TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 219, 1976

EXTREME POINTS OF UNIVALENT FUNCTIONS WITH TWO FIXED POINTS

BY

HERB SILVERMAN

ABSTRACT. Univalent functions of the form $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$, where $a_n \geqslant 0$, are considered. We examine the subclasses for which $f(z_0) = z_0$ or $f'(z_0) = 1$, z_0 real. The extreme points of these classes that are starlike of order α are determined.

1. **Introduction.** In [7], Schild examined the class of polynomials of the form $f(z) = z - \Sigma_{n=2}^{N} a_n z^n$, where $a_n \geqslant 0$ and $f(z)$ is univalent in the disk $|z| < 1$. In [5], Piłat studied the class of univalent polynomials of the form $f(z) = a_1 z - \Sigma_{n=2}^{N} a_n z^n$, where $a_n \geqslant 0$ and $f(z_0) = z_0 > 0$. This paper deals with functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

where either

$$a_n \geqslant 0, \quad f(z_0) = z_0 \qquad (-1 < z_0 < 1; z_0 \neq 0) \tag{2}$$

or

$$a_n \geqslant 0, \quad f'(z_0) = 1 \qquad (-1 < z_0 < 1). \tag{3}$$

As special cases, some of our results reduce to those of Schild or Piłat. A function $f(z)$ is said to be starlike of order α, $0 \leqslant \alpha < 1$, if

$$\operatorname{Re}\{zf'(z)/f(z)\} > \alpha \qquad (|z| < 1)$$

and is said to be convex of order α if

$$\operatorname{Re}\{1 + zf''(z)/f'(z)\} > \alpha \qquad (|z| < 1).$$

Given α and z_0 fixed, let $S_0^*(\alpha, z_0)$ be the subclass of functions starlike of order α that satisfy (2), and $S_1^*(\alpha, z_0)$ be the subclass of functions starlike of order α that satisfy (3). Also denote by $K_0(\alpha, z_0)$ and $K_1(\alpha, z_0)$ the subclasses

Presented to the Society, January 24, 1976; received by the editors May 7, 1975.
AMS (MOS) subject classifications (1970). Primary 30A36; Secondary 30A32.
Key words and phrases. Univalent, starlike, convex, extreme point.

of functions convex of order α that satisfy, respectively, (2) and (3).

In §2, we determine necessary and sufficient conditions for functions to be in these classes. In §3, we find the extreme points for each of these classes. In §4, we give a necessary and sufficient condition for a subset B of the real interval (0, 1) to have the property that $\bigcup_{z_\gamma \in B} S_0^*(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_0(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} S_1(\alpha, z_\gamma)$, $\bigcup_{z_\gamma \in B} K_1(\alpha, z_\gamma)$ each forms a convex family. The extreme points of each of these classes is then determined. Many of the results in this paper reduce to those in [8] in the special case $z_0 = 0$.

I would like to thank Professor Eligiusz Złotkiewicz for some interesting discussions concerning this topic.

2. **The main subclasses.** In [4] it is shown that a necessary and sufficient condition for functions of the form $f(z) = z - \Sigma_{n=2}^{\infty} a_n z^n$ $(a_n \geqslant 0)$ to be starlike of order α is that $\Sigma_{n=2}^{\infty}(n-\alpha)a_n \leqslant 1-\alpha$.

The proof of the comparable result for functions of the form (1) is essentially the same. See also [8]. We include the proof for the sake of completeness.

THEOREM 1. *A function $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ $(a_n \geqslant 0)$ is starlike of order α if and only if $\Sigma_{n=2}^{\infty}(n-\alpha)a_n \leqslant a_1(1-\alpha)$.*

PROOF. Assume that

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leqslant a_1(1-\alpha). \tag{4}$$

It suffices to show that the values for $zf'(z)/f(z)$ lie in a circle centered at $w = 1$ whose radius is $1-\alpha$. We have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\Sigma_{n=2}^{\infty}(n-1)a_n z^n}{a_1 z - \Sigma_{n=2}^{\infty} a_n z^n}\right| \leqslant \frac{\Sigma_{n=2}^{\infty}(n-1)a_n}{a_1 - \Sigma_{n=2}^{\infty} a_n}.$$

This last expression is bounded above by $1-\alpha$ if (4) is satisfied.

Conversely, assume that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{a_1 z - \Sigma_{n=2}^{\infty} n a_n z^n}{a_1 z - \Sigma_{n=2}^{\infty} a_n z^n}\right\} > \alpha. \tag{5}$$

Choose values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (5) and letting $z \longrightarrow 1$ through real values, we obtain (4).

COROLLARY. *A function $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ $(a_n \geqslant 0)$ is convex of order α if and only if $\Sigma_{n=2}^{\infty} n(n-\alpha)a_n \leqslant a_1(1-\alpha)$.*

PROOF. It is well known that $f(z)$ is convex of order α if and only if $zf'(z)$ is starlike of order α. Since $zf'(z) = a_1 z - \Sigma_{n=2}^{\infty} n a_n z^n$, we may replace a_n with na_n in the theorem.

By choosing particular values for $a_1 \neq 0$, we obtain some interesting subclasses.

THEOREM 2. *Suppose $a_n \geqslant 0$ for every n. Then $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ is in $S_0^*(\alpha, z_0)$ if and only if $\Sigma_{n=2}^{\infty} a_n [(n-\alpha)/(1-\alpha) - z_0^{n-1}] \leqslant 1$.*

PROOF. Since $f(z_0)/z_0 = 1 = a_1 - \Sigma_{n=2}^{\infty} a_n z_0^{n-1}$, the result follows upon substituting

$$a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} \tag{6}$$

into the statement of Theorem 1.

REMARK. With a slight modification of the proof of Theorem 3 in [8], we can show under the conditions of Theorem 2 that $f(z)$ is univalent if and only if $\Sigma_{n=2}^{\infty} a_n (n - z_0^{n-1}) \leqslant 1$. Hence all such functions are univalent if and only if they are starlike.

COROLLARY 1. *If $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ is in $S_0^*(\alpha, z_0)$, then*

$$a_n \leqslant (1-\alpha)/((n-\alpha) - (1-\alpha) z_0^{n-1}) \qquad (n = 2, 3, \ldots),$$

with equality for $f(z) = ((n-\alpha)z - (1-\alpha)z^n)/((n-\alpha) - (1-\alpha)z_0^{n-1})$.

COROLLARY 2. *Suppose $a_n \geqslant 0$ for every n. Then $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ is in $K_0(\alpha, z_0)$ if and only if $\Sigma_{n=2}^{\infty} a_n [n(n-\alpha)/(1-\alpha) - z_0^{n-1}] \leqslant 1$.*

PROOF. This follows from the corollary to Theorem 1 as Theorem 2 followed from Theorem 1.

We now examine relationships between the classes $S_0^*(\alpha, z_0)$ and $K_0(\alpha, z_0)$.

THEOREM 3. *If $f(z) \in K_0(\alpha, z_0)$, then $f(z) \in S_0^*(2/(3-\alpha), z_0)$.*

PROOF. If $f(z) \in K_0(\alpha, z_0)$, then $\Sigma_{n=2}^{\infty} a_n [n(n-\alpha)/(1-\alpha) - z_0^{n-1}] \leqslant 1$. In view of Theorem 2, we wish to find the largest β for which $\Sigma_{n=2}^{\infty} a_n [(n-\beta)/(1-\beta) - z_0^{n-1}] \leqslant 1$. It suffices to find the range of values of β for which $(n-\beta)/(1-\beta) \leqslant n(n-\alpha)/(1-\alpha)$ for every n. Solving, we find $\beta \leqslant 2/(3-\alpha)$. This result is sharp, with extremal function

$$f(z) = \frac{2(2-\alpha)z - (1-\alpha)z^2}{2(2-\alpha) - (1-\alpha)z_0}.$$

REMARK. The conclusion of Theorem 3 is true even if α is negative. Thus if $f(z) \in K_0(\alpha, z_0)$ for *any* real $\alpha \leqslant 1$, then $f(z)$ is also starlike and univalent. Note that a function in the class $K(\alpha)$ of normalized univalent convex functions need not be starlike for $\alpha < 0$ and need be univalent (and close-to-convex) only if $\alpha \geqslant -\frac{1}{2}$. This latter statement follows from a criterion of Kaplan [3], because

for $z = re^{i\theta}$, $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{1 + \frac{zf''}{f'}\right\} d\theta \geqslant \int_{\theta_1}^{\theta_2} -\frac{1}{2}\, d\theta \geqslant -\pi \qquad (\alpha \geqslant -\tfrac{1}{2}).$$

Observe that $f(z) = \int_0^z (1-t)^{2(\alpha-1)}\, dt$ is in $K(\alpha)$ but, by a lemma of Royster [6], is not univalent for $\alpha < -\frac{1}{2}$.

THEOREM 4. *If $f(z) \in S_0^*(\alpha, z_0)$, then $f(z)$ is convex in the disk*

$$|z| < r = r(\alpha) = \inf_n \left[\frac{n-\alpha}{n^2(1-\alpha)}\right]^{1/(n-1)} \qquad (n = 2, 3, \ldots).$$

The result is sharp, with the extremal function being of the form

$$f_n(z) = \frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

PROOF. It suffices to show that $|zf''(z)/f'(z)| \leqslant 1$ for $|z| \leqslant r(\alpha)$. We have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leqslant \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus $|zf''(z)/f'(z)| \leqslant 1$ if

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leqslant 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} - \sum_{n=2}^{\infty} na_n|z|^{n-1},$$

or

$$\sum_{n=2}^{\infty} a_n(n^2|z|^{n-1} - z_0^{n-1}) \leqslant 1. \tag{7}$$

According to Theorem 2, $\sum_{n=2}^{\infty} a_n((n-\alpha)/(1-\alpha) - z_0^{n-1}) \leqslant 1$. Hence (7) will be true if

$$n^2|z|^{n-1} - z_0^{n-1} \leqslant (n-\alpha)/(1-\alpha) - z_0^{n-1} \qquad (n = 2, 3, \ldots). \tag{8}$$

Solving (8) for $|z|$ we obtain

$$|z| \leqslant [(n-\alpha)/n^2(1-\alpha)]^{1/(n-1)} \qquad (n = 2, 3, \ldots),$$

and the result follows.

REMARK. The conclusions in Theorems 3 and 4 are independent of the fixed point z_0. In [8], these results were proved for the special case $z_0 = 0$.

We turn next to the classes $S_1^*(\alpha, z_0)$ and $K_1(\alpha, z_0)$. If $f(z)$ has the form (1) with condition (3) satisfied, then a_1 may be expressed as

$$a_1 = 1 + \sum_{n=2}^{\infty} na_n z_0^{n-1}. \tag{9}$$

Upon substituting (9) into the statement of Theorem 1 and its corollary, we obtain respectively

THEOREM 5. *Suppose* $a_n \geqslant 0$ *for every* n. *Then* $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ *is in* $S_1^*(\alpha, z_0)$ *if and only if* $\Sigma_{n=2}^{\infty} a_n[(n-\alpha)/(1-\alpha) - nz_0^{n-1}] \leqslant 1$.

COROLLARY. *Suppose* $a_n \geqslant 0$ *for every* n. *Then* $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ *is in* $K_1(\alpha, z_0)$ *if and only if* $\Sigma_{n=2}^{\infty} a_n [n(n-\alpha)/(1-\alpha) - nz_0^{n-1}] \leqslant 1$.

3. **Extreme points.** There have been numerous papers recently dealing with the extreme points for the closed convex hull of several compact families of univalent functions. See for example [1] and [2]. The importance of determining the extreme points of a compact family F lies in the fact that the maximum or minimum value of any continuous linear functional defined over the set of analytic functions occurs at one of the extreme points of the closed convex hull of F. Unlike the classes most often considered, $S_0^*(\alpha, z_0)$ is a convex family. For if $f_1(z)$ and $f_2(z)$ are in $S_0^*(\alpha, z_0)$, then $f(z) = \lambda f_1(z) + (1-\lambda) f_2(z)$ can be shown to satisfy the coefficient inequality of Theorem 2 and $f(z_0) = \lambda f_1(z_0) + (1-\lambda) f_2(z_0) = z_0$. We will now show that the extreme points of $S_0^*(\alpha, z_0)$ are

$$z \quad \text{and} \quad z\left[\frac{(n-\alpha)-(1-\alpha)z^{n-1}}{(n-\alpha)-(1-\alpha)z_0^{n-1}}\right] \qquad (n = 2, 3, 4, \ldots).$$

THEOREM 6. *Set*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = \frac{(n-\alpha)z-(1-\alpha)z^n}{(n-\alpha)-(1-\alpha)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

Then $f(z) \in S_0^*(\alpha, z_0)$ *if and only if it can be expressed in the form* $f(z) = \Sigma_{n=1}^{\infty} \lambda_n f_n(z)$, *where* $\lambda_n \geqslant 0$ *and* $\Sigma_{n=1}^{\infty} \lambda_n = 1$.

PROOF. Suppose $f(z) = \Sigma_{n=1}^{\infty} \lambda_n f_n(z)$, where $\lambda_n \geqslant 0$ and $\Sigma_{n=1}^{\infty} \lambda_n = 1$. Then

$$f(z) = \left[\lambda_1 + \sum_{n=2}^{\infty} \frac{\lambda_n(n-\alpha)}{(n-\alpha)-(1-\alpha)z_0^{n-1}}\right] z - \sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha)z^n}{(n-\alpha)-(1-\alpha)z_0^{n-1}}.$$

Note that $f(z_0) = [\Sigma_{n=1}^{\infty} \lambda_n] z_0 = z_0$. We also have

$$\sum_{n=2}^{\infty} \frac{\lambda_n(1-\alpha)}{(n-\alpha)-(1-\alpha)z_0^{n-1}} \left[\frac{(n-\alpha)-(1-\alpha)z_0^{n-1}}{1-\alpha}\right] = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leqslant 1.$$

Hence by Theorem 2, $f(z) \in S_0^*(\alpha, z_0)$.

Conversely, suppose $f(z) \in S_0^*(\alpha, z_0)$. Since

$$a_n \leqslant (1-\alpha)/((n-\alpha)-(1-\alpha)z_0^{n-1}) \qquad (n = 2, 3, \ldots),$$

we may set

$$\lambda_n = [((n-\alpha)-(1-\alpha)z_0^{n-1})/(1-\alpha)]a_n \qquad (n = 2, 3, \ldots),$$

and $\lambda_1 = 1 - \Sigma_{n=2}^{\infty}\lambda_n$. Then $f(z) = \Sigma_{n=1}^{\infty}\lambda_n f_n(z)$.

In like manner, the coefficient bounds on $K_0(\alpha, z_0)$, $S_1^*(\alpha, z_0)$, and $K_1(\alpha, z_0)$ enable us to prove

THEOREM 7. *Set*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = \frac{n(n-\alpha)z-(1-\alpha)z^n}{n(n-\alpha)-(1-\alpha)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

Then $f(z) \in K_0(\alpha, z_0)$ if and only if it can be expressed in the form $f(z) = \Sigma_{n=1}^{\infty}\lambda_n f_n(z)$, where $\lambda_n \geqslant 0$ and $\Sigma_{n=1}^{\infty}\lambda_n = 1$.

THEOREM 8. *Set*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = \frac{(n-\alpha)z-(1-\alpha)z^n}{(n-\alpha)-(1-\alpha)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

Then $f(z) \in S_1^*(\alpha, z_0)$ if and only if it can be expressed in the form $f(z) = \Sigma_{n=1}^{\infty}\lambda_n f_n(z)$, where $\lambda_n \geqslant 0$ and $\Sigma_{n=1}^{\infty}\lambda_n = 1$.

THEOREM 9. *Set*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = \frac{(n-\alpha)z-((1-\alpha)/n)z^n}{(n-\alpha)-(1-\alpha)z_0^{n-1}} \qquad (n = 2, 3, \ldots).$$

Then $f(z) \in K_1(\alpha, z_0)$ if and only if it can be expressed in the form $f(z) = \Sigma_{n=1}^{\infty}\lambda_n f_n(z)$, where $\lambda_n \geqslant 0$ and $\Sigma_{n=1}^{\infty}\lambda_n = 1$.

REMARK. Since the operator L defined by $Lf(z) = \int_0^z f(t)/t\,dt$ is an isomorphism from $S_0^*(\alpha, z_0)$ to $K_1(\alpha, z_0)$, Theorem 9 is also seen to be a consequence of Theorem 6.

4. Convex families. Suppose B is a nonempty subset of the real interval $(0, 1)$. We define $S_0^*(\alpha, B)$ by

$$S_0^*(\alpha, B) = \bigcup_{z_\gamma \in B} S_0^*(\alpha, z_\gamma).$$

As previously mentioned, if B consists of a single element then $S_0^*(\alpha, B)$ is a convex family. It is of interest to investigate this class for other subsets B. We shall make use of the following

LEMMA. *If $f(z) \in S_0^*(\alpha, z_0) \cap S_0^*(\alpha, z_1)$, where z_0 and z_1 are distinct positive numbers, then $f(z) = z$.*

PROOF. Setting $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ $(a_n \geqslant 0)$, we must have

$$a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} = 1 + \sum_{n=2}^{\infty} a_n z_1^{n-1}.$$

But this means that $a_n \equiv 0$ for $n \geqslant 2$, and the proof is complete.

REMARK. If the condition that the fixed points be positive is relaxed, the conclusion need not follow. For instance if $f(z) \in S_0^*(\alpha, z_0)$ and $f(z)$ is odd, then $f(z) \in S_0^*(\alpha, -z_0)$.

THEOREM 10. *If B is contained in the interval* $(0, 1)$ *and* $0 \leqslant \alpha < 1$, *then $S_0^*(\alpha, B)$ is a convex family if and only if B is connected.*

PROOF. We first assume B is connected. Suppose $z_0, z_1 \in B$ with $z_0 \leqslant z_1$. If $f(z) = a_1 z - \Sigma_{n=2}^{\infty} a_n z^n$ is in $S_0^*(\alpha, z_0)$, $g(z) = b_1 z - \Sigma_{n=2}^{\infty} b_n z^n$ is in $S_0^*(\alpha, z_1)$, and $0 \leqslant \lambda \leqslant 1$, we will show that there exists a z_2 $(z_0 \leqslant z_2 \leqslant z_1)$ such that $h(z) = \lambda f(z) + (1 - \lambda) g(z)$ is in $S_0^*(\alpha, z_2)$. Set

$$t(z) = \frac{h(z)}{z} = \lambda a_1 + (1 - \lambda) b_1 - \lambda \sum_{n=2}^{\infty} a_n z^{n-1} - (1 - \lambda) \sum_{n=2}^{\infty} b_n z^{n-1}$$

(10)

$$= 1 + \lambda \sum_{n=2}^{\infty} a_n (z_0^{n-1} - z^{n-1}) + (1 - \lambda) \sum_{n=2}^{\infty} b_n (z_1^{n-1} - z^{n-1}),$$

and note that $t(z)$ is real when z is real with $t(z_0) \geqslant 1$ and $t(z_1) \leqslant 1$. Hence $t(z_2) = 1$ for some z_2, $z_0 \leqslant z_2 \leqslant z_1$. Since z_1, z_2, and λ are arbitrary, the family $S_0^*(\alpha, B)$ is convex.

Conversely if B is not connected we can choose $z_0, z_1 \in B$, $z_2 \notin B$, with $z_0 < z_2 < z_1$. Assume that $f(z)$ and $g(z)$ are not both the identity function. Then, using the notation of (10) except that we fix $z = z_2$ and allow λ to vary,

$$t(\lambda) = t(z_2, \lambda) = 1 + \lambda \sum_{n=2}^{\infty} a_n (z_0^{n-1} - z_2^{n-1}) + (1 - \lambda) \sum_{n=2}^{\infty} b_n (z_1^{n-1} - z_2^{n-1}).$$

Since $t(z_2, 0) > 1$ and $t(z_2, 1) < 1$, there must exist a λ_0, $0 < \lambda_0 < 1$, for which $t(z_2, \lambda_0) = 1$. Thus $h(z) \in S_0^*(\alpha, z_2)$ for $\lambda = \lambda_0$. Since $z_2 \notin B$, an application of the lemma shows that $h(z) \notin S_0^*(\alpha, B)$. Hence the family $S_0^*(\alpha, B)$ is not convex.

THEOREM 11. *If* $[z_0, z_1] \subset (0, 1)$, *then the extreme points of* $S_0^*(\alpha, [z_0, z_1])$ *are* z,

$$f_n(z) = ((n - \alpha)z - (1 - \alpha)z^n)/((n - \alpha) - (1 - \alpha)z_0^{n-1}) \qquad (n = 2, 3, \ldots),$$

and

$$g_n(z) = ((n - \alpha)z - (1 - \alpha)z^n)/((n - \alpha) - (1 - \alpha)z_1^{n-1}) \qquad (n = 2, 3, \ldots).$$

PROOF. A function $h(z) \in S_0^*(\alpha, z_2)$, $z_0 \leqslant z_2 \leqslant z_1$, can only be an extreme point of $S_0^*(\alpha, [z_0, z_1])$ if it is an extreme point of $S_0^*(\alpha, z_2)$. So we may assume that $h(z)$ is an extreme point of $S_0^*(\alpha, z_2)$ and show that $h(z)$ is an extreme point of $S_0^*(\alpha, [z_0, z_1])$ if and only if $z_2 = z_0$ or $z_2 = z_1$. To show that

$$h_n(z) = ((n-\alpha)z - (1-\alpha)z^n)/((n-\alpha) - (1-\alpha)z_2^{n-1}) \qquad (n = 2, 3, \ldots)$$

can be expressed as a convex linear combination of $f_n(z)$ and $g_n(z)$ when $z_0 < z_2 < z_1$, we set

$$h_n(\lambda, z) = \lambda\left(\frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_0^{n-1}}\right) + (1-\lambda)\left(\frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_1^{n-1}}\right).$$

For z real and positive, we have $h_n(0, z) < h_n(z) < h_n(1, z)$. Hence there must exist a λ_0, $0 < \lambda_0 < 1$, for which $h_n(\lambda_0, z) = h_n(z)$. In fact, for λ_0 such that

$$\frac{\lambda_0}{(n-\alpha) - (1-\alpha)z_0^{n-1}} + \frac{1-\lambda_0}{(n-\alpha) - (1-\alpha)z_1^{n-1}} = \frac{1}{(n-\alpha) - (1-\alpha)z_2^{n-1}}$$

we have the coefficients for $h_n(\lambda_0, z)$ agreeing with the coefficients of $h_n(z)$. That is, $h_n(\lambda_0, z) = h_n(z)$ throughout the unit disk when

$$\lambda_0 = \frac{(n-\alpha) - (1-\alpha)z_0^{n-1}}{(n-\alpha) - (1-\alpha)z_2^{n-1}}\left(\frac{z_1^{n-1} - z_2^{n-1}}{z_1^{n-1} - z_0^{n-1}}\right).$$

Thus $h_n(z)$ cannot be an extreme point.

On the other hand, we can show for z positive and $0 \leqslant \lambda \leqslant 1$ that

$$f_n(z) < \lambda\left(\frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_3^{n-1}}\right) + (1-\lambda)\left(\frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_4^{n-1}}\right)$$

$$(z_0 < z_3 \leqslant z_1, z_0 < z_4 \leqslant z_1),$$

and

$$g_n(z) > \lambda\left(\frac{(n-\alpha)z - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_5^{n-1}}\right) + (1-\lambda)\left(\frac{(n-\alpha) - (1-\alpha)z^n}{(n-\alpha) - (1-\alpha)z_6^{n-1}}\right)$$

$$(z_0 \leqslant z_5 < z_1, z_0 \leqslant z_6 < z_1).$$

This completes the proof.

Using the method of proof in Theorem 11, we obtain the following

COROLLARY. *If* $0 < z_0 < z_1 < 1$, *the closed convex hull of* $S_0^*(\alpha, \{z_0, z_1\})$ *is* $S_0^*(\alpha, [z_0, z_1])$.

We may similarly expand on the classes $K_0(\alpha, z_0)$, $S_1^*(\alpha, z_0)$, and $K_1(\alpha, z_0)$. Our results may be summed up as

THEOREM 12. *Let* $T(\alpha, [z_0, z_1])$, $0 < z_0 < z_1 < 1$, *denote any of the classes* $S_0^*(\alpha, [z_0, z_1])$, $K_0(\alpha, [z_0, z_1])$, $S_1^*(\alpha, [z_0, z_1])$, *or* $K_1(\alpha, [z_0, z_1])$. *Then the extreme points of* $T(\alpha, [z_0, z_1])$ *are* $\{$*extreme points of* $T(\alpha, z_0)\} \cup \{$*extreme points of* $T(\alpha, z_1)\}$, *and the closed convex hull of* $T(\alpha, \{z_0, z_1\})$ *is* $T(\alpha, [z_0, z_1])$.

REFERENCES

1. L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken, *Convex hulls and extreme points of families of starlike and convex mappings*, Trans. Amer. Math. Soc. **185** (1973), 413–428 (1974). MR **49** #3102.

2. L. Brickman, T. H. MacGregor and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91–107. MR **43** #494.

3. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185 (1953). MR **14**, 966.

4. E. P. Merkes, M. S. Robertson and W. T. Scott, *On products of starlike functions*, Proc. Amer. Math. Soc. **13** (1962), 960–964. MR **26** #316.

5. Barbara Piłat, *Sur une classe de fonctions normées univalentes dans le cercle unité*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **17** (1963), 69–73 (1965). MR **33** #2803.

6. W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. **12** (1965), 385–387. MR **32** #1342.

7. A. Schild, *On a class of functions schlicht in the unit circle*, Proc. Amer. Math. Soc. **5** (1974), 115–120. MR **15**, 694.

8. Herb Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711