ON THE TRIVIAL EXTENSION 
OF EQUIVALENCE RELATIONS 
ON ANALYTIC SPACES 

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ABSTRACT. In this paper, we shall consider the problem: let $X$ be a 
(reduced) analytic space and $A$ a nowhere dense analytic set in $X$. And let $R$ 
be a proper equivalence relation on $A$ such that the quotient space $A/R$ is an 
analytic space, and $\tilde{R}$ the trivial extension of $R$ to $X$. Then, is $X/\tilde{R}$ an 
analytic space? To this, we have three sufficient conditions. Moreover, using 
this result we shall extend Satz 1 of H. Kerner [8]. 

1. Introduction. Let $(X, \mathcal{O})$ be an analytic space and $R$ an equivalence 
relation on $X$. Then the local ringed quotient space $(X/R, \mathcal{O}/R)$ is defined and 
the problem, whether $(X/R, \mathcal{O}/R)$ is an analytic space, is studied by H. Cartan, 
H. Holmann, B. Kaup and others. 

In this paper, we shall consider the problem: let $X$ be a (reduced) analytic 
space and $A$ a nowhere dense analytic set in $X$. And let $R$ be a proper equiva-
lence relation on $A$ such that the quotient space $A/R$ is an analytic space, and 
$\tilde{R}$ the trivial extension of $R$ to $X$. Then, is $X/\tilde{R}$ an analytic space? To this, we 
have 

Theorem. $X/\tilde{R}$ is an analytic space, if one of the following three state-
ments is satisfied: 

(1) $R$ is finite. 
(2) $A$ is contractible in $X$ and the canonical mapping $j: A/R \to X/\tilde{R}$ is 
    quasi-finite. 
(3) $A$ is contractible and retractable in $X$. 

Next, using Theorem, (3), we shall extend Satz 1 of H. Kerner [8]: let $X_k$ 
be a connected complex manifold, $A_k$ a contractible and retractable analytic set 
in $X_k$ and $R_k$ a proper equivalence relation on $A_k$ such that $A_k/R_k$ is an analy-
tic space and $\dim_a R_k(a) > 0$ for any $a \in A_k$ ($k = 1, 2$). Then, we have the 
following diagrams of analytic spaces: 

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of equivalence relation.
Here $p_k : A_k \to A_k/R_k$, $\tilde{p}_k : X_k \to X_k/\tilde{R}_k$ are natural projections, $i_k : A_k \to X_k$ is the injection and $j_k : A_k/R_k \to X_k/\tilde{R}_k$ is the canonical mapping. Let $r_k : X_k \to A_k$ be the holomorphic retraction. Then, we have

**Theorem.** Suppose that f.m.d. $r_2 \geq \dim A_1 + 2$. If $X_1/\tilde{R}_1$ and $X_2/\tilde{R}_2$ are analytically equivalent, then the above two diagrams are analytically equivalent.

H. Kerner has treated the case that $r_k : X_k \to A_k$ is a weakly negative vector bundle and $R_k(a) = A_k$ for any $a \in A_k$.

2. Trivial extension of equivalence relations. Let $L$ be the category of local ringed spaces [6]: objects in $L$ are local ringed spaces and morphisms in $L$ are morphisms of local ringed spaces.

**Definition 1.** A commutative diagram of morphisms in $L$:

\[
\begin{array}{ccc}
Z & \xrightarrow{b} & P \\
\downarrow{s} & & \downarrow{a} \\
X & \xrightarrow{r} & Y
\end{array}
\]

is called a pushout (and $P$ is called the pushout for $r$ and $s$), if for any object $A$ and morphisms $u : Y \to A$, $v : Z \to A$ in $L$ with $v \circ s = u \circ r$, there exists the unique morphism $p : P \to A$ such that $p \circ b = v$ and $p \circ a = u$.

Let $(X, X^0)$ be a (reduced) analytic space and $R$ an equivalence relation on $X$. Then there exists the local ringed quotient space $(X/R, X^0/R)$ and the natural projection $p : X \to X/R$ is a morphism of local ringed spaces, where $X/R$ is the quotient topological space of $X$ by $R$ and $X^0/R$, the structure sheaf on $X/R$, is defined as follows: for any open set $U \subseteq X/R$, $(X^0/R)(U) := \{ f : U \to \mathbb{C}, f \circ p \in \Gamma(p^{-1}(U), X^0) \}$.

**Definition 2.** An equivalence relation $R$ on $X$ is called proper if for any compact set $K \subseteq X$, the $R$-saturated set $R(K)$ (i.e. the union of all equivalence classes meeting $K$) is also compact.

This condition is equivalent that $X/R$ is locally compact and the natural projection $p : X \to X/R$ is proper.

**Definition 3.** Let $A$ be a subset of $X$ and $R$ an equivalence relation on $A$. The trivial extension $\tilde{R}$ of $R$ to $X$, an equivalence relation on $X$, is defined by
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\[ \widetilde{R}(x) := \begin{cases} R(x), & \text{for } x \in A, \\ \{x\}, & \text{for } x \notin A, \end{cases} \]

where \( R(x), x \in A, \) denotes the equivalence class by \( R \) containing \( x. \)

Let \( (A, A^0) \) be a nowhere dense analytic set of \( (X, x^0) \) and \( R \) an equivalence relation on \( A. \) Then we have the local ringed quotient spaces \( (A/R, A^0/R), (X/R, x^0/R). \) Let \( p: A \to A/R, \) \( \widetilde{p}: X \to X/R \) be natural projections and \( i: A \to X \) the injection. Then there exists the canonical mapping \( j: A/R \to X/R \) \( (\widetilde{p} \circ i = j \circ p) \) and \( j \) is a morphism in \( L. \)

**Lemma 1.** \( X/R \) is the pushout for \( i \) and \( p \) in \( L. \)

**Proof.** For any object \( Z \) and morphisms \( u: A/R \to Z, \) \( v: X \to Z \) in \( L \) with \( v \circ i = u \circ p, \) we define the mapping as follows: for any \( \widetilde{x} \in X/R, \) we put \( \varphi(\widetilde{x}) := v(x) \) \( (x \in \widetilde{p}^{-1}(\widetilde{x})). \) Then this is well defined. In fact \( \widetilde{p}(x) = \widetilde{p}(x') \) \( (x, x' \in X) \) implies \( v(x) = v(x'). \) Now \( \varphi \) is continuous with \( v = \varphi \circ \widetilde{p}, \) and \( u = \varphi \circ j \) since \( u \circ p = \varphi \circ j \circ p \) and \( p \) is surjective.

For any \( f \in \mathcal{O}_X(X/R), \) there exists \( f \in (\mathcal{O}_X(X/R))_x \) with \( u_x^*(f) = \widetilde{f} \circ \widetilde{p}. \) And we put \( \varphi_x^*(f) := \widetilde{f}. \) Then \( \varphi^* \) holds commutativity and is unique. Hence \( X/R \) is the pushout in \( L \) for \( i \) and \( p. \) Q.E.D.

**Definition 4.** An analytic set \( A \subset X \) is called contractible in \( X \) if \( A \) is nowhere discrete, compact and if there exist an analytic space \( Y \) and a surjective proper holomorphic mapping \( \psi: X \to Y \) such that \( \psi(A) =: y_A \in Y \) and the restriction \( \psi(A - A) \to (Y - \{y_A\}) \) is biholomorphic.

**Definition 5.** An analytic set \( A \subset X \) is called retractable if there exists a holomorphic retraction of \( X \) to \( A \) (i.e. a surjective holomorphic mapping \( r: X \to A \) with \( r|A = \text{id}_A). \)

**Definition 6.** A morphism \( f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \) in \( L \) is called quasi-finite if for any \( x \in X, \mathcal{O}_{X,x}/(f_x^*(M_{f(x)})) \) is a finite dimensional vector space over \( \mathbb{C}, \) where \( M_{f(x)} \) is the maximal ideal of \( \mathcal{O}_{Y,f(x)}. \)

Let \( (A, \mathcal{O}_A) \) be an analytic set in \( (X, \mathcal{O}_X) \) and \( R \) a proper equivalence relation on \( A \) such that \( A/R \) is an analytic space. Using the results by B. Kaup [6] and the method of H. Kerner [8], we shall show the sufficient conditions under which \( X/R \) is an analytic space.
Theorem 1. \( X/\tilde{R} \) is an analytic space, if one of the following statements is satisfied:

1. \( R \) is finite (i.e. every equivalence class of \( A \) by \( R \) is a finite set).
2. \( A \) is contractible in \( X \) and the canonical mapping \( j: A/R \rightarrow X/R \) is quasi-finite.
3. \( A \) is contractible and retractable in \( X \).

Proof. (1) From Lemma 1, \( X/R \) is the pushout for the injection \( i: A \rightarrow X \) and the natural projection \( p: A \rightarrow A/R \). Hence, by B. Kaup [6, Satz 1.8], \( X/R \) is an analytic space.

(2) If \( A \) is contractible in \( X \), \( A \) is exceptional in \( A! \) in the sense of B. Kaup [6]. Hence, by Lemma 1 and B. Kaup [6, Aussage 1.11], \( X/R \) is an analytic space.

(3) \( \tilde{R} \) is proper since, for any compact set \( K \subset X \), \( \tilde{R}(K) = K \cup R(K) \) is also compact in \( X \).

By the assumption, there exist an analytic space \( Y \), a surjective proper holomorphic mapping \( \psi: X \rightarrow Y \) and a holomorphic retraction \( r: X \rightarrow A \). Then we have a surjective morphism \( \widetilde{r}: X/\tilde{R} \rightarrow A/R \) with \( \widetilde{r} \circ \widetilde{p} = p \circ r \). In fact, for any \( \tilde{x} \in X/\tilde{R} \), we put

\[ \widetilde{r}(\tilde{x}) := p \circ r(x) \quad (x \in \widetilde{p}^{-1}(\tilde{x})). \]

Then \( \widetilde{r}: X/\tilde{R} \rightarrow A/R \) is well defined.

Now, we claim that \((X/\tilde{R}, \chi 0/\tilde{R})\) is locally morph-separable (i.e. for any \( \tilde{x} \in X/\tilde{R} \), there exists an open neighborhood \( U \subset X/\tilde{R} \) such that \( \Gamma(U, \chi 0/\tilde{R}) \) separates points of \( U \)). Then \((X/\tilde{R}, \chi 0/\tilde{R})\) is an analytic space by H. Cartan [1, Main Theorem].

Let \( \tilde{x} \) be a point of \( X/\tilde{R} \). We may assume that \( \tilde{x} \in \phi(A/R) \). Then there exists an open neighborhood \( V \) of \( x := \tilde{r}(\tilde{x}) \) such that \( \Gamma(V, \chi 0/\tilde{R}) \) separates points of \( V \) and also there exists an open neighborhood \( O \subset Y \) of \( \psi(A) \) such that \( \Gamma(O, \chi 0) \) separates points of \( O \). Since \( W := \psi^{-1}(O) \subset X \) is an open neighborhood of \( A \), we have \( \tilde{p}^{-1}(\tilde{p}(W)) = W \), hence \( \tilde{p}(W) \) is an open neighborhood of \( \tilde{x} \). Thus, so is \( U := \tilde{p}(W) \cap \tilde{r}^{-1}(V) \subset X/\tilde{R} \). We can show that \( U \) satisfies the above statement. Let \( \tilde{y}, \tilde{z} \) be any distinct points in \( U \). Then there exist two distinct points \( y, z \) in \( X \) such that \( \tilde{p}(y) = \tilde{y}, \tilde{p}(z) = \tilde{z} \). If \( \psi(y) \neq \psi(z) \), we have \( f \in \Gamma(O, \chi 0) \) with \( f \circ \psi(y) \neq f \circ \psi(z) \). And \( f \circ \psi \in \Gamma(W, \chi 0) \) is constant on \( A \). Put
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Then \( F(\tilde{w}) = \begin{cases} f \circ \psi \circ (p|W - A)^{-1}(\tilde{w}), & \text{for } \tilde{w} \in \tilde{p}(W - A), \\ f(y_A), & \text{for } \tilde{w} \in \tilde{p}(A). \end{cases} \)

Therefore \( F(\tilde{y}) \neq F(\tilde{z}) \). If \( \psi(y) = \psi(z) \), then \( y, z \in A \) and \( p(y) \neq p(z) \). Hence we have \( g \in \Gamma(V, A \circ \rho|/R) \) with \( g \circ p(y) \neq g \circ p(z) \). Put in \( U, G := g \circ \tilde{r} \); then \( G \in \Gamma(U, \chi 0|/R) \) with \( G(\tilde{y}) \neq G(\tilde{z}) \), since \( r: X \rightarrow A \) is a holomorphic retraction. Thus \((X/R, \chi 0|/R)\) is locally morph-separable. Q.E.D.

Remark 1. We can easily find the examples such that \( X/R \) is not an analytic space, in the case that \( R \) is not finite in (1), or \( A \) is not contractible in (2), (3) respectively.

Corollary 1. Let \((X, \chi 0), (A, \chi 0)\) and \( R \) be as in Theorem 1, (1) or (3). Then \( A/R \) is embedded in \( X/R \). In particular, in the case of (3), \( A/R \) is contractible and retractable in \( X/R \).

Proof. The canonical mapping \( j: A/R \rightarrow j(A/R) \) is a holomorphic homeomorphism since \( j \) is proper. We assert that for any \( \tilde{a} \in A/R \), \( j^\star_a: (\chi 0|/R)|\tilde{a}) \rightarrow (A \circ \rho|/R)|\tilde{a}) \) is surjective.

(1) For any \( f \in (A \circ \rho|/R)|\tilde{a}) \), we have \( p^a_\star(f) \in A \circ \rho_a \) \( (a \in p^{-1}(\tilde{a})) \). Then there exists \( g \in \chi 0_a \) with \( j^\star_a(g) = p^a_\star(f) \). Since \( p \) is finite proper, we have \( G \in (\chi 0|/R)|\tilde{a}) \) with \( p^\star_a(G) = g \). Then it follows that \( j^\star_a(G) = f \).

(3) Since \( \tilde{r} \circ j = \text{id}_{A/R} \), surjectiveness of \( j^\star_a \) is evident and in particular \( \tilde{r} \) is a holomorphic retraction. Therefore \( A/R \) is retractable and contractible in \( X/R \). Q.E.D.

3. Applications. We now consider the following problem: Let \((X, \chi 0)\) and \((M, \chi 0)\) be analytic spaces, \( A \) a nowhere dense analytic set in \( X \) and \( h: A \rightarrow M \) a surjective proper holomorphic mapping. Then, does an analytic space \( Y \) exist with the following property (P)?

(P) There exist a surjective proper holomorphic mapping \( \tilde{h}: X \rightarrow Y \) and an injection \( j: M \rightarrow Y \) such that the restriction \( \tilde{h}|A = j \circ h \) and \( \tilde{h}|(X - A) \rightarrow (Y - \tilde{A}) \) \( (\tilde{A} := \tilde{h}(A)) \) is biholomorphic.

Definition 7. We say that a reduced analytic space \( X \) is maximal if, for any open set \( U \subset X \) and a nowhere dense analytic set \( S \subset U \), every continuous function on \( U \) which is holomorphic on \( U - S \) is actually holomorphic on \( U \).

Remark 2. If an analytic space \((X, \chi 0)\) is maximal, \( \chi 0 \) is the maximal reduced complex structure on \( X \).

Let \( X, A \) and \( R \) be as in Theorem 1 (1) or (2) or (3). If \( X \) is maximal, so is \( X/R \).
Let $R_h$ be the equivalence relation on $A$ defined by $h: A \to M$ (i.e. for any $u, v \in A$, $u R_h v$ means $h(u) = h(v)$). Then $R_h$ is proper and, if $M$ is maximal we can show that $A/R_h, M$ are isomorphic. Thus from Theorem 1 and Corollary 1, we have

**Theorem 2.** If (1) or (3) in Theorem 1 is satisfied for $X, A, R_h$ and $M$ is maximal, there exists an analytic space $Y$ with the property (P).

**Corollary 2.** Let $X, A, M$ and $R_h$ be as in Theorem 2. Suppose that $X$ is maximal. Then any maximal analytic space $Y'$ with the property (P) is biholomorphically equivalent to $X/R_h$.

**Proof.** Let $\widetilde{p}' : X \to Y'$ be a surjective proper holomorphic mapping and $j': A/R_h \to Y'$ an injection such that the restriction $\tilde{p}'|A = j' \circ p$ and $\widetilde{p}'|(X - A) \to (Y' - \tilde{p}'(A))$ is biholomorphic. Then, from Lemma 1, we have the unique holomorphic mapping $\psi: X/R_h \to Y'$ with $\tilde{p}' = \psi \circ \tilde{p}, j' = \psi \circ j$.

![Diagram](License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use)
$M_2$, $\text{pr}(x_1, x_2) = (x_1, x_2^0)$ for any $(x_1, x_2) \in M_1 \times M_2$, $x_2^0$ is a fixed point).

If $\varphi$ is a holomorphic mapping of an analytic space $X$ into an analytic space $Y$, we put f.m.d. $\varphi := \min_{x \in X} \text{dim}_x \varphi^{-1}(\varphi(x))$. Then using Lemma 2 and the assumption $\dim R_k(a) > 0$, we can prove the next lemma in almost like manner as in [8].

**Lemma 3.** Suppose that f.m.d. $r_k \geq 2$. Then $\tilde{A}_k := \tilde{p}_k(A_k)$ is the set of all singular points of $X_k/R_k$.

**Theorem 3.** Suppose that f.m.d. $r_2 \geq \dim A_1 + 2$. If $X_1/R_1$ and $X_2/R_2$ are analytically equivalent, the following diagrams $(k = 1, 2)$ are analytically equivalent.

**Proof.** We first show that

\[ \text{f.m.d. } r_1 \geq \dim A_2 + 2 \]

in some open neighborhood of $A_1$.

By assumption, any point of $A_k$ $(k = 1, 2)$ has an open neighborhood with the property stated in Lemma 2. Let $O_k$ be the union of all such open neighborhoods. Then

\[ \dim O_2 - \dim A_2 \geq \text{f.m.d. } r_2 \geq \dim A_1 + 2. \]

Since $\dim O_1 = \dim O_2$, it follows that

\[ \text{f.m.d. } (r_1 | O_1) = \dim O_1 - \dim A_1 \geq \dim A_2 + 2. \]

Hence, by Lemma 3, $\tilde{A}_k := \tilde{p}_k(A_k)$ $(k = 1, 2)$ is the set of all singular points of $X_k/R_k$. Let $\psi : X_1/R_1 \rightarrow X_2/R_2$ be the biholomorphic mapping. Then $\psi(\tilde{A}_1) = \tilde{A}_2$, and there exists an open neighborhood $U_k \subset X_k/R_k$ of $\tilde{A}_k$ with $U_k := \tilde{p}_k^{-1}(U_k) \subset O_k$.

We now assert that there exists a holomorphic mapping $\psi^- : U_1^- \rightarrow U_2^-$ such that $\psi \circ \tilde{p}_1 = \tilde{p}_2 \circ \psi^-$. We put

\[ \psi^- := \psi | (U_1^- \setminus \tilde{A}_1) \rightarrow (U_2^- \setminus \tilde{A}_2), \]

\[ \tilde{p}_k^- := \tilde{p}_k | (U_k^- \setminus A_k) \rightarrow (U_k^- \setminus \tilde{A}_k) \quad (k = 1, 2). \]
These mappings are biholomorphic. And we put, on $U_1^* - A_1$, $\tau := r_2 \circ (\widetilde{P}_2)^{-1} \circ \psi^* \circ \widetilde{P}_1^*$. Then $\tau: (U_1^* - A_1) \to A_2$ is also holomorphic. Since f.m.d. $\tau \geq \dim A_1 + 2$ on $U_1^* - A_1$, we have the holomorphic mapping $\widetilde{\tau}: U_1^* \to A_2$ such that $\widetilde{\tau}(U_1^* - A_1) = \tau$ [9, Satz 2]. Define the mapping $\psi^*: U_1^* \to U_2^*$ as follows:

$$
\psi^*(x) = \begin{cases} 
(\widetilde{P}_2)^{-1} \circ \psi^* \circ \widetilde{P}_1^*(x), & \text{for } x \in U_1^* - A_1, \\
\iota_2 \circ \tau(x), & \text{for } x \in A_1,
\end{cases}
$$

where $\iota_2: A_2 \to U_2^*$ is the injection. Remark that $\widetilde{\tau} = r_2 \circ \psi^*$ on $U_1^*$.

Then we can show that $\psi^*: U_1^* \to U_2^*$ is continuous. To show this, it suffices to say that $\psi^*$ is continuous at any $a \in A_1$, and hence, for any sequence $\{a_n\} \subset U_1^* - A_1$ which converges to $a$, $\{\psi^*(a_n)\}$ converges and $\lim_{n \to \infty} \psi^*(a_n) = \psi^*(a)$.

$\{\psi^*(a_n)\} = \{\widetilde{P}_2^{-1}(\psi \circ \widetilde{P}_1(a_n))\} \subset U_2^* - A_2$ has cluster points in $U_2^*$ since $\widetilde{P}_2$ is proper, and they must be contained in $A_2$. Furthermore, the cluster points are unique and coincide with $\psi^*(a)$. In fact, if $\alpha$ is a cluster point of $\{\psi^*(a_n)\}$, we have a subsequence $\{a'_n\}$ of $\{a_n\}$ with $\lim_{n \to \infty} \widetilde{P}_2^{-1} \circ \psi \circ \widetilde{P}_1(a'_n) = \alpha$. Then

$$
\alpha = r_2(\alpha) = r_2 \left( \lim_{n \to \infty} \widetilde{P}_2^{-1} \circ \psi \circ \widetilde{P}_1(a'_n) \right) = \lim_{n \to \infty} r_2 \circ \widetilde{P}_2^{-1} \circ \psi \circ \widetilde{P}_1(a'_n) = \lim_{n \to \infty} \tau(a'_n) = \lim_{n \to \infty} \widetilde{\tau}(a'_n) = \widetilde{\tau}(a) = \psi^*(a).
$$

Hence $\lim_{n \to \infty} \psi^*(a_n) = \psi^*(a)$. Therefore $\psi^*$ is continuous. Since $U_k^*$ is a complex manifold ($k = 1, 2$) and $\psi^*| (U_1^* - A_1)$ is holomorphic on $U_1^* - A_1$, $\psi^*$ is holomorphic on $U_1^*$. Further, $\psi \circ \widetilde{P}_1 = \widetilde{P}_2 \circ \psi^*$ on $U_1^*$.

To complete the proof of the theorem, it suffices to show that $\psi^*$ is bijective and its inverse is holomorphic. By ($\ast$), we also have the holomorphic mapping $(\psi^{-1})^*: U_2^* \to U_1^*$ such that $\psi^{-1} \circ \widetilde{P}_2 = \widetilde{P}_1 \circ (\psi^{-1})^*$ on $U_2^*$. Then it follows that
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Hence \( \psi^\wedge : U_1^\wedge \to U_2^\wedge \) is biholomorphic and, in particular, \( \psi^\wedge (A_1) = A_2 \). Therefore \( A_k, X_k \) and \( A_k/R_k \) \((k = 1, 2)\) are analytically equivalent respectively, and the two diagrams are analytically equivalent. Q.E.D.

REMARK 3. H. Kerner [8] has treated the case that \( r_k : X_k \to A_k \) \((k = 1, 2)\) is a weakly negative vector bundle and \( R_k(a) = A_k \) for any \( a \in A_k \).

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