q}, \quad 1 \leq r \leq \infty,$$

such that:

1. $d_r$ is a left (graded) $H^*(T)$-module homomorphism:
(2) \( \theta_{r+1} \) is induced from \( \theta_0 \) by passing to subquotients;

(3) \( \theta_2 = \mu_2 \);

(4) the following diagram commutes, where \( Y' \) denotes the restriction of the composite

\[
H^*(T) \otimes \text{Ext}_A(M, F) \xrightarrow{H^* \otimes 1} H^*(A) \otimes \text{Ext}_A(M, F) \rightarrow \text{Ext}_A(M, F)
\]

to \( H^*(T) \otimes F^p \):

\[
\begin{array}{ccc}
H^*(T) \otimes F^p & \xrightarrow{Y'} & F^p + s \\
\downarrow & & \downarrow \\
H^*(T) \otimes E^p_{s+t} & \xrightarrow{\theta} & E^p_{s+t+s}
\end{array}
\]

(2.8)

Theorem 2.2 is the analog of Theorem 6.1 (1a) of Cartan and Eilenberg [5, p. 349] with Tor replaced by Ext. If products are not required, the construction in the proof of (2.5) carries over with trivial modifications for \( \text{Ext}_A(M, C) \), so a separate proof will be omitted.

For the proof of (2.5), let \( X = \text{B}(A, M) = \text{B}(A, A) \otimes_A M \). Define \( \epsilon: X \rightarrow M \) by \( \epsilon[a_1 \cdots a_s m] = 0 \) if \( s \geq 1 \) and \( \epsilon(a[m]m) = am \). Then \( X \) with \( \epsilon \) is a free \( A \)-resolution of \( M \).

Let

\[
0 \rightarrow F \xrightarrow{\eta} Y^0 \xrightarrow{\delta} Y^1 \rightarrow \cdots
\]

be a resolution of \( F \) by bigraded, injective \( T \)-modules. In assigning degrees of maps we follow the usual convention \( Y^{s,t} = Y_{-s,-t} \). Form the sum \( \text{Hom}_A(X, Y) = \sum_{i,j} \text{Hom}_A(X_i, Y^j) \) and define coboundaries

\[
\begin{align*}
\delta &: \text{Hom}_A(X_i, Y^j) \xrightarrow{\delta} \text{Hom}_A(X_{i+1}, Y^j) \\
d &: \text{Hom}_A(X_i, Y^j) \xrightarrow{\alpha} \text{Hom}_A(X_i, Y^{j+1})
\end{align*}
\]

by

\[
(\delta f)(x) = (-1)^{\deg f + 1} f(\partial_M x), \quad (df)(x) = \delta(f(x))
\]

where \( \partial_M = \partial \otimes_A 1 \) and \( \deg f = -(s + s' + t + t') \) if \( f: X_{s,t} \rightarrow Y^{s',t'} \).

With this definition the squares

\[
\begin{array}{ccc}
\text{Hom}_A(X_{i+1}, Y^j) & \xrightarrow{d} & \text{Hom}_A(X_{i+1}, Y^{j+1}) \\
\downarrow & & \downarrow \\
\text{Hom}_A(X_i, Y^j) & \xrightarrow{d} & \text{Hom}_A(X_i, Y^{j+1})
\end{array}
\]

anticommute, so that \( \text{Hom}_A(X, Y) \) is a bicomplex. The total differential \( \Delta = \delta + d \) makes \( \text{Hom}_A(X, Y) \) into a cochain complex.
The bicomplex $\text{Hom}_A(X, Y)$ has row and column filtrations,

$$G^p \text{Hom}_A(X, Y) = \sum_{i \geq p, t} \text{Hom}_A(X^p_t, Y^i),$$

$$F^p \text{Hom}_A(X, Y) = \sum_{j \geq p, t} \text{Hom}_A(X^p_t, Y^j).$$

The row filtration gives a spectral sequence with

$$E^{p,q}_2 = H_p(H_q(\text{Hom}_A(X, Y); d); \delta).$$

Since the $X_i$ are $A$-free, the rows are acyclic, so that

$$H_q(\text{Hom}_A(X, Y); d) = \begin{cases} \text{Hom}_A(X, F), & q = 0, \\ 0, & q \neq 0, \end{cases}$$

and hence

$$E^{p,q}_2 = \begin{cases} \text{Ext}_A^p(M, F), & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Thus the spectral sequence of the row filtration collapses with trivial extensions and identifies $H_q(\text{Hom}_A(X, Y); \Delta)$ with $\text{Ext}_A^p(M, F)$. We say $x \in F^p \text{Ext}_A^p(M, F)$ if it has a representing cocycle in $F^p \text{Hom}_A(X, Y)$.

The spectral sequence of Theorem 2.2. is the one corresponding to the column filtration $F^p$. Its $E_2$ term will be identified in the form (2.4). Define a map

$$\gamma: \text{Hom}_A(X_q, Y^p) \to \text{Hom}_{T^*}(Y^p, \text{Hom}_A(X_q, F))$$

by

$$\gamma(f)(y^*) = (-1)^{\text{deg} f \text{deg} y^*} y^*: f.$$

We have

$$E^{p,q}_2 = H_p(H_q(\text{Hom}_A(X, Y); \delta); d).$$

Note that $X$ with augmentation $\epsilon$ is a projective $S$-resolution of $M$, by the hypothesis of the $S$-action on $A$. Thus the homology of $\text{Hom}_S(X, F)$ with the induced coboundary is $\text{Ext}_S^p(M, F)$. Since $Y^p$ is a projective $T^*$-comodule, it is easily verified that $\gamma$ induces an isomorphism

$$H_q(\text{Hom}_A(X, Y); \delta) \cong \text{Hom}_{T^*}(Y^p, \text{Ext}_S^q(M, F)).$$

Hence $E^{p,q}_2 = \text{Ext}_{T^*}^q(F, \text{Ext}_S^q(M, F))$.

For the products, we use an equivalent formulation of the bicomplex. Let $\alpha$ be the composite

$$(\overline{A}^*)^i \otimes M^* \otimes Y^i \to \text{Hom}(\overline{A}^i \otimes M, Y^i)$$

$$\cong \text{Hom}_A(A \otimes \overline{A}^i \otimes M, Y^i) = \text{Hom}_A(X^p_t, Y^i)$$
where
\[ (\alpha' (a^* \otimes m^* \otimes y), a \otimes m) = (-1)^{\mu \deg y (a^* \otimes m^*, a \otimes m)} y, \]
\[ \mu = \deg (a^* \otimes m^*), \]
and \((, )\) is the dual pairing.

We have commutative diagrams
\[ (\mathcal{A}^*)^j \otimes M^* \otimes Y^j \xrightarrow{\alpha} \text{Hom}_A(X^j, Y^j) \]
\[ (\mathcal{A}^*)^j \otimes M^* \otimes Y^j \xrightarrow{\alpha} \text{Hom}_A(X^j, Y^j) \]
\[ 1 \otimes d \]

so that the bicomplex \((\text{Hom}_A(X, Y), \delta, d, \Delta)\) may be replaced by \((C(F, M) \otimes Y, \delta', d', \Delta')\) where \(\delta' = \delta_M \otimes 1, d' = 1 \otimes d\) and \(\Delta' = \delta' + d'\) with the usual sign conventions.

Now \(C(F, M) \otimes Y\) is a left differential \(\overline{C}(A)\)-module with respect to the cup product and each of its differentials; that is, if \(a \in \overline{C}(A), b \in C(F, M) \otimes Y\) we have
\[ \delta'(ab) = \delta(a)b + (-1)^{\deg a \delta'(b)}, \quad d'(ab) = (-1)^{\deg a} ad'(b) \]
where \(\delta\) is the coboundary in \(\overline{C}(A)\).

In particular the identification of \(\text{Ext}_A(M, F)\) with \(H_*(\text{Hom}_A(X, Y); \Delta) \cong H_*(C(F, M) \otimes Y; \Delta')\) given by the row spectral sequence is an \(H^*(A)\)-module homomorphism. Suppose \(h \in C(F, M)\) is a cocycle identified with a cocycle \(\Sigma \alpha_i\) of \(C(F, M) \otimes Y\), then
\[ \eta'(h) - \sum \alpha_i = \Delta'(w) \quad \text{for some } w \in C(F, M) \otimes Y, \]
where \(\eta': C(F, M) \to C(F, M) \otimes Y^0\) corresponds to \(\eta_*\) under the isomorphism \(\alpha\). If \(z \in \overline{C}(A)\) is a cocycle, then
\[ \Delta'((-1)^{\deg z} zw) = z\Delta'(w) = \eta'(zh) - z \left( \sum \alpha_i \right) \]
so that \(zh\) is identified with \(z(\Sigma \alpha_i)\).

If \(z \in \overline{C}(A)\) is a cocycle, its action on \(C(F, M) \otimes Y\) commutes in the graded sense with the coboundaries, and it clearly preserves filtration, so the cup-product action induces the structure maps \(\lambda_r\) of Theorem 2.5. In particular the differentials are left graded \(H^*(A)\)-module homomorphisms.

To identify the product on \(E_2\), suppose \(f: A \otimes \overline{A^q} \otimes M \to F\) represents
Let $x \in \text{Ext}_{G}^{q}(M, F)$ and $g: \tilde{A}^{e} \to F$ represents $z \in H^{q}(A)$, then $z$ acts on $x$ through $H^{1}(A)$, and $zx$ is represented by $g \cdot f$, where
\[ g \cdot f(a \otimes a' \otimes a'') = (-1)^{\deg f \deg(a' \otimes a'')} g(a \otimes a') f(a'') \]
for $a' \in \tilde{A}^{e}$, $a'' \in \tilde{A}^{q}$. We have for $z \in \tilde{C}(A), w \in C(F, M) \otimes Y$, $\gamma_{g}(zw) = z \cdot \gamma_{g}(w)$ from which it follows that $\delta_{2} = \mu_{1}$. This completes (2.5).

**Corollary 2.11.** Let $A = S \otimes T$ and let $e: T \to F$ be the augmentation. Then $H^{*}(S)$ acts on the spectral sequence by the structure maps $\lambda_{r}(1 \otimes e)^{r}$. For $r = 2$ this action is induced on $\text{Ext}_{r} F$, from the Yoneda product.

We note that the maps $\lambda_{r}$ annihilate the image of $H^{*}(T) \otimes E_{r}$ in $H^{*}(A) \otimes E_{r}$ for $r \geq 2$ since $H^{*}(A)$ acts on $\text{Ext}_{S}(M, F)$ through $H^{1}(\phi)$. Thus the structure maps $\theta_{r}$ are needed to “detect” the action of $H^{*}(T)$.

For the proof of (2.7), it will be convenient to describe the Yoneda products in terms of anticommutative diagrams. Let $N$ be a $T$-module and let $Y$ be the injective resolution used in (2.5), then an element $x \in \text{Ext}_{T}^{p}(N, F)$ is represented by a cocycle $f: N \to Y^{p}$. Let $g: F \to Y^{q}$ represent $z \in H^{q}(T)$. Form the anticommutative diagram:

\[ (2.12) \]

\[ 0 \to F \to Y^{0} \to Y^{1} \to \cdots \to Y^{p} \]

\[ \downarrow g^{0} \downarrow g^{1} \downarrow g^{p} \]

\[ Y^{q} \to Y^{q+1} \to \cdots \to Y^{q+p} \]

Then $zx$ is represented by $(-1)^{\deg f \deg g} g^{p} f$. There is an analogous construction on the bicomplex $\text{Hom}_{A}(X, Y)$.

**Lemma 2.13.** Suppose $\Sigma_{i=0}^{+q} f_{i} X_{i} \to Y^{p+q-i}$ is a cocycle representing $x \in \text{Ext}_{q}^{p+q}(M, F)$. Suppose $g: F \to Y^{q}$ represents $z \in H^{q}(T)$. Then we may construct the top squares in the following anticommutative diagram since $\Sigma f_{i}$ is a cocycle, and the bottom since the $Y^{i}$ are injective:

\[ (2.14) \]

\[ 0 \leftarrow M \leftarrow X_{0} \leftarrow \cdots \leftarrow X_{q} \leftarrow X_{q+1} \leftarrow \cdots \leftarrow Y_{q+p} \]

\[ f_{0} \downarrow f_{q} \downarrow f_{q+1} \downarrow f_{p+q} \downarrow f \]

\[ Y^{p+q} \leftarrow \cdots \leftarrow Y^{p} \leftarrow Y^{p-1} \leftarrow \cdots \leftarrow Y^{0} \leftarrow F \leftarrow 0 \]

\[ \downarrow g^{p+q} \downarrow g^{p} \downarrow g^{p-1} \downarrow g^{0} \downarrow g \]

\[ Y^{p+q+r} \leftarrow \cdots \leftarrow Y^{p+r} \leftarrow Y^{p+r-1} \leftarrow \cdots \leftarrow Y^{r} \]

Then $zx$ is represented by $(-1)^{\deg z \deg x} g^{p+q-r} f_{l}$.

**Proof.** By the identification of $H_{q}(\text{Hom}_{A}(X, Y); \Delta)$ with $\text{Ext}_{A}(M, F)$
using the row spectral sequence (equivalently, by a chain homotopy argument) there exist \( f^i: X_{j-1} \rightarrow Y^{p+q-i} \) \((j = 1, \ldots, p+q)\), \( f: X_{p+q} \rightarrow F \) such that \( \Sigma f_i - \eta f = \Delta(\Sigma f_i^j) \). Then

\[
\Delta \left( (-1)^{\deg x} \sum g^{p+q-i} f_i \right) = \sum g^{p+q-i} f_i - g f,
\]

so \( \Sigma g^{p+q-i} f_i \) represents the same class as \( g f \).

Now extend (2.14) to the right as follows: Let \( 0 \leftarrow F \leftarrow Z_0 \leftarrow Z_1 \leftarrow \cdots \) be a \( T \)-projective resolution of \( F \) and construct \( A \)-maps \( h_i: X_{p+q-i} \rightarrow Z_i \) to make the top squares of (2.15) below anticommute, and by a chain homotopy argument construct \( g''_r, g'_r \) so that \( \Delta(\Sigma g'_r) = g e_0 - r g' \):

\[
\begin{array}{ccccccc}
X_{p+q} & \leftarrow & X_{p+q+1} & \leftarrow & \cdots & \leftarrow & X_{p+q-r} & \leftarrow & X_{p+q+r} \\
f & & h_0 & & h_1 & & h_{r-1} & & h_r \\
& & & & \downarrow & & \downarrow & & \downarrow \\
& & & & Z_0 & \leftarrow & Z_1 & \leftarrow & \cdots & \leftarrow & Z_{r-1} & \leftarrow & Z_r \\
g & & & & & & \downarrow & & & & & & \downarrow \\
& & & & & & & & g' & & \downarrow g' \\
Y^r & \leftarrow & Y^{r-1} & \leftarrow & \cdots & \leftarrow & Y^0 & \leftarrow & F & \leftarrow & 0
\end{array}
\]

Then \( \Delta(\Sigma g''_r h_{r-1}) = g f - g'h_r \), so \( g f \) represents the same class as \( g'h_r \). The result follows.

To show that the differentials are left \( H^*(T) \)-morphs, consider the diagram:

\[
\begin{array}{ccccccc}
H_{p+q-1}(F^p/r+1/F^p) & \xrightarrow{D^1_{r+1}} & H_{p+q}(F^p/r+1/F^p+1) & \xrightarrow{D^2_{r+1}} & H_{p+q+1}(F^p+1/r^p+1) \\
D_{r+1} & & j & & i \\
H_{p+q+1}(F^p+1/r^p+r+1) & & & & & & & & & \downarrow \end{array}
\]

The maps are those of the exact sequences of the respective triples.

By a standard construction of the spectral sequence of a filtered chain complex [9], we have \( E_{r+q}^r = \text{Ker } D^2_r \text{Im } D^1_r \). If \( x \in \text{Ker } D^2_r \) represents \( \bar{x} \in E_{r+q}^r \), then \( d_r(\bar{x}) = \bar{y} \) where \( i(y) = D_{r+1}(x) \). In turn \( x \) is represented by a map \( f_q^i: X_q \rightarrow Y^p \) with \( \delta f = 0 \), and \( D_{r+1}(x) \) is represented by \( df_q^i: X_q \rightarrow Y^{p+1} \). By exactness \( D_{r+1}(x) \in \text{Im } i \), so there exist \( f'_{q-r-1}: X_{q-r} \rightarrow Y^{p+r} \) \((j = 0, \ldots, r)\), \( f_{q-r+1}: X_{q-r+1} \rightarrow Y^{p+r} \) with \( \Delta(\Sigma f'_{q-j}) = f_{q-r+1} - d f_q^i \). Let \( z \in H^q(T) \) be represented by \( g, g^0, \ldots, g^p \) as in (2.13), then \( xz \) is represented by

\[
(-1)^\deg z \deg x g^p f_q^i.
\]
But
\[ \Delta \left( \sum g^{p+1} f_{q-r} \right) = (-1)^{\deg x} g^{p+1} f_{q-r} - d(g f_q), \]
so \( d_q(x) = (-1)^{\deg x} d_q(x) \).

This completes (1) of (2.7). Now (2) and (3) are clear, and (4) follows from (2.13) since \( \rho_\pi([\Sigma f_i]) = [f_q] \).

The following result, needed for §3, may be established by a simple diagram.

**Proposition 2.17.** Let \( \mu \) denote the \( \ast \)-coaction on \( \Ext_S(M, F) \) in the change of rings spectral sequence, and let \( Y \) denote Yoneda product and \( t \) the twist map. The following diagram commutes:

\[
\begin{array}{ccc}
1 \otimes \mu & \rightarrow & H^\ast(S) \otimes T^\ast \otimes \Ext_S(M, F) \\
H^\ast(S) \otimes \Ext_S(M, F) & \downarrow & t \otimes 1 \\
\Ext_S(M, F) & \rightarrow & T^\ast \otimes H^\ast(S) \otimes \Ext_S(M, F) \\
& \downarrow & 1 \otimes Y \\
& \mu & \rightarrow T^\ast \otimes \Ext_S(M, F)
\end{array}
\]

3. The factorization theorem. This section is devoted to the proof of Theorem 1.9. For purposes of induction we prove the more general form below.

For \( s, t \) nonnegative integers define \( E(s, t) = A_0 \cdots A_r \) if \( t > s \) and \( E(s, t) = Z_2 \) for \( t < s \). Let \( E(s, \infty) = \bigcup_{t \geq s} E(s, t) \). Let \( M = M(n) \) be as in (1.9).

**Theorem 3.1.** With the notation as above, there is an isomorphism

\[ \Ext_{E(s, \infty)}(M, Z_2) \cong Z_2[q_{n+s}, q_{n+s+1}, \ldots] \otimes \Ext_{E(s, n+s-1)}(M, Z_2) \]

of \( Z_2[q_{n+s}, q_{n+s+1}, \ldots] \)-modules, where the action on \( \Ext_{E(s, n+s-1)} \) is trivial.

The proof of (3.1) relies on the spectral sequence of §2, taking as \( \varphi \) the inclusion \( E(s, t) \subset E(s, t + 1) \). Thus

(3.2) \[ E_2^{s, q} = \Ext_{A(\beta_{t+2})}^p(Z_2, \Ext_{E(s, t)}^{0, q}(M, Z_2)). \]

The argument will show that the coaction

(3.3) \[ \Ext_{E(s, t)}(M, Z_2) \rightarrow \Lambda(\beta_{t+2}) \otimes \Ext_{E(s, t)}(M, Z_2) \]

is trivial for \( t \geq n + s - 1 \); it follows that

\[ E_2 \cong H^\ast(\Lambda(\beta_{t+1})) \otimes \Ext_{E(s, t)}(M, Z_2) \]

\[ = Z_2[q_{t+1}] \otimes \Ext_{E(s, t)}(M, Z_2) \text{ for } t \geq n + s - 1. \]

It will follow from the succeeding arguments that the spectral sequence collapses. The abelian group extensions are trivial since the Ext groups are vector...
spaces, and the action of $q_{t+1}$ corresponds on both sides, so the theorem follows.

The required triviality of (3.3) will be established in two steps.

**Lemma 3.4.** The restriction of the coaction (3.3) to $\text{Ext}_{E(s,t)}^0(M, Z_2)$ is trivial for $t \geq n + s - 1$.

**Lemma 3.5.** $\text{Ext}_{E(s,t)}^0(M, Z_2)$ is decomposable as an $H^*(E(s, t))$-module in terms of $\text{Ext}_{E(s,t)}^0(M, Z_2)$.

The triviality of (3.3) follows by (2.17).

In what follows, all coefficients in the homology or cohomology of a space are in $Z_2$.

We prove (3.4) and (3.5) for $M(n) = \tilde{H}^*(BO(n))$; the results for $\tilde{H}^*(MO(n))$ follow since the latter is a direct summand of $\tilde{H}^*(BO(n))$ as $E(s, t)$-modules.

Turning to the proof of (3.4), consider the identifications

$$H^*(MO(n)) \subset H^*(BO(n)) \subset H^*(BD(n)),$$

where $D(n)$ is the set of diagonal matrices in $O(n)$. The first inclusion is the Thom isomorphism which identifies $H^*(MO(n))$ with the ideal generated by $w_n$ in $H^*(BO(n)) = Z_2[w_1, \ldots, w_n]$. The second is induced by the inclusion $i$: $D(n) \subset O(n)$. Recall that $(B_i)^*$ is a monomorphism whose image is the algebra of symmetric polynomials in $H^*(BD(n)) = Z_2[V_1, \ldots, V_n]$, and $(B_i)^*(w_j) = \sigma_j(V_1, \ldots, V_n)$. Define the “exterior degree” of a monomial in the $V_i$ to be the number of $V_i$ which occur to an odd power. Similarly define the exterior degree of a polynomial if all of its monomial terms have a common exterior degree.

Dually, $H^*_{\text{odd}}(BO(n))$ has a basis $x_{i_1} \cdots x_{i_n}$ for $i_j \geq 0$ (where $x_0 = 1$ in the multiplication on $H^*_{\text{odd}}(BO)$) or equivalently $x_{i_1} \cdots x_{i_k}$, $1 < k < n$, $i_j \geq 1$, where $0 \neq x_j \in H_j(\mathbb{RP}^\infty)$. Define the exterior degree of $x_{i_1} \cdots x_{i_k}$ as the number of odd $i_j$ which are odd. Also note that the image of $H^*_{\text{odd}}(BO(n - 1))$ in $H^*_{\text{odd}}(BO(n))$ is the span of monomials $x_{i_1} \cdots x_{i_k}$ for $k < n - 1$.

Write $M_{\text{odd}}$ for $\text{Hom}(M, Z_2)$ and define $D_i: M_{\text{odd}} \to M_{\text{odd}}$ to be the coefficient of $\beta_i$ in the coaction over $E_{\text{odd}}$. Note $D_i$ is a derivation with respect to the multiplication on $H^*_{\text{odd}}(BO)$.

The triviality of the coaction (3.3) for $t \geq n + s - 1$ may now be restated: if $x \in M(n)_{\text{odd}}$ and $D_{s+1}(\alpha) = 0, \ldots, D_{t+1}(\alpha) = 0$, then $D_{t+2}(\alpha) = 0$. Since $M(n)_{\text{odd}} \subset M(n + 1)_{\text{odd}}$ it will be sufficient to show

**Lemma 3.7.** If $x \in M(n)_{\text{odd}}$ and $D_{s+1}(x) = 0, \ldots, D_{s+n}(x) = 0$, then $D_{s+n+1}(x) = 0$.

For use in (3.5) we state a more general version.

**Lemma 3.8.** Suppose $x \in M(n)_{\text{odd}}$ is in the span of monomials of exterior
degree \leq k - 1, and suppose \( D_{s+1}(x) = 0, \ldots, D_{s+k}(x) = 0 \). Then \( D_{s+k+1}(x) = 0 \).

Note that (3.8) implies (3.7) by taking \( k = n \), since monomials of exterior degree \( n \) in \( M(n)_* \) are in fact primitive.

Lemma 3.8 will be proved by dualizing to take advantage of the ring structure of \( H^*(BD(n)) \). We have

**Lemma 3.9.** If \( y \in \tilde{H}^*(BO(n)) \subseteq H^*(BD(n)) \) is spanned by monomials of exterior degree \( \leq k \), then \( Q_{s+k} y = \Sigma_{i=0}^{s+k-1} Q_{i} y_{i} \), where \( y_{i} \in \tilde{H}^*(BO(n)) \).

Next recall that the \( Q_{i} \) are primitive in the Steenrod algebra [11] and hence act as \((\text{mod } 2)\) derivations, so they commute with squared elements. Now \( H^*(BD(n)) \) is a free module over its subring \( \mathbb{Z}_2[\nu_1, \ldots, \nu_n] \) by multiplication, and the indecomposables of the form \( \nu_1 \cdots \nu_n \) may be identified, after renumbering, with the Thom class \( u_{p} \in H^p(MO(n)) \). We have

**Proposition 3.10.** To establish (3.9) it suffices to consider the case where \( y = u_{k} \in \tilde{H}^k(MO(k)) \) and \( y_{s}, \ldots, y_{s+k-1} \in \tilde{H}^*(MO(k)) \) have exterior degree \( k \).

To prove (3.10), let \( e_{m}: S^{1} \wedge MO(m) \to MO(m+1) \) denote the \( m \)th structure map of the Thom spectrum \( MO \), and let \( \Sigma \) be the cohomology suspension. Then the composite

\[
\tilde{H}^{q+1}(MO(m+1)) \xrightarrow{m^*} \tilde{H}^{q+1}(S^{1} \wedge MO(m)) \xrightarrow{\Sigma} \tilde{H}^{q}(MO(m))
\]

is an \( E \)-homomorphism, maps \( u_{m+1} \) to \( u_{m} \), and reduces the exterior degree of each monomial by 1. Thus (3.9) for \( k \) implies its analog for each \( p < k \).

Next note that \( Q_{i} \) reduces the exterior degree of each monomial by 1, so \( Q_{s+k} u_{k} \) has exterior degree \( k - 1 \), and the \( y_{i} \) may be assumed to have exterior degree \( k \).

Since \( y \) is symmetric and spanned by monomials of exterior degree \( \leq k \) it may be written

\[
y = \sum_{p=1}^{k} \sum_{\sigma(1) < \cdots < \sigma(p)} f_{p, \sigma} \nu_{\sigma(1)} \cdots \nu_{\sigma(p)}
\]

where \( \sigma \in S_{n} \) and \( f_{1, \sigma} = f_{1}(\nu_{\sigma(1)}^{2}, \ldots, \nu_{\sigma(n)}^{2}) \). By the first part of the proof, we may write

\[
Q_{s+k} V_{1} \cdots V_{p} = \sum_{i=1}^{s+k-1} Q_{i}^{p}(V_{1}, \ldots, V_{p});
\]

let \( y_{i}^{p} = y_{i}^{p}(V_{\sigma(1)}, \ldots, V_{\sigma(p)}) \). By the above remarks,

\[
Q_{s+k} y = \sum_{i=1}^{s+k-1} Q_{i} \left( \sum_{p=1}^{k} \sum_{\sigma(1) < \cdots < \sigma(p)} f_{p, \sigma} y_{i}^{p} \right).
\]
It remains to show the polynomials in parentheses are symmetric so that they represent elements $y_e \in \widetilde{H}^*(BO(n))$.

By the symmetry of $y$ each sum $\Sigma_{\alpha f_{pe}} V_{\alpha(1)} \cdots V_{\alpha(p)}$ for fixed $p$ is symmetric and so consists of the sum of the terms in the orbit of $f_{pe} V_1 \cdots V_p$ where $e \in S_n$ is the identity. Now $y^p_{fe} \in \widetilde{H}^*(MO(p))$ has exterior degree $p$ so may be written $y^p_{fe} = g^p(x_1^p, \ldots, x_p^p)V_1 \cdots V_p$ where $g^p$ is symmetric in its $p$ indeterminates. Thus if $\sigma$ fixes $f_{pe} V_1 \cdots V_p$ it also fixes $f_{pe} y^p_{fe}$, so (writing $C$ for stabilizer) $C_{f_{pe}} V_1 \cdots V_p \subseteq C_{f_{pe}} y^p_{fe}$ hence

$$|S_n : C_{f_{pe}} y^p_{fe}| \leq |S_n : C_{f_{pe}} V_1 \cdots V_p| = \binom{n}{p}.$$

On the other hand $f_{pe} y^p_{fe}$ has at least the \binom{n}{p} elements in its orbit corresponding to the combinations of indeterminates. Thus $\Sigma_{\alpha f_{pe}} y^p_{fe}$ is symmetric, completing (3.10).

To prove the condition of (3.10) it will be convenient to use the notation of §1, where $\widetilde{H}^*_e(MO(k))$ is the span of monomials $b^E = b_1 b_2^2 \cdots$ of degree $\leq k$. If $E = (e_1, e_2, \ldots)$ let $|E| = \Sigma e_i$ and grade $b^E = \Sigma e_i$, that is $b^E$ is assigned its "stable grade." Note that $1 \in H_0(MO)$ is the unique element dual to each Thom class $u_k$. If we form a minimal resolution

$$0 \to H_*(MO(k)) \to E(s, s + k)_* \otimes V \to \cdots$$

then the existence of the relation of (3.10) is equivalent to the statement $\beta_{s+k+1} \otimes 1 \notin \text{Im } e$, where $e(1) = 1 \otimes 1_V$. In particular it is not necessary to exhibit the elements $y_e$.

Denote by $H \subseteq V$ the span of images of monomials in $b_2, b_4, \ldots, b_{2r}, \ldots$ and identify elements of $H$ by their polynomial pre-images. Let $h : V \to H$ be the projection induced through $e$ from the one in $\widetilde{H}_e(MO(k))$ which preserves monomials in the $b_{2r}$ and annihilates all others. Let $e' = (1 \otimes h)e$, then by the coaction formula (1.8) it will suffice to show $\beta_{s+k+1} \otimes 1 \notin \text{Im } e'|_W$, where $W \subset \widetilde{H}_e(MO(k))$ (with stable grading) is the span of monomials containing only one factor $b_i$ of odd degree.

**Proposition 3.11.** Let $y = b_2 m^-1 b^E \in W_{2s+k+1-1}$, then $e'(y)$ is either zero or the sum of precisely two "monomials," i.e. terms of the form $\beta_i \otimes z$ where $z$ is a monomial of $H$.

For the proof, suppose $e'(y) \neq 0$, then $b^E \in H$. Let $M$ denote the term in $e'(y)$ containing the $\beta_i$ of highest grade; it has the form $\beta_i \otimes b^E$ or $\beta_i \otimes b_{2r}b^E$.

If $M = \beta_i \otimes b^E$, then $D_i(b_2 m^-1) = 1$, so $2m - 1 = 2^l - 1$. Since $D_i(b_2 m^-1) \in H, 2m - 2^q$ must be zero or a positive power of 2; i.e. $q = i$ or $q = i - 1$. Thus for $i \geq s + 2, s + 1 \leq i - 1, i \leq s + k + 1$ and (3.11) holds.
Suppose $i = s + 1$, then $|E| = (2^{s+k+1} - 1) - (2^{s+1} - 1)$ so $\alpha(|E|) \geq k$, where $\alpha$ denotes the number of 1's in the dyadic expansion. Thus $\deg b^E \geq k$, $\deg y \geq k + 1$, a contradiction since $y \in \widehat{H}_*(MO(k))$.

If $M = \beta_i \otimes b_{2r} b^E$, then $r \neq i$, otherwise the term $\beta_{i+1} \otimes b^E$ would also appear, since $i \neq s + k + 1$ by the grade of $y$. Now $r \leq s + k + 1$ by the grade of $y$, so $e'(y) = \beta_i \otimes b_{2r} b^E + \beta_r \otimes b_{2i} b^E$ and (3.11) holds, provided $r \geq s + 1$.

Assume $r \leq s$. Note $i \leq s + k$ by the grade of $y$. Then

$$2^{s+k+1} - 1 > |E| = (2^{s+k+1} - 1) - (2m - 1)$$

$$= 2^{s+k+1} - 2i - 2r > 2^{s+k+1} - 2^{s+k} - 2r$$

$$= 2^{s+k} - 2r > 2^{s+k} - 2^s,$$

so $\alpha(|E|) = \alpha(2^{s+k} - 2^r) = k$. Thus $\deg b^E \geq k$, so $\deg y \geq k + 1$, a contradiction.

**Corollary 3.12.** Every element of $\text{Im } e'|_w$ is the sum of an even number of "monomials." In particular $\beta_{s+k+1} \otimes 1 \notin \text{Im } e'$.

This completes (3.11), (3.9) and (3.4).

The proof of (3.5) is by induction on $t$. For $t < s$ we have $E(s, t) = Z_2$ so the result holds trivially.

Assume (3.5) for $t$ and all $s \geq 0$; this requires only finitely many steps. Apply the spectral sequence (3.2). By (2.5) and (2.12) it has a module structure over $H^*(E(s, t)) = Z_2[q_s, \ldots, q_t]$ which is induced from the Yoneda product on $\text{Ext}_{E(s, t)}(M, Z_2)$ in $E_2$. It also has the structure $\{\theta_r\}$ over $Z_2[q_{t+1}]$. By naturality these structures commute and induce a structure over $H^*(E(s, t + 1)) = Z_2[q_s, \ldots, q_{t+1}]$ which restricts to the given ones. We show $E_2$ is decomposable over $H^*(E(s, t + 1))$.

Let $T = \wedge (\beta_{t+2})$. By the cobar construction, $\text{Ext}_{T*}(Z_2, N)$ is decomposable over $H^*(T)$ in terms of $\text{Ext}_{T*}^0$ for any $T*$-comodule $N$. Thus $E_2$ is decomposable over $H^*(T)$ in terms of $\text{Ext}_{T*}^0(Z_2, \text{Ext}_{E(s, t)}(M, Z_2)) \subset \text{Ext}_{E(s, t)}(M, Z_2)$.

By the inductive hypothesis, any $x \in \text{Ext}_{E(s, t)}^0$ is decomposable over $H^*(E(s, t))$ in terms of $x_i \in \text{Ext}_{E(s, t)}^0$. The $x_i$, however, may not lie in

$$\text{Ext}_{T*}^0(Z_2, \text{Ext}_{E(s, t)}^0(M, Z_2)) = E_2^{0,0}.$$

**Lemma 3.13.** Let $z \in \text{Ext}_{E(s, t)}^0(M, Z_2)$. Then $z$ may be written $z = z_1 + z_2$ where $z_1 \in \text{Ext}_{E(s, t)}^0$, $z_2 \in \text{Ext}_{E(s, t+1)}^0 \subset \text{Ext}_{E(s, t)}^0$ and $q_s z_1 = 0, \ldots, q_t z_1 = 0$.

Using (3.13) we can replace the $x_i$ by $x_i' \in \text{Ext}_{E(s, t+1)}^0$ since the terms cor-
responding to \( z_1 \) are annihilated by \( H^*(E(s, t)) \). Note that the \( x_i' \) are permanent cycles. The differentials commute (in the graded sense) with the composite action of \( H^*(E(s, t + 1)) \) on \( E_s \) so the spectral sequence collapses. The abelian group extensions are trivial and the spectral sequence preserves products in the sense of (2.5) and (2.7), so \( \operatorname{Ext}_{E(s, t+1)} \) is decomposable over \( H^*(E(s, t + 1)) \) by induction over the filtration.

To prove (3.13), write \( z \in M(n)^* = \widetilde{H}_*(BO(n)) \) in terms of the basis \( \{x_{i_1} \ldots x_{i_p} \}, 1 \leq p \leq n, i_p \geq 1 \), and let \( z_2 \) denote the sum of terms of exterior degree \( \leq t - s \). By (3.8) (with \( k = t - s + 1 \)) we have \( D_{t+2}(z_2) = 0 \), so \( z_2 \in \operatorname{Ext}_{E(s,t+1)}^{0} \). Let \( z_1 = z - z_2 \).

**Lemma 3.14.** Suppose \( x \in \operatorname{Ext}_{E(s,t)}^{0}(M(n), Z_2) \) is spanned by monomials \( x_{i_1} \ldots x_{i_p} \) of exterior degree \( \geq t - s + 1 \). Then \( q_s x = 0, \ldots, q_t x = 0 \) in \( \operatorname{Ext}_{E(s,t)}(M(n), Z_2) \).

The proof will be by induction on \( n \).

**Remark 3.15.** Let \( s < i < t \) and let \( N \) be an \( E(s, t) \)-module. Examination of a resolution shows that \( q_s x = 0 \) in \( \operatorname{Ext}_{E(s,t)}(N, Z_2) \) if and only if there exists \( y \in N^* \) with \( D_{i+1}(y) = x, D_j(y) = 0 \) for \( j = s + 1, \ldots, t + 1, j \neq i + 1 \).

As in (3.8) the proof will exploit ring structure, in this case the multiplicity of \( H^*(BO(n)) = Z_2[w_1, \ldots, w_n] \). Let \( U = Z_2[w_2^n] \). The \( Q_i \) commute with the multiplicative action of \( U \), so \( H^*(BO(n)) \) and \( M(n) = \widetilde{H}_*(BO(n)) \) are \( E(s, t) \otimes U \)-modules. The following description of \( \operatorname{Ext}_{U}^{0}(M(n), Z_2) \) will be used in a spectral sequence argument.

**Proposition 3.16.** There is an isomorphism
\[
f: \operatorname{Ext}_{U}^{0}(M(n), Z_2) \rightarrow M(n-1)^* \oplus \widetilde{H}_*(MO(n-1))
\]
of \( E(s, t)^* \)-comodules.

For the proof, consider \( E(s, t)^* \)-maps \( p_1: M(n)^* \rightarrow M(n-1)^*, p_2: M(n)^* \rightarrow \widetilde{H}_*(MO(n)) \) defined by \( p_1(x_{i_1} \ldots x_{i_k}) = x_{i_1} \ldots x_{i_k} \) for \( k < n, p_1(x_{i_1} \ldots x_{i_n} = 0; p_2(x_{i_1} \ldots x_{i_n}) = x_{i_1} \ldots x_{i_n} \), \( p_2(x_{i_1} \ldots x_{i_n}) = 0 \) for \( k < n \) (\. Throughout). Let \( \Delta_n: M(n)^* \rightarrow M(n)^* \) be the map dual to multiplication by \( w_n \).

Then \( \operatorname{Ker} \Delta_n = M(n-1)^* \) and \( \operatorname{Ker} \Delta_n^2 = \operatorname{Ext}_{U}^{0}(M(n), Z_2) \). Thus if \( x \in \operatorname{Ext}_{U}^{0}, \Delta_n^2 p_2(x) = 0 \), so \( \Delta_n p_2(x) \in M(n-1)^* \). Since \( \Delta_n(x_{i_1} \ldots x_{i_n}) = x_{i_1} \ldots x_{i_n} \) (\. Through), \( p_2(x) \) has the form \( x_1 h + u \) where \( h \in \widetilde{H}_*(MO(n-1)) \), \( u \in \operatorname{Ker} \Delta_n \). By definition of \( p_2 \), \( u \) is a sum of terms \( x_{i_1} \ldots x_{i_n}, i_j \geq 1 \), so \( u \in \widetilde{H}_*(MO(n)) \). But also \( u \in M(n-1)^* \), so \( u = 0, p_2(x) = x_1 h \). Multiplication by \( x_1 \) is an \( E(s, t)^* \)-monomorphism, so define \( f(x) = (p_1(x), h) \). The inverse map is given by \( (y, z) \mapsto y + x_1 z \).

For the proof of (3.14), note that \( D_i \) reduces the exterior degree of each
term $x_{k_1} \cdots x_{k_t}$ by 1 so we may assume $x$ of (3.14) has homogeneous exterior degree $q \geq t - s + 1$. For $n = 1$ the highest exterior degree that can occur is 1, so $t = 0$. For $s > 0$, $E(s, 0) = Z_2$ so the result is trivial. For $s = 0$, if $D_1(x) = 0$, $x \in \widetilde{H}_s(BO(1)) = \widetilde{H}_s(RP^\infty)$ has exterior degree 1, so $x = x_{2r-1}$ for some $r$, and $x = D_1(x_{2r})$. Thus $q_x x = 0$ by (3.15).

Assume (3.14) for $M(n - 1)$ and suppose $x \in Ext^0_{E(s, t)}(M(n), Z_2)$ has exterior degree $q \geq t - s + 1$ and $q_x \not= 0$ for some $i$, $s < i < t$. Let $m$ be the largest integer $r$ such that $A^{2r}(q_x x)$ $\not= 0$, then

$$u = \Delta_{2r}^m(q_x x) \in Ext^0_{U^*}(Z_2, Ext^1_{E(s, t)}(M(n), Z_2)),$$

which is $E_{2, \cdot}^{0, 1}$ in the change of rings spectral sequence converging to $Ext_{E(s, t)}(M(n), Z_2)$. No differentials can hit $u$, and the only possible differential on $E_{2, \cdot}^{0, 1}$ is

$$d_2: E_{2, \cdot}^{0, 1} \to E_{2, \cdot}^{2, 0} = Ext^2_{U^*}(Tor_0^E(s, t)(Z_2, M), Z_2)$$

and $E_{2, \cdot}^{2, 0} = 0$ since $U$ is a PID. Let $0 \not= u' \in Ext^1_{E(s, t)}(M(n), Z_2)$ represent the class of $u$ in $E_{\infty}$.

CLAIM. $Ext^0_{E(s, t)}(M(n), Z_2)$ is decomposable by Yoneda products over $H^*(E(s, t))$ (acting by the homomorphism induced from the projection $E(s, t) \otimes U \to E(s, t)$) in terms of $Ext^0_{E(s, t)}(M, U)$. We have the spectral sequence

$$Ext^a_{E(s, t)}(Z_2, Ext^b_{U^*}(M(n), Z_2)) \Rightarrow Ext_{E(s, t)}^{a + b}(M(n), Z_2)$$

with the product structure $\{\theta_x\}$. Since $U$ acts freely on $M(n)$, $E_{2, \cdot}^{a, b} = 0$ for $b \not= 0$ and the spectral sequence collapses with trivial extensions. By the inductive hypothesis, (3.14) holds for $\widetilde{H}^*(BO(n - 1))$ and thus for $\widetilde{H}^*(MO(n - 1))$, so by (3.16) $E_2 = E_{\infty}$ is decomposable, and the claim follows.

Thus $u' = \Sigma_{r=0}^t z_r$, $z_r \in Ext^0_{E(s, t)}(M(n), U)$, and by the coaction formula (1.8) we may assume each $z_r$ has exterior degree $q$. Let $j: E(s, t) \to E(s, t) \otimes U$ be the inclusion, then $0 \not= u = j^*(u') = \Sigma_{r=0}^t q_r j^*(z_r)$, so for some $z_w, q_w j^*(z_w) \not= 0$ in $Ext^1_{E(s, t)}(M, Z_2)$, where $w = j^*(z_w)$ has exterior degree $q$ and $\Delta_n^2(w) = 0$. This will be shown to be impossible if $q \geq t - s + 1$.

Let $y = p_1(w)$, $z = p_2(w)$ where $p_1, p_2$ are as in (3.16). Then $y \in M(n - 1)$ has exterior degree $q$, so by induction $q_x y = 0$, ..., $q_{t-1} y = 0$ in $Ext^1_{E(s, t)}(M(n - 1), Z_2)$. By naturality the images of the $q_j y$ are zero in $Ext^1_{E(s, t)}(M(n), Z_2)$.

As in (3.16), $z = x_1 h$, where $h \in \widetilde{H}_s(MO(n - 1))$ has exterior degree $q - 1$ $\geq (t - s + 1) - 1 = t - (s + 1) + 1$. By induction on $n$, we have $q_{s+1} h = 0$, ..., $q_t h = 0$ in
where the inclusion is induced by the projection of $M(n - 1)$ onto the direct summand $\tilde{H}^*(MO(n - 1))$. Thus by (3.15) there exist $z_{s+1}, \ldots, z_t \in \tilde{H}_*(MO(n - 1))$ with $D_{i+1}(z_i) = 0$ for $j = s + 2, \ldots, t + 1, i \neq i + 1$. Let
\[ z'_{x+1} = x_{2s+1}h \in M(n)_0, \]
\[ z'_k = x_1z_k + x_{2s+1}D_{s+1}(z_k) \in M(n)_0, \quad k = s + 2, \ldots, t + 1. \]
Recall that the $D_i$ are derivations satisfying $D_iD_j = D_jD_i$, $D_i^2 = 0$. By an easy calculation,
\[ D_{i+1}(z'_i) = z', \quad i = s, \ldots, t, \]
\[ D_j(z'_j) = 0, \quad j = s + 1, \ldots, t + 1, j \neq i + 1. \]
Thus $q_iw = q_iy + q_iz = 0$ in $\text{Ext}_{E(s,t)}(M(n), Z_2)$, contradicting $q_iw \neq 0$. This completes (3.14), (3.5) and (3.1).

**Note.** The last calculation relies on the fact that $D_i(x_{2s+1}) = 0$ for $i > s + 1$, which requires the inductive step to treat the lowest dimensional generator, $q_s$. This is the reason for the generality of (3.1).

We note in closing that Theorem 3.1 is the best possible result of its type, i.e. $n + s - 1$ cannot be replaced by $n + s - 2$. Let
\[ x = D_{s+1}D_{s+2} \cdots D_{s+n-1}(x_{2s+1} \cdots x_{2s+n}). \]
Then $x \in \text{Ext}_{E(s,n+s-2)}^0(\tilde{H}^*(BO(n); Z_2), Z_2)$, but $D_{s+n}(x) = x_1^t \neq 0$, so $x \notin \text{Ext}_{E(s,n+s-1)}^0$.

**REFERENCES**


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